Title: Fault tolerance of "bad" quantum low-density parity check codes

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Abstract: In my talk, I will discuss various families of quantum
low-density parity check
(LDPC) codes and their fault tolerance. Such codes yield finite code rates and
at the same time
simplify error correction and encoding due to low-weight stabilizer
generators. As an example, a large family of

hypergraph-product codes is considered. Of particular interest are families of
quantum LDPC codes with finite rate and distance scaling as square root of
blocklength since this represents the best known exponent in
distance scaling, even for codes of dimensionality 1. In relation to such
codes, we show that any family of LDPC codes, quantum or classical,
where distance scales as a positive power of the block length, $d
\propto n^\alpha$, $\alpha>0$ ($\alpha<1$ for "bad"

codes), can correct all errors with certainty if the
error rate per qubit is sufficiently small. We specifically
analyze the case of LDPC version of the quantum
hypergraph-product codes recently suggested by Tillich and Z\'emor. These codes are a
finite-rate generalization of the toric codes, and, for sufficiently
large quantum computers, offer an advantage over the toric codes.

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FAULT TOLERANCE OF "BAD" QUANTUM LOW-DENSITY PARITY CHECK CODES

arXiv:1208.2317

Alexey A. Kovalev and Leonid Pryadko
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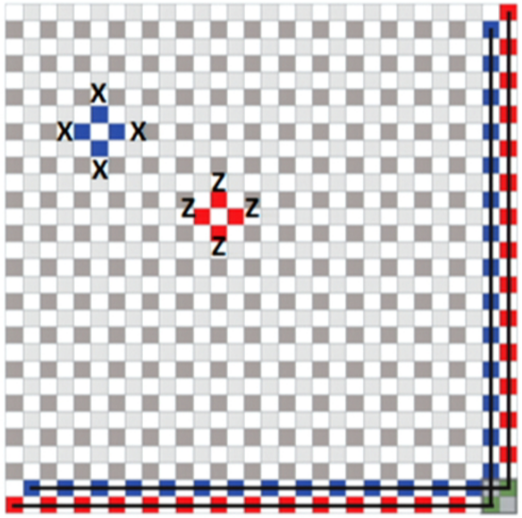
OUTLINE

- Introduction: classical and quantum *LDPC codes*
- *Tanner graph* representation of *LDPC codes*
- LDPC code construction from two binary matrices
  - *Toric codes* and *hypergraph product codes*
  - Examples of quantum LDPC (generalized toric) codes with *finite rate*
- Generalizations of *hypergraph product codes* corresponding to rotation of boundary conditions (e.g. rotated *toric code* and checker board code)
- Asymptotic lower bound on LDPC quantum codes
- Fault tolerance of LDPC codes and relation between error correction on a Tanner graph representation

THE SIMPLEST LDPC CODE-TORIC CODE

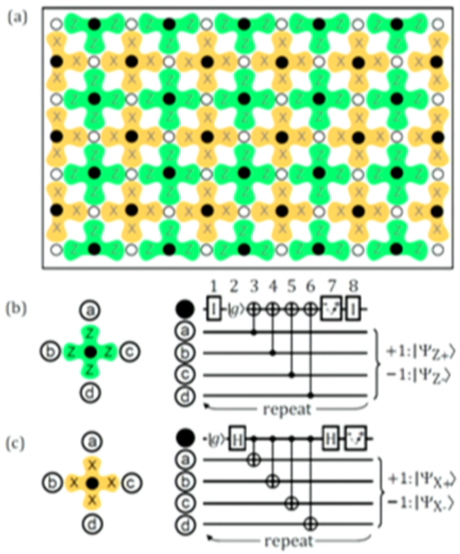


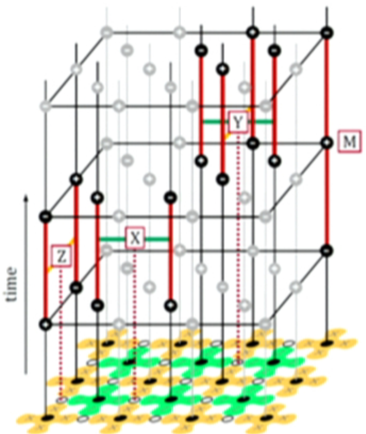
1. Stabilizer generators commute with line like logical operators.
2. Stabilizer generators commute with each other.
3. Combinations of stabilizer generators form cycles that are topologically different from the logical operators.
4. Logical operators can be deformed by stabilizer generators.

Two stabilizer generators and two pairs of anticommuting logical operators of a [[450; 2; 15]] toric code (red and blue, respectively, X and Z operators, green - overlap of Z and X operators, dark and light gray - dual sublattices of physical qubits). Other stabilizer generators are obtained by shifts over the same sublattice with periodic boundaries.

E. Dennis, A. Kitaev, A. Landahl, and J. Preskill, J. Math. Phys. 43, 4452 (2002).

WHY SUCH CODES ARE INTERESTING?





Unfortunately:

$$kd^2 = O(n)$$

S. Bravyi, D. Poulin, and B. Terhal, Phys. Rev. Lett. 104, 050503 (2010)

D. A. Meyer, M. H. Freedman, and F. Luo, in Mathematics of Quantum Computation (CRC Press, 2002), pp. 287–320.

A. G. Fowler, M. Mariantoni, J. M. Martinis, A. N. Cleland, arXiv:1208.0928

1. Overhead of ancillary qubits is small.
2. Threshold is very high p=0.57%.
3. All we need to do is to measure ancillas and keep track of errors.

CLASSICAL LDPC CODES

Classical LDPC codes are exceptional for error correction,
e.g. **Gallager codes, IRE Trans. Info. Theory IT-8: 21-28 (1962)**.

Number of 1s in every row (=r), and in every column (=c) for the parity check matrix is fixed.

H2 and H3 are formed from H1 by column permutations,
e.g. c= 4, r = 3 for [20,7,6] code

$$
\begin{pmatrix} \mathrm{H1} \\ \mathrm{H2} \\ \mathrm{H3} \end{pmatrix} ==
\left(\begin{array}{cccccccccccccccccccc}
1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1\\
=&=&=&=&=&=&=&=&=&=&=&=&=&=&=&=&=&=&=&=\\
1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0\\
0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0&1&0&0&0\\
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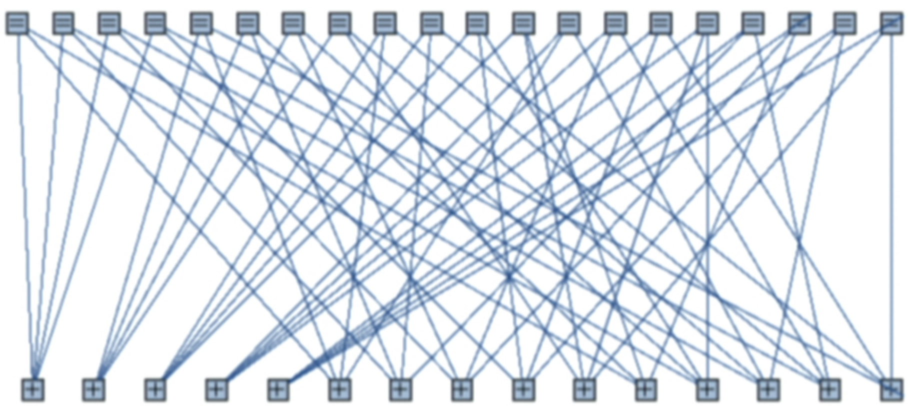
=&=&=&=&=&=&=&=&=&=&=&=&=&=&=&=&=&=&=&=\\
1&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&1&0&0\\
0&1&0&0&0&0&1&0&0&0&1&0&0&0&0&1&0&0&0&0\\
0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&0&1&0\\
0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&1&0&0&0\\
0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1
\end{array}\right)
$$

Convenient to represent by bipartite graph:



STABILIZER CODES: BINARY REPRESENTATION

To this end, we want to construct binary quantum stabilizer codes with low weight stabilizer generators. We consider Pauli group $\mathscr{P}_n = i^m\{I,X,Y,Z\}^{\otimes n},\ m=0,\ldots,3$

> An $[[n,k,d]]$ stabilizer code $\mathcal{Q}$ is a 2^k-dimensional subspace of the Hilbert space $\mathcal{H}_2^{\otimes n}$ stabilized by an Abelian stabilizer group $\mathscr{S} = \langle G_1,\ldots,G_{n-k}\rangle$, $-\mathbb{1}\notin\mathscr{S}$; $\mathcal{Q}=\{|\psi\rangle : S|\psi\rangle = |\psi\rangle, \forall S\in\mathscr{S}\}$.

Pauli operators are mapped to two binary strings, $\mathbf{v},\mathbf{u}\in\{0,1\}^n$, $U\equiv i^{m'}X^{\mathbf{v}}Z^{\mathbf{u}} \to (\mathbf{v},\mathbf{u})$, where $X^{\mathbf{v}} = X_1^{v_1}X_2^{v_2}\ldots X_n^{v_n}$ and $Z^{\mathbf{u}} = Z_1^{u_1}Z_2^{u_2}\ldots Z_n^{u_n}$. A product of two quantum operators corresponds to a sum $(\mathrm{mod}\, 2)$ of the corresponding pairs $(\mathbf{v}_i,\mathbf{u}_i)$.

In this representation, a stabilizer code is represented by parity check matrix written in binary form for X and Z Pauli operators so that, e.g. XIYZYI=-(XIXIXI)x(IIZZZI) -> (101010)|(001110).

$$\boldsymbol{H}=\begin{array}{c} \begin{array}{cc} \text{Ax} & \text{Az} \end{array} \\ \left(\begin{array}{ccccc|ccccc} 1&1&0&0&0&0&0&1&0&1\\ 0&1&1&0&0&1&0&0&1&0\\ 0&0&1&1&0&0&1&0&0&1\\ 0&0&0&1&1&1&0&1&0&0 \end{array}\right) \end{array}$$

Example of a parity check matrix *H* of [[5,1,3]] code written in X-Z form.

$$\boxed{A_X A_Z^T + A_Z A_X^T = 0}$$

Necessary and sufficient condition for existence of stabilizer code with stabilizer commuting operators corresponding to ***H***.

$$\boxed{(z|x)\odot(z'|x')^T = zx'^T + xz'^T}$$

Row orthogonality with respect to symplectic product.

STABILIZER CODES: ERROR CORRECTION

1. Measure stabilizer generators to obtain syndrome of error *E* ⟺ $H \odot E^T$

2. Correct error according to syndrome.

- The correctable error set ***Ec*** is defined by:

  If E_1 and E_2 are in *Ec*, then one of the two conditions hold:

  1. $E_2^\dagger E_1 \notin H_\perp \setminus \mathscr{S}$ distinct error syndromes
  2. $E_2^\dagger E_1 \in \mathscr{S}$ degenerate code

- The detectable error set ***Ed*** is defined by:

  If *E* is in *Ed*, then one of the two conditions hold:

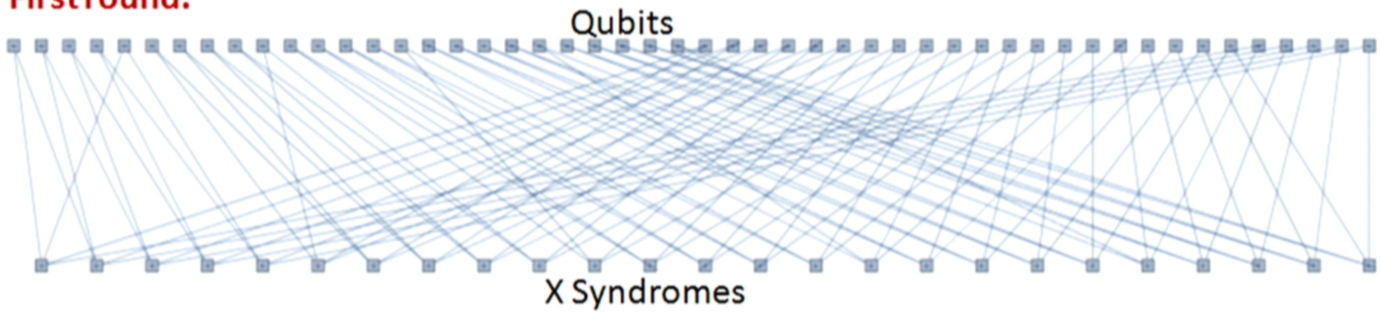
  1. $E \notin H_\perp \setminus \mathscr{S}$ distinct error syndromes
  2. $E \in \mathscr{S}$ degenerate code

Syndrome of (*I I I Y I*) error: $\left(\begin{array}{ccccc|ccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \end{array}\right) \odot (\overset{E_X}{00010}|\overset{E_Z}{00010})^T = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

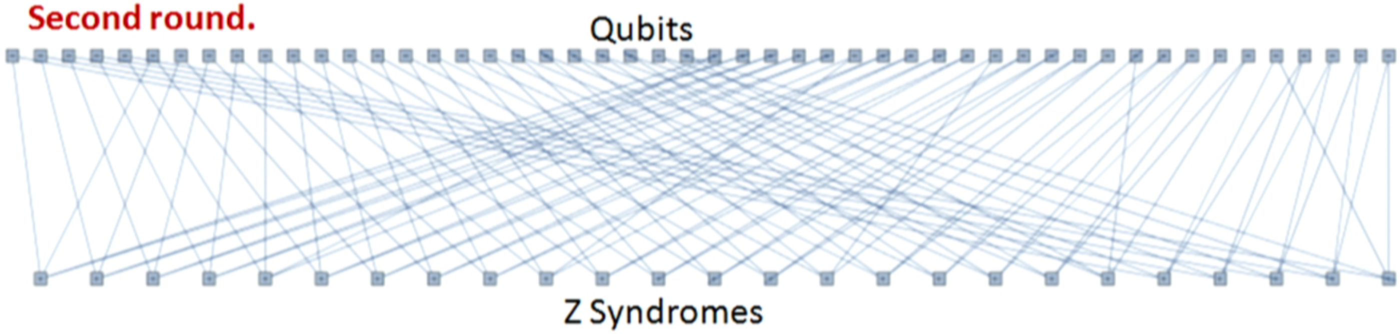
The distance of a quantum stabilizer code is defined as the minimal weight of all detectable errors, i.e. Hamming weight of $E_X \vee E_Z$

SYNDROME MEASUREMENTS AND TANNER GRAPH

First round.



Second round.



Two round syndrome measurement for [[50,2,5]] toric code.

- **Hardware wise LDPC codes can be realized on superconducting qubits A. De, L.P. Pryadko, arXiv:1209.2764.**
- **Efficient decoders have to be found as believe propagation decoders do not work due to loops.**

SYNDROME MEASUREMENTS AND TANNER GRAPH

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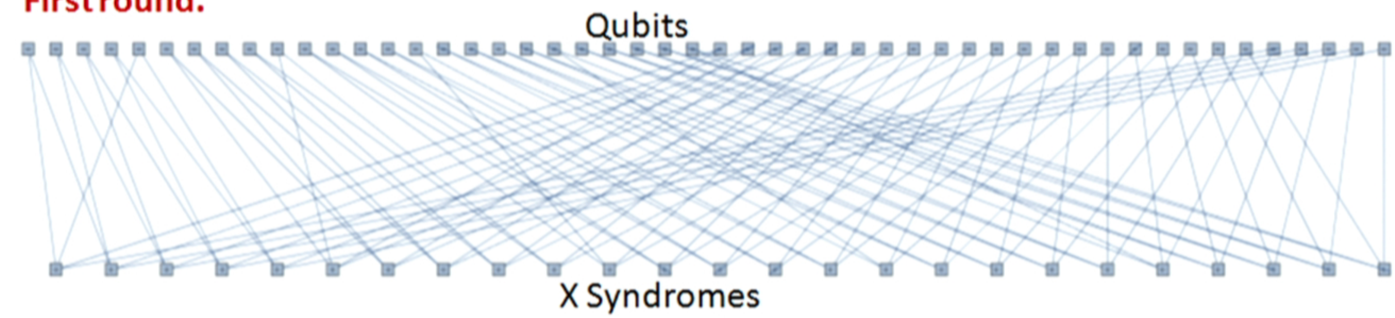

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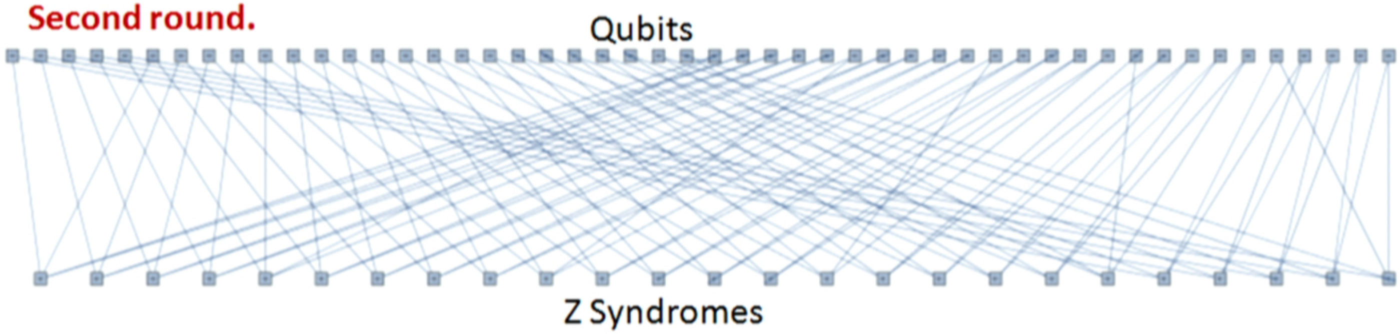

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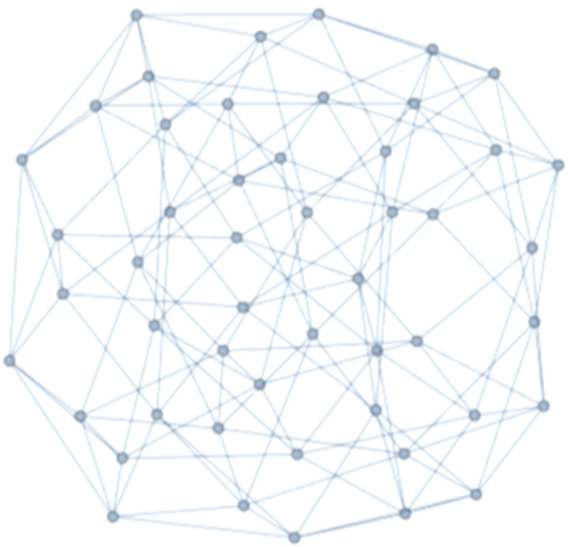
LDPC CODE IS A LOCAL CODE ON SOME GRAPH



Graph representation of [[50,2,5]] toric code.

Qubit connectivity matrix:

H=(Ax, Az)

G = Ax V Az

R – row weight,

C – column weight

Vertex degree: $z=(R-1)C$

For syndrome connectivity:

$$z_s=(C-1)R$$

We need *zs* rounds of measurements instead of two for toric code, *zs* does not depend on blocklength.

It should be possible to characterize the topological order associated with local on graphs Hamitonians:

Xie Chen, Zheng-Cheng Gu, Xiao-Gang Wen, Phys. Rev. B 82, 155138 (2010)

S. Bravyi, M. B. Hastings, and F. Verstraete, Phys. Rev. Lett. 97, 050401 (2006)

QUANTUM LDPC CODES

Advantages:

- Easy error correction for such codes: simple quantum measurements, easy classical processing, and parallelism.
- Such codes allow fault tolerant error correction. **Kovalev & Pryadko, arXiv:1208.2317**

Disadvantages:

- Hard to achieve good code parameters compared to non-LDPC codes.
- There are no or very few known bounds for quantum LDPC codes so explicit constructions are important.

QUANTUM CSS CODES FROM (HYPER)GRAPHS

Parity check matrix for a Calderbank-Shor-Steane (CSS) code: $H = \left(\begin{array}{c|c} G_X & 0 \\ 0 & G_Z \end{array} \right), \; G_X G_Z^T = 0$ ⟸ commutativity

SURFACE QUANTUM CODES: Qubits correspond to edges of a graph, rows of *Gx* are vertices, rows of *Gz* are faces.

In such construction each column has only 2 entries.

Advantages: infinite family with unbounded distance, small weight and locality of measurements, fault tolerance; **A. Y. Kitaev, Ann. Phys., vol. 303, p. 2, (2003); H. Bombin and M. A. Martin-Delgado, Phys. Rev. A 76, 012305 (2007).**

Can generalize to arbitrary graphs but not clear how to find the dual graph!

When (hyper)graph is a product of two (hyper)graphs we can identify generalized faces and find a dual object which is a hypergraph -> HYPERGRAPH PRODUCT CODES

Tillich & Zemor, in Information Theory, (2009), arxiv:0903.0566.

Example: Toric code represents the graph-product construction

QUANTUM CODE FROM TWO CLASSICAL

Stabilizer code is constructed from two classical codes with parity check matrices (may have linearly dependent rows/columns):

$$r_1 \updownarrow \overset{\overset{n_1}{\longleftrightarrow}}{\mathcal{H}_1} \qquad r_2 \updownarrow \overset{\overset{n_2}{\longleftrightarrow}}{\mathcal{H}_2}$$

Constructed code has parameters [[N,K,D]]

$$N = n_1 r_2 + n_2 r_1$$

$$K = 2k_1 k_2 - k_1(n_2 - r_2) - k_2(n_1 - r_1)$$

$$D \geq \mathrm{Min}[\mathrm{dist}(\mathcal{H}_1), \mathrm{dist}(\mathcal{H}_2), \mathrm{dist}(\mathcal{H}_1^T), \mathrm{dist}(\mathcal{H}_2^T)]$$

J.P. Tillich, G. Zemor, in Proc. IEEE Int. Symp. Inf. Theory (ISIT), 799 (2009).

Algebraic form for hypergraph-product codes!

$$\boxed{H = \left(\begin{array}{c|c} G_X & 0 \\ 0 & G_Z \end{array}\right), \quad \begin{array}{c} G_X = (E_2 \otimes \mathcal{H}_1 \;\; \mathcal{H}_2 \otimes E_1), \\ G_Z = (\mathcal{H}_2^T \otimes \widetilde{E}_1 \;\; \widetilde{E}_2 \otimes \mathcal{H}_1^T). \end{array}}$$

Gx and *Gz* correspond to two dual hypergraphs.

E – unit matrix and $\otimes$ – Kronecker product.

Commutativity follows from:

$$(A \otimes B)(C \otimes D) = AC \otimes BD$$

Kovalev & Pryadko, IEEE International Symposium on Information Theory Proceedings (ISIT 2012) pp. 348 – 352, arXiv:1202.0928v3.

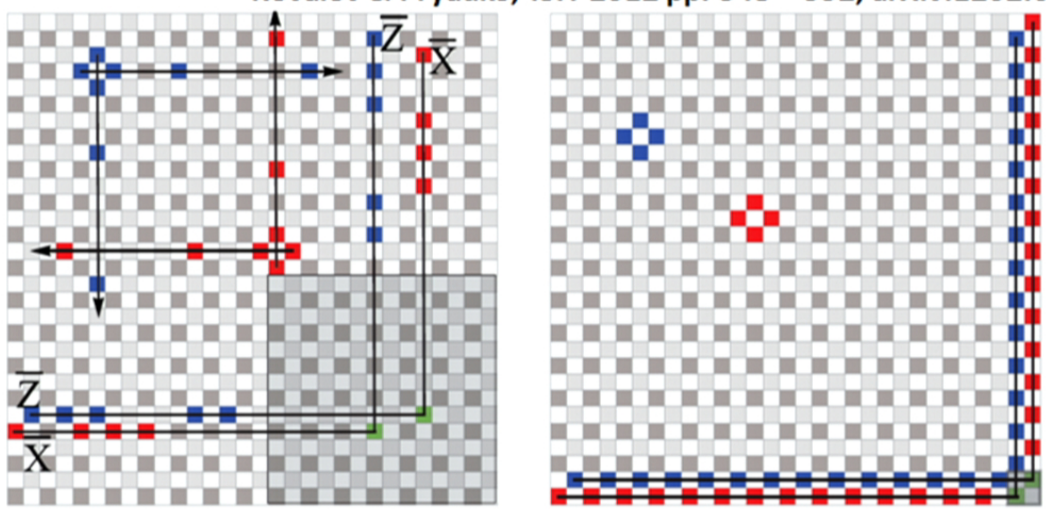
LDPC HYPERGRAPH-PRODUCT CODES

Example: Suppose we take LDPC code [n,k,d] with full rank matrix $\mathcal{H}_1, \mathcal{H}_2^T = \mathcal{H}_1$, $n-k$, n
then parameters of the quantum code are: $[[(n-k)^2 + n^2, k^2, d]]$

Tillich & Zemor, in Information Theory, (2009), arxiv:0903.0566.

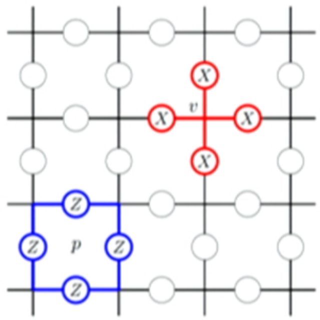
Example: If $\mathcal{H}_1$ is a (nxn) circulant matrix of a LDPC cyclic code ([n,k,d]) and $\mathcal{H}_2^T = \mathcal{H}_1$
then parameters of the corresponding quantum codes are [[2n^2, 2k^2, d]],

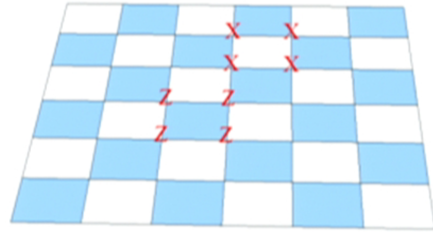
Kovalev & Pryadko, ISIT 2012 pp. 348 – 352, arXiv:1202.0928v3 (2012).



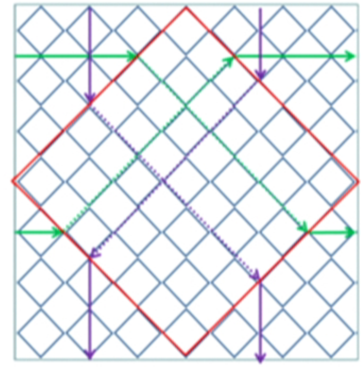
Left: Two stabilizer generators (marked by arrows) and two pairs of anticommuting logical operators (marked by lines) of a $[[450, 98, 5]]$ code formed by circulant matrices $\mathcal{H}_1 = \mathcal{H}_2$ corresponding to coefficients of a polynomial $h(x) = 1 + x + x^3 + x^7$ (red – X operators, blue – Z operators, green – overlap of Z and X operators, dark and light gray – dual sublattices of physical qubits). All other stabilizer generators are obtained by shifts over the same sublattice with periodic boundaries. In the shaded region, each gray square uniquely corresponds to a different logical operator, thus 98 encoded logical qubits. Right: same for the toric code $[[450, 2, 15]]$.

ROTATED TORIC CODES

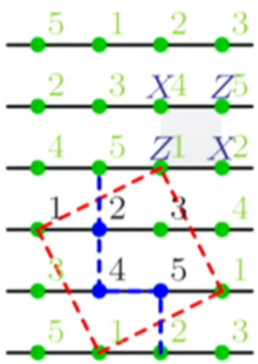




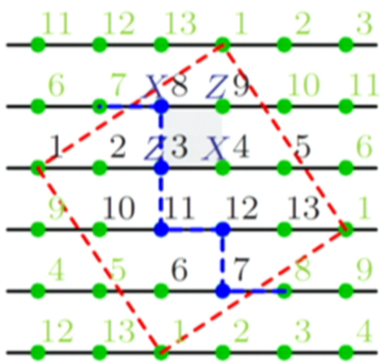
Even case!



Toric code can be broken into two rotated toric codes by the procedure on the right; rotated by 45 degrees codes have the same distance but twice smaller blocklength.
Examples: [[9,1,3]], [[25,1,5]], [[16,2,4]] and [[36,2,6]].



[[5,1,3]] Toric code



[[13,1,5]] Toric code

H. Bombin and M. A. Martin-Delgado, Phys. Rev. A, vol. 76, no. 1, p.012305, 2007

When the translation vectors are (a,b) and (b,-a) (orthogonal), then n=a^2+b^2, d=|a|+|b|, and k=1 if d is odd or 2 if d is even.
The example is for a=t+1, b=t, with t=1,2.

Wen-plaquette model, **Su-Peng Kou, M. Levin, and Xiao-Gang Wen, Phys. Rev. B 78, 155134 (2008).**

A. A. Kovalev, I. Dumer, and L. P. Pryadko, Phys. Rev. A, vol. 84, p. 062319, 2011

ROTATED HYPERGRAPH-PRODUCT CODES: NON-BIPARTITE CASE

Codes are constructed from two symmetric matrices:

Non CSS construction!

$$H = (E_2 \otimes \mathcal{H}_1 | \mathcal{H}_2 \otimes E_1), \quad \mathcal{H}_i^T = \mathcal{H}_i, \quad i = 1, 2.$$

Constructed code has parameters [[N,K,D]]

$$N = n_1 n_2$$

$$K = \dim(C_{\mathcal{H}_1}) \dim(C_{\mathcal{H}_2})$$

$$D \geq \mathrm{Min}[\mathrm{dist}(\mathcal{H}_1), \mathrm{dist}(\mathcal{H}_2)]$$

LDPC constructions with asymptotically finite K/N and D^2/N are possible!

LDPC constructions:

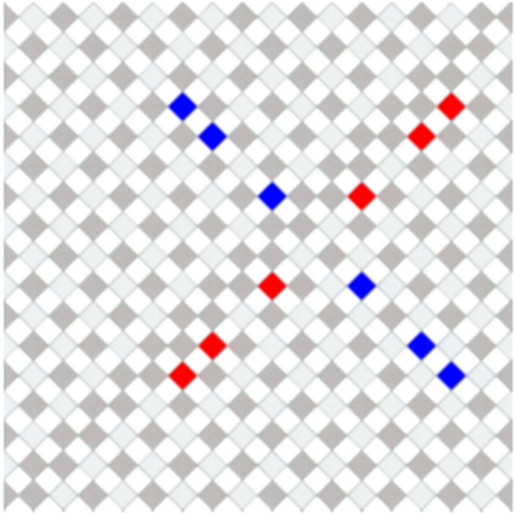
1. Any classical LDPC [n,k,d] code with a parity matrix *P* can be symmetrized:

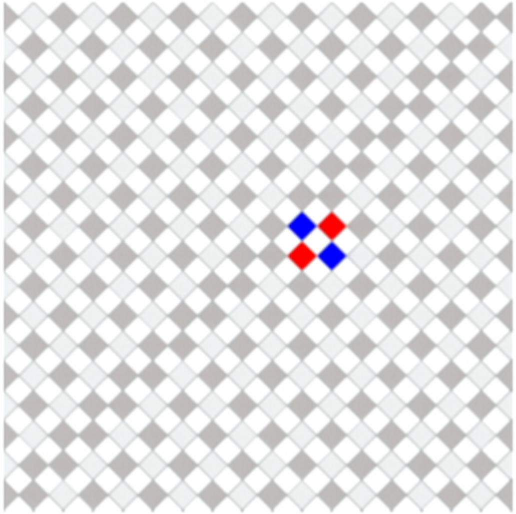
$\mathcal{H}_1^{\mathrm{sym}} = \begin{pmatrix} \mathbb{1} & P \\ P^T & 0 \end{pmatrix}$, which leads to [2n-k,k,d] LDPC classical code.

2. Start from a cyclic LDPC [n,k,d] code with circulant parity check matrix generated by palindromic polynomial, i.e. $x^{\deg h(x)} h(1/x) = h(x), \; n - \deg h(x)$ is even. Circulant symmetric matrix of [n,k,d] code can be generated from polynomial:

$$h_1(x) = x^{[n - \deg h(x)]/2} h(x)$$

NON-BIPARTITE CODE EXAMPLES





Left: a stabilizer generator of a $[[289, 81, 5]]$ non-CSS rotated code formed by circulant matrices $\mathcal{H}_1 = \mathcal{H}_2$ corresponding to coefficients of a polynomial $h(x) = 1 + x + x^3 + x^6 + x^8 + x^9$. The division into two sublattices is impossible and all other stabilizer generators are obtained by shifts over the light and dark gray qubits with periodic boundaries. Right: same for the toric code $[[289, 1, 17]]$ and $h(x) = 1 + x$.

ROTATED CODES: BIPARTITE CASE

We construct CSS codes from matrices *a* and *b* corresponding to classical codes:

$\mathcal{H}_1 = \binom{10}{01} \otimes a_1 + \binom{01}{10} \otimes b_1, \mathcal{H}_2^p = a_2 \otimes \binom{10}{01} + b_2 \otimes \binom{01}{10},\ E^{(1/2)}$- unit matrix of half size.

$$\boxed{H = \left(\begin{array}{c|c} G_X & 0 \\ 0 & G_Z \end{array} \right), \quad \begin{array}{c} G_X = (E_2^{(1/2)} \otimes \mathcal{H}_1, \mathcal{H}_2^p \otimes E_1^{(1/2)}), \\ G_Z = (\mathcal{H}_2^{pT} \otimes \widetilde{E}_1^{(1/2)}, \widetilde{E}_2^{(1/2)} \otimes \mathcal{H}_1^T). \end{array}}$$

Constructed code has parameters [[N,K,D]].

$$N = n_1 r_2/2 + n_2 r_1/2$$
$$K = 2k_1^a k_2^a + 2k_1^s k_2^s - k_1(n_2 - r_2)/2 - k_2(n_1 - r_1)/2$$
$$D \geq \frac{1}{2}\mathrm{Min}[\mathrm{dist}(\mathcal{H}_1), \mathrm{dist}(\mathcal{H}_2), \mathrm{dist}(\mathcal{H}_1^T), \mathrm{dist}(\mathcal{H}_2^T)]$$

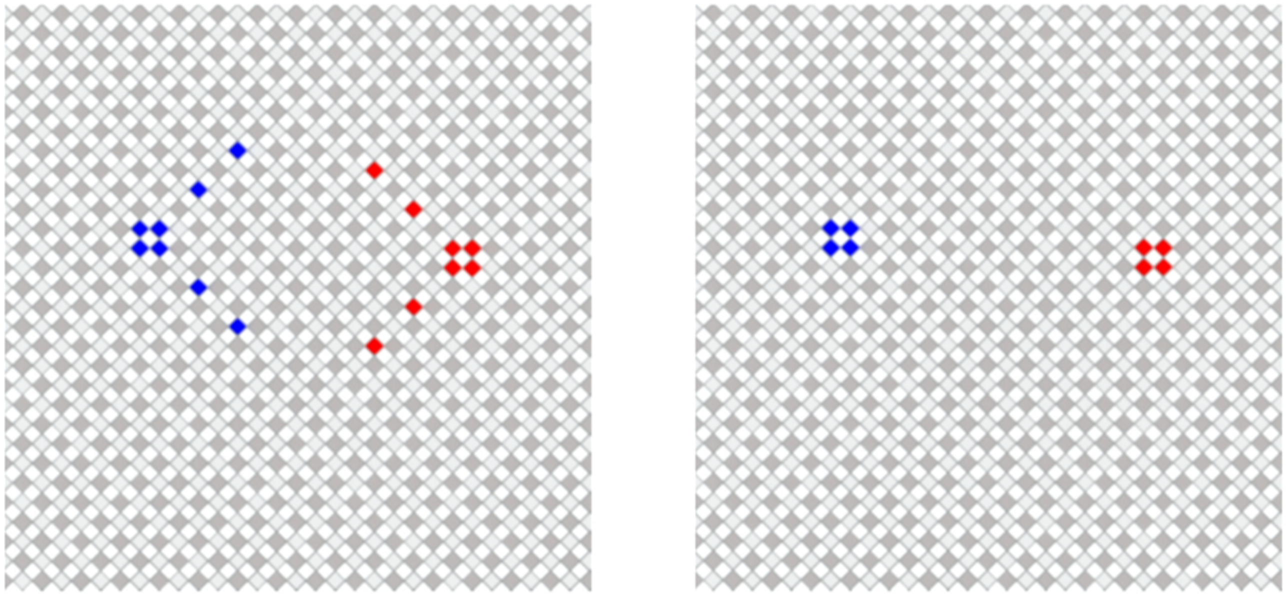
Improves over hypergraph-product codes, i.e. half blocklength with the same number of encoded quibits and distance.

1. For square matrices when $b_i = E_i^{(1/2)}$
2. Construction from square curculant matrices when $h(x)$ divides $1 - x^{n/2}$

$$N = n_1 n_2$$
$$K = 2k_1 k_2$$
$$D \geq \mathrm{Min}[\mathrm{dist}(\mathcal{H}_1), \mathrm{dist}(\mathcal{H}_2), \mathrm{dist}(\mathcal{H}_1^T), \mathrm{dist}(\mathcal{H}_2^T)]$$

LDPC constructions with asymptotically finite K/N and D^2/N are possible!

BIPARTITE CODE EXAMPLES



Left: X and Z stabilizer generators for the CSS rotated code $[[900, 50, 14]]$ formed by circulant matrices $\mathcal{H}_1^{(1/2)} = \mathcal{H}_2^{(1/2)}$ corresponding to coefficients of a polynomial $h(x) = 1 + x + x^3 + x^5$. Right: same for the toric code $[[900, 2, 30]]$ and $h(x) = 1 + x$.

ASYMPTOTIC PROPERTIES

Suppose we take LDPC code [n,k,d] with full rank matrix $\mathcal{H}_1; \mathcal{H}_2^T = \mathcal{H}_1$
and obtain the quantum hypergraph-product code $[[(n-k)^2 + n^2, k^2, d]]$
Number of 1s in every row (=*r*), and in every column (=*c*) for $\mathcal{H}_1$ is fixed (or bounded).

For such classical codes there exists Gilbert-Varshamov (GV) bound:

$$\boxed{H(\delta) + (1-R)p_r(R,\delta) \leq 0}$$

S. Litsyn and V. Shevelev, Information Theory, IEEE Transactions on 48, 887 (2002).

$p_r(R,\delta) = \ln\left[(1+y)^k/2 + (1-y)^k/2\right] - \delta r \ln y - rH(\delta),\ R = 1 - c/r,$
y is the positive root of $(1+y)^{k-1} + (1-y)^{k-1} = (1-\delta)\left[(1+y)^k + (1-y)^k\right],$
and $H(\delta) = -\delta \ln \delta - (1-\delta)\ln(1-\delta),\ \delta = d/n.$

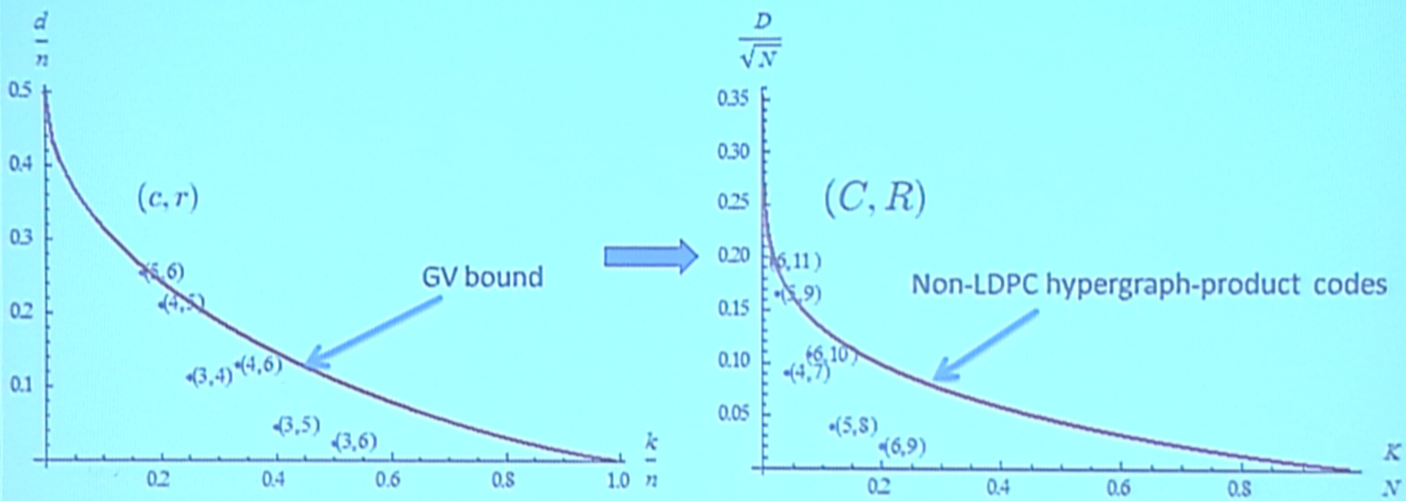
For large *n* there exist quantum LDPC codes with parameters:

$$\boxed{\mathcal{Q}^{HPC} = [[n, \frac{(r-c)^2}{r^2+c^2}n, \frac{r\delta(r,c)}{\sqrt{r^2+c^2}}\sqrt{n}]],}$$

for which the parity check matrix *H* has number of 1s in rows bounded by $R = r + c,$
and in columns by $C = r.$

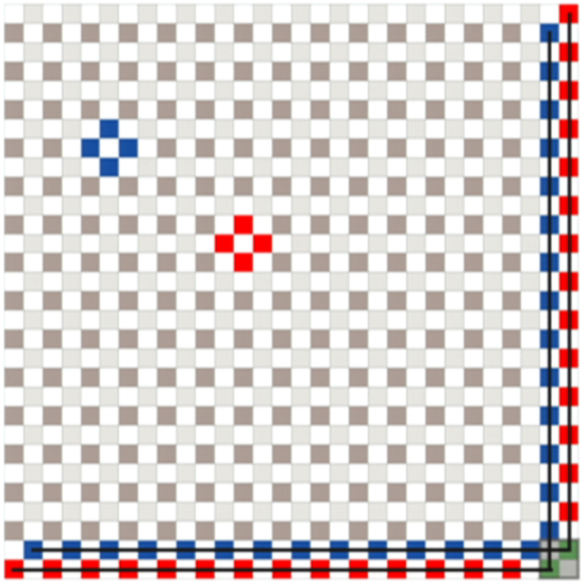
INFINITE FAMILIES OF QUANTUM LDPC CODES

Plots below show that there is a penalty on code parameters due to LDPC structure.



By setting *r* and *c* for classical codes we can generate families of quantum LDPC codes with the characteristics given in the plot.
(C, R) stands for the column and row weight.

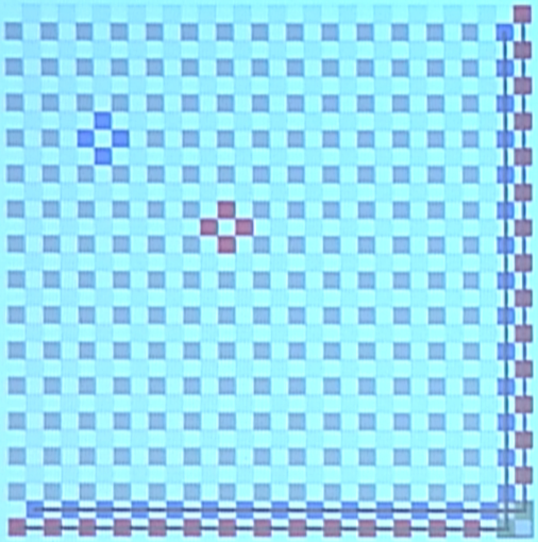
FAULT TOLERANCE OF LDPC CODES



$$Prob_{fail} \leq L^2 \sum_{l \geq d} \frac{2D}{2D-1}(2D-1)^l [4p(1-p)]^{l/2}$$

E. Dennis, A. Kitaev, A. Landahl, and J. Preskill,
J. Math. Phys. 43, 4452 (2002).

FAULT TOLERANCE OF LDPC CODES



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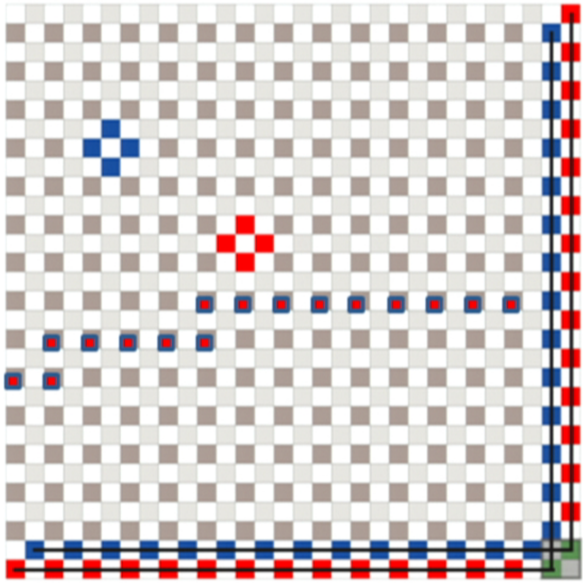

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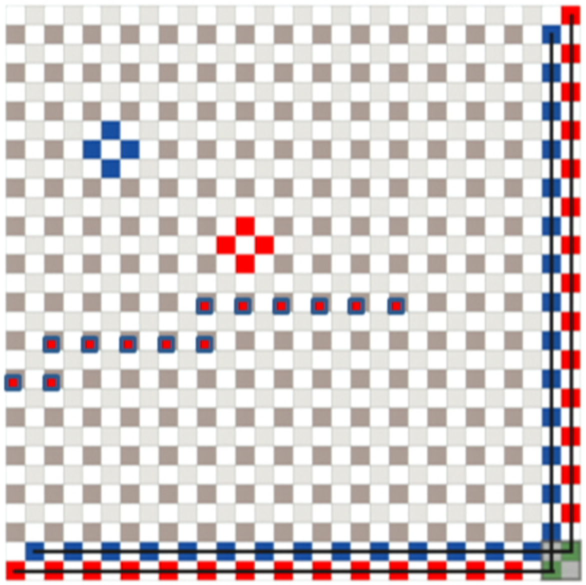

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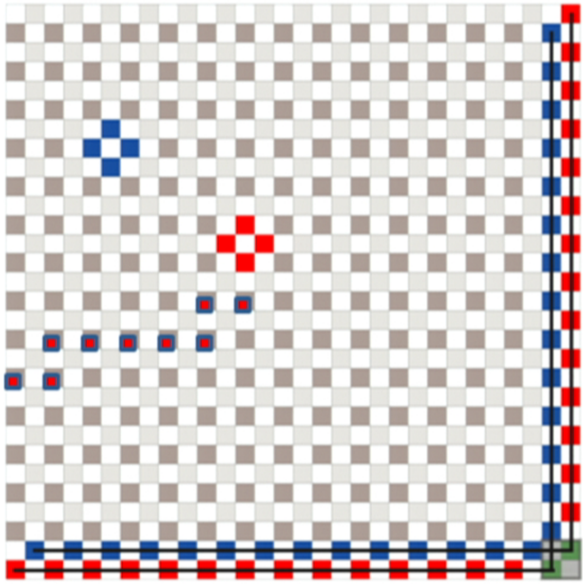

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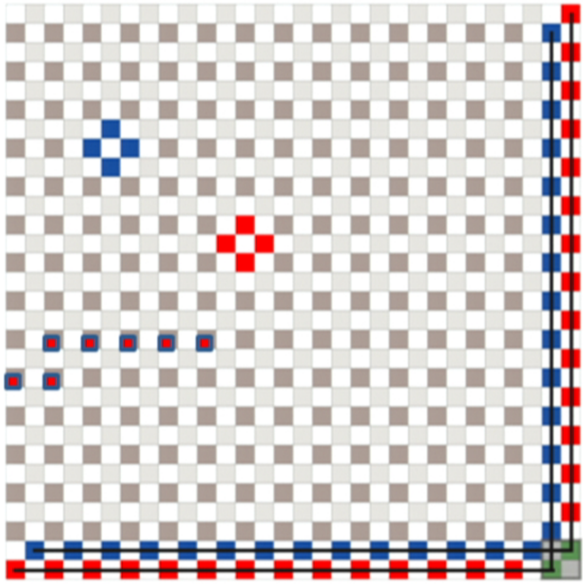

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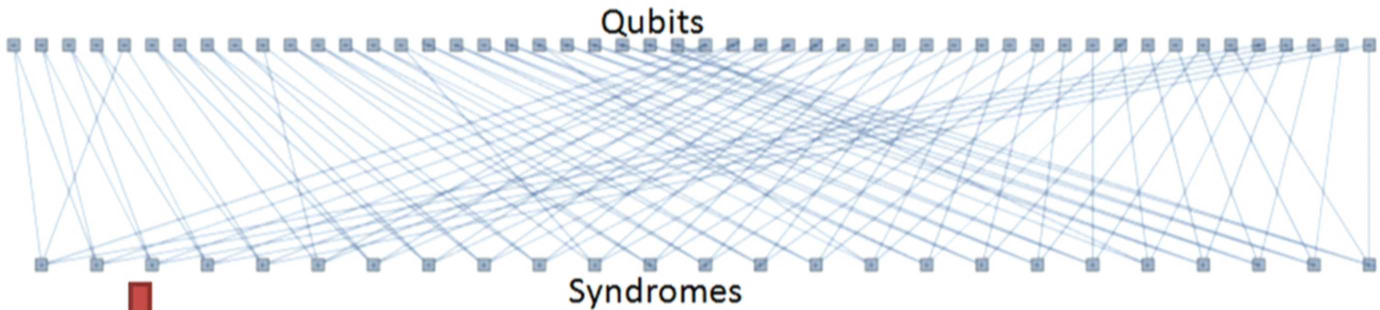
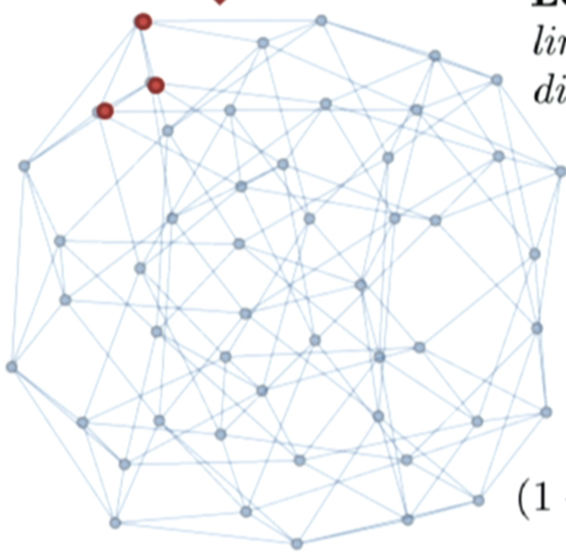
COUNTING CONNECTED REGIONS





Lemma 1. *For any graph $\mathcal{G}$ with vertex degrees limited by z, the site- or bond-percolation cluster size distribution has exponential tail for $p < p_0 \equiv \frac{1}{z-1}$.*

We use Cayley Tree to tackle arbitrary graph.

$$t_z(s) \equiv s(z-2)+2$$

$$t \le t_z(s)$$

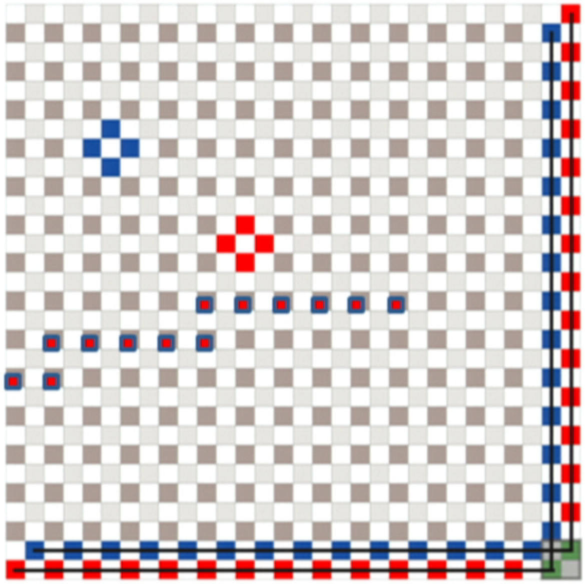
Cluster-size distribution: $n_s^{(p)}(x) = \sum_t a_{s,t}(x) p^s (1-p)^t$

$$(1-p)^t \le (1-p_0)^t \frac{(1-p)^{t_z(s)}}{(1-p_0)^{t_z(s)}}$$

$$\Downarrow$$

$$n_s^{(p)}(x) \le n_s^{(p_0)}(x) \frac{(1-p)^2}{(1-p_0)^2} \alpha_z^s, \quad \alpha_z \equiv \frac{p(1-p)^{z-2}}{p_0(1-p_0)^{z-2}}$$

FAULT TOLERANCE OF LDPC CODES



$$Prob_{fail} \leq L^2 \sum_{l \geq d} \frac{2D}{2D-1}(2D-1)^l[4p(1-p)]^{l/2}$$

E. Dennis, A. Kitaev, A. Landahl, and J. Preskill,
J. Math. Phys. 43, 4452 (2002).

COUNTING UNCORRECTABLE CLUSTERS

Uncorrectable cluster has to have errors in more than a half of its qubits.

$$f_s^{(p)}(x) \equiv \sum_{t,l \geq d} a_{st}(x) \sum_{m=\lceil l/2 \rceil}^{u} \binom{l}{m} p^m (1-p)^{l-m+t} \leq \sum_{t,l \geq d} a_{st}(x) 2^l p^{l/2} (1-p)^{l/2+t}$$

$$\leq [4(1-p)/p]^{s/2} n_s^{(p)}(x)$$

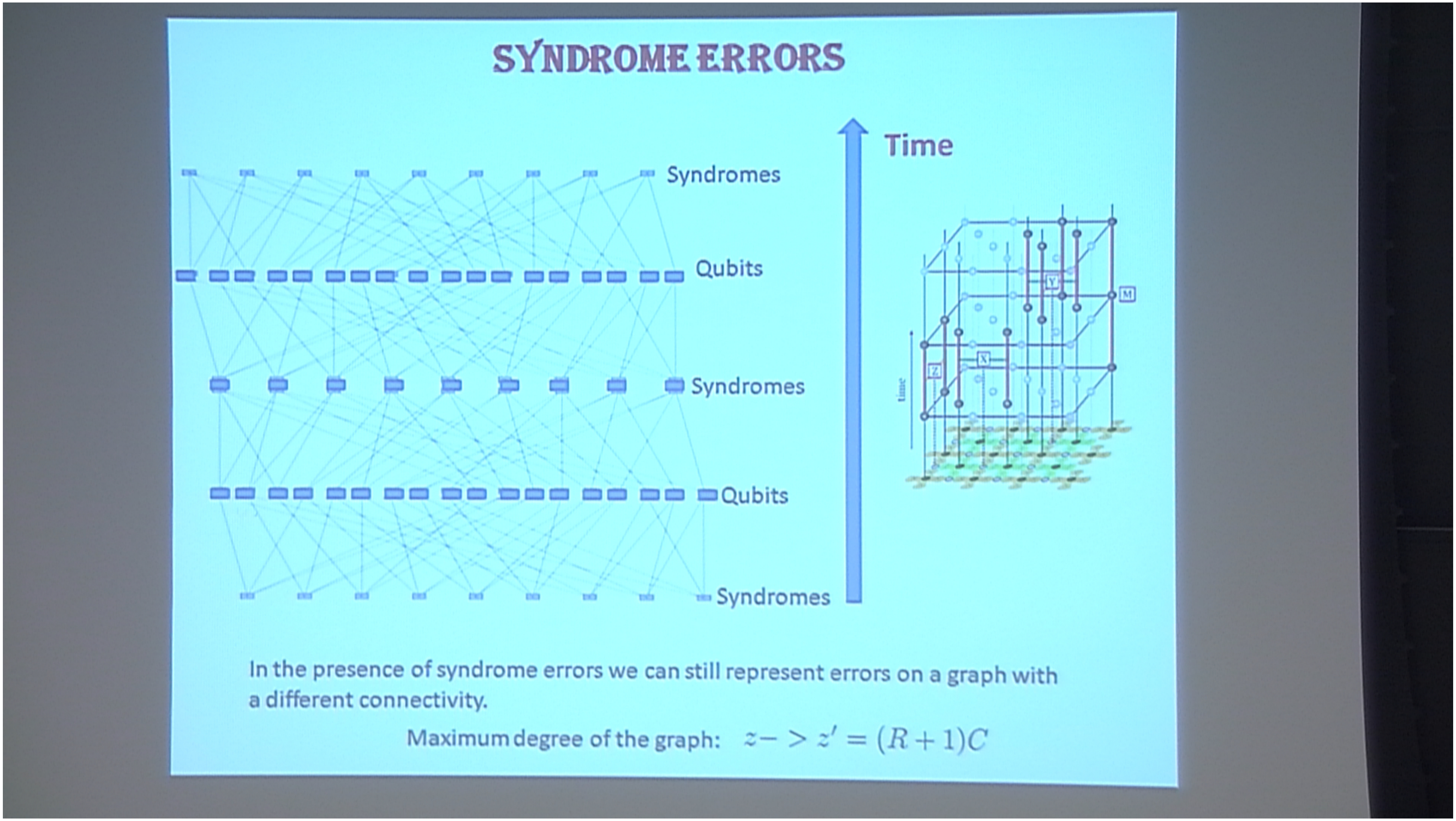
$$\Downarrow$$

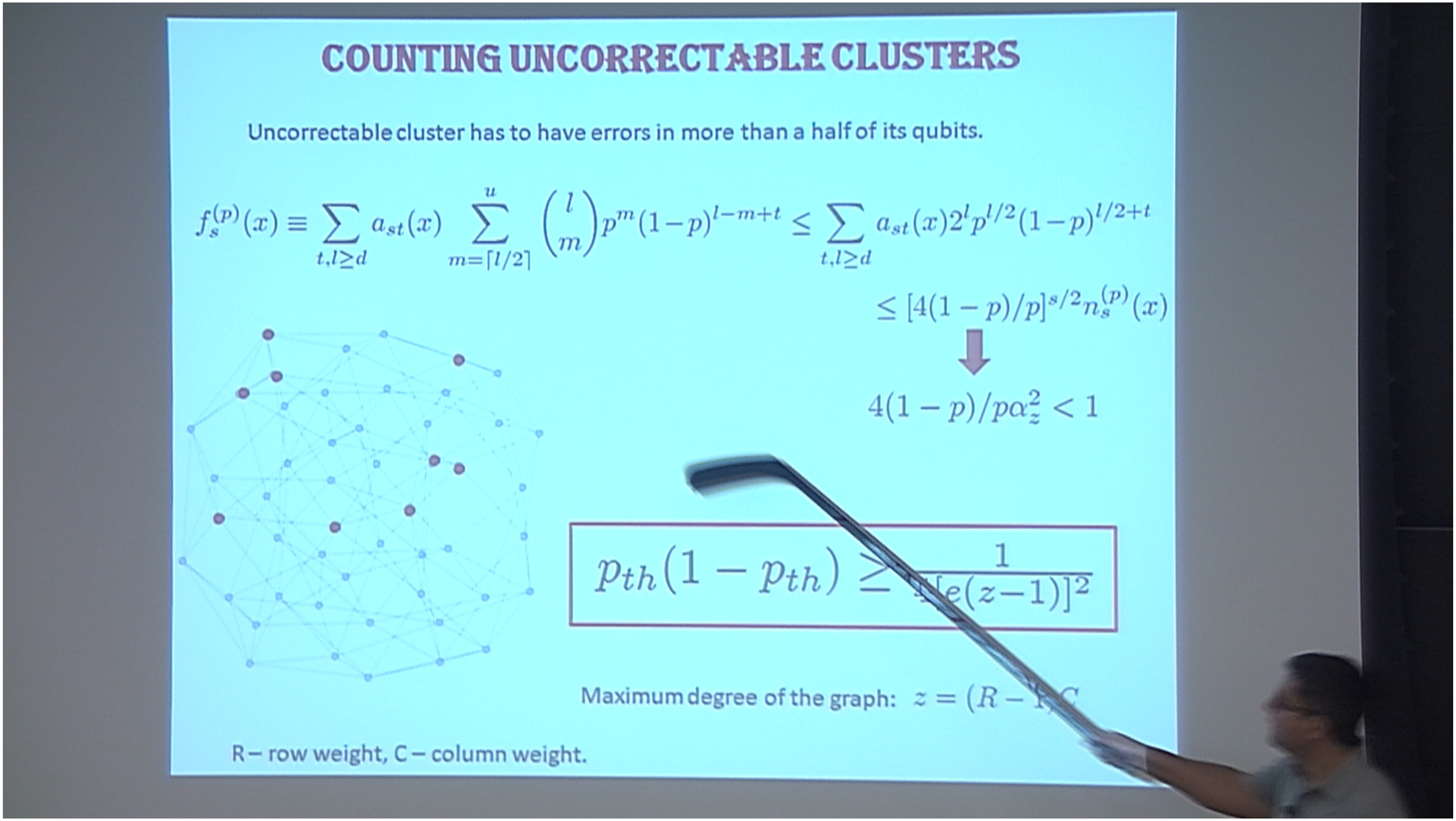
$$4(1-p)/p\alpha_z^2 < 1$$

$$\boxed{p_{th}(1-p_{th}) \geq \frac{1}{4[e(z-1)]^2}}$$

Maximum degree of the graph: $z = (R-1)C$

R – row weight, C – column weight.


SYNDROME ERRORS
Time
Syndromes
Qubits
Syndromes
Qubits
Syndromes
Y
M
X
Z
time
In the presence of syndrome errors we can still represent errors on a graph with a different connectivity.
Maximum degree of the graph: $z -> z' = (R+1)C$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Uncorrectable cluster has to have errors in more than a half of its qubits.
$f_s^{(p)}(x) \equiv \sum_{t,l\geq d} a_{st}(x) \sum_{m=\lceil l/2 \rceil}^{u} \binom{l}{m} p^m (1-p)^{l-m+t} \leq \sum_{t,l\geq d} a_{st}(x) 2^l p^{l/2} (1-p)^{l/2+t}$
$\leq [4(1-p)/p]^{s/2} n_s^{(p)}(x)$
$4(1-p)/p\alpha_z^2 < 1$
$p_{th}(1-p_{th}) \geq \frac{1}{4[e(z-1)]^2}$
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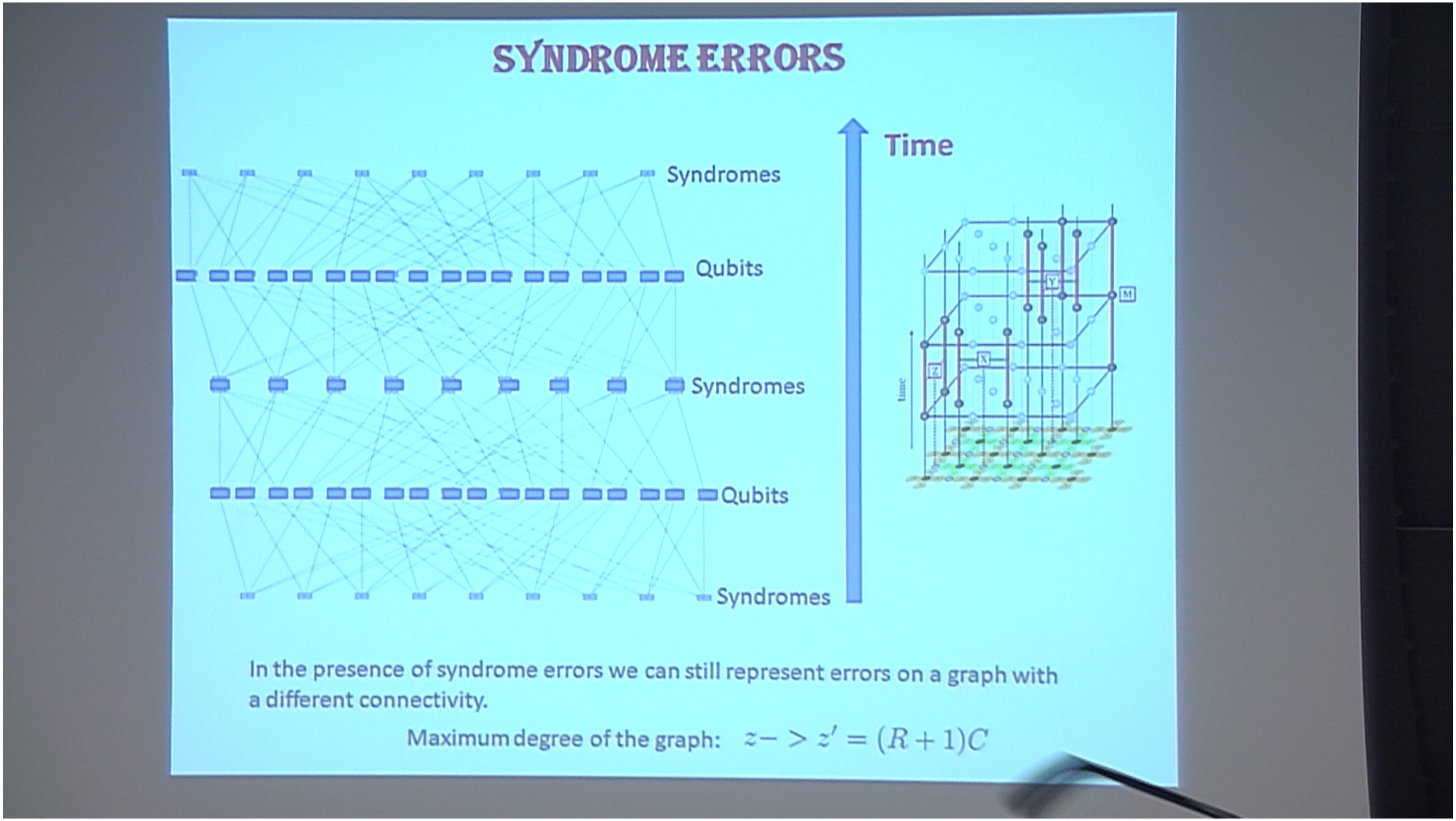
SYNDROME ERRORS
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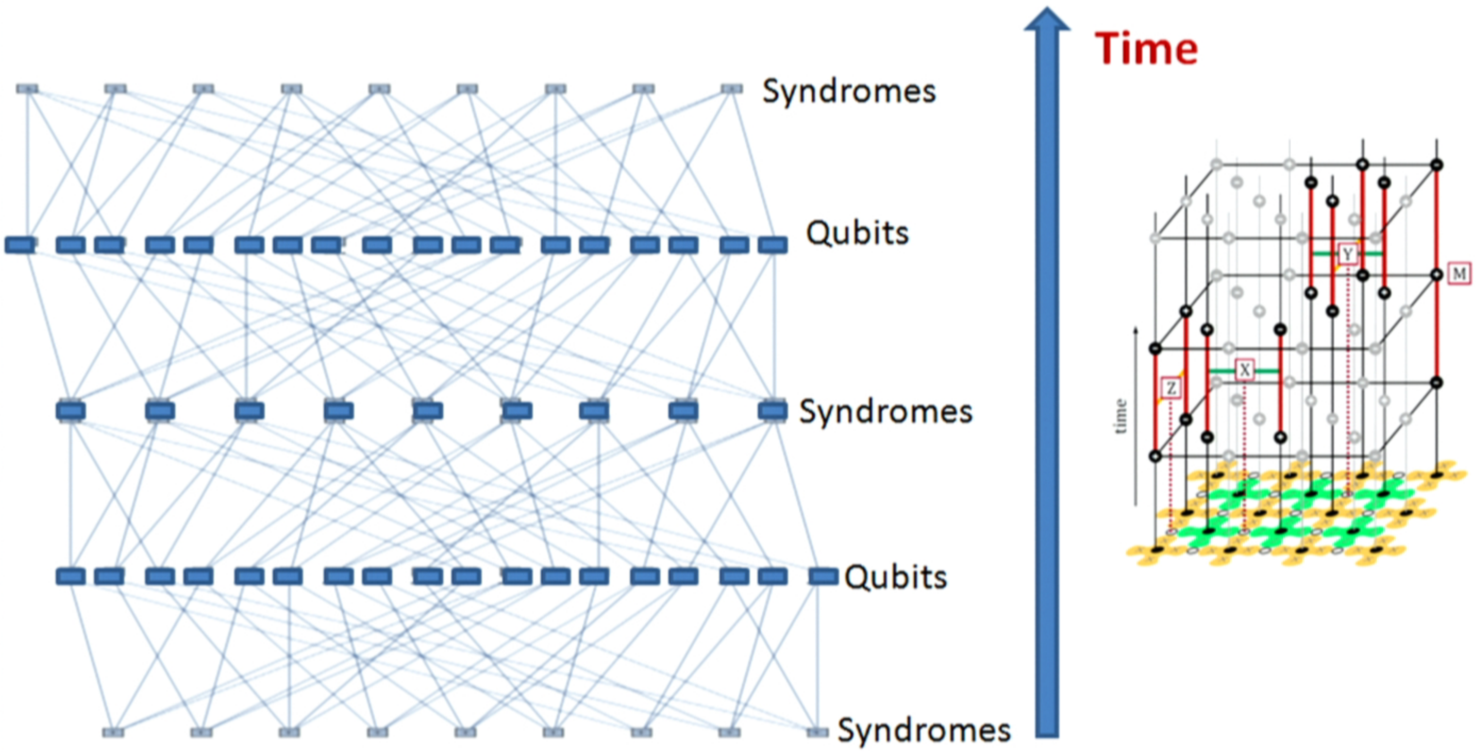
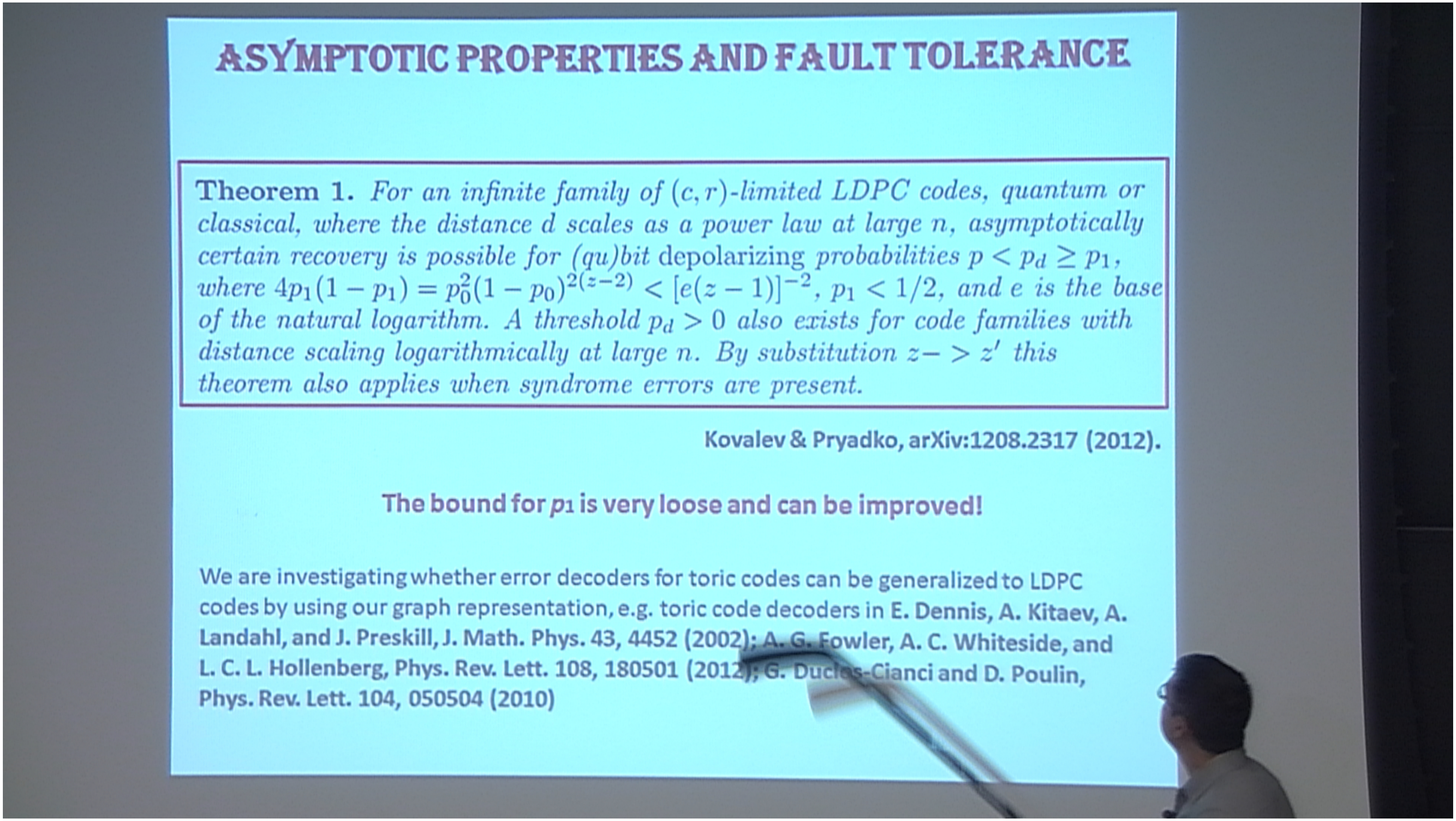
SYNDROME ERRORS



In the presence of syndrome errors we can still represent errors on a graph with a different connectivity.

Maximum degree of the graph: $z -> z' = (R+1)C$


ASYMPTOTIC PROPERTIES AND FAULT TOLERANCE
Theorem 1. *For an infinite family of (c,r)-limited LDPC codes, quantum or classical, where the distance d scales as a power law at large n, asymptotically certain recovery is possible for (qu)bit* depolarizing *probabilities $p < p_d \geq p_1$, where $4p_1(1-p_1) = p_0^2(1-p_0)^{2(z-2)} < [e(z-1)]^{-2}$, $p_1 < 1/2$, and e is the base of the natural logarithm. A threshold $p_d > 0$ also exists for code families with distance scaling logarithmically at large n. By substitution $z- > z'$ this theorem also applies when syndrome errors are present.*
Kovalev & Pryadko, arXiv:1208.2317 (2012).
The bound for p_1 is very loose and can be improved!
We are investigating whether error decoders for toric codes can be generalized to LDPC codes by using our graph representation, e.g. toric code decoders in E. Dennis, A. Kitaev, A. Landahl, and J. Preskill, J. Math. Phys. 43, 4452 (2002); A. G. Fowler, A. C. Whiteside, and L. C. L. Hollenberg, Phys. Rev. Lett. 108, 180501 (2012); G. Duclos-Cianci and D. Poulin, Phys. Rev. Lett. 104, 050504 (2010)

CONCLUSIONS

- We suggest to construct LDPC codes with locality on graphs which generalizes local LDPC codes (e.g. surface codes).
- We establish fault-tolerance of quantum LDPC codes by tanner graph representation of error correction.
- The explicit construction based on hypergraph-product codes leads to lower bound on parameters of quantum LDPC codes.
- Questions:

  1) Topological order for hypergraph-product codes?

  2) Design of efficient decoders for quantum LDPC codes?

  3) Existence of error threshold as a phase transition?