Title: Statistical Mechanics - Lecture 11

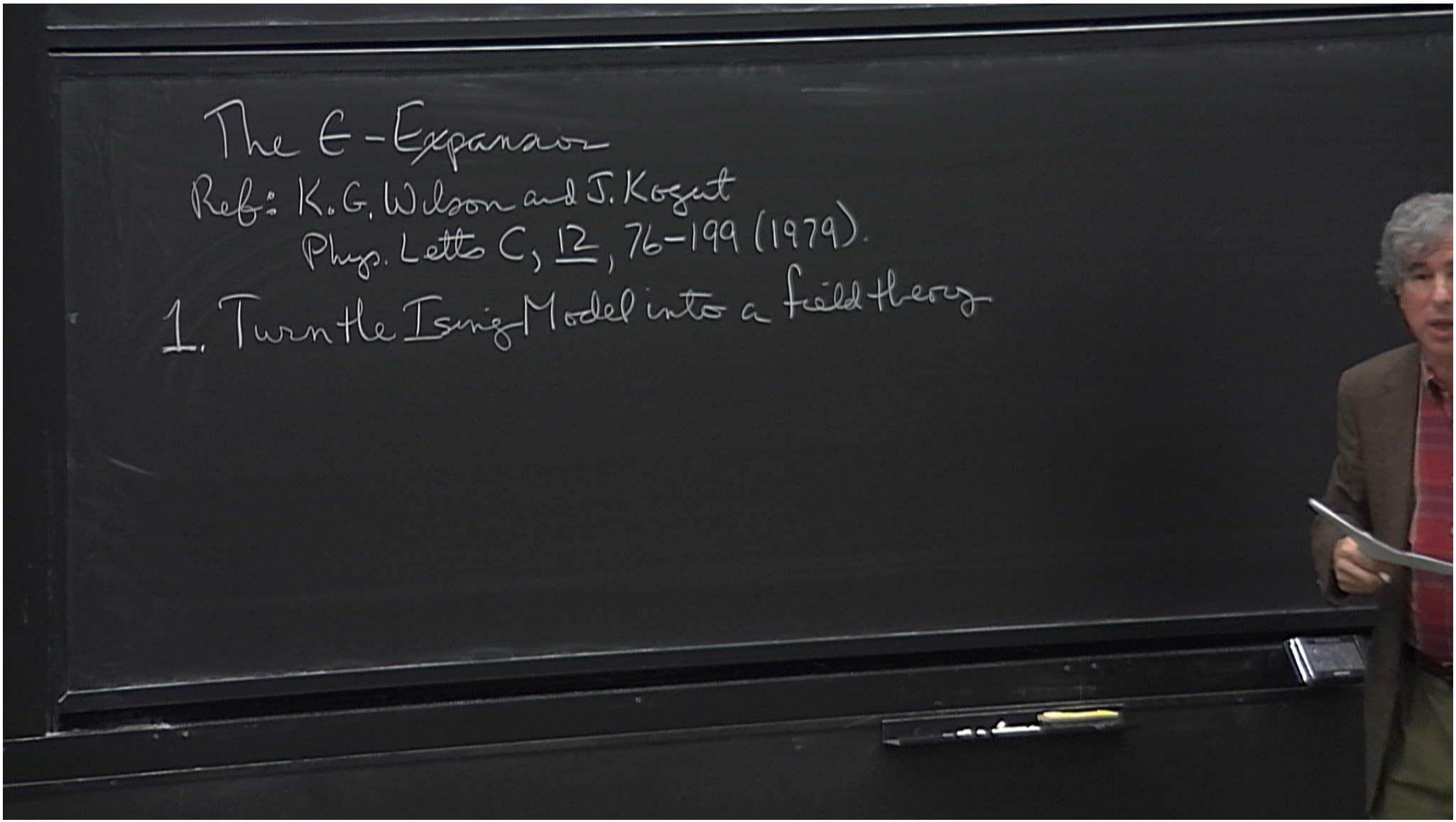
Date: Nov 26, 2012 10:30 AM

URL: http://pirsa.org/12110033

Abstract:

The ϵ-Expansion

Ref: K. G. Wilson and J. Kogut
Phys. Letts C, 12, 76–199 (1979).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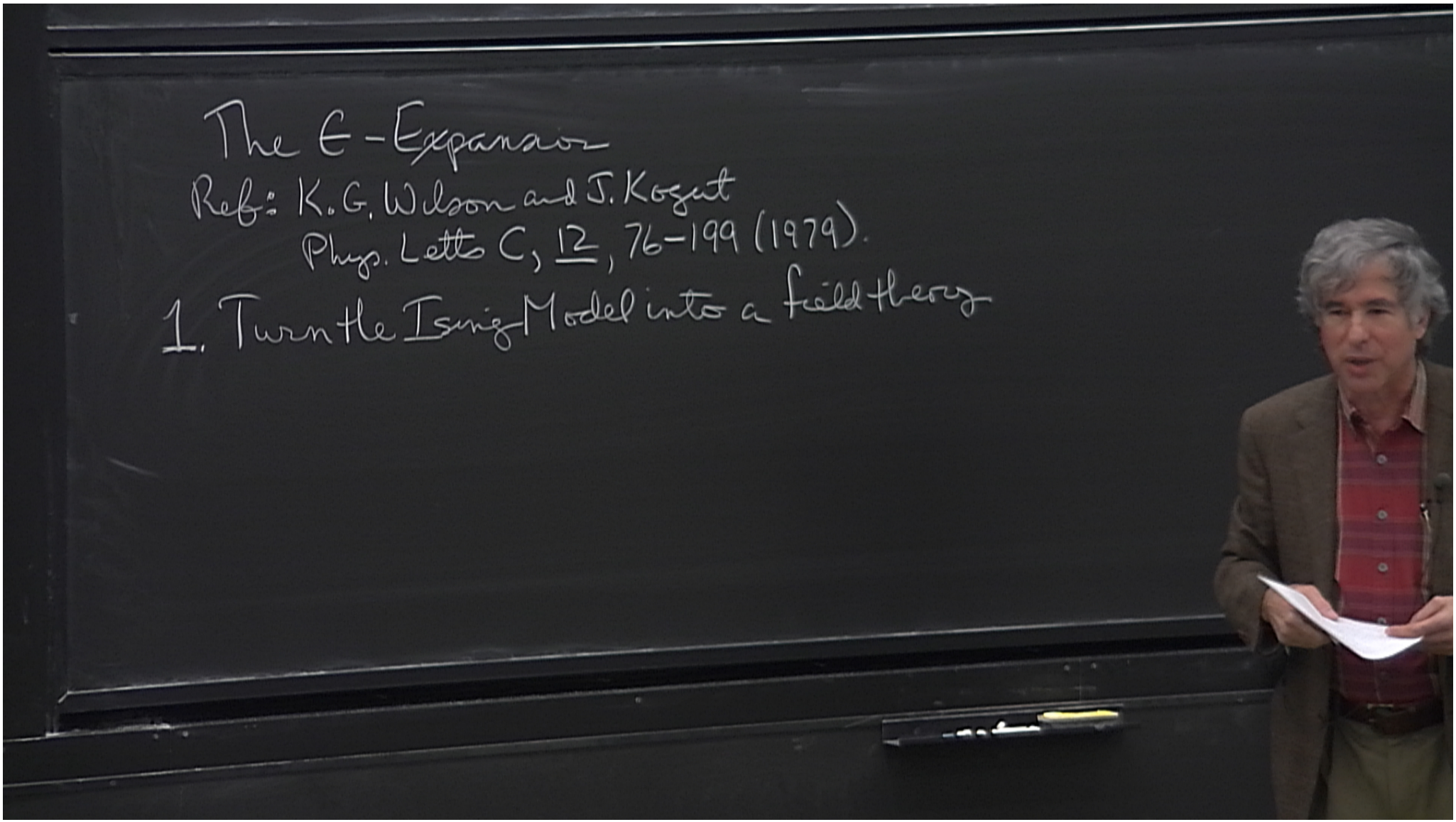
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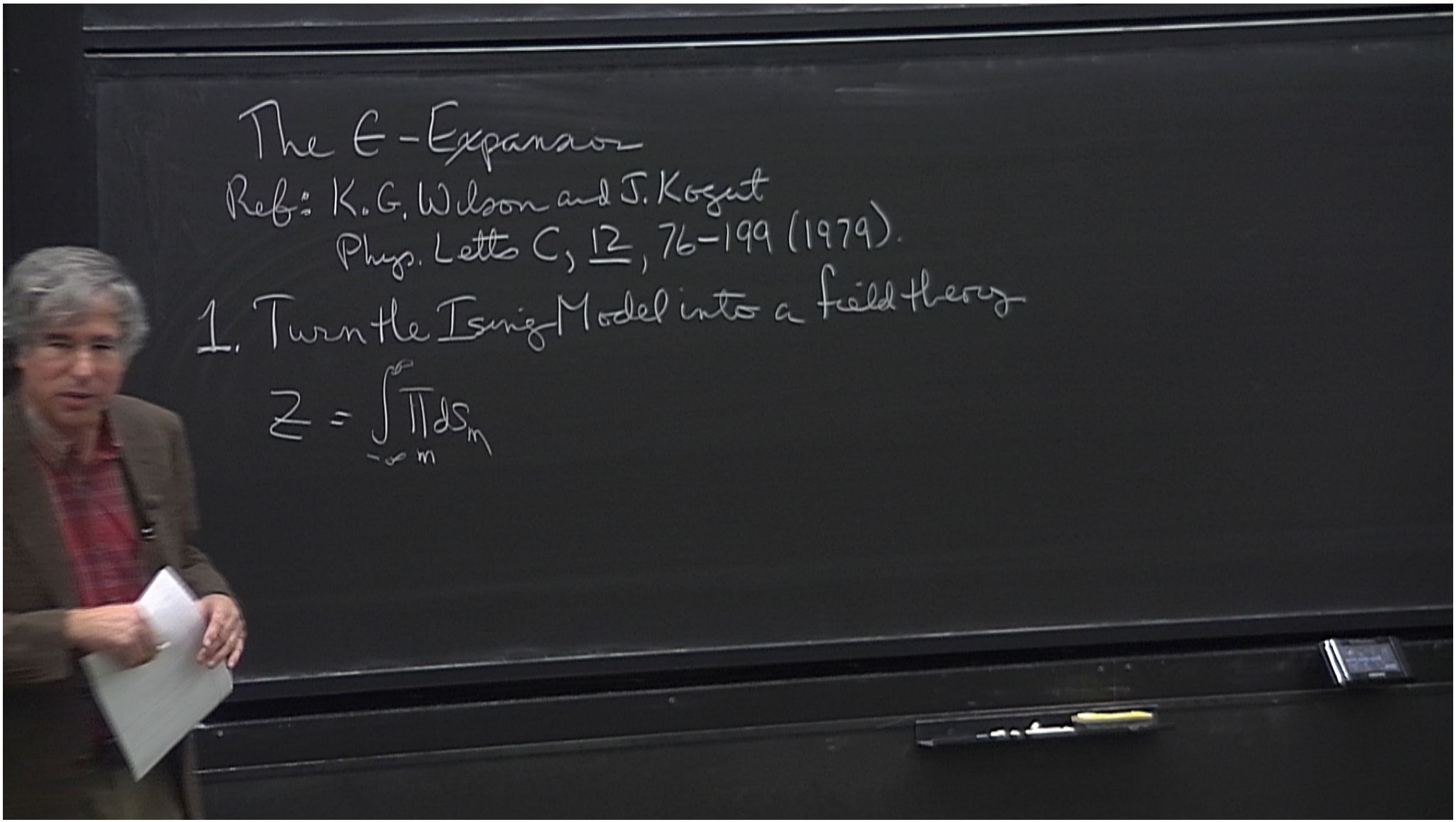
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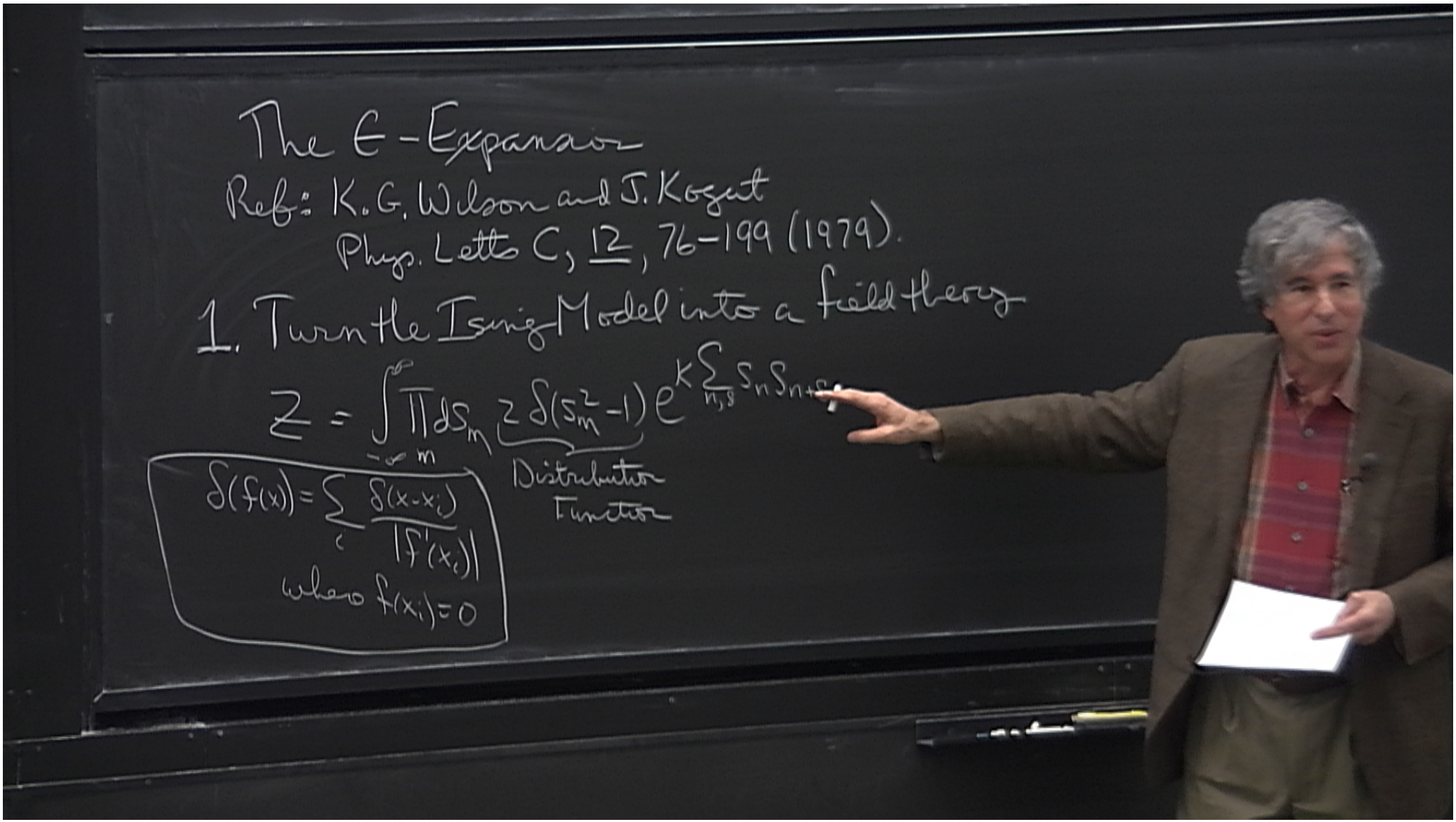
$$\boxed{\delta(f(x)) = \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|} \quad \text{where } f(x_i) = 0}$$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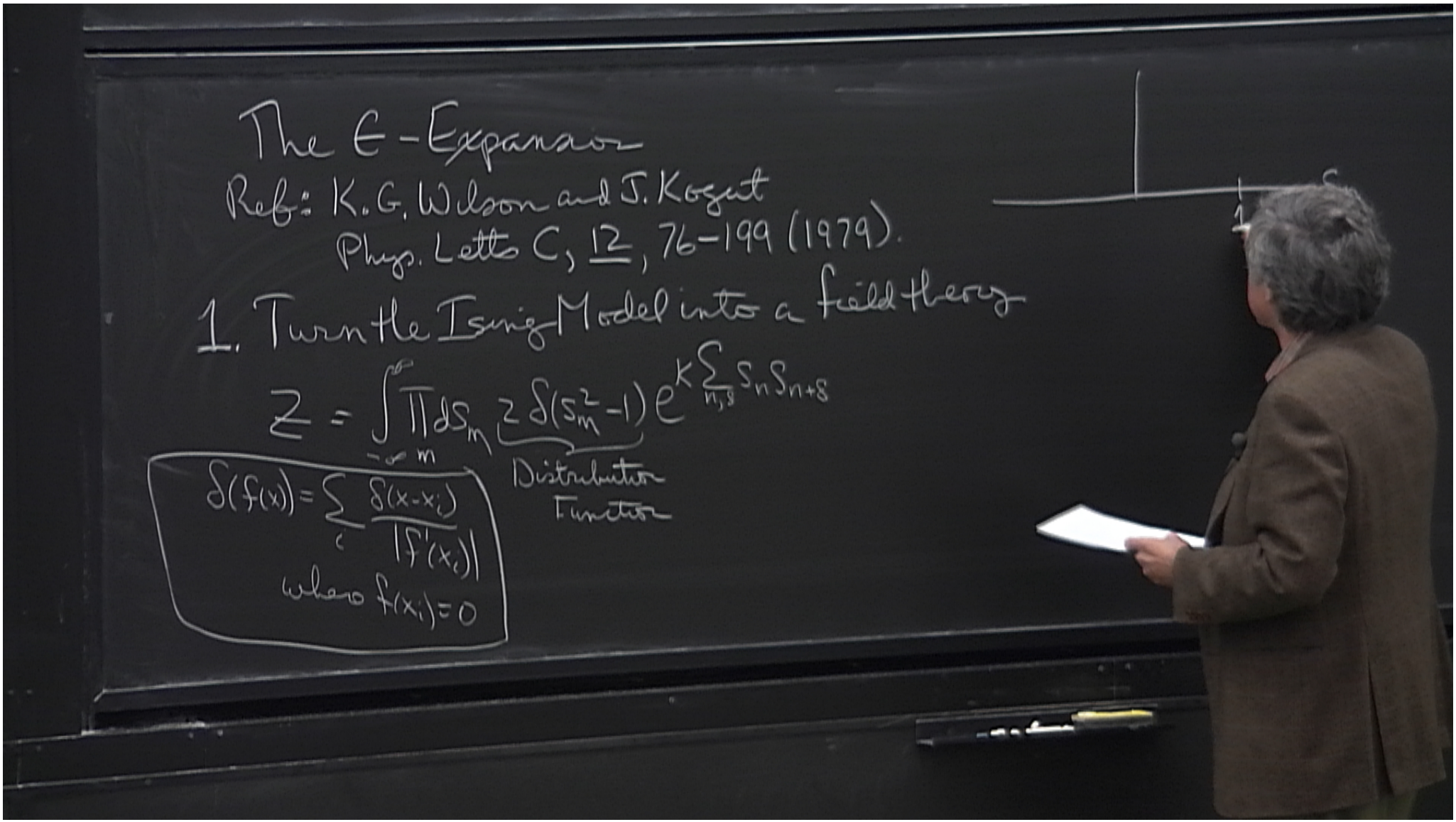
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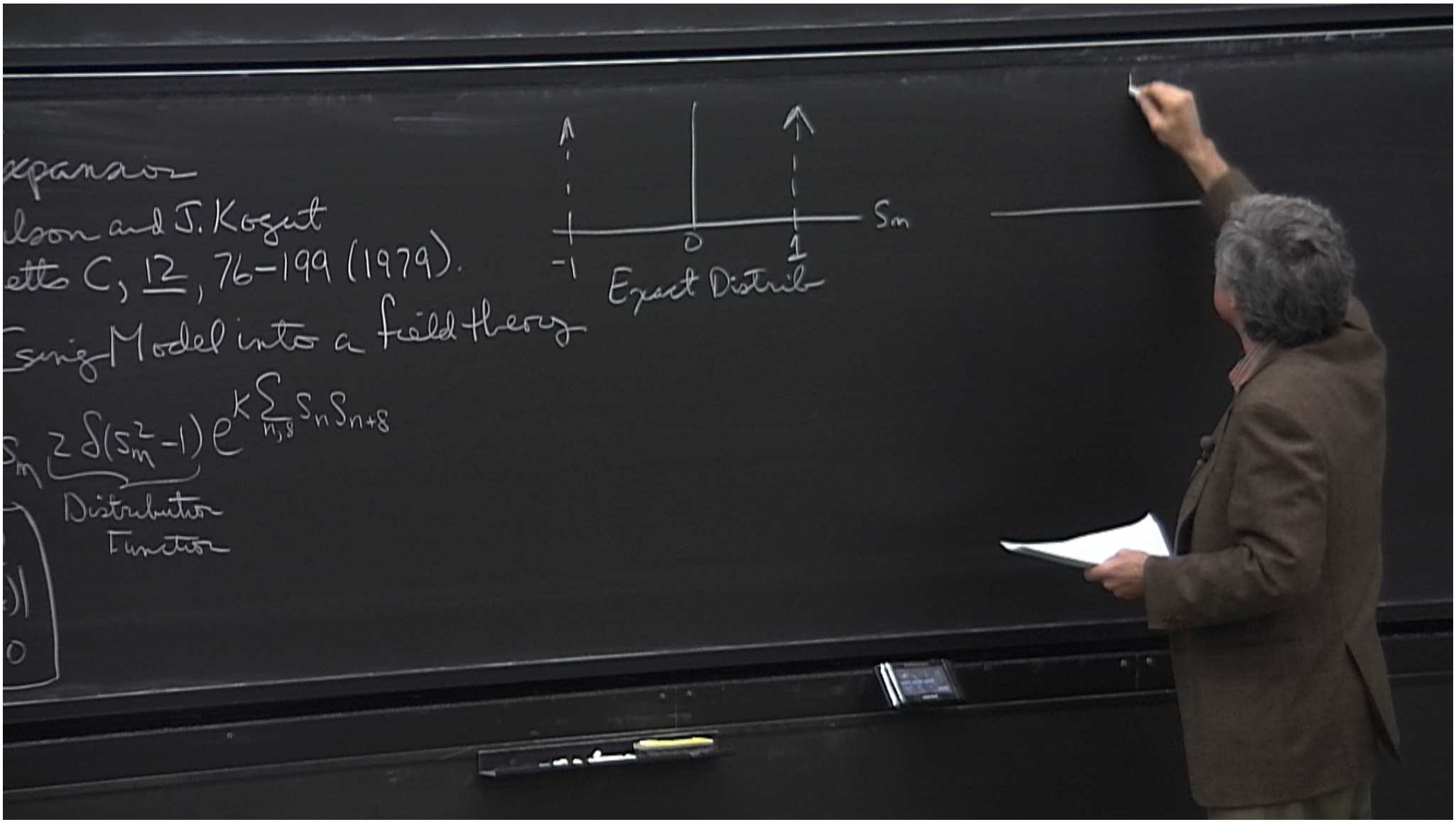
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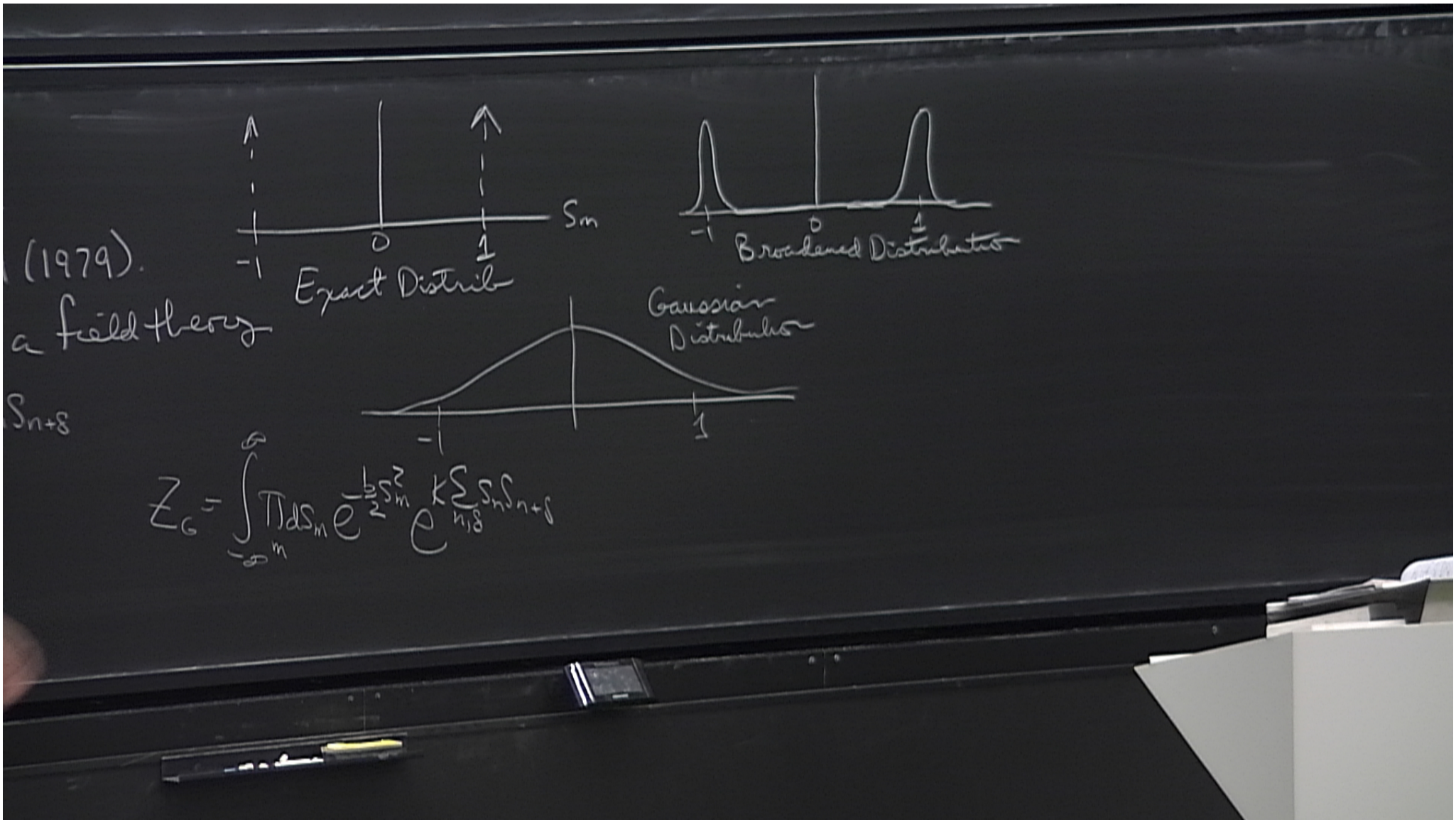
Ising Model into a field theory
$S_m \; \underbrace{z\,\delta(S_m^2-1)}_{\text{Distribution Function}} \; e^{K\sum_{n,s} S_n S_{n+s}}$
S_m
-1
0
1
Exact Distrib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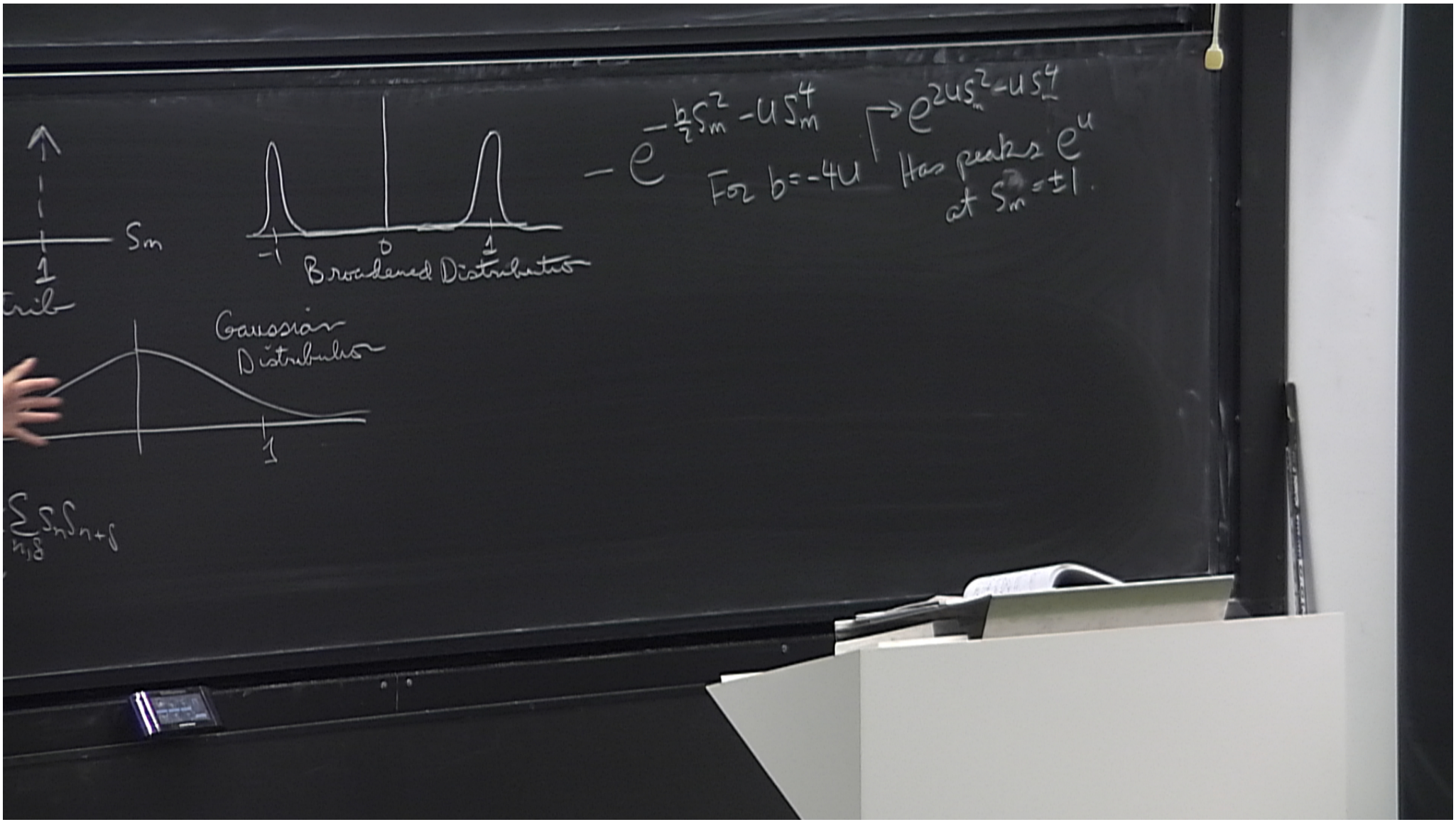
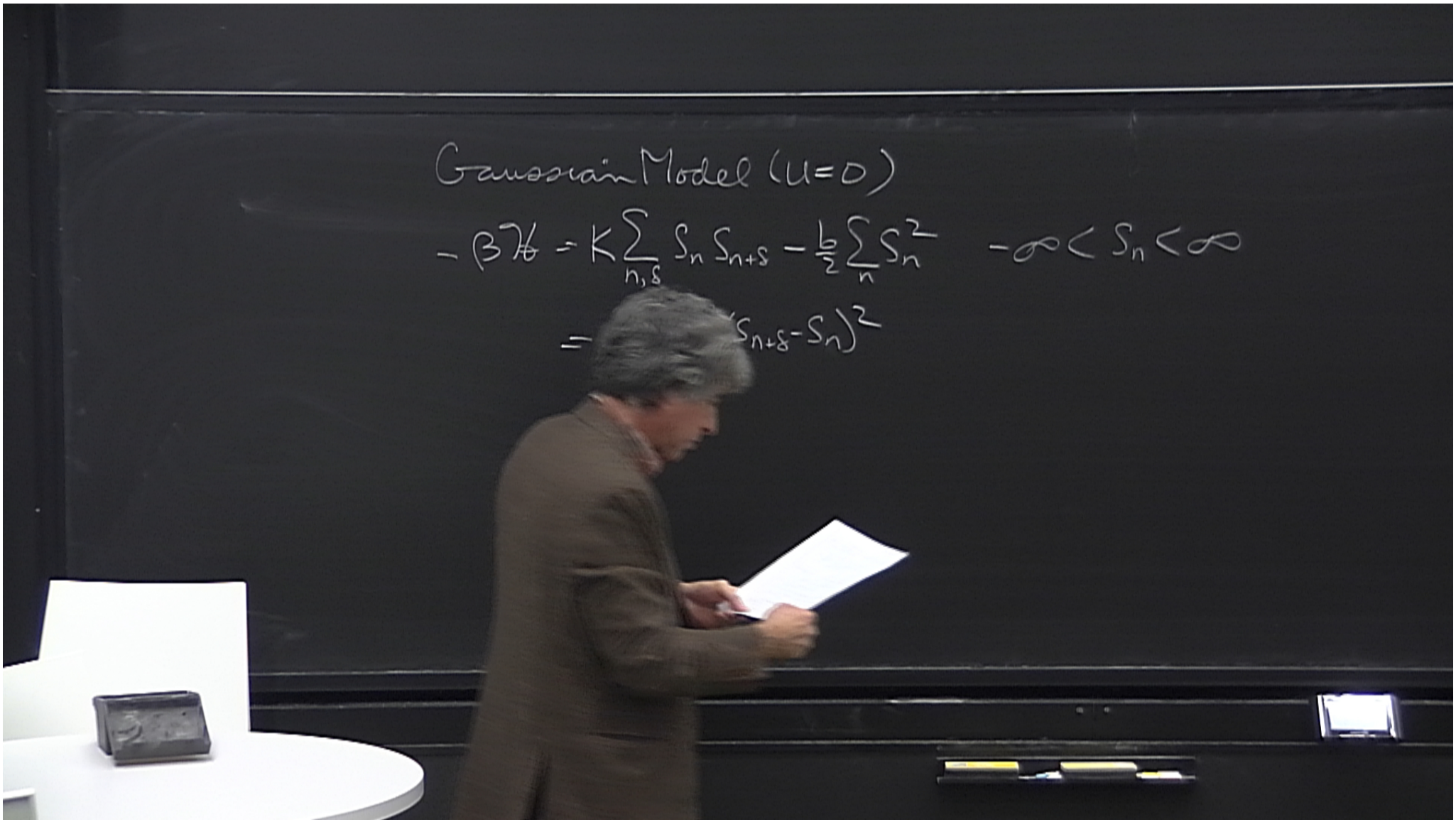
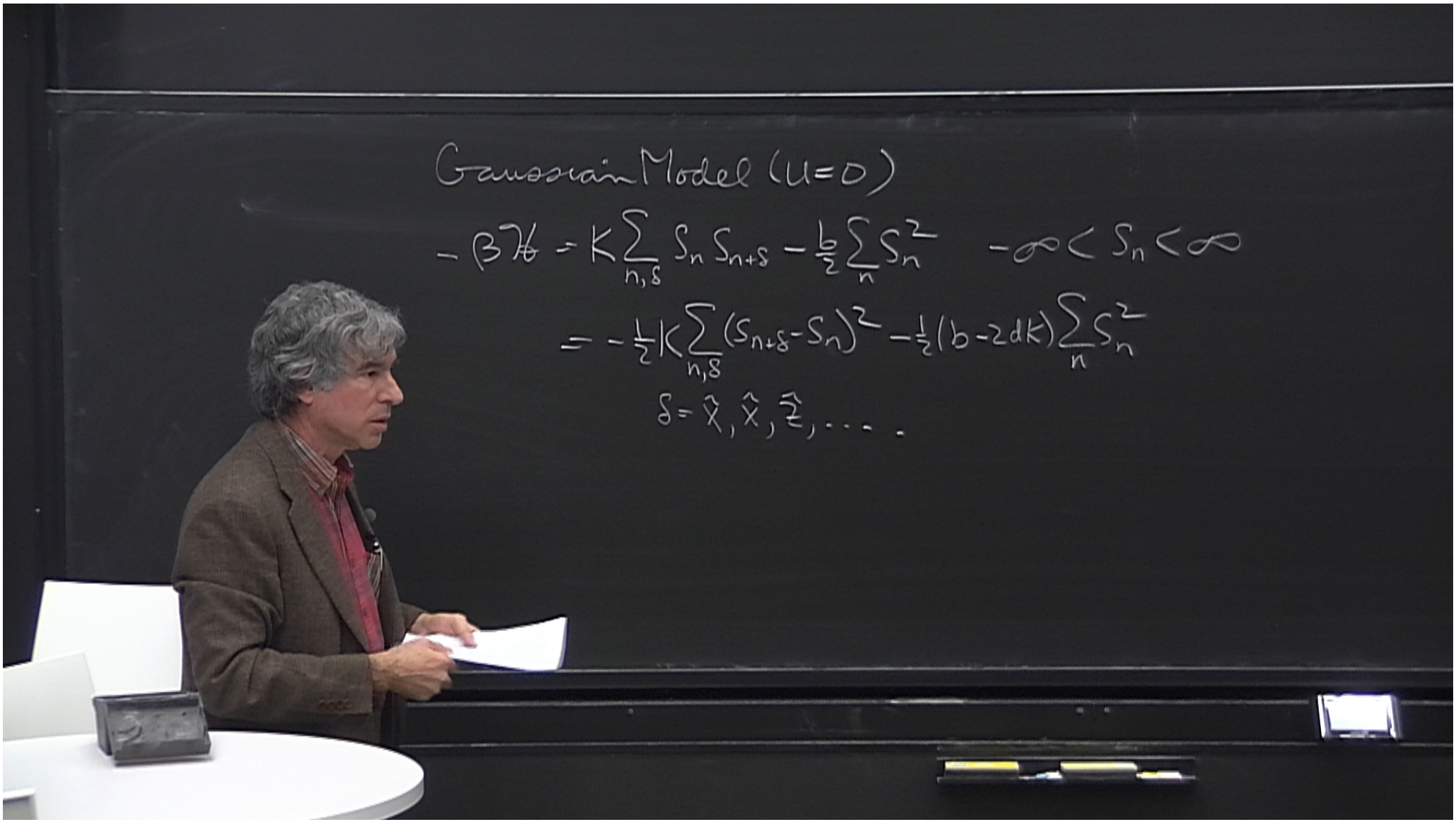
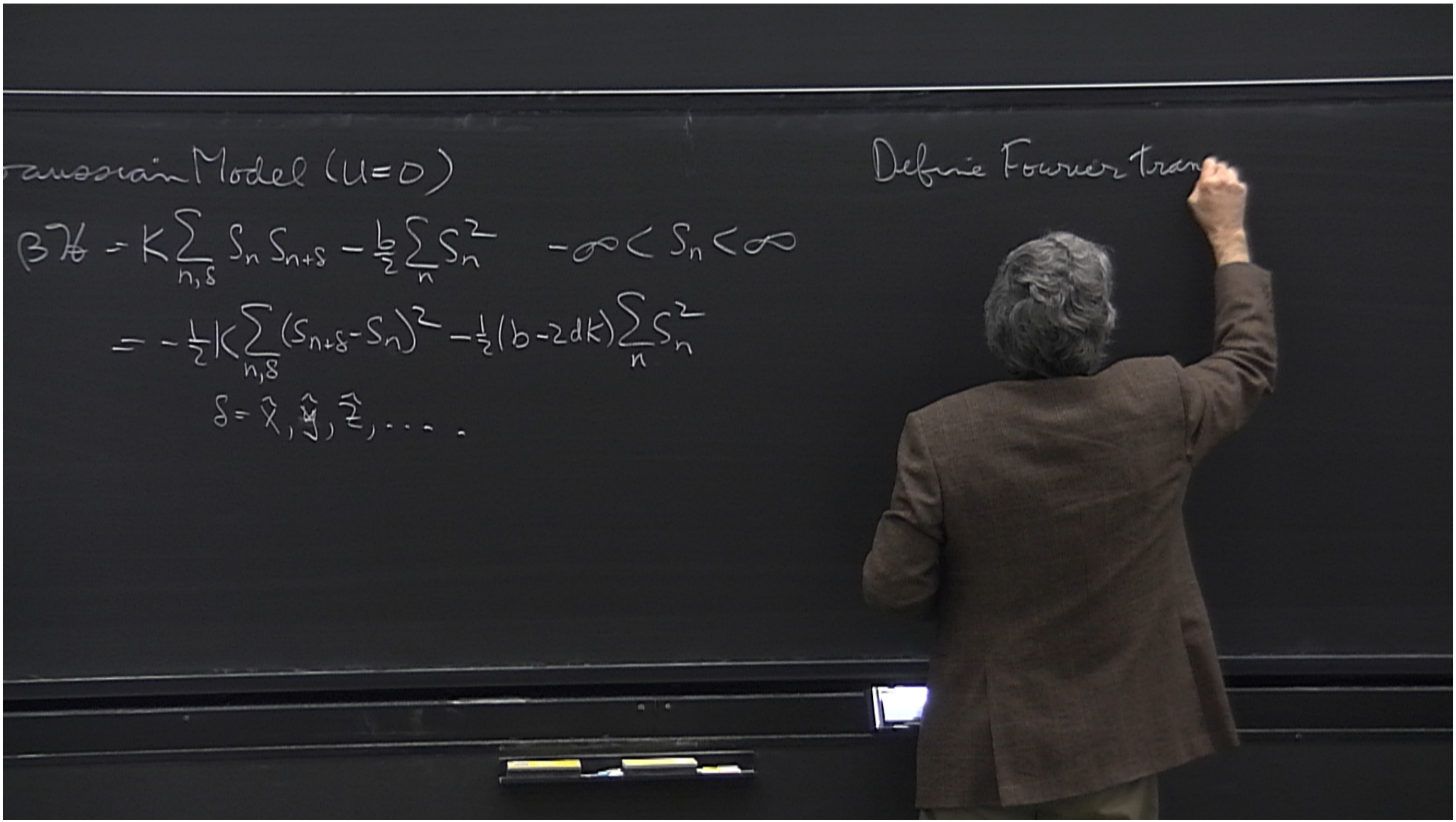
a field theory
$S_{n+\delta}$
S_m
-1
0
1
Exact Distrib
Broadened Distribution
Gaussian Distribution
$Z_G = \int_{-\infty}^{\infty} \prod_m dS_m \, e^{-\frac{b}{2} S_m^2} \, e^{K \sum_{n,\delta} S_n S_{n+\delta}}$


S_m
1
-1
0
1
Broadened Distribution
$-e^{-\frac{b}{2}S_m^2 - uS_m^4}$
For $b=-4u$
$\rightarrow e^{2uS_m^2 - uS_m^4}$
Has peaks e^u
at $S_m = \pm 1$.
Gaussian
Distribution
1
$\sum_{n,\delta} S_n S_{n+\delta}$


Gaussian Model (u=0)
$-\beta\mathcal{H} = K\sum_{n,\delta} S_n S_{n+\delta} - \frac{b}{2}\sum_n S_n^2$
$-\infty < S_n < \infty$
$= (S_{n+\delta} - S_n)^2$


Gaussian Model (u=0)
$-\beta\mathcal{H} = K\sum_{n,\delta} S_n S_{n+\delta} - \frac{b}{2}\sum_n S_n^2 \quad -\infty < S_n < \infty$
$= -\frac{1}{2}K\sum_{n,\delta}(S_{n+\delta}-S_n)^2 - \frac{1}{2}(b-2dK)\sum_n S_n^2$
$\delta = \hat{x}, \hat{x}, \hat{z}, \ldots$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$\delta = \hat{x}, \hat{y}, \hat{z}, \ldots$
Define Fourier Tran

$u=0)$

$-\frac{b}{2}\sum_n S_n^2 \qquad -\infty < S_n < \infty$

$)^2 - \frac{1}{2}(b-2dk)\sum_n S_n^2$

. . . .

Define Fourier transformed spins

$S_n = \int_q e^{i\vec{q}\cdot\vec{n}}\sigma_q$ where

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integrated over the B.Z

$-\infty < S_n < \infty$

$$H(k) \sum_n S_n^2$$

$\infty < S_n < \infty$

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Then

$$\sum_{n,\delta} (S_{n+\delta} - S_n)^2 = \sum_{n,\delta} \iint_{\xi\,\xi'} \left(e^{i\vec{\xi}\cdot\vec{\delta}} - 1\right)\left(e^{-i\vec{\xi}'\cdot\vec{\delta}} - 1\right) e^{i(\vec{\xi}+\vec{\xi}')\cdot\vec{n}} \sigma_{\xi}\sigma_{\xi'}$$

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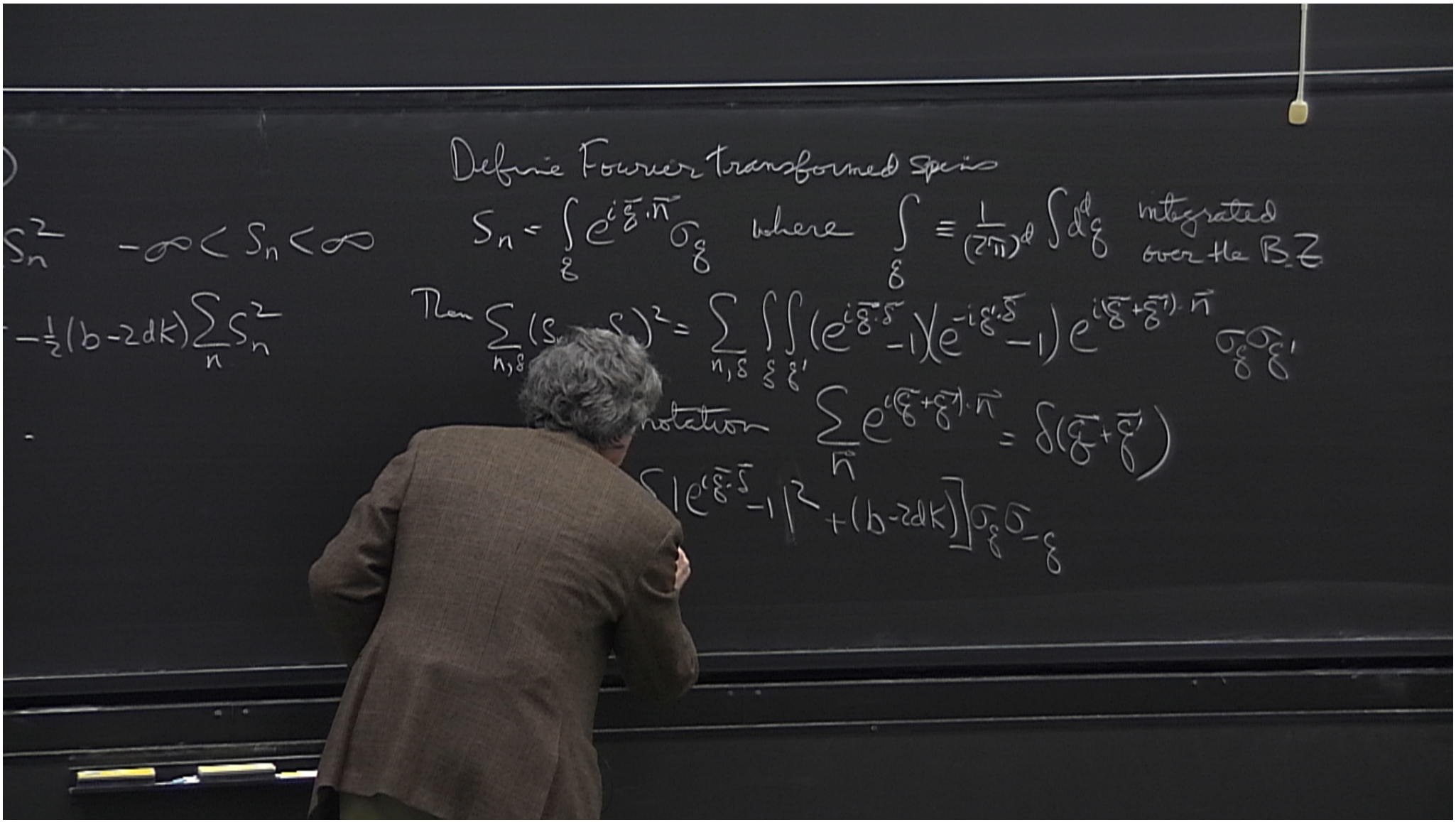
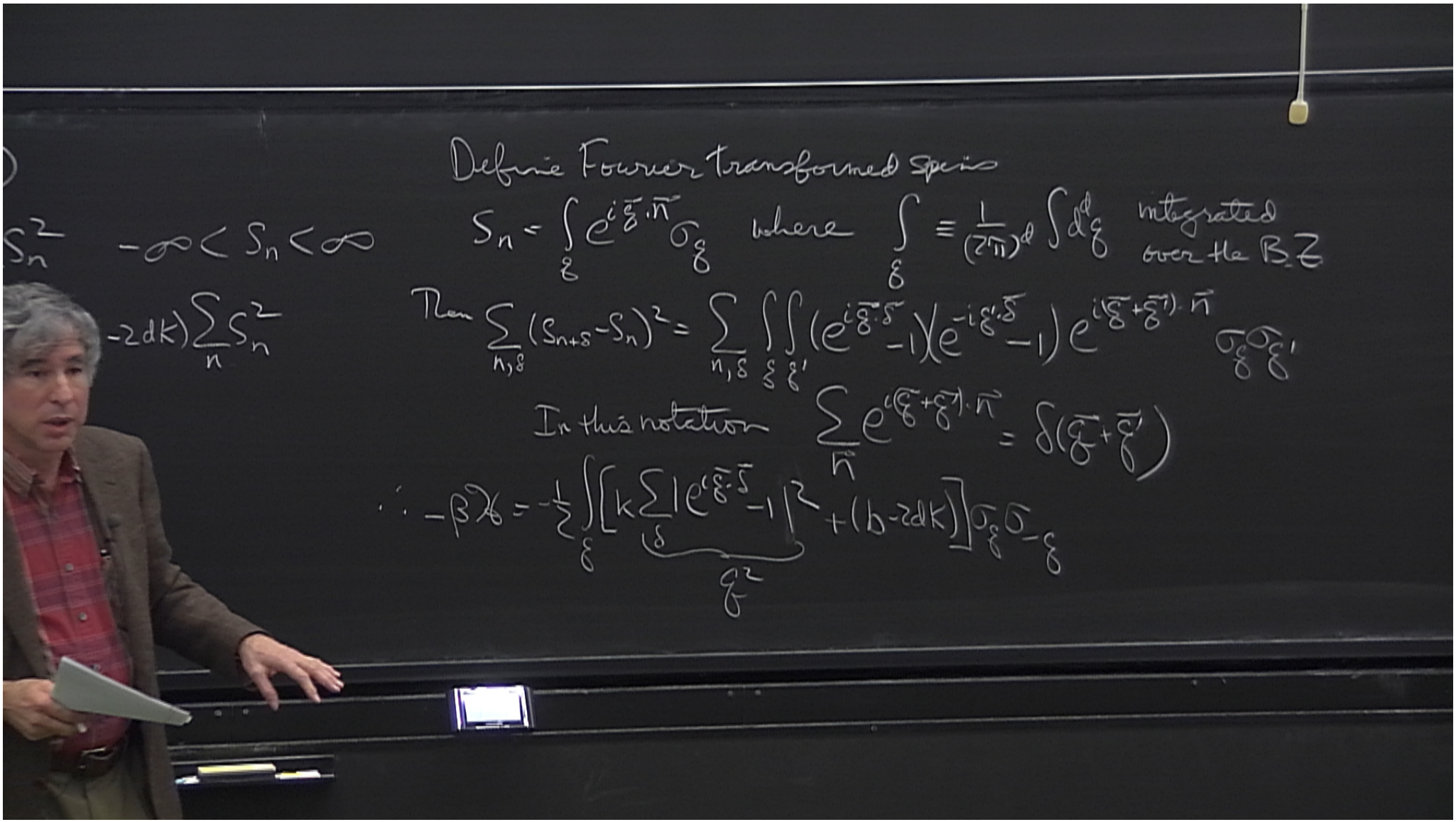
In this notation $$\sum_{\vec{n}} e^{i(\vec{\xi}+\vec{\xi}')\cdot\vec{n}} = \delta(\vec{\xi}+\vec{\xi}')$$


Define Fourier transformed spins
$S_n = \int_q e^{i\vec{q}\cdot\vec{n}} \sigma_q$ where $\int_q \equiv \frac{1}{(2\pi)^d}\int d^d q$ integrated over the B.Z
S_n^2 $-\infty < S_n < \infty$
$-\frac{1}{2}(b-2dK)\sum_n S_n^2$
Then $\sum_{n,\delta}(S_n - S_{n+\delta})^2 = \sum_{n,\delta}\iint_{q\,q'}(e^{i\vec{q}\cdot\vec{\delta}}-1)(e^{-i\vec{q}'\cdot\vec{\delta}}-1)\, e^{i(\vec{q}+\vec{q}')\cdot\vec{n}}\, \sigma_q \sigma_{q'}$
notation $\sum_{\vec{n}} e^{i(\vec{q}+\vec{q}')\cdot\vec{n}} = \delta(\vec{q}+\vec{q}')$
$|e^{i\vec{q}\cdot\vec{\delta}}-1|^2 + (b-2dK)]\sigma_q \sigma_{-q}$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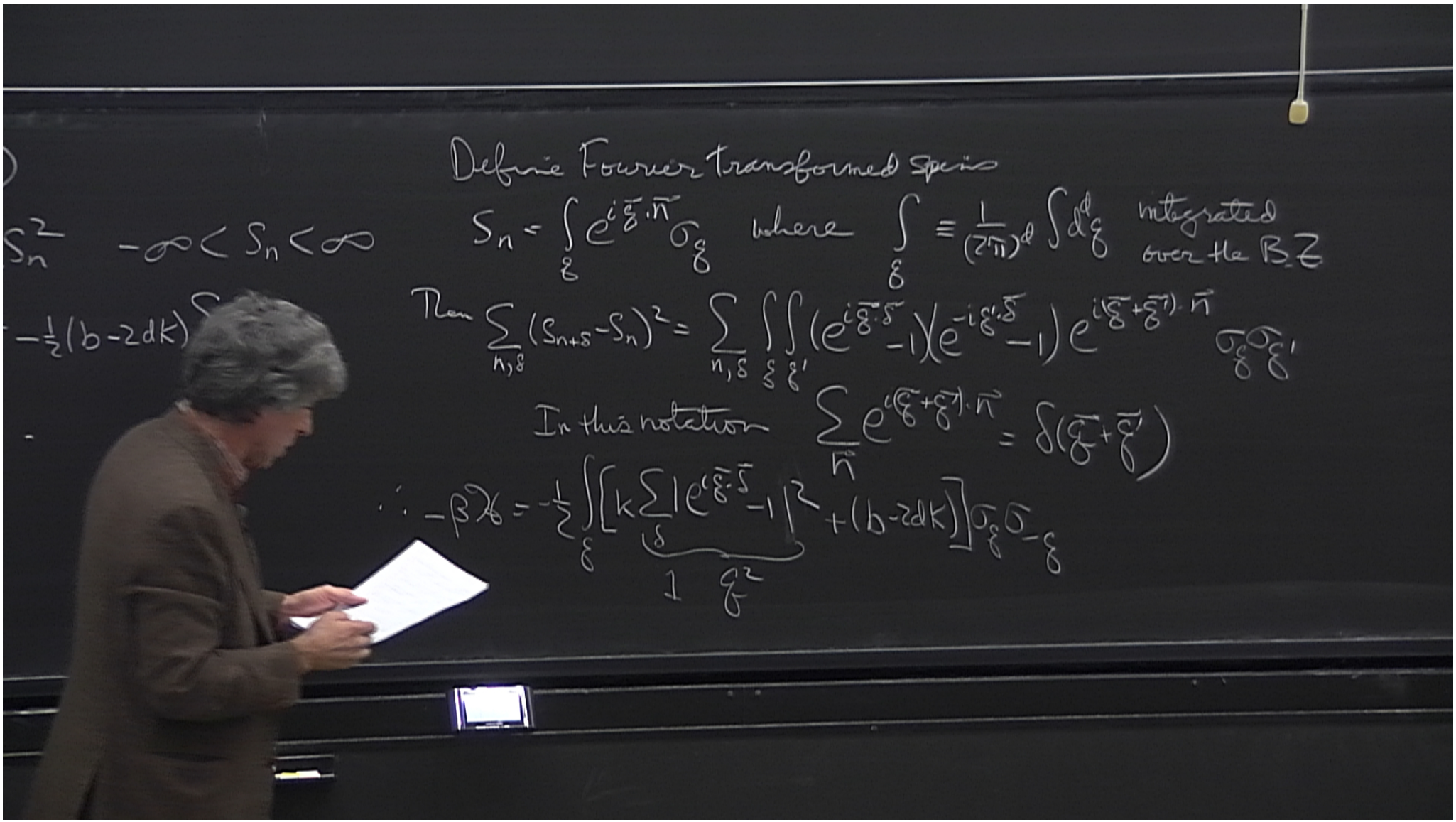
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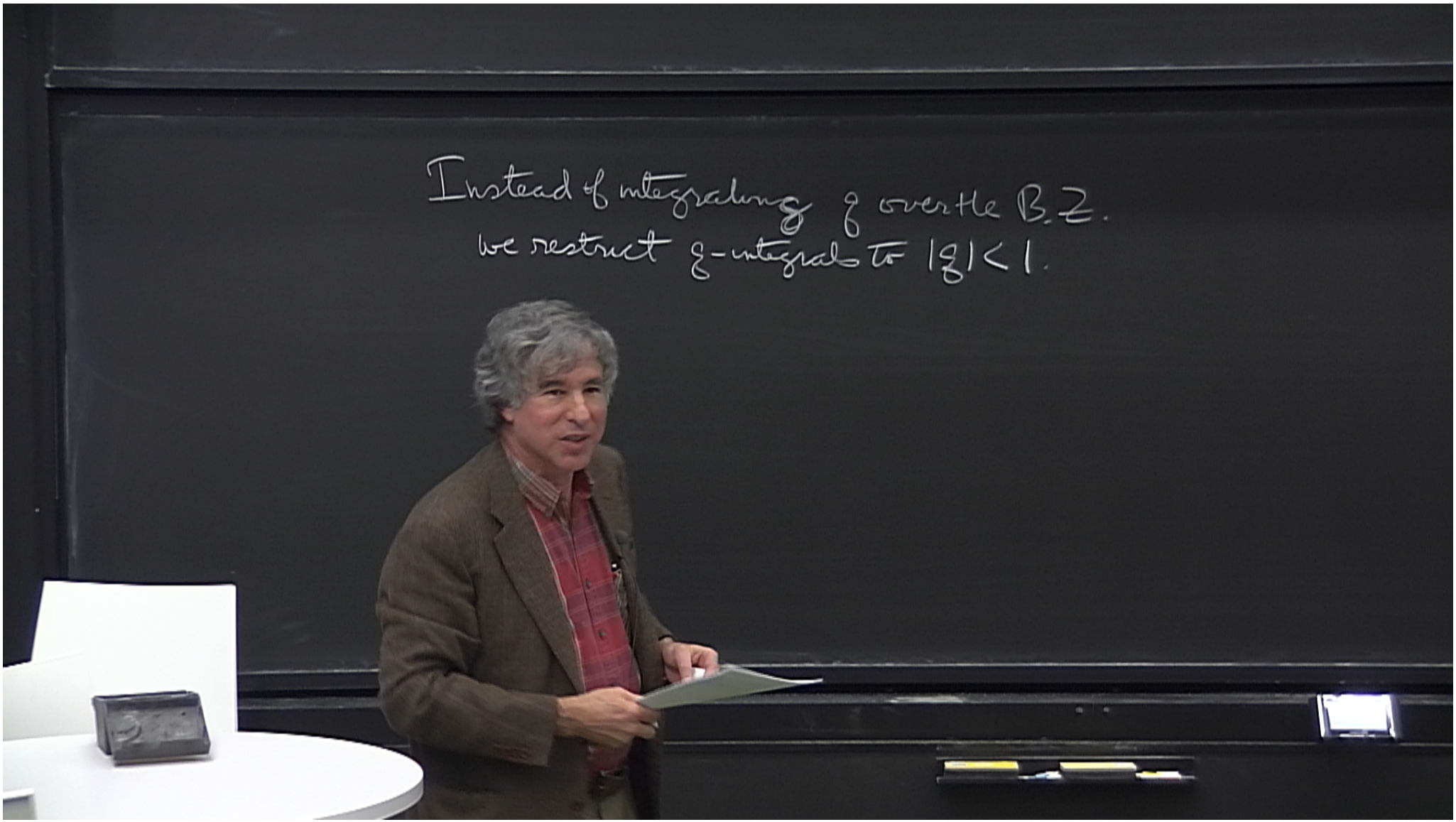
In this notation $$\sum_{\vec{n}} e^{i(\vec{q}+\vec{q}')\cdot\vec{n}} = \delta(\vec{q}+\vec{q}')$$

$$\mathcal{H} = -\frac{1}{2}\int_q\Big[k\underbrace{\sum_\delta\left|e^{i\vec{q}\cdot\vec{\delta}}-1\right|^2}_{1.\ q^2} + \underbrace{(b-2dk)}_{2.\ kr}\Big]\sigma_q\sigma_{-q}$$

3. Absorb k into the σ_q's

$$-\beta\mathcal{H} = -\frac{1}{2}\int_q (q^2+r)\sigma_q\sigma_{-q}$$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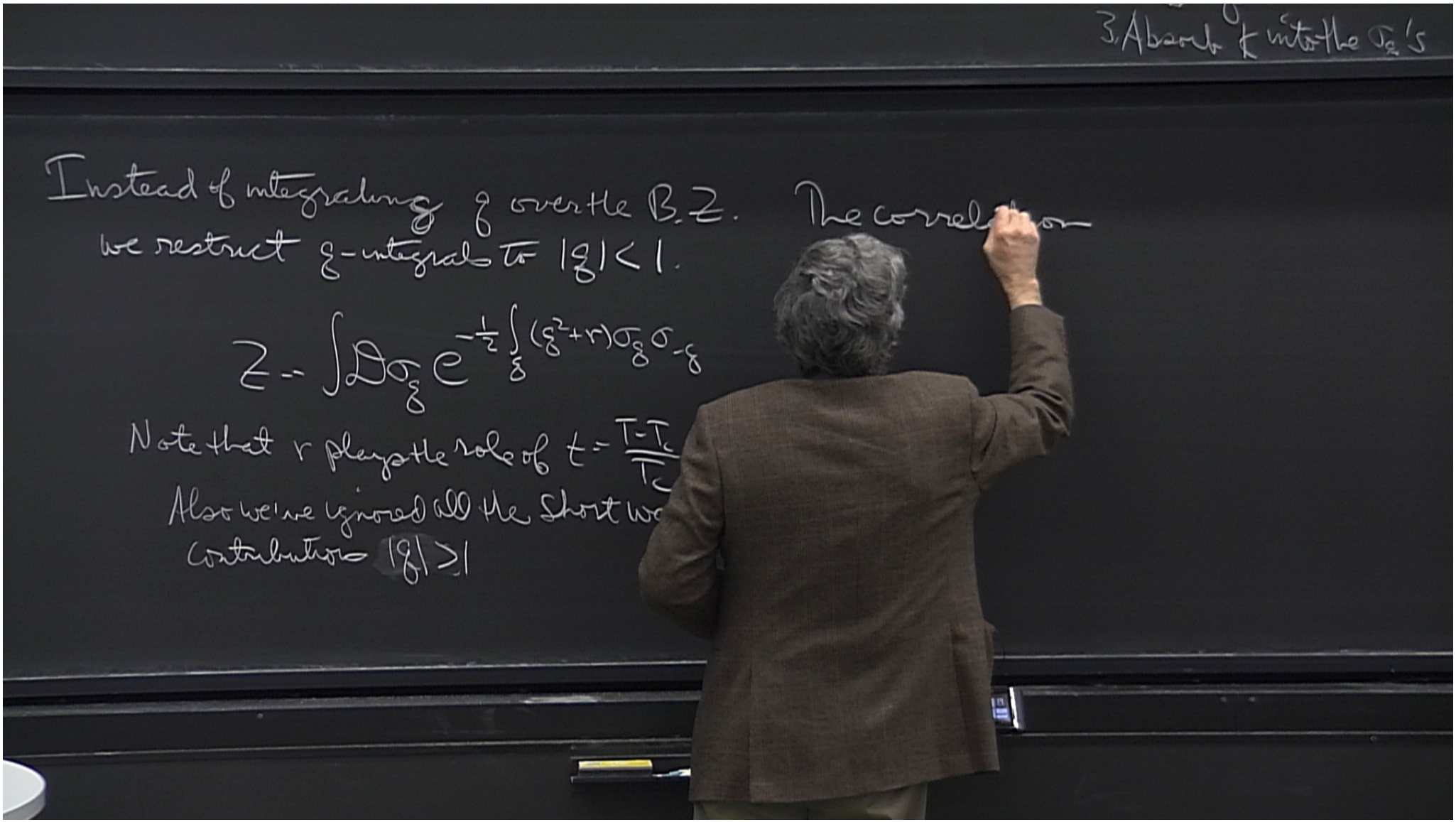
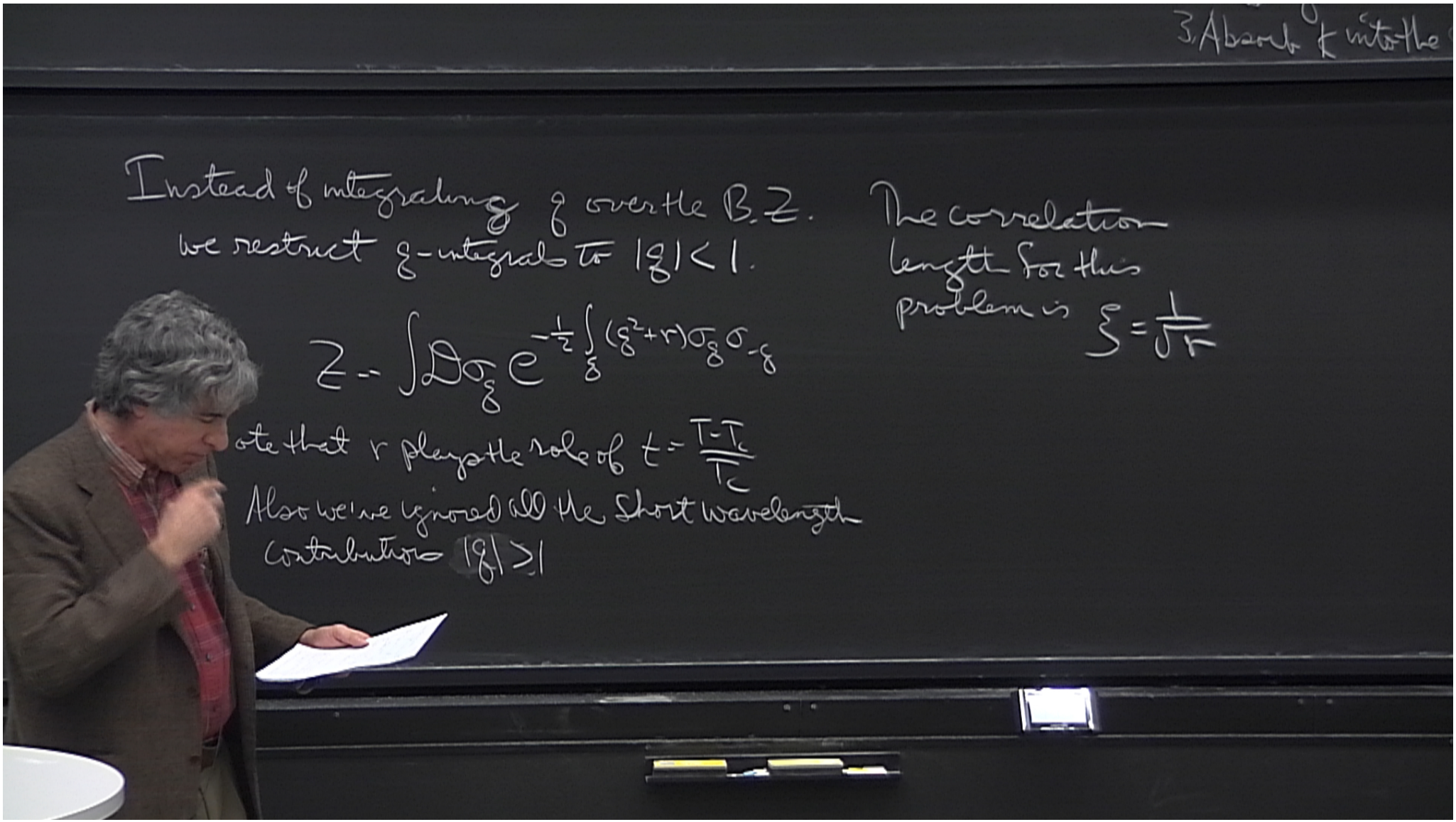

3. Absorb k into the σ_q's
Instead of integrating q over the B.Z. The correlation
we restrict q-integrals to |q| < 1.
Z = ∫ Dσ_q e^{-½ ∫_q (q²+r) σ_q σ_{-q}}
Note that r plays the role of t = (T-T_c)/T_c
Also we've ignored all the short wa
contributions |q| ≥ 1


3. Absorb k into the
Instead of integrating q over the B.Z.
we restrict q-integrals to |q|<1.
The correlation length for this problem is $\xi = \frac{1}{\sqrt{r}}$
$Z = \int \mathcal{D}\sigma_q \, e^{-\frac{1}{2}\int_q (q^2+r)\sigma_q \sigma_{-q}}$
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Note that r plays the role of $t = \frac{T-T_c}{T_c}$

Also we've ignored all the short wavelength contributions $|q|>1$

The correlation length for this problem is $\xi = \frac{1}{\sqrt{r}}$

Define Fourier transformed spins

$<\infty$ $\quad S_n = \int_q e^{i\vec{q}\cdot\vec{n}}\sigma_q$ where $\int_q \equiv \frac{1}{(2\pi)^d}\int d^d q$ integrated over the B.Z.

Then $\sum_{n,\delta}(S_{n+\delta}-S_n)^2 = \sum_{n,\delta}\iint_{q\,q'}(e^{i\vec{q}\cdot\vec{\delta}}-1)(e^{-i\vec{q}'\cdot\vec{\delta}}-1)\,e^{i(\vec{q}+\vec{q}')\cdot\vec{n}}\,\sigma_q\sigma_{q'}$

In this notation $\sum_{\vec{n}} e^{i(\vec{q}+\vec{q}')\cdot\vec{n}} = \delta(\vec{q}+\vec{q}')$

$$\therefore -\beta\mathcal{H} = -\frac{1}{2}\int_q\Big[k\underbrace{\sum_\delta|e^{i\vec{q}\cdot\vec{\delta}}-1|^2}_{1.\ q^2}+\underbrace{(b-2dk)}_{2.\ kr}\Big]\sigma_q\sigma_{-q}$$

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$$\rightarrow -\beta\mathcal{H} = -\frac{1}{2}\int_q(q^2+r)\sigma_q\sigma_{-q}$$

The correlation length for this

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In this notation $\sum_{\vec{n}} e^{i(\vec{q}+\vec{q}')\cdot\vec{n}} = \delta(\vec{q}+\vec{q}')$

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3. Absorb k into the σ_q's

$$\rightarrow -\beta\mathcal{H} = -\frac{1}{2}\int_q(q^2+r)\sigma_q\sigma_{-q}$$

The correlation length for this

1. ξ^2 2. Kr

3. Absorb K into the σ_ξ's

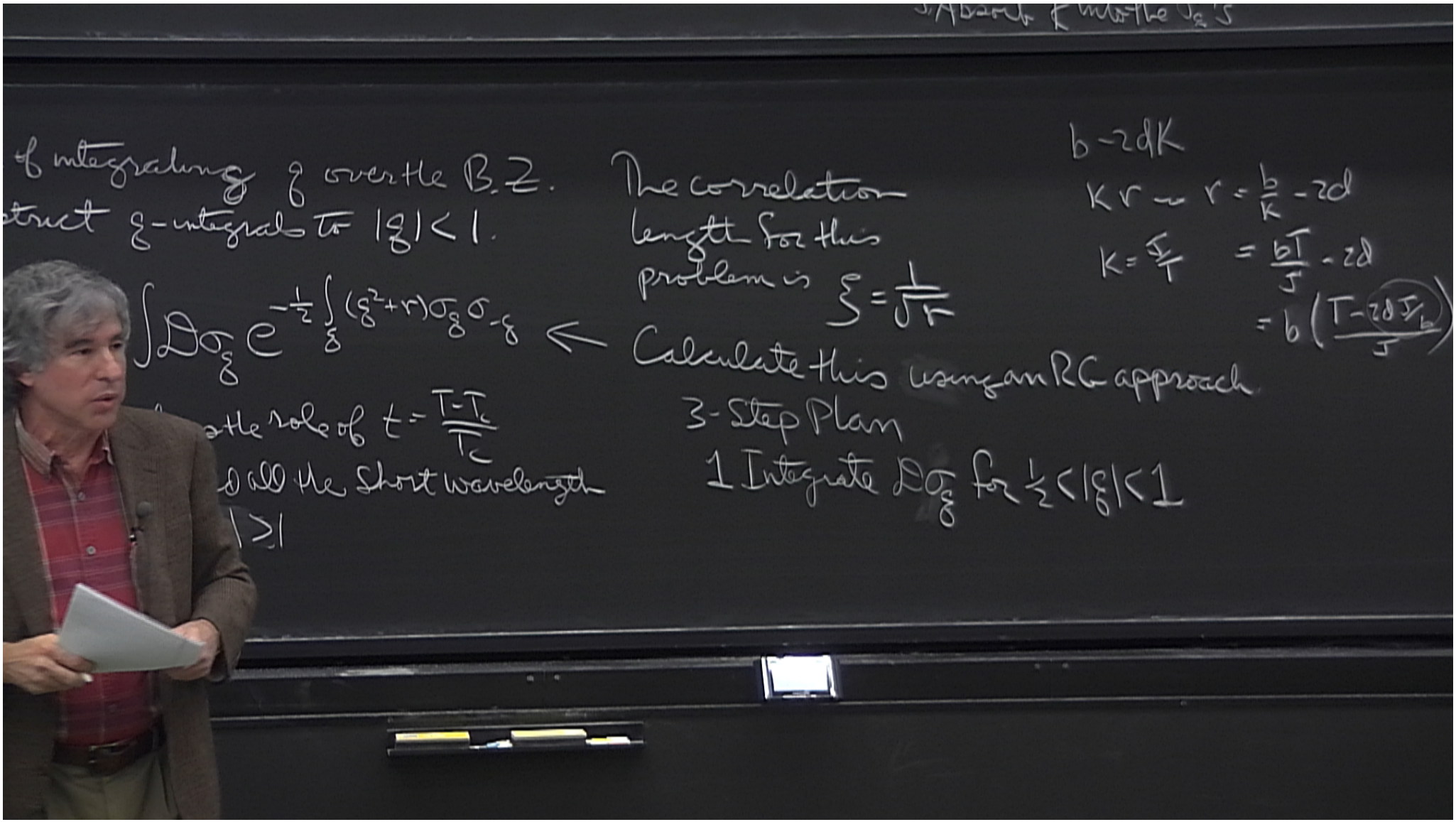
$b - 2dK$

$Kr \sim r = \frac{b}{K} - 2d$

$K = \frac{J}{T} \quad = \frac{bT}{J} - 2d$

$= b\left(\frac{T - 2dJ/b}{J}\right)$

The correlation length for this problem is $\xi = \frac{1}{\sqrt{r}}$


of integrating q over the B.Z.
tract q-integrals to |q|<1.
$\int \mathcal{D}\sigma_q \, e^{-\frac{1}{2}\int_q (q^2+r)\sigma_q \sigma_{-q}}$
the role of $t = \frac{T-T_c}{T_c}$
all the short wavelength
|>1
The correlation length for this problem is $\xi = \frac{1}{\sqrt{r}}$
Calculate this using an RG approach
3-Step Plan
1 Integrate $\mathcal{D}\sigma_q$ for $\frac{1}{2}<|q|<1$
$b-2dK$
$Kr \sim r = \frac{b}{K} - 2d$
$K = \frac{J}{T}$
$= \frac{bT}{J} - 2d$
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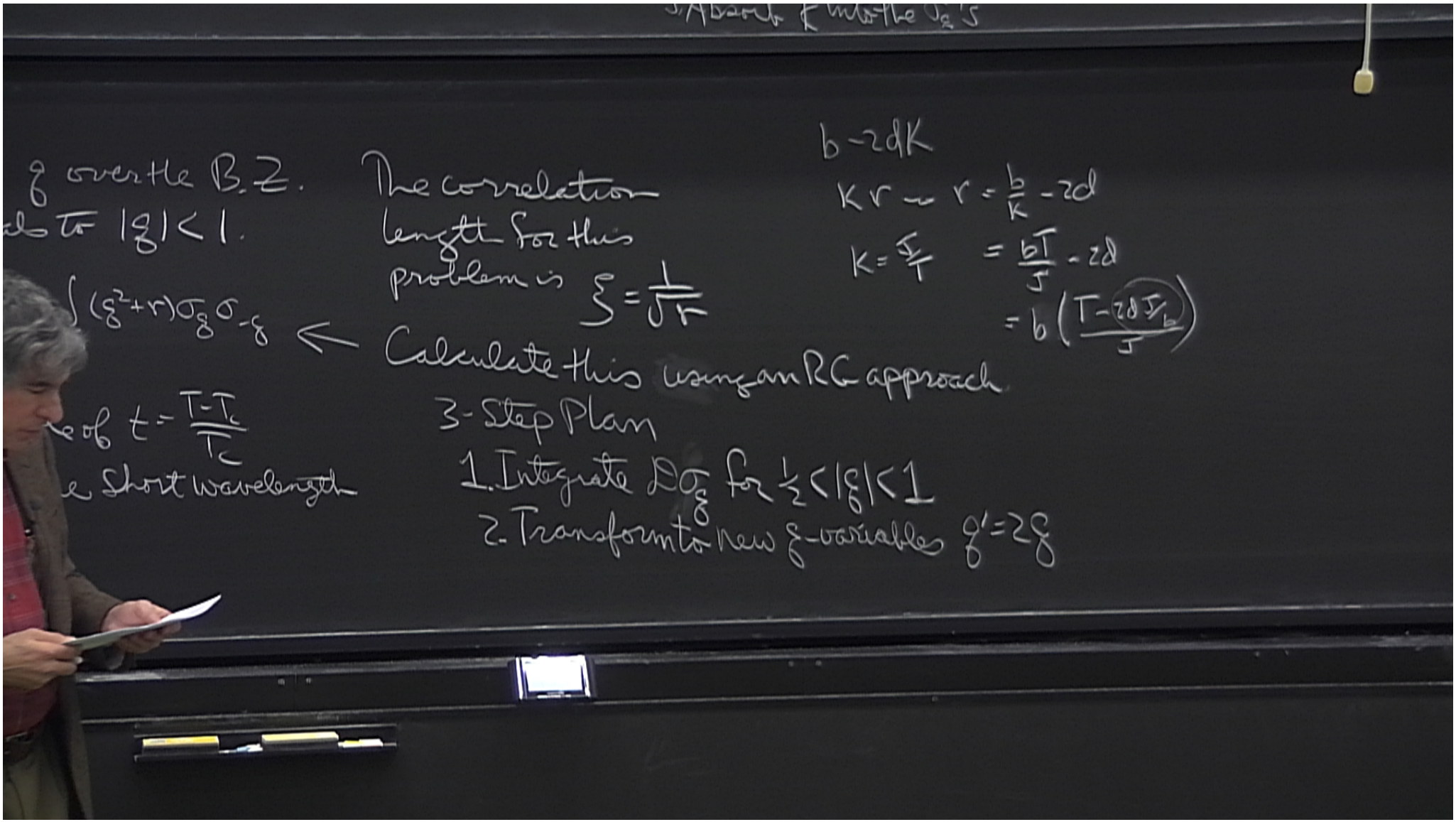
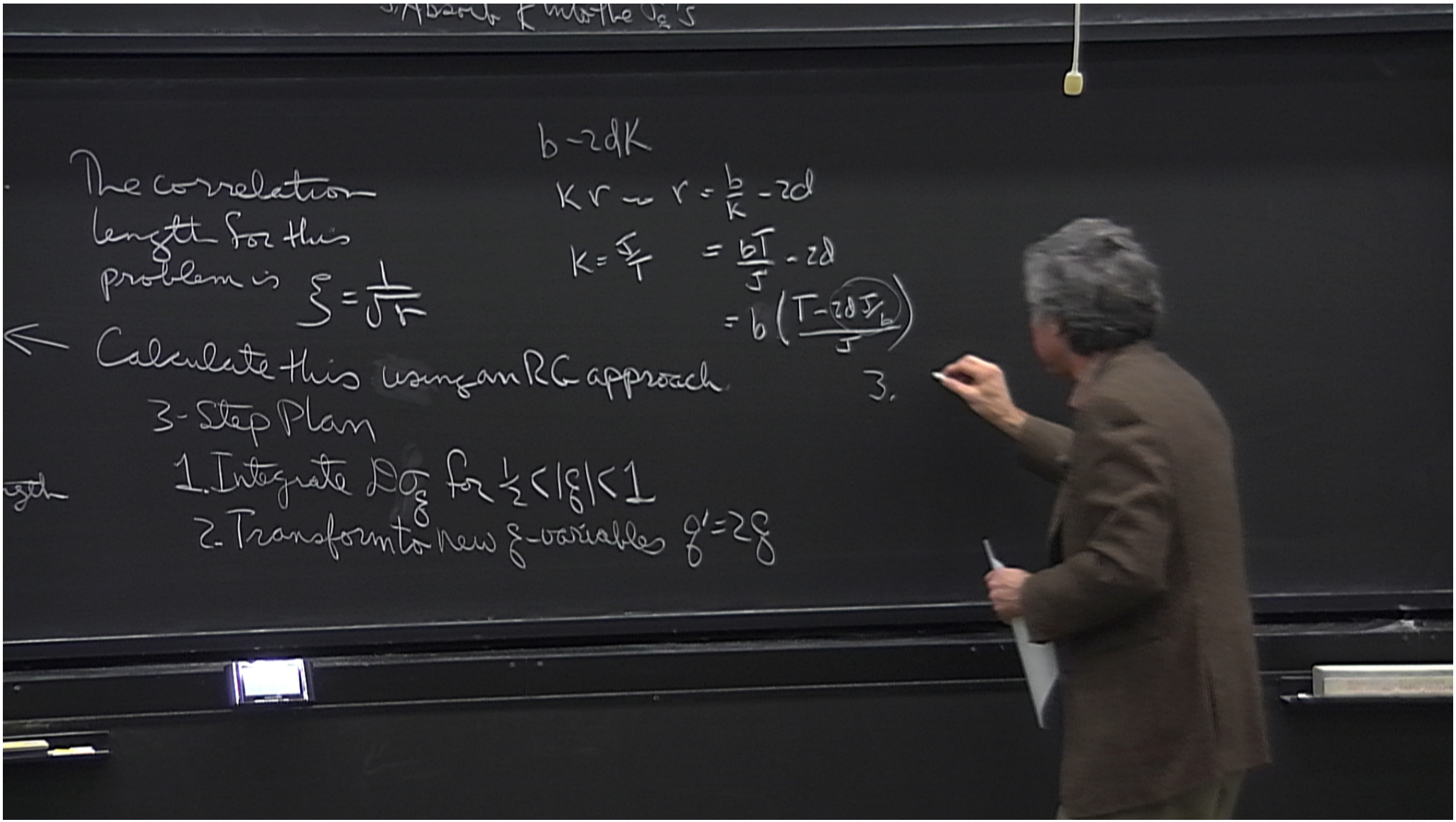

q over the B.Z.
ds to |q|<1.
∫(q²+r)σ_q σ_-q
e of t = (T-T_c)/T_c
e short wavelength
The correlation length for this problem is ξ = 1/√r
Calculate this using an RG approach
3-Step Plan
1. Integrate σ_q for 1/2<|q|<1
2. Transform to new q-variables q'=2q
b-2dK
Kr → r = b/K - 2d
K = J/T = bT/J - 2d
= b(T - 2dJ/b)/J


The correlation length for this problem is $\xi = \frac{1}{\sqrt{r}}$
Calculate this using an RG approach
3-Step Plan
1. Integrate $\mathcal{D}\sigma_\xi$ for $\frac{1}{2} < |\xi| < 1$
2. Transform to new ξ-variables $\xi' = 2\xi$
$b - 2dK$
$Kr \sim r = \frac{b}{K} - 2d$
$K = \frac{J}{T}$
$= \frac{bT}{J} - 2d$
$= b\left(\frac{T - 2dJ/b}{J}\right)$
3.

3. Absorb K into the σ_ξ's

$b - 2dK$

$Kr \sim r = \frac{b}{K} - 2d$

$K = \frac{J}{T} \quad = \frac{bT}{J} - 2d$

$= b\left(\frac{T - 2dJ/b}{J}\right)$

The correlation length for this problem is $\xi = \frac{1}{\sqrt{r}}$

← Calculate this using an RG approach.

3-Step Plan

1. Integrate $\mathcal{D}\sigma_\xi$ for $\frac{1}{2} < |\xi| < 1$
2. Transform to new ξ-variables $\xi' = 2\xi$
3. Rescale the $\sigma_{\xi'}$ so that $\sigma_{\xi'} = \zeta \sigma'_{\xi'}$ where ζ

$b - 2dK$

$Kr \sim r = \frac{b}{K} - 2d$

$K = \frac{J}{T}$ $\quad = \frac{bT}{J} - 2d$

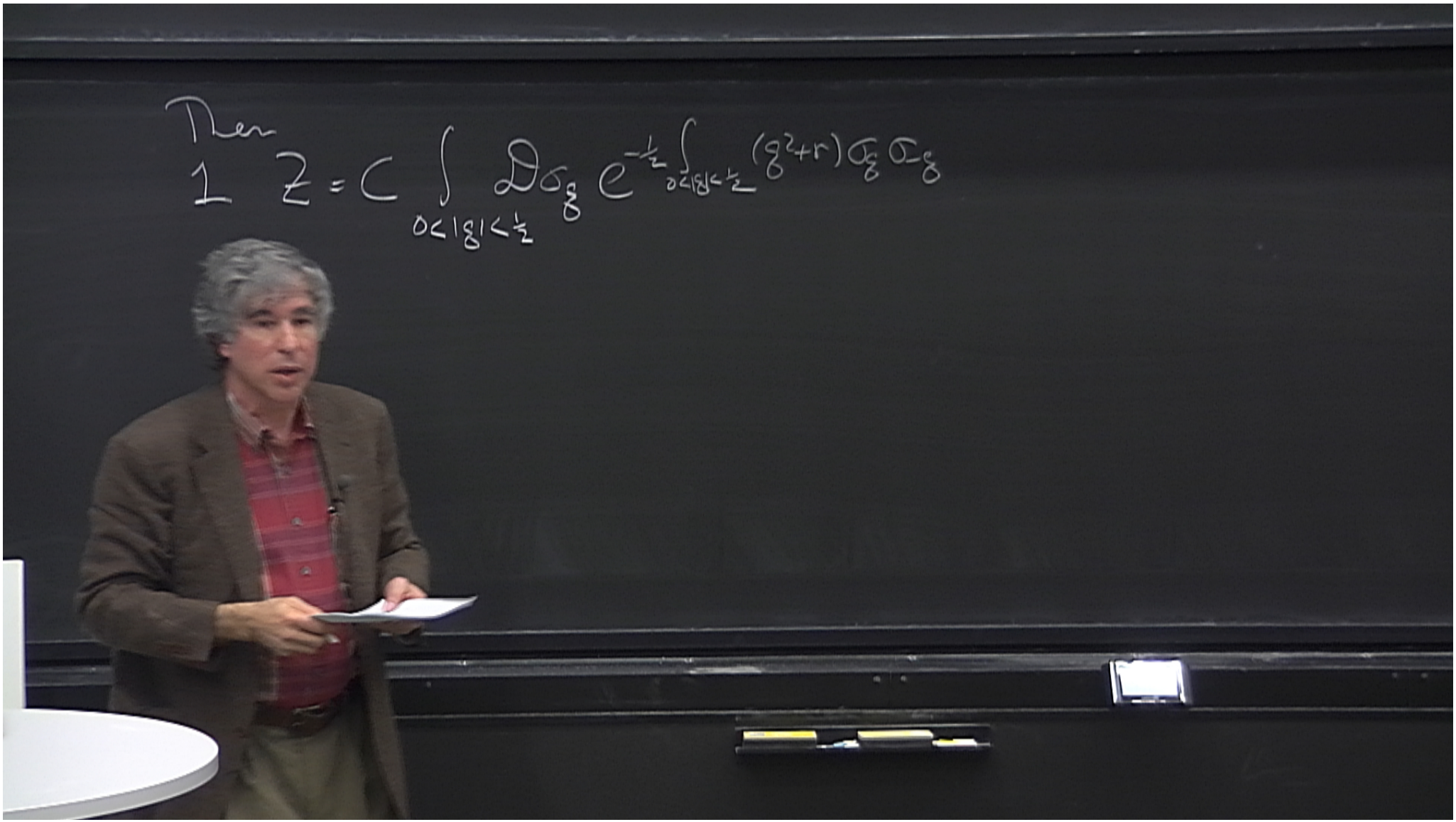
$= b\left(\frac{T - 2dJ/b}{J}\right)$

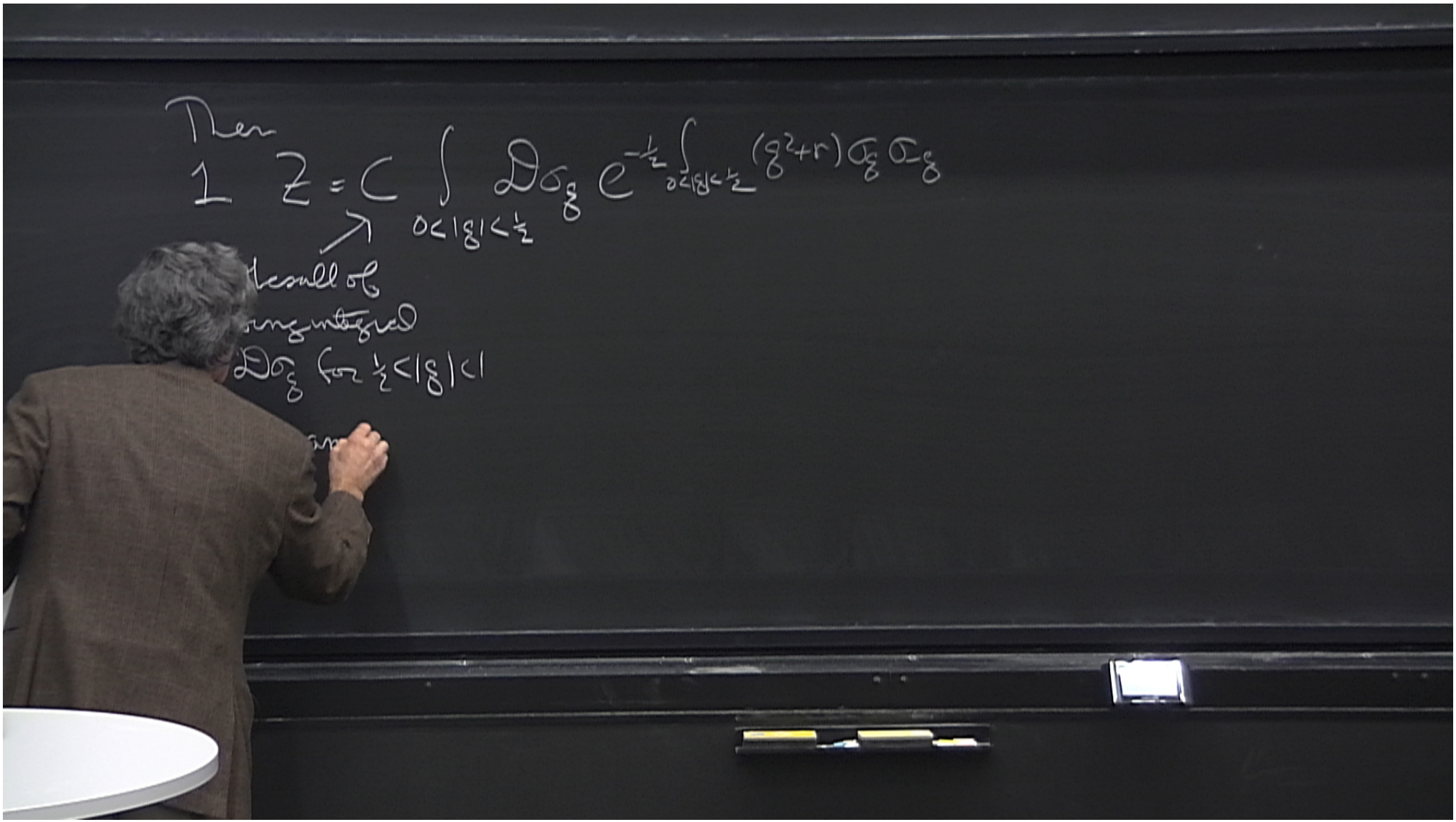
e correlation
ngth for this
roblem is $\xi = \frac{1}{\sqrt{r}}$

alculate this using an RG approach

3-Step Plan

1. Integrate $\mathcal{D}\sigma_\xi$ for $\frac{1}{2} < |\xi| < 1$
2. Transform to new ξ-variables $\xi' = 2\xi$
3. Rescale the $\sigma_{\xi'}$ so that $\sigma_{\xi'} = \zeta\sigma'_{\xi'}$ where ζ is defined so that the coefficient of $\xi'^2\sigma'_{\xi'}\sigma'_{-\xi'}$ is 1.


Then
1 $Z = C \int_{0<|g|<\frac{1}{2}} \mathcal{D}\sigma_g \, e^{-\frac{1}{2}\int_{0<|g|<\frac{1}{2}} (g^2+r)\sigma_g \sigma_g}$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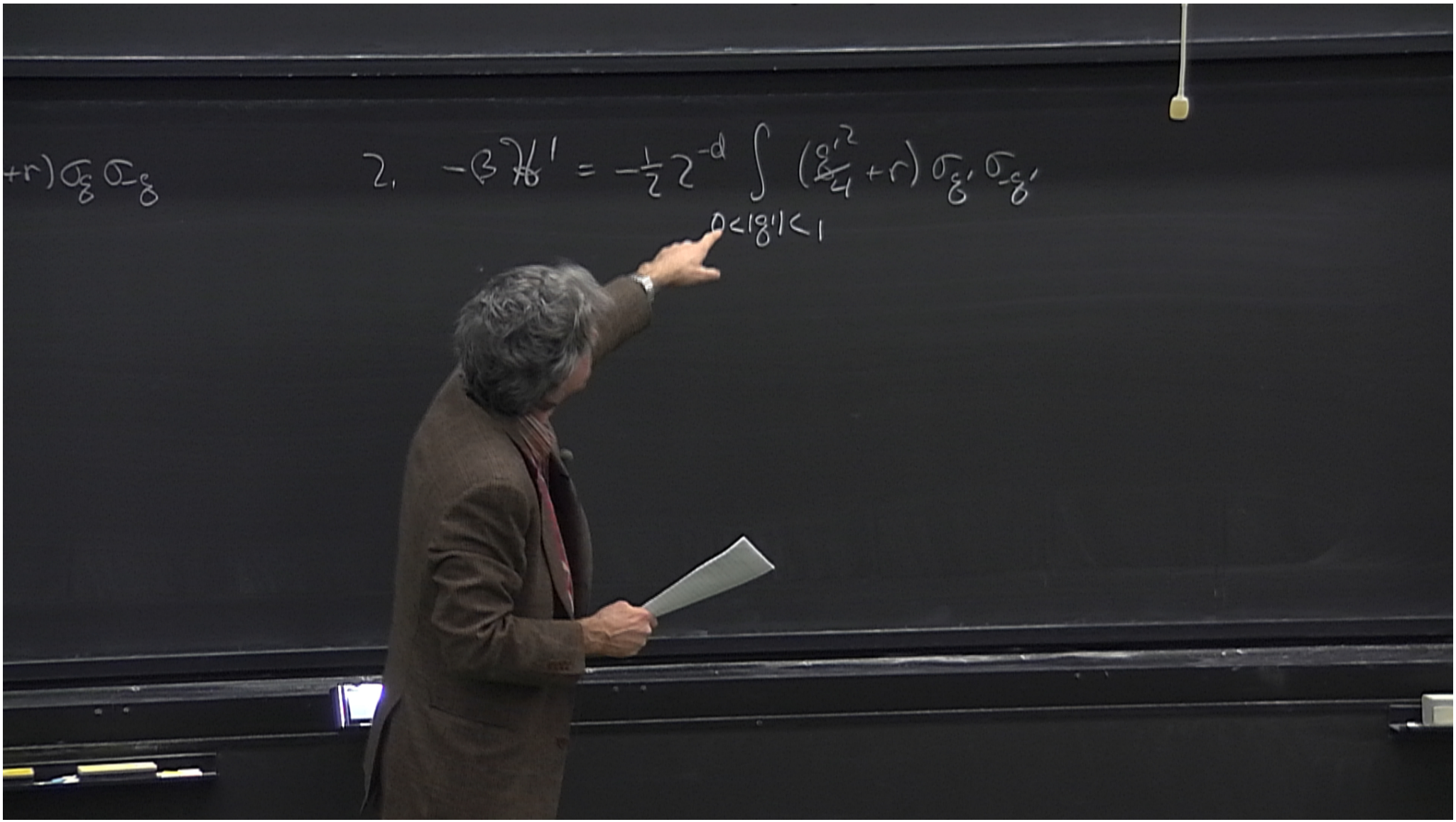
1 $Z = C \int_{0<|g|<\frac{1}{2}} \mathcal{D}\sigma_g \, e^{-\frac{1}{2}\int_{0<|g|<\frac{1}{2}} (g^2+r)\sigma_g \sigma_{-g}}$
result of
integral
$\mathcal{D}\sigma_g$ for $\frac{1}{2}<|g|<1$

Then

$$1 \quad Z = C \int_{0<|g|<\frac{1}{2}} \mathcal{D}\sigma_g \, e^{-\frac{1}{2}\int_{0<|g|<\frac{1}{2}} (g^2+r)\sigma_g \sigma_g}$$

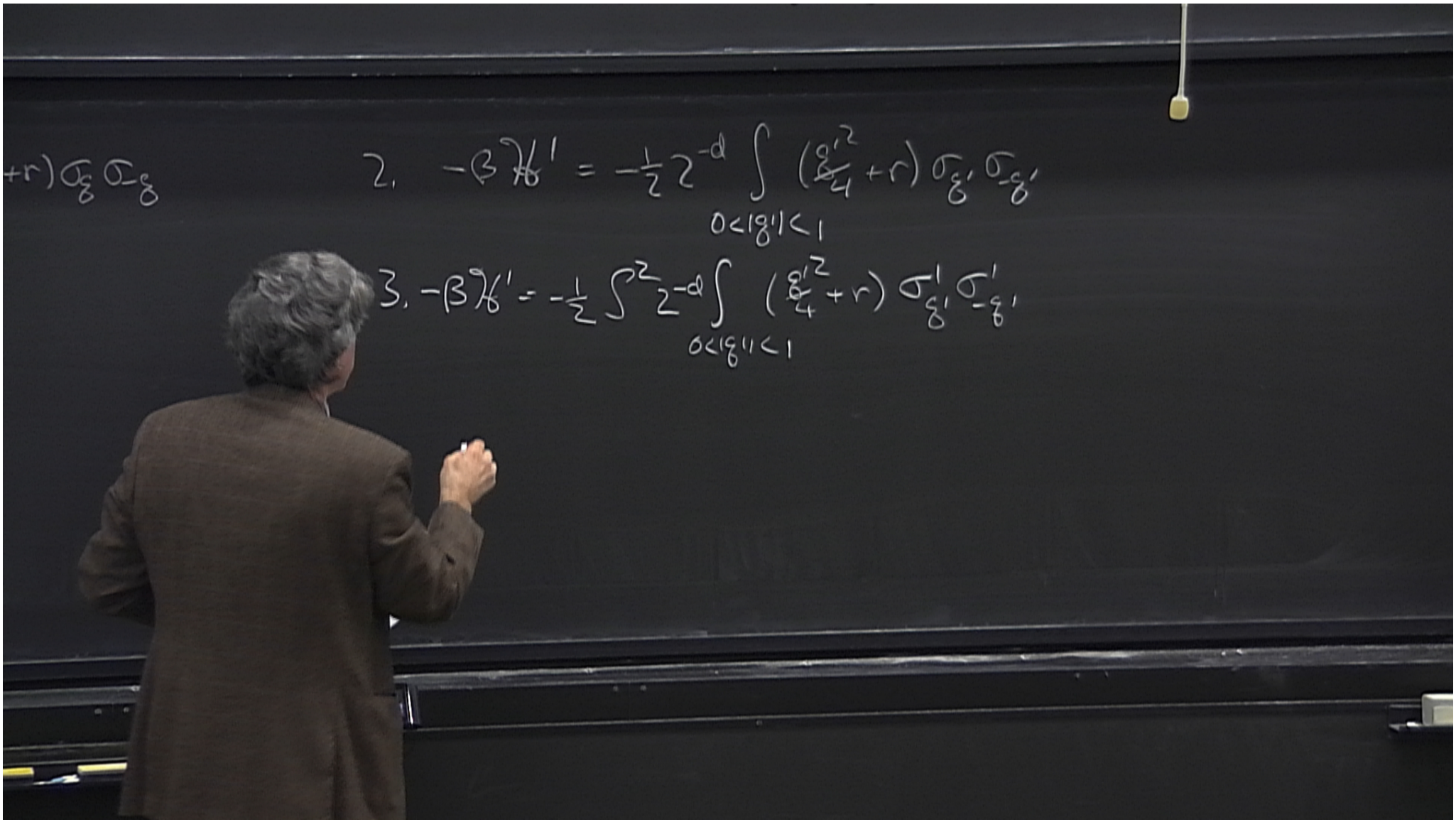
the result of doing integral $\mathcal{D}\sigma_g$ for $\frac{1}{2}<|g|<1$

C is an analytic function of r near $r=0$


+r) σq σq
2. −β H' = −½ 2^−d ∫ (q'²/4 + r) σq' σ−q'
0<|q'|<1

$+r)\sigma_q \sigma_q$

2. $-\beta \mathcal{H}' = -\frac{1}{2} 2^{-d} \int_{0<|q'|<1} \left(\frac{q'^2}{4} + r\right) \sigma_{q'} \sigma_{-q'}$


+r) σ_q σ_{-q}
2. −β𝒱' = −½ 2^{−d} ∫_{0<|q'|<1} (q'²/4 + r) σ_{q'} σ_{−q'}
3. −β𝒱' = −½ Λ² 2^{−d} ∫_{0<|q'|<1} (q'²/4 + r) σ'_{q'} σ'_{−q'}

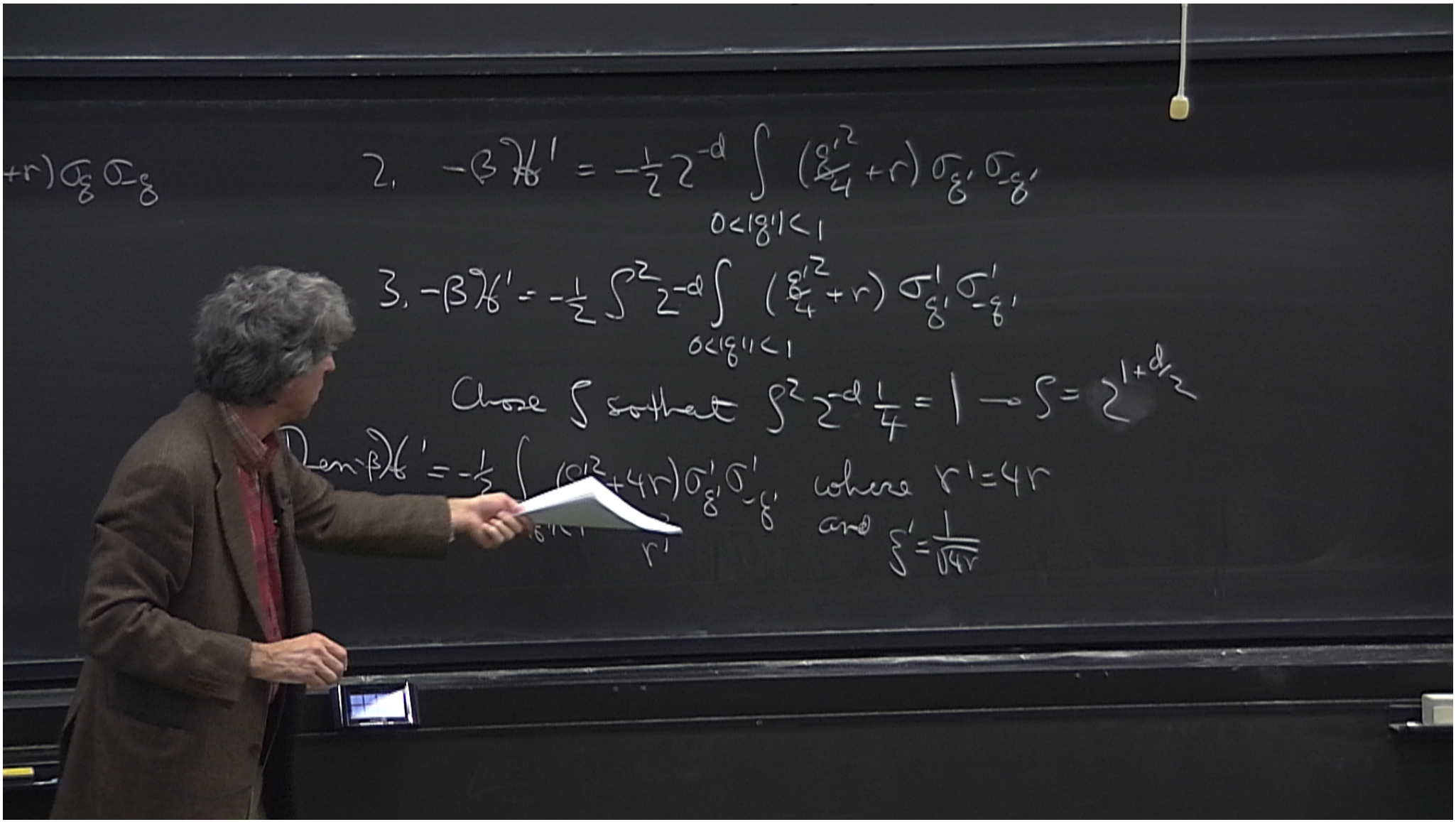
$+r)\sigma_q\sigma_{-q}$

2. $$-\beta\mathcal{H}' = -\frac{1}{2}z^{-d}\int_{0<|q'|<1}\left(\frac{q'^2}{4}+r\right)\sigma_{q'}\sigma_{-q'}$$

3. $$-\beta\mathcal{H}' = -\frac{1}{2}\zeta^2 z^{-d}\int_{0<|q'|<1}\left(\frac{q'^2}{4}+r\right)\sigma'_{q'}\sigma'_{-q'}$$

Chose ζ so that $\zeta^2 z^{-d}\frac{1}{4} = 1 \rightarrow \zeta = z^{1+d/2}$

Then $$-\beta\mathcal{H}' = -\frac{1}{2}\int_{0<|q'|<1}(q'^2+4r)\sigma'_{q'}\sigma'_{-q'}$$


+r) σ_q σ_q
2. −βℋ' = −½ 2^{−d} ∫ (q'²/4 + r) σ_{q'} σ_{−q'}
0<|q'|<1
3. −βℋ' = −½ ζ² 2^{−d} ∫ (q'²/4 + r) σ'_{q'} σ'_{−q'}
0<|q'|<1
Chose ζ so that ζ² 2^{−d} 1/4 = 1 → ζ = 2^{1+d/2}
Then −βℋ' = −½ ∫ (q'² + 4r) σ'_{q'} σ'_{−q'} where r' = 4r
and ξ' = 1/√4r

2. $-\beta\mathcal{H}' = -\frac{1}{2} z^{-d} \int_{0<|q'|<1} \left(\frac{q'^2}{4} + r\right) \sigma_{q'} \sigma_{-q'}$

3. $-\beta\mathcal{H}' = -\frac{1}{2} \zeta^2 z^{-d} \int_{0<|q'|<1} \left(\frac{q'^2}{4} + r\right) \sigma'_{q'} \sigma'_{-q'}$

Chose ζ so that $\zeta^2 z^{-d} \frac{1}{4} = 1 \rightarrow \zeta = z^{1+d/2}$

$= -\frac{1}{2} \int_{0<|q'|<1} (q'^2 + \underbrace{4r}_{r'}) \sigma'_{q'} \sigma'_{-q'}$ where $r' = 4r$ and $\xi' = \frac{1}{\sqrt{4r}} = \frac{1}{2}\xi$

$r = 0$ is a fixed point

the flow is away from $r = 0$.

Consider the $u\sigma^4$ model $(u \neq 0)$

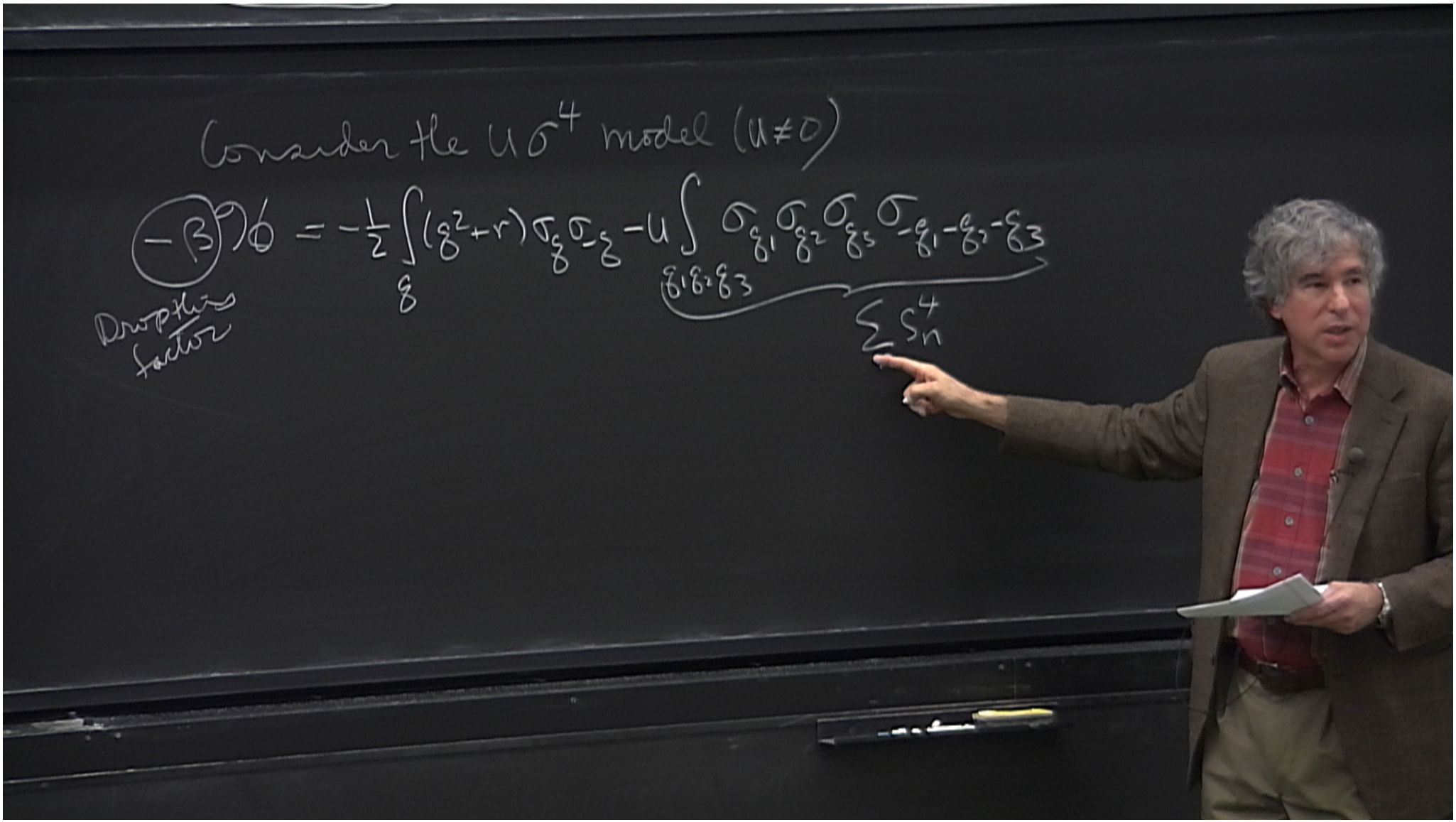
$$-\beta \mathcal{H} = -\frac{1}{2}\int_{q} (q^2 + r)\,\sigma_q \sigma_{-q} - u \int_{q_1 q_2 q_3} \sigma_{q_1} \sigma_{q_2} \sigma_{q_3} \sigma_{-q_1 - q_2 - q_3}$$

Drop this factor

Consider the $u\sigma^4$ model $(u \neq 0)$

$$-\beta \mathcal{H} = -\frac{1}{2}\int_{q} (q^2+r)\,\sigma_q \sigma_{-q} - u \underbrace{\int_{q_1 q_2 q_3} \sigma_{q_1}\sigma_{q_2}\sigma_{q_3}\sigma_{-q_1-q_2-q_3}}_{\sum_n S_n^4}$$

Dropping factor


Consider the $u\sigma^4$ model ($u \neq 0$)
$-\beta \mathcal{H} = -\frac{1}{2}\int_{q} (q^2+r)\sigma_q \sigma_{-q} - u \underbrace{\int_{q_1 q_2 q_3} \sigma_{q_1}\sigma_{q_2}\sigma_{q_3}\sigma_{-q_1-q_2-q_3}}_{\sum S_n^4}$
Drop this factor

Consider the $u\sigma^4$ model ($u \neq 0$)

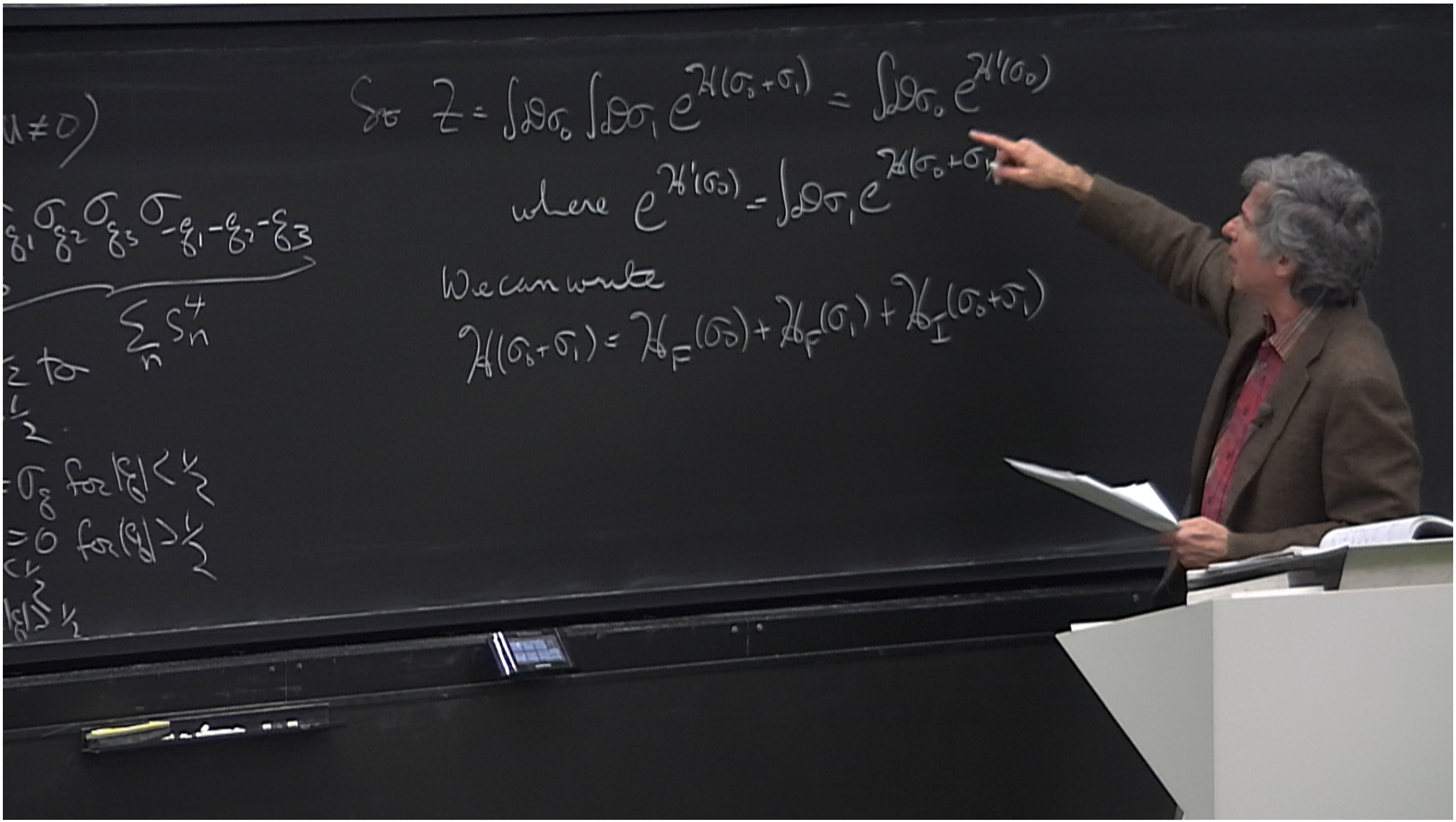
$$-\beta \mathcal{H} = -\frac{1}{2}\int_{\xi} (\xi^2 + r)\,\sigma_{\xi}\sigma_{-\xi} - u \underbrace{\int_{\xi_1\xi_2\xi_3} \sigma_{\xi_1}\sigma_{\xi_2}\sigma_{\xi_3}\sigma_{-\xi_1-\xi_2-\xi_3}}_{\sum_n S_n^4}$$

Drop this factor

Couples σ_ξ with $|\xi| < \frac{1}{2}$ to others with $|\xi| > \frac{1}{2}$.

Write $\sigma_\xi = \sigma_{0\xi} + \sigma_{1\xi}$ where $\sigma_{0\xi} = \sigma_\xi$ for $|\xi| < \frac{1}{2}$
$= 0$ for $|\xi| > \frac{1}{2}$

$\sigma_{1\xi} = 0$ for $|\xi| < \frac{1}{2}$
$= \sigma_\xi$ for $|\xi| > \frac{1}{2}$


So $Z = \int \mathcal{D}\sigma_0 \int \mathcal{D}\sigma_1 \, e^{\mathcal{H}(\sigma_0+\sigma_1)} = \int \mathcal{D}\sigma_0 \, e^{\mathcal{H}'(\sigma_0)}$
where $e^{\mathcal{H}'(\sigma_0)} = \int \mathcal{D}\sigma_1 \, e^{\mathcal{H}(\sigma_0+\sigma_1)}$
We can write
$\mathcal{H}(\sigma_0+\sigma_1) = \mathcal{H}_F(\sigma_0) + \mathcal{H}_F(\sigma_1) + \mathcal{H}_I(\sigma_0+\sigma_1)$
$\sum_n S_n^4$

where $f(x_i) = 0$

$$-(\beta)\lambda_0 = -\frac{1}{2}\int_{q}(q^2+r)\sigma_q\sigma_{-q} - u\int_{q_1,q_2,q_3}\sigma_{q_1}\sigma_{q_2}\sigma_{q_3}\sigma_{-q_1-q_2-q_3}$$

Dropping factor

$\sum_n S_n^4$

where

We can w

$\mathcal{H}(\sigma_q + \sigma$

where $\mathcal{H}$

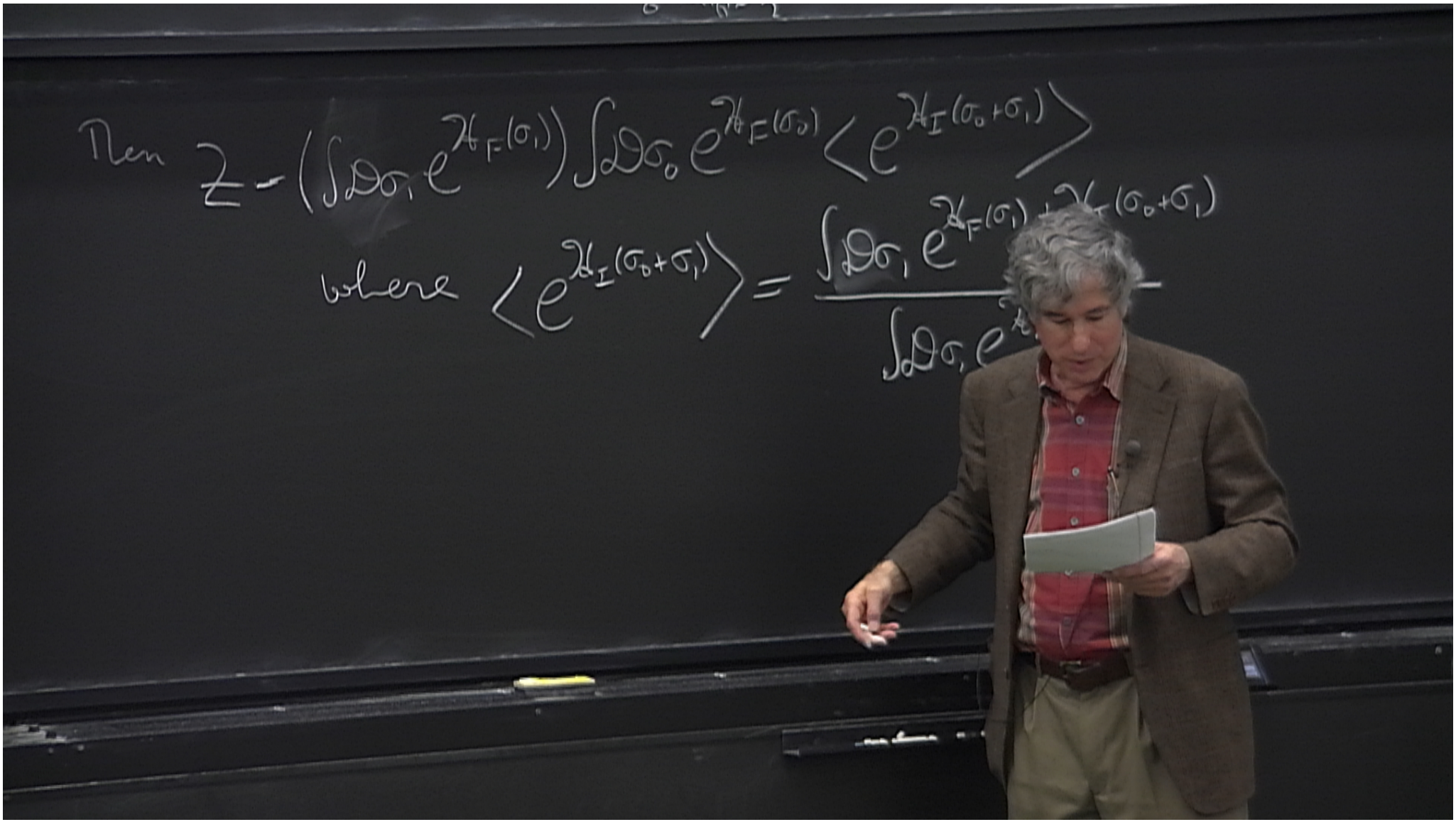
Couples σ_q with $|q| < \frac{1}{2}$ to others with $|q| > \frac{1}{2}$

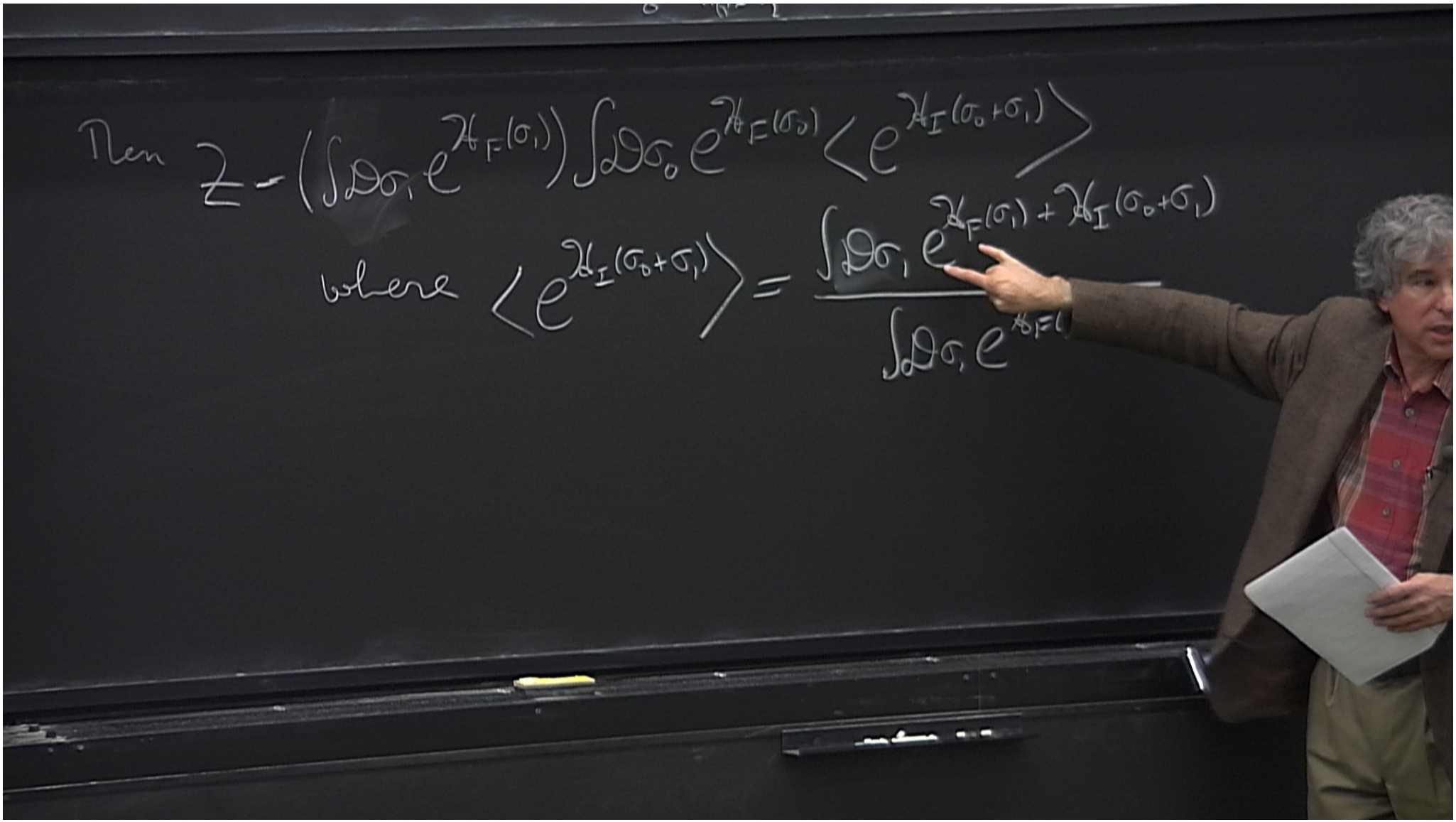
Write $\sigma_q = \sigma_{0q} + \sigma_{1q}$ where $\sigma_{0q} = \sigma_q$ for $|q| < \frac{1}{2}$

$= 0$ for $|q| > \frac{1}{2}$

$\sigma_{1q} = 0$ for $|q| < \frac{1}{2}$

$= \sigma_q$ for $|q| > \frac{1}{2}$


Then $Z = \left(\int \mathcal{D}\sigma_1 e^{\mathcal{H}_F(\sigma_1)}\right) \int \mathcal{D}\sigma_0 e^{\mathcal{H}_F(\sigma_0)} \langle e^{\mathcal{H}_I(\sigma_0+\sigma_1)} \rangle$
where $\langle e^{\mathcal{H}_I(\sigma_0+\sigma_1)} \rangle = \frac{\int \mathcal{D}\sigma_1 e^{\mathcal{H}_F(\sigma_1)+\mathcal{H}_I(\sigma_0+\sigma_1)}}{\int \mathcal{D}\sigma_1 e^{\mathcal{H}}}$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where $\left\langle e^{\mathcal{H}_I(\sigma_0+\sigma_1)} \right\rangle = \dfrac{\int \mathcal{D}\sigma_1 \, e^{\mathcal{H}_F(\sigma_1) + \mathcal{H}_I(\sigma_0+\sigma_1)}}{\int \mathcal{D}\sigma_1 \, e^{\mathcal{H}_F(}}$

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Then $Z = \left(\int \mathcal{D}\sigma_1 \, e^{\mathcal{H}_F(\sigma_1)}\right) \int \mathcal{D}\sigma_0 \, e^{\mathcal{H}_F(\sigma_0)} \left\langle e^{\mathcal{H}_I(\sigma_0+\sigma_1)} \right\rangle$

where $\left\langle e^{\mathcal{H}_I(\sigma_0+\sigma_1)} \right\rangle = \dfrac{\int \mathcal{D}\sigma_1 \, e^{\mathcal{H}_F(\sigma_1) + \mathcal{H}_I(\sigma_0+\sigma_1)}}{\int \mathcal{D}\sigma_1 \, e^{\mathcal{H}_F(\sigma_1)}}$

Use cumulant expansion

$$\left\langle e^{\mathcal{H}_I} \right\rangle = e^{\langle \mathcal{H}_I \rangle + \frac{1}{2}\left(\langle \mathcal{H}_I^2 \rangle - \langle \mathcal{H}_I \rangle^2\right)}$$

Then $Z = \left(\int \mathcal{D}\sigma_1 \, e^{\mathcal{H}_F(\sigma_1)}\right) \int \mathcal{D}\sigma_0 \, e^{\mathcal{H}_F(\sigma_0)} \left\langle e^{\mathcal{H}_I(\sigma_0+\sigma_1)} \right\rangle$

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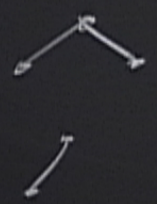
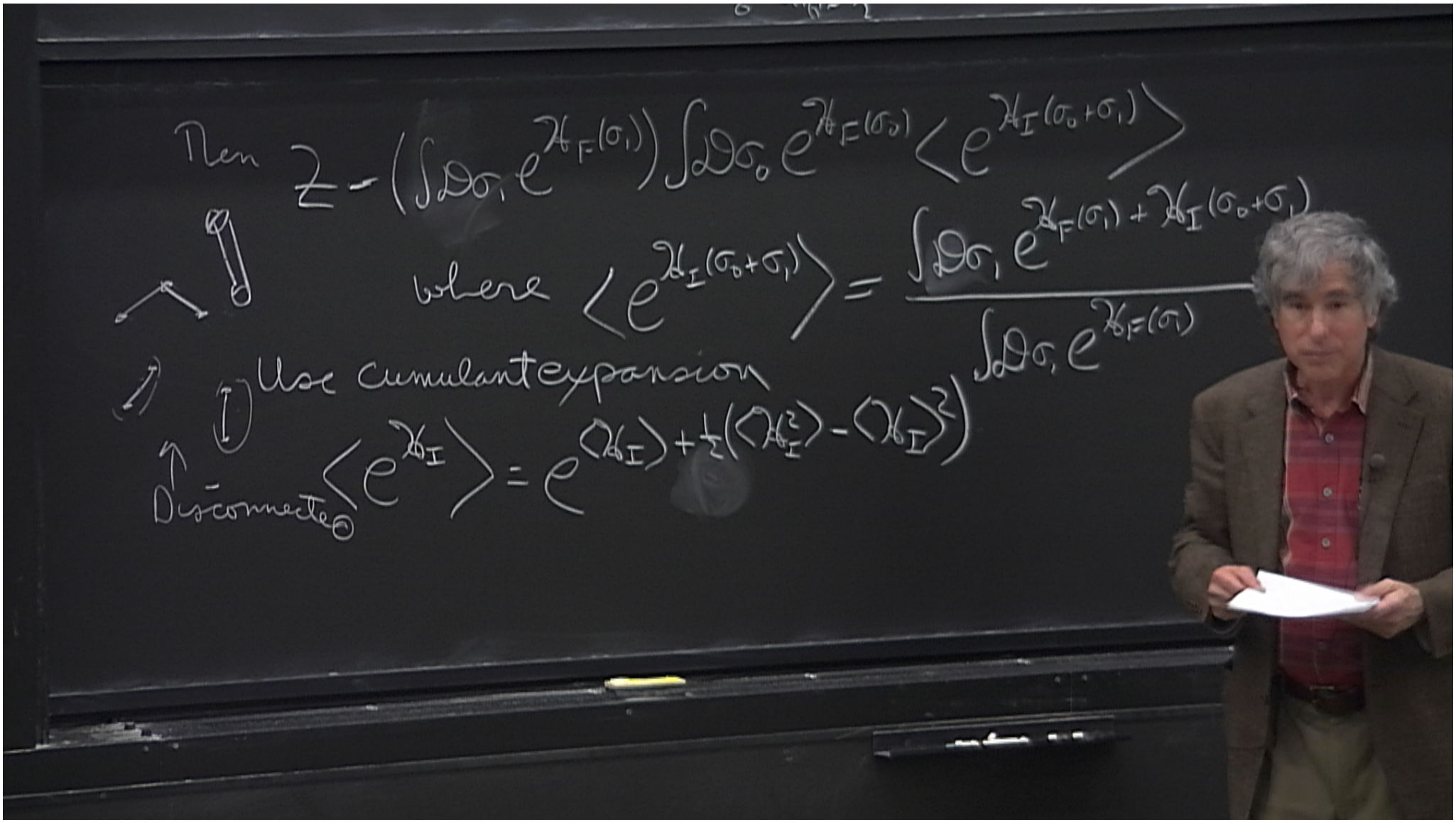


Use cumulant expansion

Disconnected $\left\langle e^{\mathcal{H}_I} \right\rangle = e^{\langle \mathcal{H}_I \rangle + \frac{1}{2}\left(\langle \mathcal{H}_I^2 \rangle - \langle \mathcal{H}_I \rangle^2\right)}$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where $\left\langle e^{\mathcal{H}_I(\sigma_0+\sigma_1)} \right\rangle = \frac{\int \mathcal{D}\sigma_1 e^{\mathcal{H}_F(\sigma_1) + \mathcal{H}_I(\sigma_0+\sigma_1)}}{\int \mathcal{D}\sigma_1 e^{\mathcal{H}_F(\sigma_1)}}$
Use cumulant expansion
Disconnected
$\left\langle e^{\mathcal{H}_I} \right\rangle = e^{\langle \mathcal{H}_I \rangle + \frac{1}{2}\left(\langle \mathcal{H}_I^2 \rangle - \langle \mathcal{H}_I \rangle^2\right)}$