

Title: Statistical Mechanics - Lecture 9

Date: Nov 22, 2012 10:30 AM

URL: <http://pirsa.org/12110029>

Abstract:

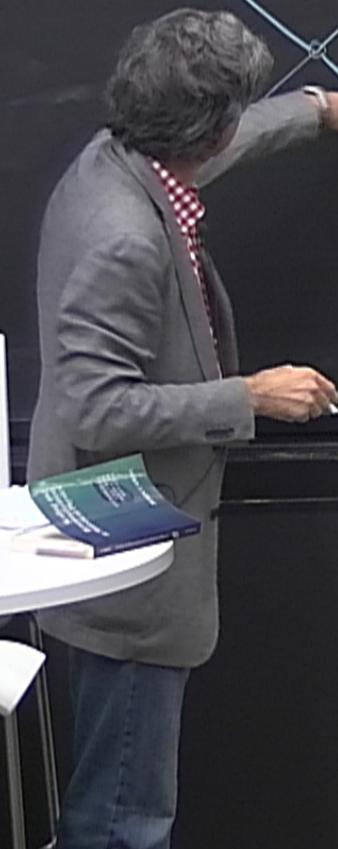


Higher Dimension



Higher Dimension

Trace over X -spins
to obtain new
Hamiltonian for O -spins

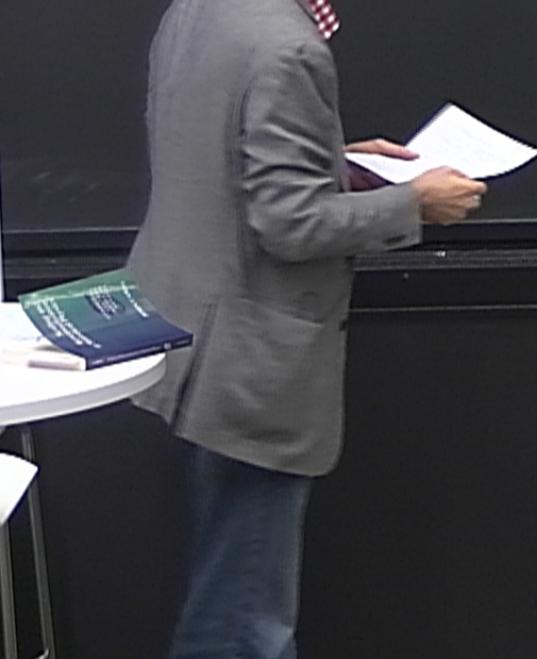




Higher Dimension

Trace over X-spins
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Hamiltonian for O-spins

$$-\beta \mathcal{H}_N = K \sum_{\langle i,j \rangle} S_i S_j$$





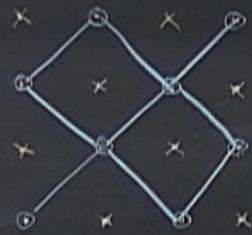
Higher Dimension

Trace over X-spins
to obtain new
Hamiltonian for O-spins

$$-\beta \mathcal{H}_N = K_1 \sum_{\langle i,j \rangle} S_i S_j$$

$$-\beta \mathcal{H}_{N/2}^I = K'_1 \sum_{\substack{\langle i,j \rangle \\ \text{nn Ospins}}} N_i N_j$$





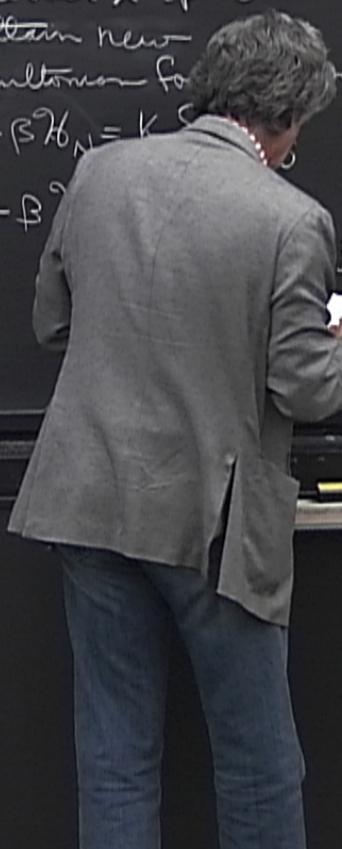
Higher Dimension

Trace over X -spins
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Hamiltonian for

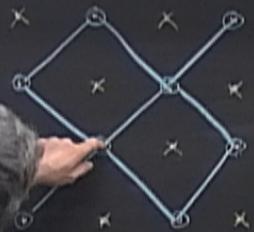
$$-\beta \mathcal{H}_N = K_1$$

$$-\beta^2$$

$$K_1 + K_2$$



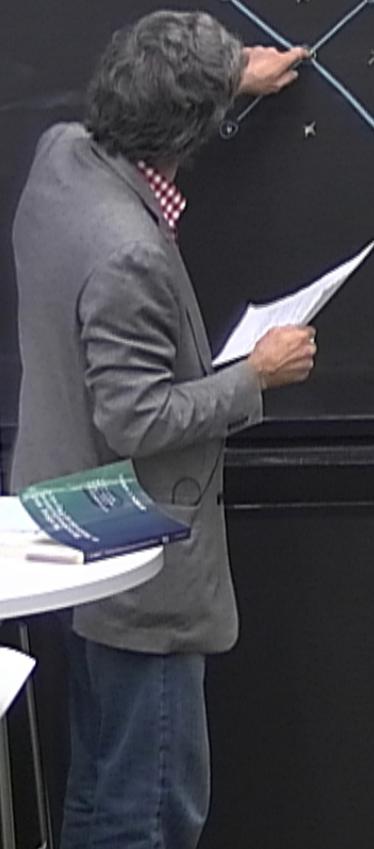
Higher Dimension



Trace over x -spins
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$$-\beta \mathcal{H}_N = K_1 \sum_{\langle i,j \rangle} S_i S_j$$

$$-\beta \mathcal{H}'_{N/2} = K'_1 \sum_{\substack{\langle i,j \rangle \\ \text{nn spins}}} M_i M_j + K'_2 \sum_{\substack{\langle i,j \rangle \\ \text{nnn}}} M_i M_j$$





Higher Dimension

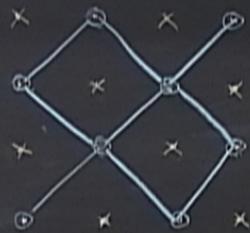
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$$-\beta \mathcal{H}_N = K_1 \sum_{\langle i,j \rangle} S_i S_j$$

$$-\beta \mathcal{H}'_{N/2} = K'_1 \sum_{\substack{\langle i,j \rangle \\ \text{nn spins}}} \mu_i \mu_j + K'_2 \sum_{\langle\langle i,j \rangle\rangle} \mu_i \mu_j + K'_3 \sum_{\langle\langle k \rangle\rangle} \mu_i \mu_j \mu_k \mu_\ell$$

Procedure does not close





Higher Dimension

Trace over X -spins
to obtain new
Hamiltonian for O -spins

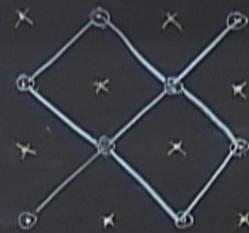
$$-\beta \mathcal{H}_N = K_1 \sum_{\langle i,j \rangle} S_i S_j$$

$$-\beta \mathcal{H}'_{N/2} = K'_1 \sum_{\substack{\langle i,j \rangle \\ i \in O \text{ or } j \in P}} N_i N_j$$

Procedure does not close

Iterations

$$\vdots \sum_{\langle i,j \rangle} M_i M_j M_k M_\ell$$



Higher Dimension

Trace over X -spins
to obtain new
Hamiltonian for O -spins

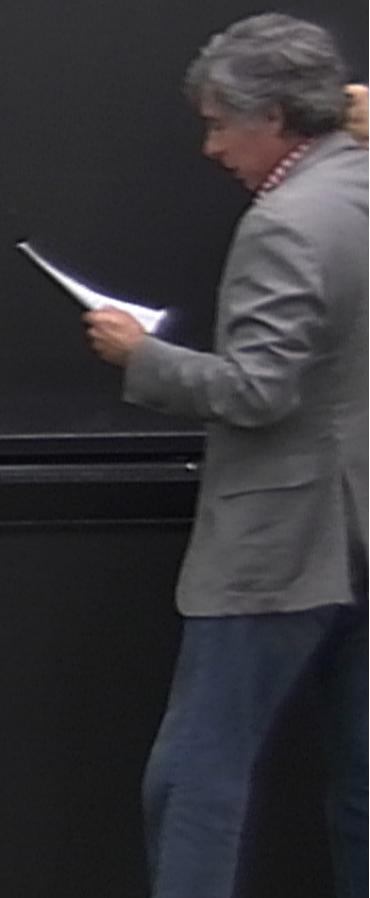
$$-\beta \mathcal{H}_N = K_1 \sum_{\langle i,j \rangle} S_i S_j$$

$$-\beta \mathcal{H}'_{N/2} = K'_1 \sum_{\substack{\langle i,j \rangle \\ \text{nn O-spins}}} \mu_i \mu_j + K'_2 \sum_{\substack{\langle i,j \rangle \\ \text{nnn O-spins}}} \mu_i \mu_j + K'_3 \sum_{\langle j,k \rangle} \mu_i \mu_j \mu_k \mu_\ell$$

Procedure does not close

Approximations can be
used to obtain closed
recursion relations

See B+H Sec 21.2 p



Procedure does not close

Approximations can be
used to obtain closed
recursion relations

See B&H Sec 21.2 pp 556-574
also Sec 21.4 pp 568-574

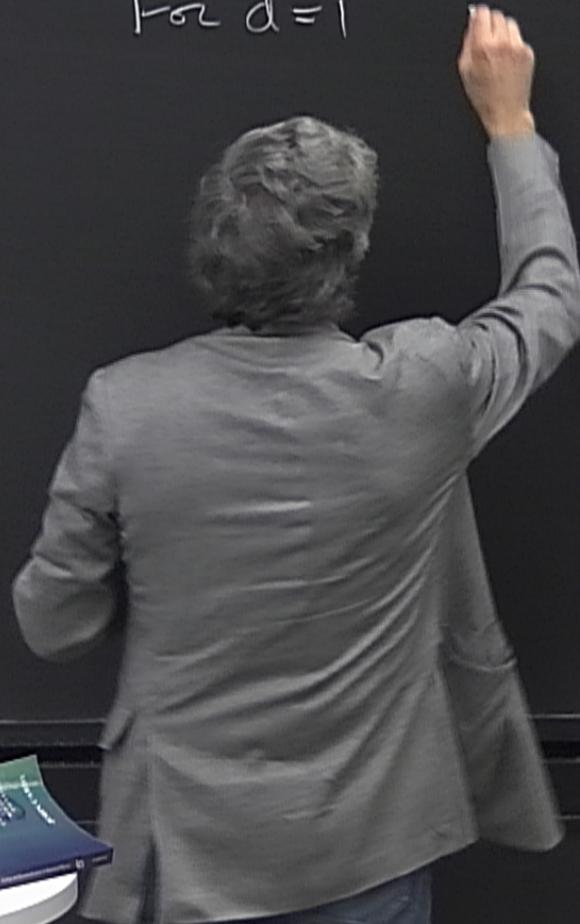
$$\sum_{i,j} \mu_i \mu_j + K_3 \sum_{\{K\}} \mu_i \mu_j \mu_k \mu_l$$

Procedure does not close
Approximations can be
used to obtain closed
recursion relations

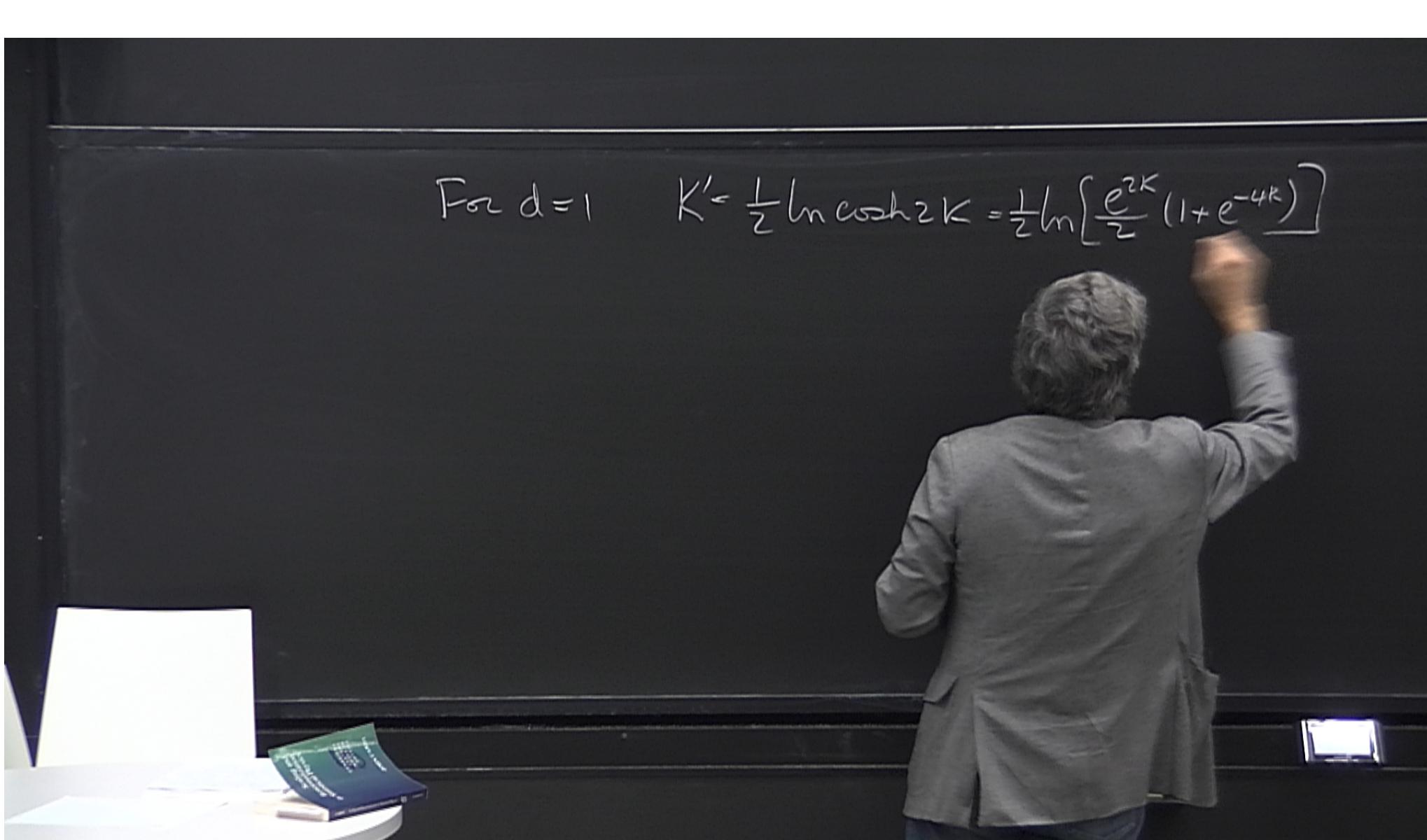
See B+H Sec 21.2 pp 556-574
also Sec 21.4 pp 568-574
Also Nauenberg+Nienhuis PRL 33 1548 (1974)

$$\sum_{i,j} M_i M_j + K_3' \sum_{\{K\}} \mu_i \mu_j \mu_k \mu_\ell$$

For $d=1$

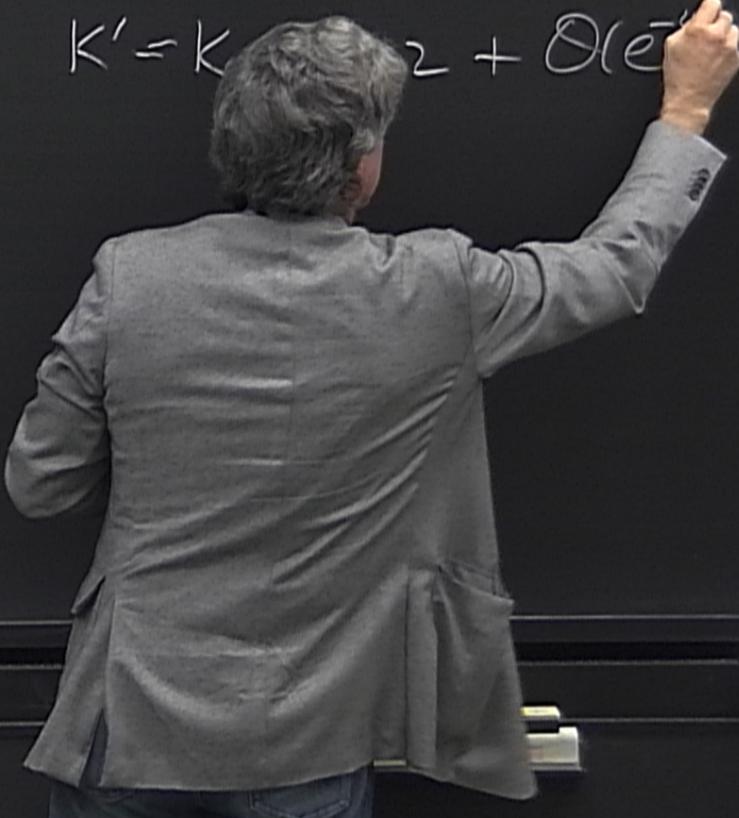


$$\text{For } d=1 \quad K' = \frac{1}{2} \ln \cosh 2K = \frac{1}{2} \ln \left[\frac{e^{2K}}{2} (1 + e^{-4K}) \right]$$



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$$\text{For large } K \quad K' = K - \frac{1}{2} + O(e^{-4K})$$



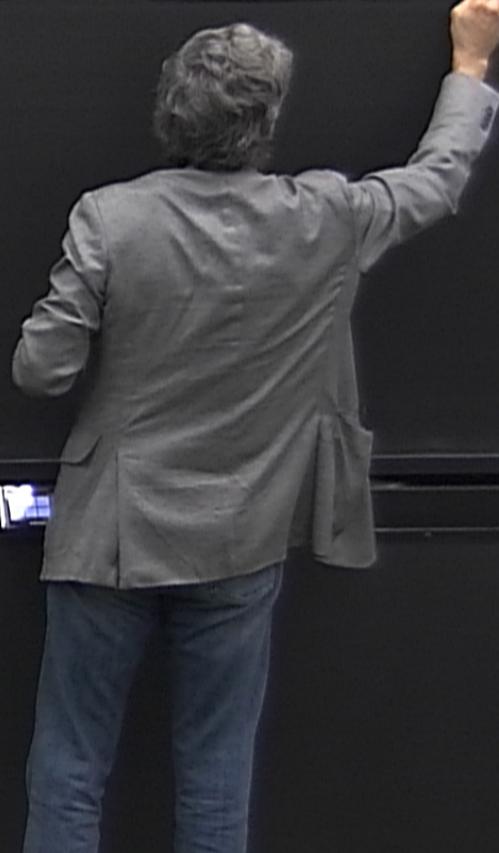
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$$\boxed{s_1^0 \quad s_2^0 \quad s_3^0} \quad \boxed{s_4^0 \quad s_5^0 \quad s_6^0}$$

$$\text{For large } K \quad K' = K - \frac{1}{2} \ln 2 + O(e^{-4K})$$

So $K' \approx K$ but smaller

Why is $K' < K$?



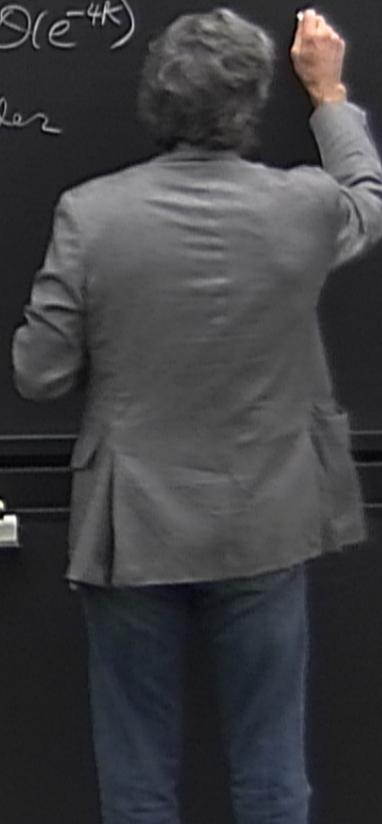
$$\text{For } d=1 \quad K' = \frac{1}{2} \ln \cosh 2K = \frac{1}{2} \ln \left[\frac{e^{2K}}{2} (1 + e^{-4K}) \right]$$

$$\boxed{\frac{U_1 - K' - U_2}{S_1 - S_2} \left[\frac{K' + S_2}{S_1 + S_2 - S_1} \right]}$$

$$\text{For large } K \quad K' = K - \frac{1}{2} \ln 2 + O(e^{-4K})$$

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$$\text{For } d=1 \quad K' = \frac{1}{2} \ln \cosh 2K = \frac{1}{2} \ln \left[\frac{e^{2K}}{2} (1 + e^{-4K}) \right]$$

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So $K' \approx K$ but smaller
Why is $K' < K$?

$$\boxed{\frac{M_1}{S_1} \frac{K'}{S_2} \frac{M_2}{S_3} S_4}$$

$$K' = K \underbrace{\langle S_3 \rangle}_{M_1=1} \underbrace{\langle S_4 \rangle}_{M_2} \lesssim 1 \text{ eV}$$

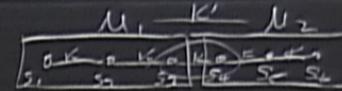


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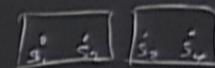
$$\text{For large } K \quad K' = K - \frac{1}{2} \ln 2 + O(e^{-4K})$$

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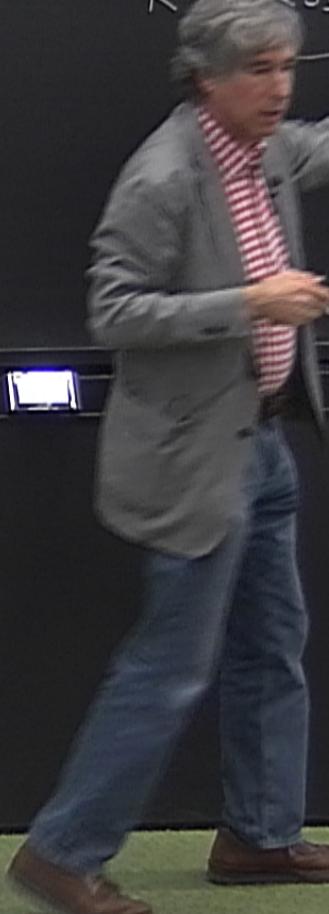


$$K' = \nu(S_3) \nu(S_4)$$



$$K' = K(S_2)_{S_1=1}$$

\rightarrow S_3 and S_4 are large

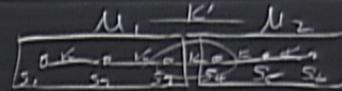


$$\text{For } d=1 \quad K' = \frac{1}{2} \ln \cosh 2K = \frac{1}{2} \ln \left[\frac{e^{2K}}{2} (1 + e^{-4K}) \right]$$

$$\text{For large } K \quad K' = K - \frac{1}{2} \ln 2 + O(e^{-4K})$$

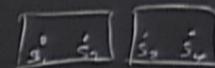
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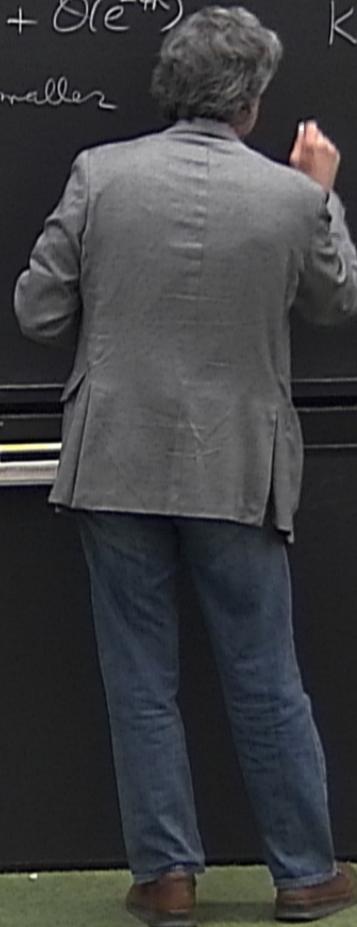


$$K' = \underbrace{K(S_3, S_4)}_{M_1=1, N_2=1}$$

$\lesssim 1$ even for K large



$$K' = K(S_2)_{S_1=1}$$

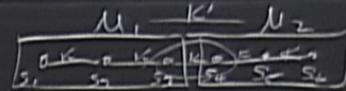


$$\text{For } d=1 \quad K' = \frac{1}{2} \ln \cosh 2K = \frac{1}{2} \ln \left[\frac{e^{2K}}{2} (1 + e^{-4K}) \right]$$

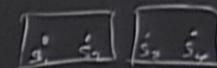
$$\text{For large } K \quad K' = K - \frac{1}{2} \ln 2 + O(e^{-4K})$$

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$$K' = \underbrace{K \langle S_3 \rangle \langle S_4 \rangle}_{M_1=1, N_2=1}$$

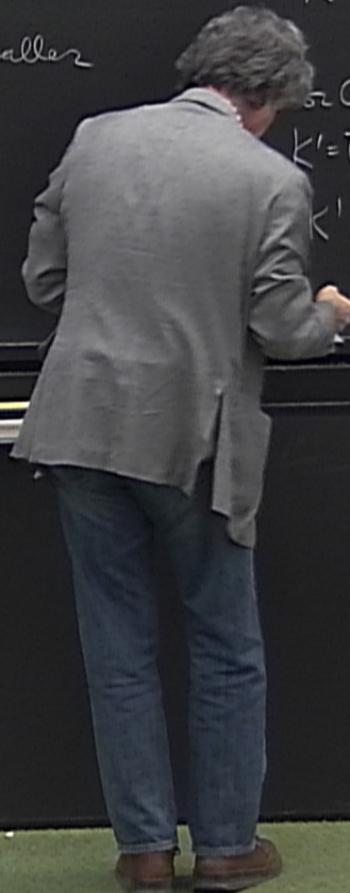


$$K' = K \langle S_2 \rangle_{S_1=1}$$

or Cardy's method

$$K' = \tanh^{-1} \left[\left(\tanh K \right)^{\frac{3}{2}} \right]$$

$$K' =$$

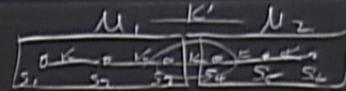


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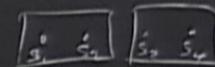
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Why is $K' < K$?



$$K' = K \langle S_3 \rangle \langle S_4 \rangle_{\substack{U_1=1 \\ U_2=1}}$$

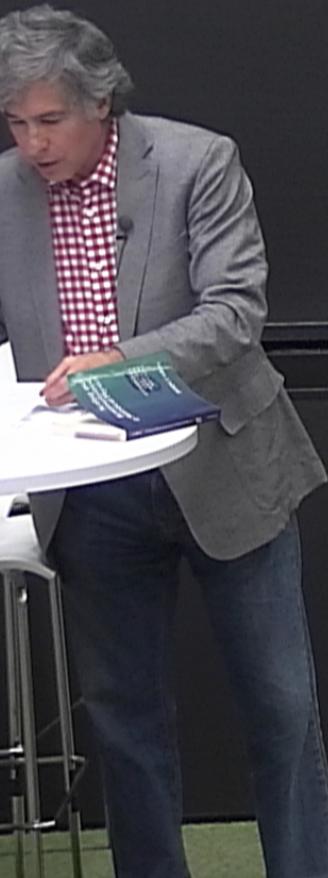


$$K' = K \langle S_2 \rangle_{S_1=1}$$

For Cardy's method

$$\text{For } K \gg 1 \quad K' = \tanh^{-1} \left[\left(\tanh K \right)^2 \right]$$

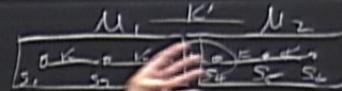
$$K' = K - \frac{1}{2} \ln 3 - \frac{3}{2} \frac{e^{-2K}}{1+e^{-2K}}$$



$$\text{For } d=1 \quad K' = \frac{1}{2} \ln \cosh 2K = \frac{1}{2} \ln \left[\frac{e^{2K}}{2} (1 + e^{-4K}) \right]$$

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$$K' = \underbrace{\langle S_3 \rangle}_{M_1=1} \underbrace{\langle S_4 \rangle}_{N_2=1}$$

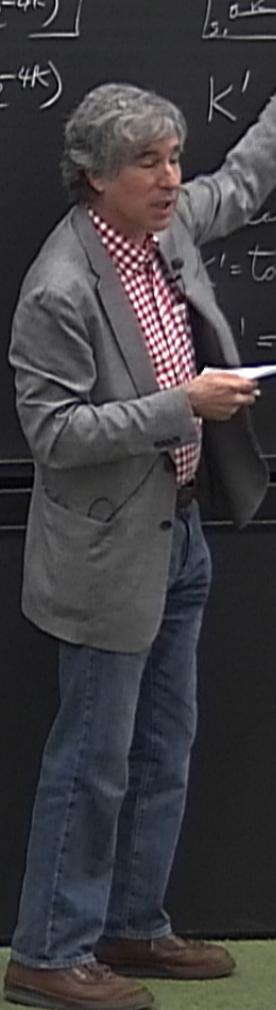


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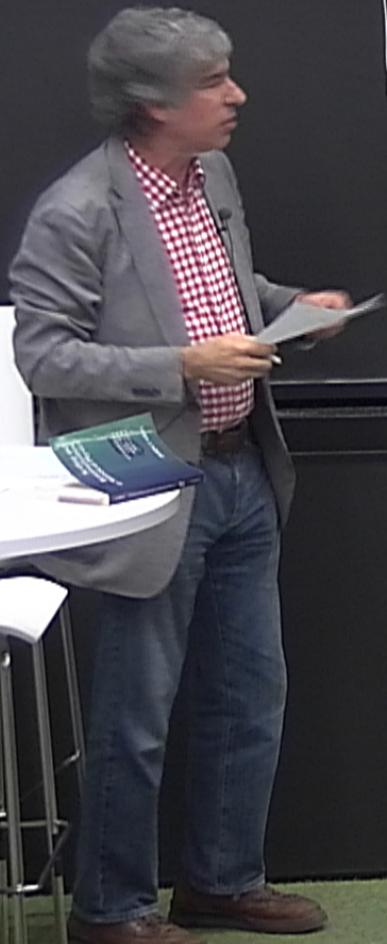
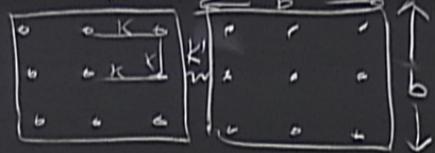
Landy's method ≤ 1 even for K large

$$K' = \tanh^{-1} \left[\left(\tanh K \right)^3 \right]$$

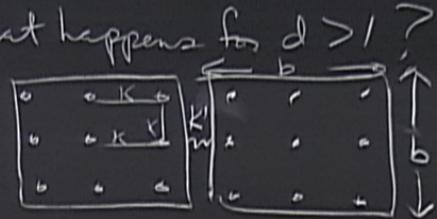
$$K' = K - \frac{1}{2} \ln 3 - \frac{3}{2} \approx 2.45$$



What happens for $d > 1$?

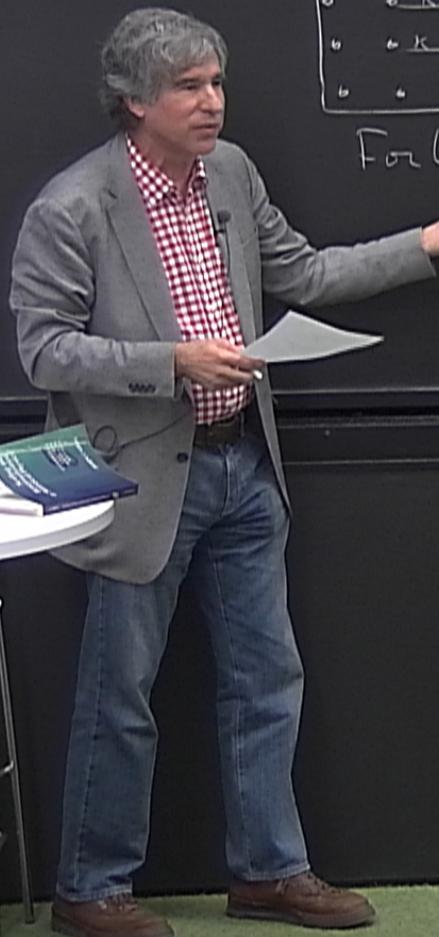


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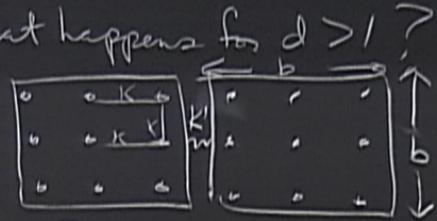


For large K

$$\sim b^{d-1} K$$

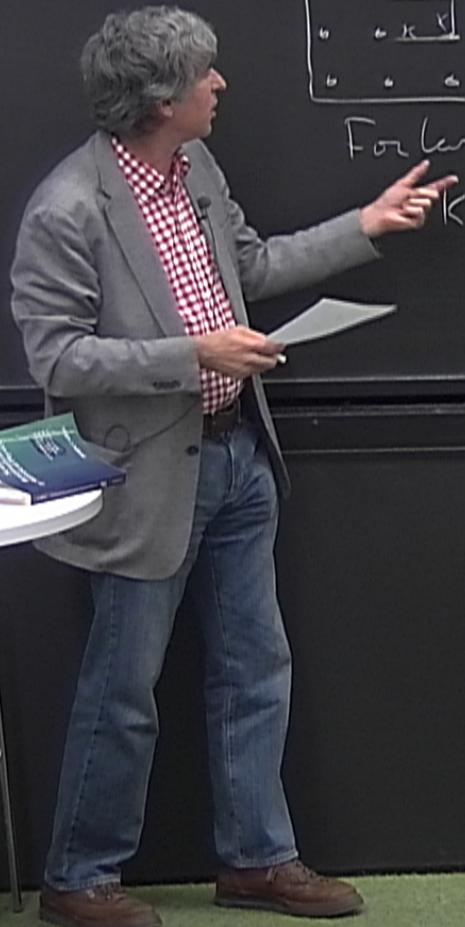


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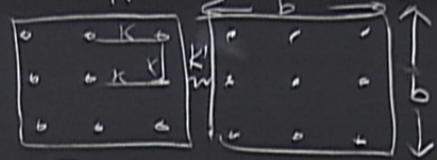


For large K

$$K^d \sim b^{d-1} K$$

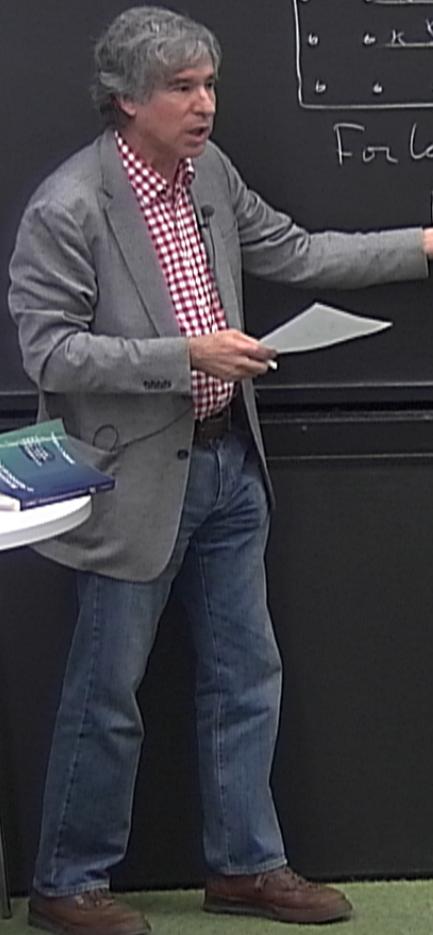


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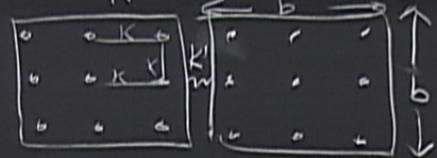


For large K

$$K^{\frac{1}{d}} b^{d-1} K$$

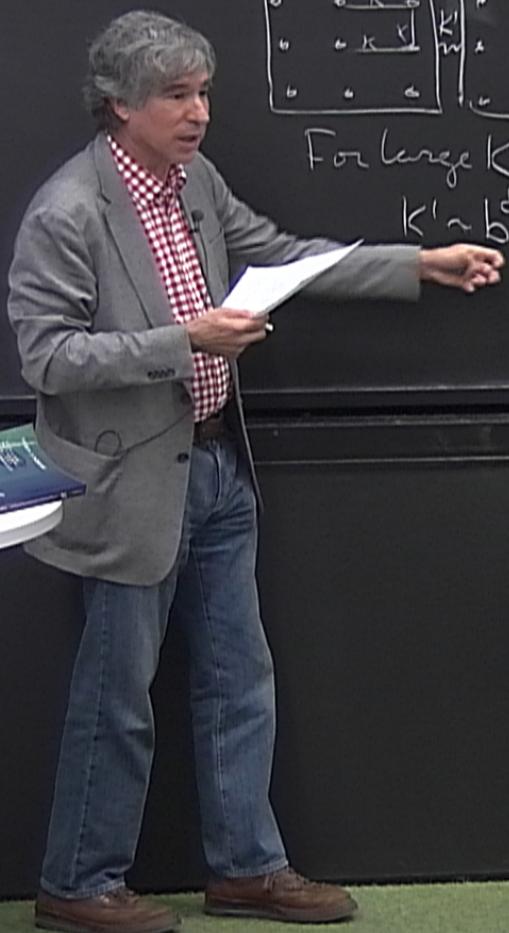


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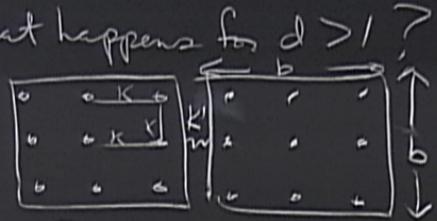


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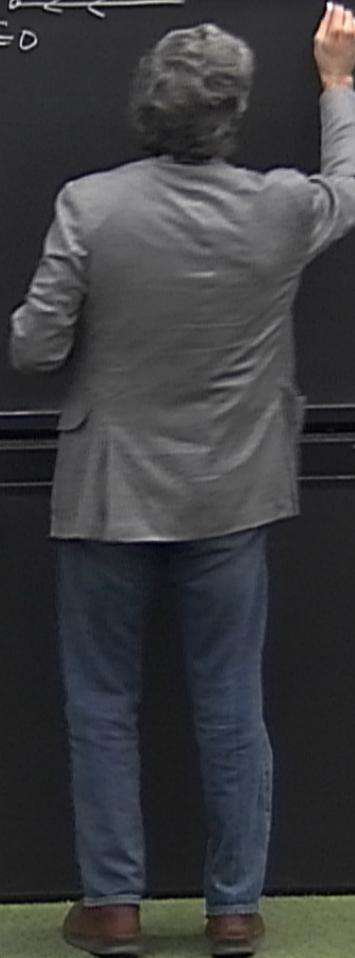
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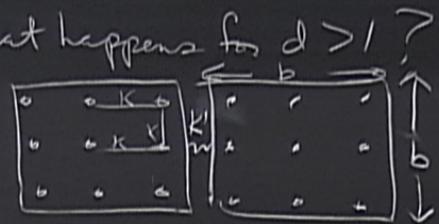
$$T=0 \quad K=\infty \quad a \leftarrow \leftarrow$$

For large K

$$K' \sim b^{d-1} K$$



What happens for $d > 1$?



For large K

$$K' \sim b^{d-1} K$$

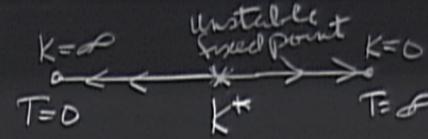
$$\begin{array}{c} K=\infty \\ \leftarrow \quad \rightarrow \\ T=\infty \end{array} \qquad \begin{array}{c} K=0 \\ \leftarrow \quad \rightarrow \\ T=\infty \end{array}$$



What happens for $d > 1$?

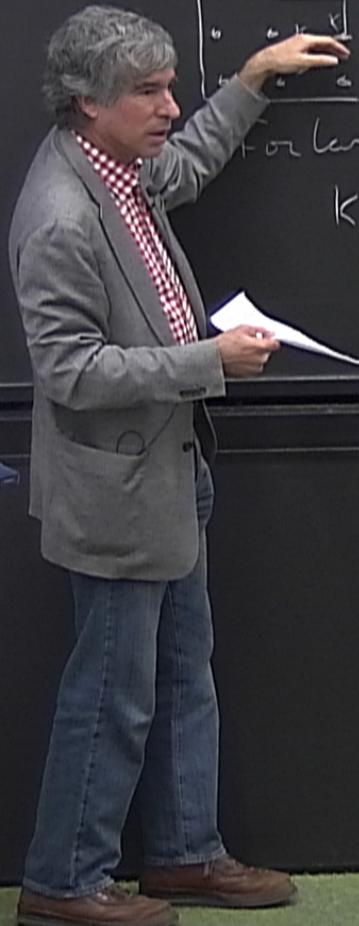
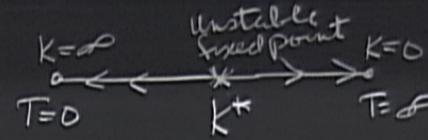
For $d > 1$

$$b^{d-1} K$$

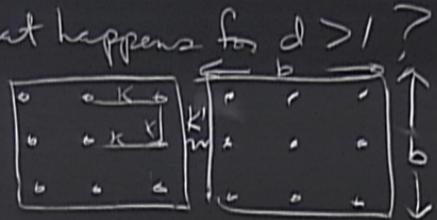


What happens for $d > 1$?

$$\text{For large } K \\ K^l \sim b^{d-1} K$$

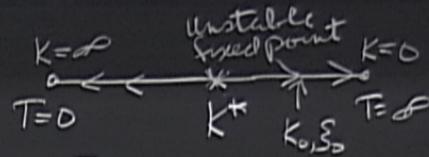


What happens for $d > 1$?



For large K

$$K' \sim b^{d-1} K$$



For small K , K flows to zero.

So $d > 1$ is qualitatively
from $d = 1$.

What is nature of point K^* ?

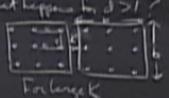
$$-\partial^2 K_{ij} = K_i \sum_{k,l} U_{klj} + K_j \sum_{k,l} U_{ikl} + K_i \sum_{k,l,m} U_{klm} K_m U_{ijl}$$

For det $K = \frac{1}{2} \ln \det(K - i\epsilon) [e^{i\theta} (1 + e^{-i\theta})]$

For large K $K \approx K - i\epsilon \omega + O(\epsilon^{1/2})$

So $K' \approx K$ (not smaller)
Why $K' < K$?

For $\text{diag}(S)$ $S \approx I$ over K large
For $K \gg 1$ $K \approx \text{diag}(S)^{-1}$
 $K' \approx K - i\omega - \frac{1}{2}\omega^2$

What happens for $d > 1$?


For large K $K \approx b^{-d} K$

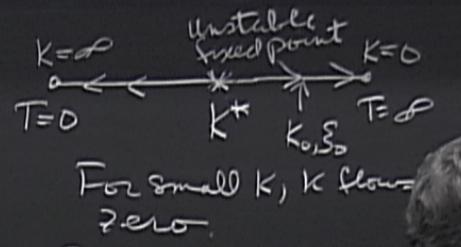
Too small K $K \approx K - i\epsilon \omega$
For small K , K becomes zero.
So for $d > 1$ is qualitatively different from $d=1$.

What is nature of point K' ?
Imaginary at some small K_0
 $S(K') = 0$
The RPA is gone $S(K) \neq S(K')$



What happens for $d > 1$?

For large K
 $K' \sim b^{d-1} K$



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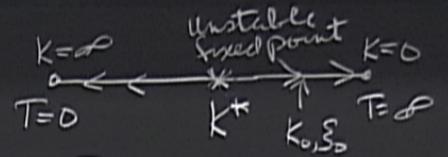
Imagine that at some small K_0
 $\xi(K_0) = \xi_0$

The RG transf gives $\xi(K) = b \xi(K')$
Start from $K^* < K < K_0$ and say that it takes

What happens for $d > 1$?

For large K

$$K' \sim b^{d-1} K$$



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$$\xi(K_0) = \xi_0$$

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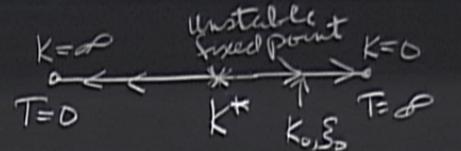
from $K^* \rightarrow K \rightarrow K_0$ and say that
be $N(K)$ steps to get to K_0 .

$$(K)$$

What happens for $d > 1$?

For large K

$$K' \sim b^{d-1} K$$



For small K , K flows to zero.

So $d > 1$ is qualitatively different from $d = 1$.

What is nature of point K^* ?

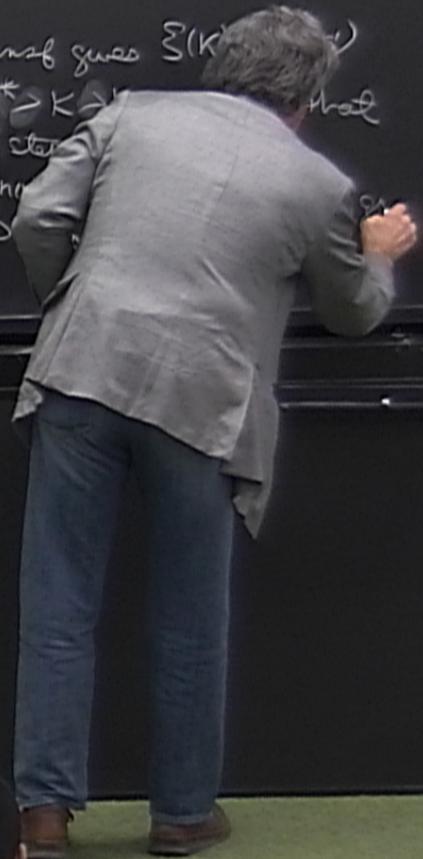
Imagine that at some small K_0

$$\xi(K_0) = \xi_0$$

The RG transf gives $\xi(K')$

Start from $K^* \rightarrow K' \rightarrow \dots$ that
(It takes $N(K)$ steps)

$$\text{Then } \xi(K) = b^{N(K)}$$



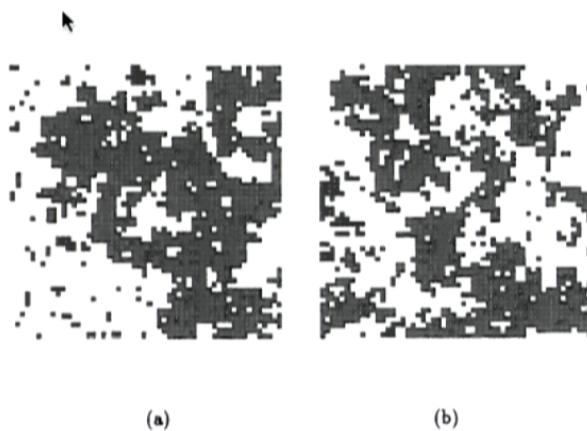


Figure 3.1. Typical configuration of the Ising model at $T = T_c$ (a), and (b) the result of a single block spin transformation.

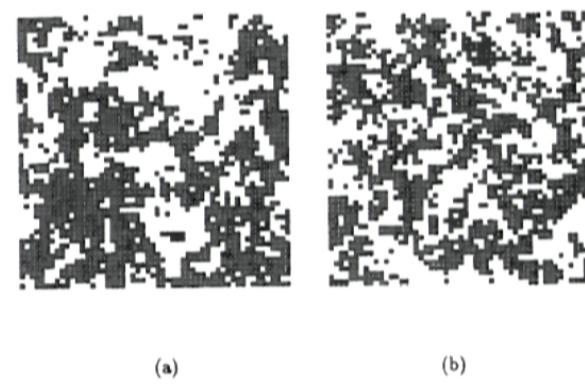
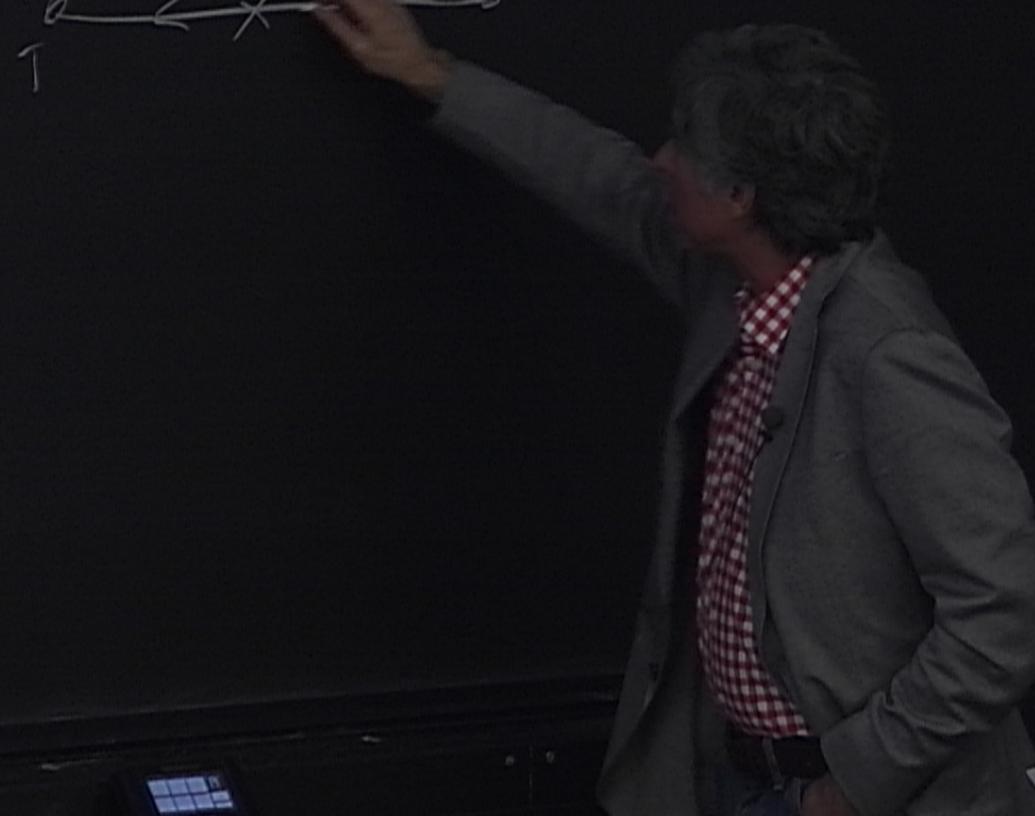
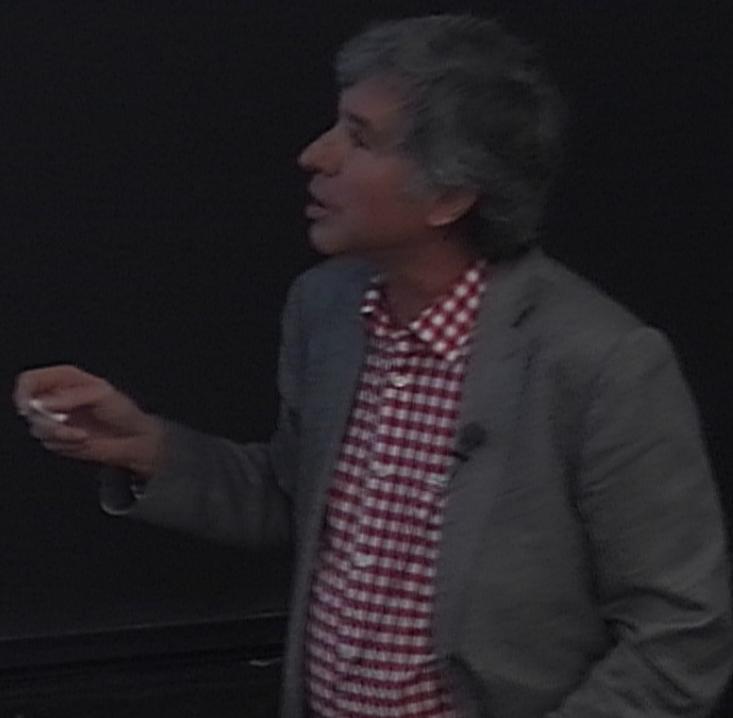


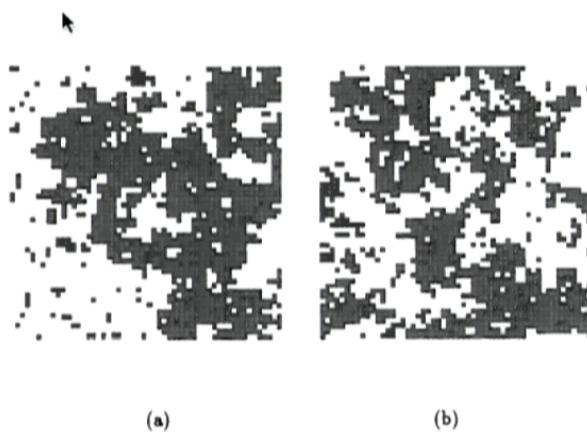
Figure 3.2. Same as Figure 3.1, slightly above the critical temperature.

$$\begin{array}{ccc} r = \infty & & k = 0 \\ q & \xleftarrow{\quad} & \end{array}$$



$$\begin{array}{ccc} r = \infty & | & k = 0 \\ \swarrow & * & \searrow \\ T \end{array}$$

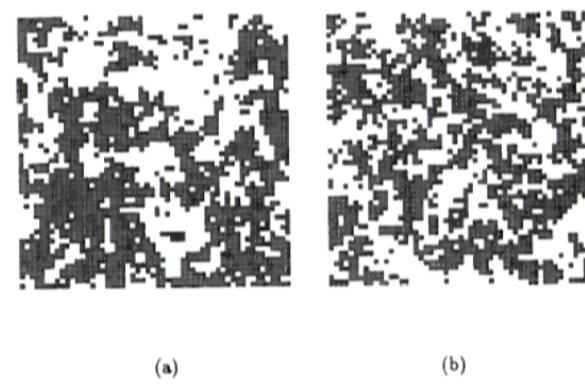




(a)

(b)

Figure 3.1. Typical configuration of the Ising model at $T = T_c$ (a), and (b) the result of a single block spin transformation.



(a)

(b)

Figure 3.2. Same as Figure 3.1, slightly above the critical temperature.

critical two-pointing
renormalization

$\langle \rangle$ = average

SIR

\rightarrow

$$\sum_{\langle i,j \rangle} u_i u_j + K_L \sum_{\langle i,j \rangle} u_i u_j + K_2 \sum_{\langle i,j \rangle}$$

$$\frac{1}{2} \ln \left\{ \frac{e^{2K}}{2} (1 + e^{-4K}) \right\}$$

$$\ln 2 + O(e^{-4K})$$

at smaller

For C

For $K \gg 1$

$K' = t$

$K' =$

$K \rightarrow \infty$

$T=0$

K^+

K^-

For small K

$\tau \rightarrow \infty$

$\int d^d k \sim \text{const}$

$S(K) =$

$K^+ - K^-$

$\propto \ln K \rightarrow \infty$

30 The renormalization group idea

3.1 Block spin transformations

31



(a)



(b)

Figure 3.1. Typical configuration of the Ising model at $T = T_c$ (a), and (b) the result of a single block spin transformation.



(a)



(b)

Figure 3.2. Same as Figure 3.1, slightly above the critical temperature.

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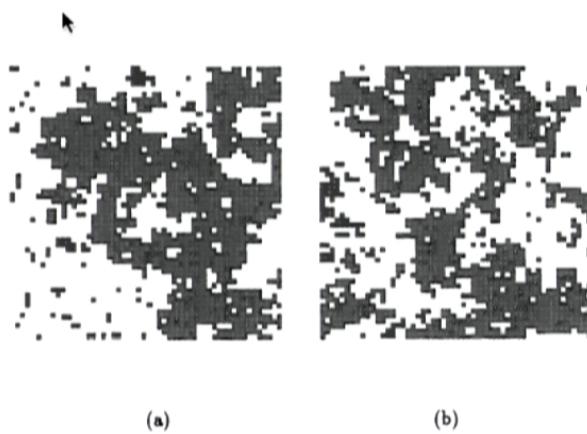


Figure 3.1. Typical configuration of the Ising model at $T = T_c$ (a), and (b) the result of a single block spin transformation.

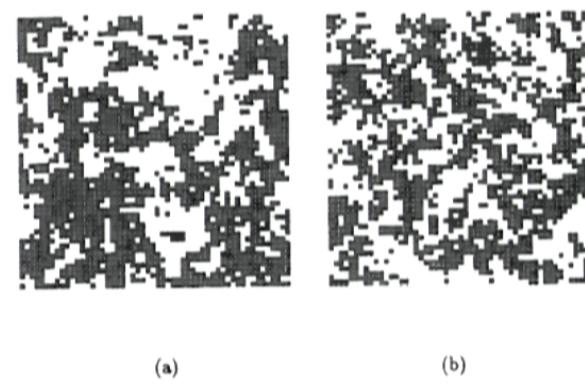


Figure 3.2. Same as Figure 3.1, slightly above the critical temperature.

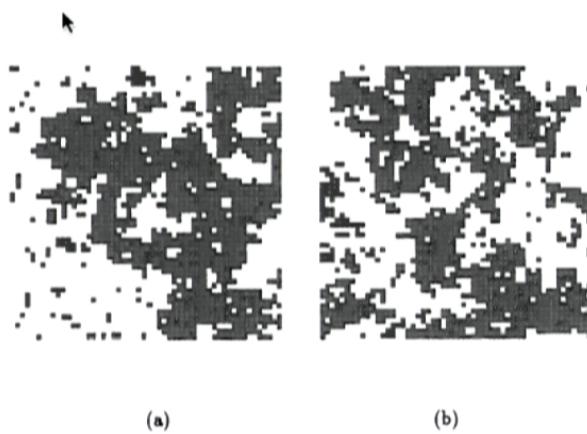


Figure 3.1. Typical configuration of the Ising model at $T = T_c$ (a), and (b) the result of a single block spin transformation.

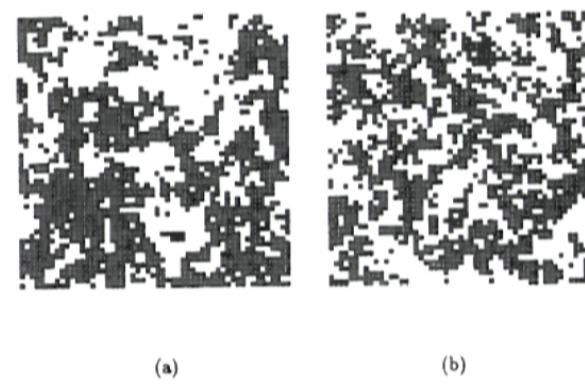


Figure 3.2. Same as Figure 3.1, slightly above the critical temperature.

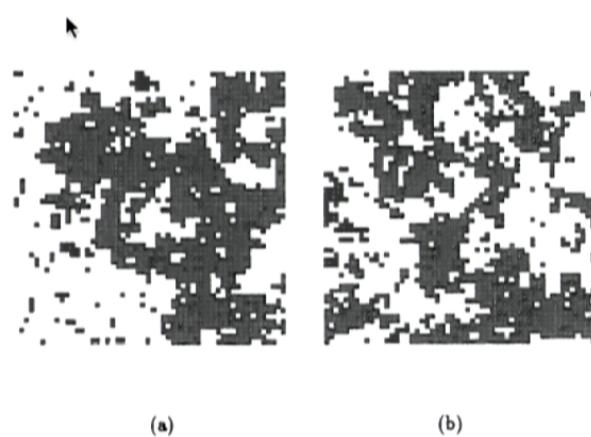


Figure 3.1. Typical configuration of the Ising model at $T = T_c$ (a), and (b) the result of a single block spin transformation.

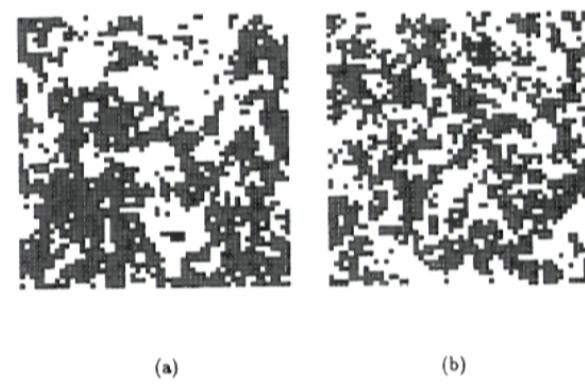
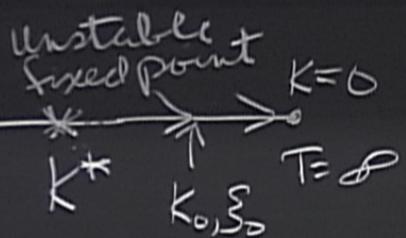


Figure 3.2. Same as Figure 3.1, slightly above the critical temperature.



For small K , K flows to zero.

$\rightarrow d > 1$ is qualitatively different from $d = 1$.

$\xi(K^*) = \infty$ means that K^* is a critical point

What is nature of point K^* ?

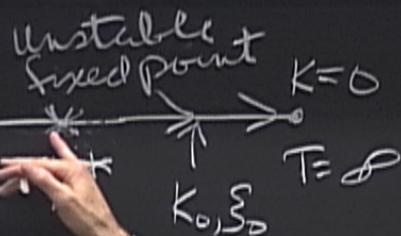
Imagine that at some small K_0

$$\xi(K_0) = \xi_0$$

The RG transf gives $\xi(K) = b \xi(K')$

Start from $K^* \rightarrow K \rightarrow K_0$ and say that it takes $N(k)$ steps to get to K_0 .

Then $\xi(K) = b^{N(k)} \xi_0$. As $K \rightarrow K^*$ $N(k)$ grows until $\xi(K) \rightarrow \infty$ as $K \rightarrow K^*$



For small δK flows to zero.

$\rightarrow d > 1$ is qualitatively different from $d = 1$.

$$\boxed{\begin{array}{l} \xi(K^*) = \infty \text{ mean} \\ K^* \text{ is a critical} \end{array}}$$

What is nature of point K^* ?

Imagine that at some small K_0

$$\xi(K_0) = \xi_0$$

The RG transf gives $\xi(K) = b \xi(K')$

Start from $K^* \rightarrow K \rightarrow K_0$ and say that it takes $N(k)$ steps to get to K_0 .

Then $\xi(K) = b^{N(k)} \xi_0$. As $K \rightarrow K^*$ $N(k)$ grows and $\xi(K) \rightarrow \infty$ as $K \rightarrow K^*$

Next, use the above to
calculate $\Sigma(K)$ and γ .

near K^*

$$K' = R(K) = R(K^*) + (K - K^*) R'(K^*) + \dots$$

Next, use the above to calculate $\Sigma(K)$ and y .

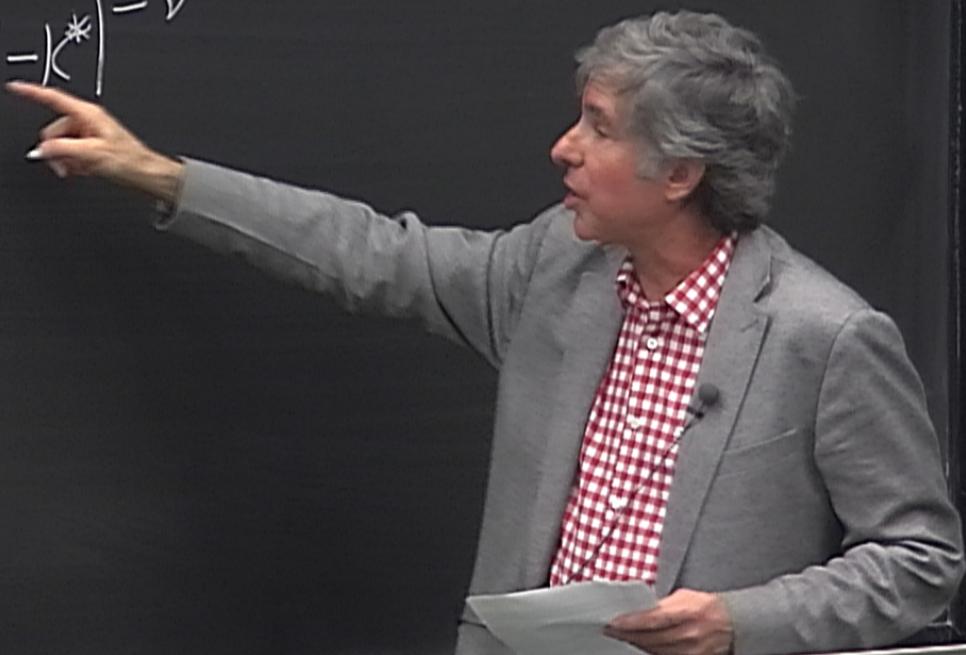
*near K^** K^*

$$K' = R(K) = R(K^*) + (K - K^*)R'(K^*) +$$
$$(K' - K^*) = b^y(K - K^*) \quad \text{where } b^y = R'$$
$$\therefore y = \ln R'$$

Close to K^*

$$\xi(K) = A |K - K^*|^{-\nu} \text{ and } \xi(K') = b \xi(K')$$

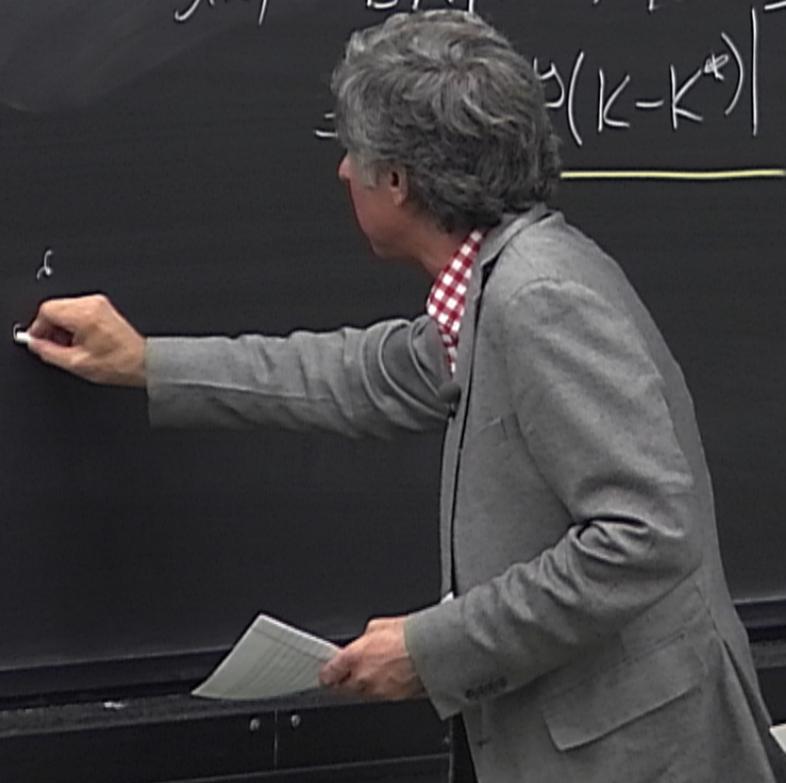
$$\xi(K) = bA |K' - K^*|^{-\nu}$$



Close to K^*

$$\xi(K) = \frac{A |K - K^*|^{-\nu}}{\text{and } \xi(K) = b \xi(K')}$$

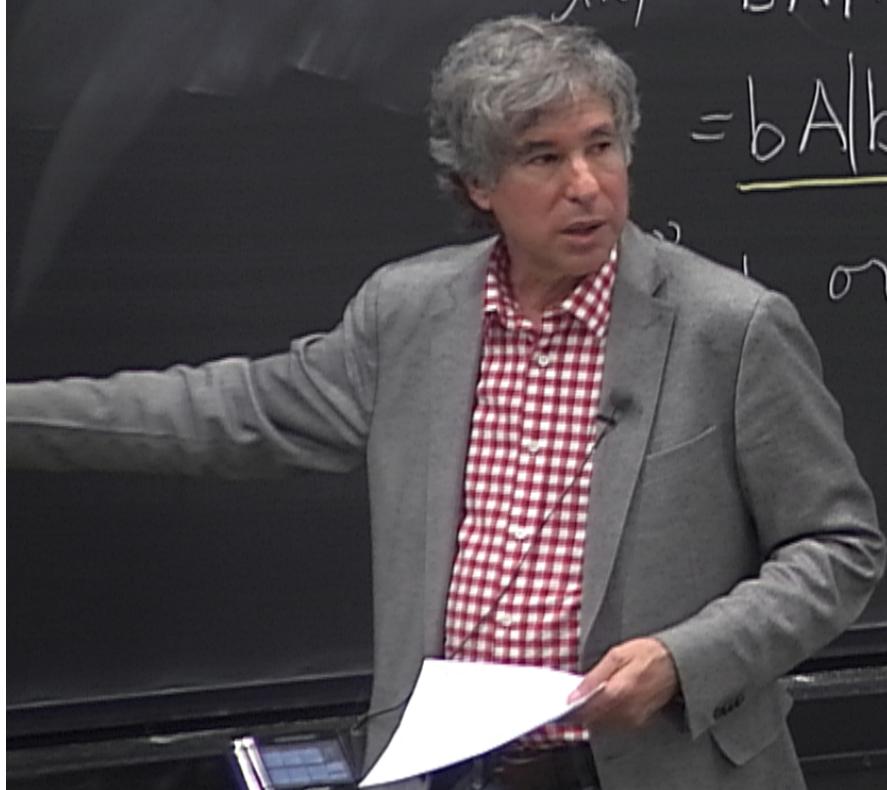
$$\begin{aligned}\xi(K) &= b A |K' - K^*|^{-\nu} \\ &= b A |(K - K^*)|^{-\nu}\end{aligned}$$



Close to K^* $\xi(K) = \frac{A|K-K^*|^{-\gamma}}{b}$ and $\xi(K) = b \xi(K')$

$$\xi(K) = b A |K'-K^*|^{-\gamma}$$
$$= b A |b^\gamma (K-K^*)|^{-\gamma}$$

$$\sigma \sim \gamma = \gamma$$



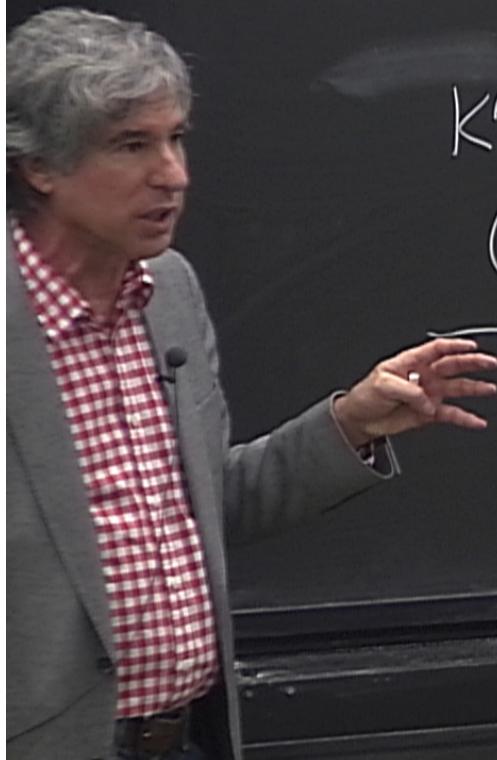
Close to K^*

Next, use the above to calculate $\zeta(K)$ and γ .

$$K' = R(K) = R(K^*) + (K - K^*)R'(K^*) + \dots$$

near K^*

$$(K' - K^*) = b^y(K - K^*) \quad \text{where } b^y = R'(K^*)$$
$$\therefore y = \ln R'(K^*) / \ln b$$



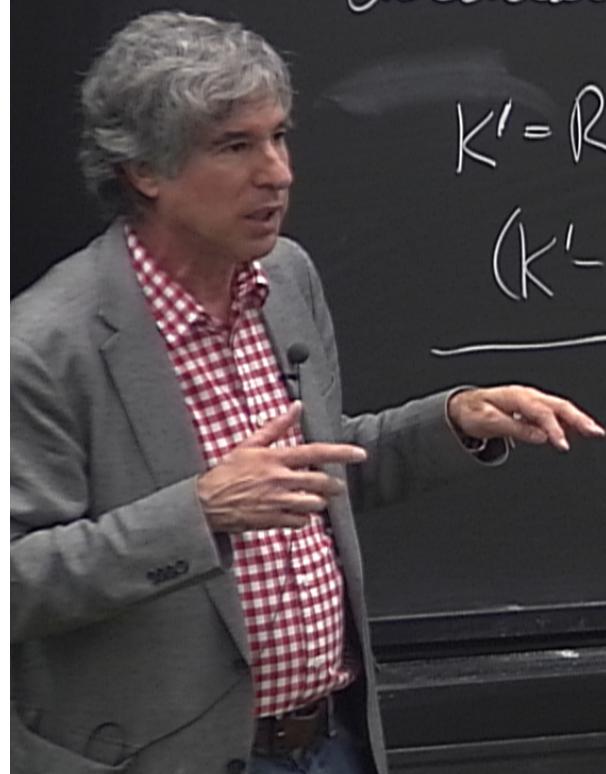
Close to K^*

Next, use the above to calculate $\zeta(K)$ and ν .

$$K' = R(K) = R(K^*) + (K - K^*)R'(K^*) + \dots$$

$$(K' - K^*) = b^y (K - K^*) \quad \text{where } b^y = R'(K^*)$$

$$\therefore y = \ln R'(K^*) / \ln b$$



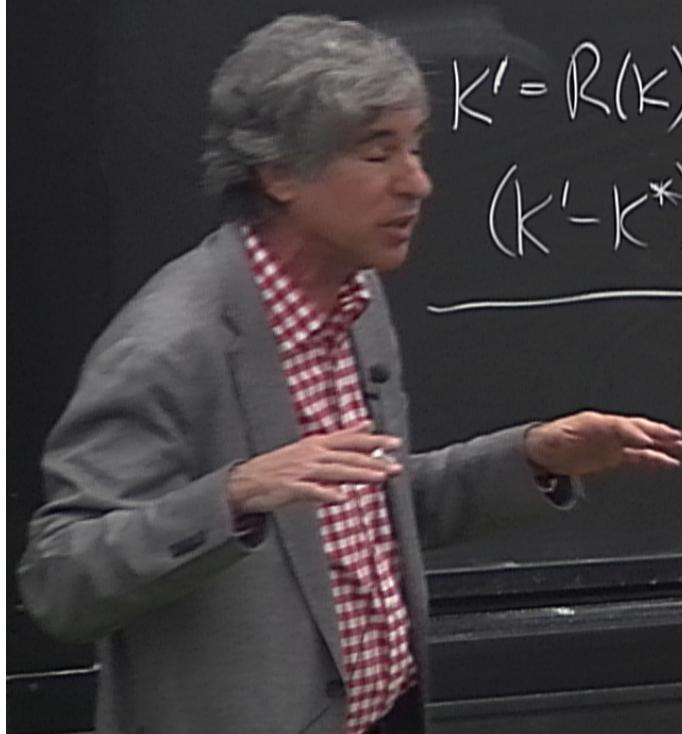
Close to K^*

Next, use the above to calculate $\zeta(K)$ and γ .

$$K' = R(K) = R(K^*) + (K - K^*)R'(K^*) + \dots$$

$$(K' - K^*) = b^y (K - K^*) \quad \text{where } b^y = R'(K^*)$$

$$\therefore y = \ln R'(K^*) / \ln b$$



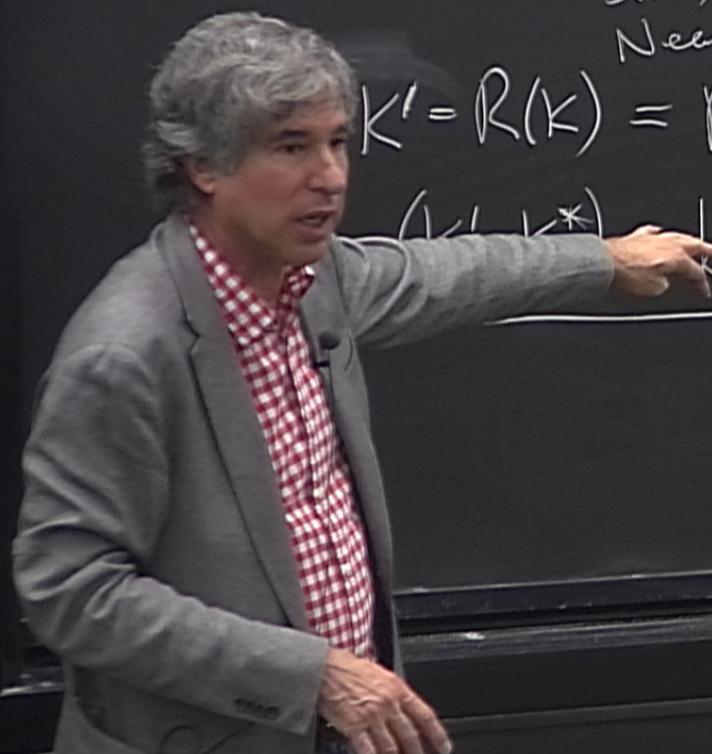
Close to K^*

Next, use the above to calculate $\zeta(K)$ and ν .

$$K' = R(K) = R(K^*) + (K - K^*)R'(K^*) + \dots$$

near K^*

$$(K - K^*) \xrightarrow{b^y(K - K^*)} \text{ where } b^y = R'(K^*)$$
$$\therefore y = \ln R'(K^*) / \ln b$$



Close to K^*

Next, use the above to calculate $\zeta(K)$ and γ .

$$K' = R(K) \underset{\text{Near } K^*}{\approx} R(K^*) + (K - K^*)R'(K^*) + \dots$$

$$\underline{(K' - K^*) = b^y(K - K^*)} \quad \text{where } b^y = R'(K^*)$$
$$\therefore y = \ln R'(K^*) / \ln b$$

