Title: Constraining RG flow in three-dimensional field theory

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Abstract: The entanglement entropy S(R) across a circle of radius R has been invoked recently in deriving general constraints on renormalization group flow in three-dimensional field theory.
At conformal fixed points, the negative of the finite part of the entanglement entropy, which is called F, is equal to the free energy on the round three-sphere. The F-theorem states that F decreases under RG flow.

Along the RG flow it has recently been shown that the renormalized entanglement entropy {\cal F}(R) = -S(R) + R S'(R), which is equal to F at the fixed points, is a monotonically decreasing function. I will review various three-dimensional field theories where we can calculate F on the three-sphere and compute its change under RG flow, including free field theories, perturbative fixed points, large N field theories with double trace deformations, gauge theories with large numbers of flavors, and supersymmetric theories with at least {\cal N} = 2 supersymmetry. I will also present calculations of the renormalized entanglement entropy along the RG flow in free massive field theory and in holographic examples.

Constraining RG flow in $(2+1)$-dimensional field theory

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Princeton University

August 2012

Table of Contents

References

This talk is based mostly on ...

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- Igor R. Klebanov, Silviu S. Pufu, B.R.S., **F-Theorem without Supersymmetry**, 1105.4598
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- Igor R. Klebanov, Tatsuma Nishioka, Silviu S. Pufu, B.R.S., **On Shape Dependence and RG Flow of Entanglement Entropy**, 1204.4160
- B.R.S., **Exact and Numerical Results on Entanglement Entropy in (5+1)-Dimensional CFT**, 1206.5025
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C-theorems in QFT

- **The C-theorem** In a $D = d + 1$ dimensional QFT, a quantity C is said to satisfy a C-theorem if ...

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  - In even D, it is natural to continue Cardy's conjecture for the "A" anomaly coefficient (see Elvang, Freedman, Hung, Kiermaier, Myers, Theisen for recent work in 6d)
  - In odd D, it is natural to consider $(-1)^{\frac{D+1}{2}} F_{S^D}$ (see Myers, Sinha, Klebanov, Pufu, B.R.S.)

$D = 2$: The Zamolodchikov c-theorem

- **Definitions**
  The metric: $ds^2 = \frac{1}{2} dz d\bar{z}$, $r^2 \equiv z\bar{z}$

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$$G(r^2) \equiv 4z^3\bar{z} \langle T_{zz}(z,\bar{z})\, T_{z\bar{z}}(0,0)\rangle$$
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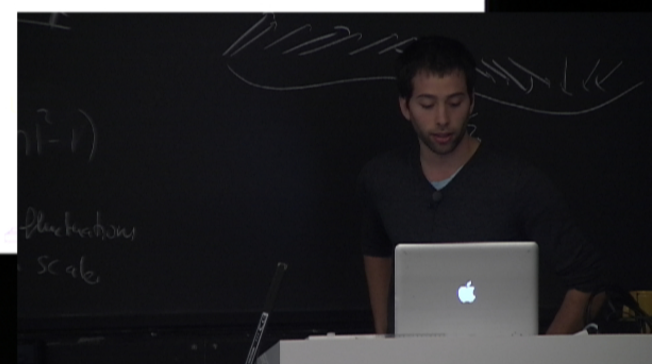

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where c is the central charge $C(r^2) = c$ and R is the curvature scalar.

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We can isolate c by putting the theory on the S^2 of radius R:

$$c = -\frac{3}{2} \int_{S^2} d^2x \sqrt{g} \langle T^\mu{}_\mu \rangle = \frac{3}{4} \frac{\partial F}{\partial \log R} \,,$$

where $F = -\log |Z_{S^2}|$.

$D = 4$: Cardy's a-theorem

- **Two Weyl anomaly coefficients in $D = 4$:**

$$\langle T^{\mu}{}_{\mu} \rangle = -\frac{c}{16\pi^2} W_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma} - 2aE_4 - \frac{a'}{16\pi^2} \nabla^2 R$$

We can isolate a by considering the integral of $\langle T^{\mu}{}_{\mu} \rangle$ on the S^4 of radius R:

$$a = -\frac{1}{4} \int_{S^4} d^4x \sqrt{g} \langle T^{\mu}{}_{\mu} \rangle = \frac{1}{16} \frac{\partial F}{\partial \log R}$$

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- **The a-theorem**
  Cardy proposed that a satisfies a C-theorem ($a_{\mathrm{UV}} > a_{\mathrm{IR}}$).
  This is called the a-theorem.
  Recently Komargodski and Schwimmer proved the a-theorem.
  They constructed a monotonic interpolating function along RG flow between the fixed points.

$D = 4$: Cardy's a-theorem

The proof of the a-theorem was preceded by more than 20 years of evidence. Much of this evidence came from studying QFT with supersymmetry.

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In supersymmetric QFT, a can be written as a function of the R-charges.
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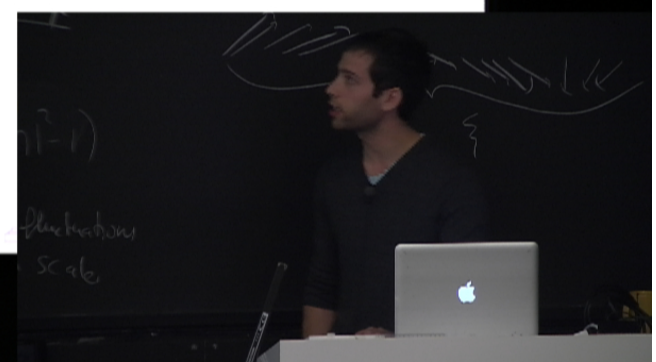
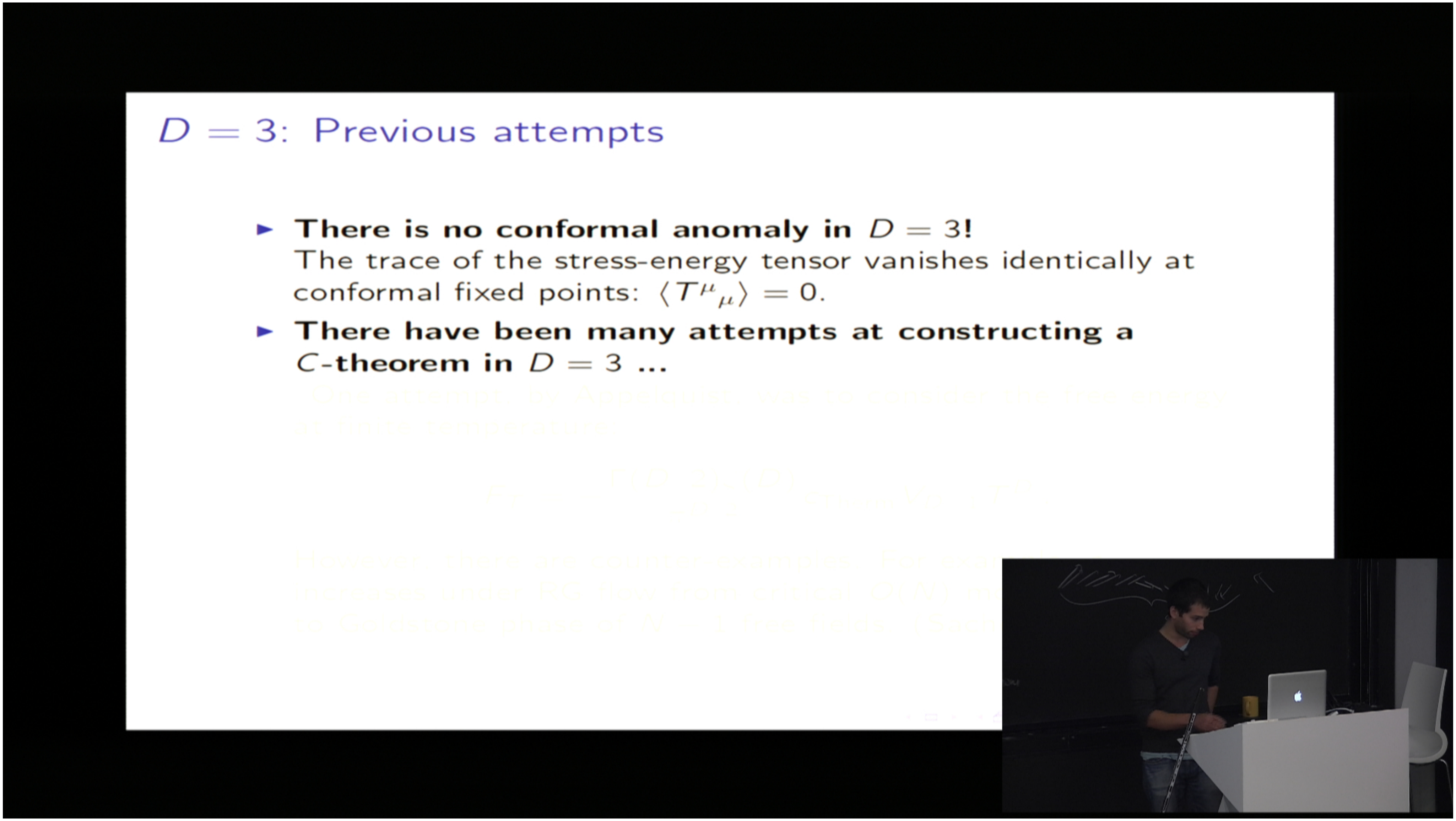
$D = 3$: Previous attempts

- **There is no conformal anomaly in $D = 3$!**
  The trace of the stress-energy tensor vanishes identically at conformal fixed points: $\langle T^{\mu}{}_{\mu} \rangle = 0$.

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  One attempt, by Appelquist, was to consider the free energy at finite temperature:

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  However, there are counter-examples. For example, c_{therm} increases under RG flow from critical $O(N)$ model fixed point to Goldstone phase of $N-1$ free fields. (Sachdev)

$D = 3$: The F-theorem

- **The F-theorem**
  The finite part of the free energy $F = -\log|Z_{S^3}|$ of CFTs on S^3 satisfies a C-theorem ($F_{\text{UV}} > F_{\text{IR}}$). (recent proof proposed by Casini and Huerta)

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- **Some motivation**
  - In D dimensions

$$\frac{\partial F}{\partial \log(R)} = -D \int_{S^D} d^D x \sqrt{g} \langle T^{\mu}{}_{\mu} \rangle$$

This vanishes at conformal fixed points in odd dimensions. The natural quantity to consider is then the finite part of F itself.

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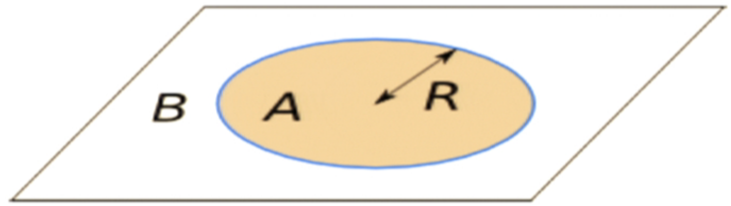
    This vanishes at conformal fixed points in odd dimensions. The natural quantity to consider is then the finite part of F itself.
  - There is a direct analogue of a-maximization: Jafferis's F-**maximization**. The F-value of the IR CFT is locally maximized by the trial R-charges.

$D = 3$: The F-theorem

F is related to the entanglement entropy.

$$S = -Tr(\rho_A \log \rho_A) \, ,$$

where ρ_A is the reduced density matrix: $\rho_A = Tr_B |0\rangle \langle 0|$.



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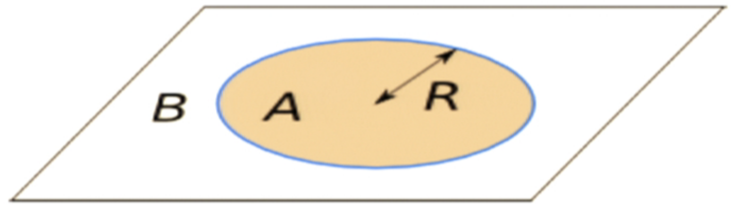

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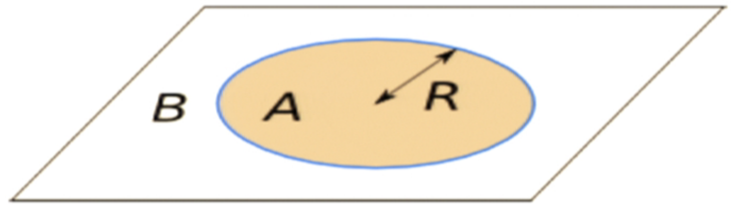

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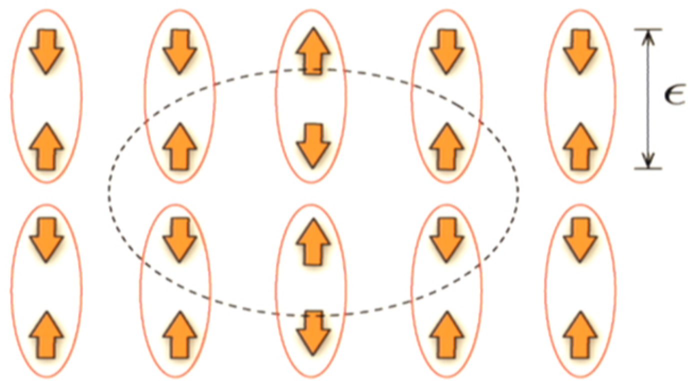
- Myer's and Sinha proposed that the finite part of the entanglement entropy should satisfy a C-theorem. **This is the same as the F-theorem**

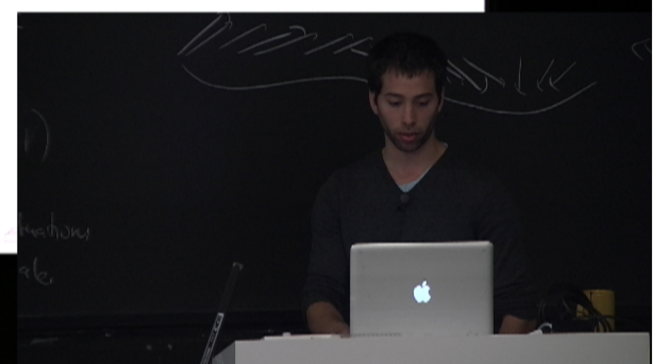
$D = 3$: The F-theorem

The entanglement entropy has a leading area law divergence. At conformal fixed points the EE across a circle of radius R is

$$S(R) = \alpha \frac{2\pi R}{\epsilon} - F \,,$$

where ϵ is the short-distance cut-off and the constant α is regularization dependent.





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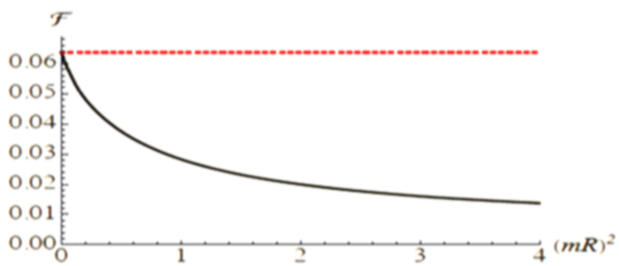
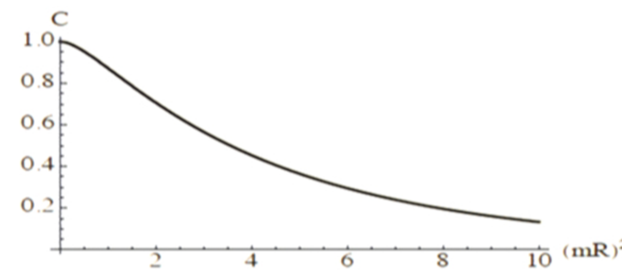
- Casini and Huerta's proposed proof of the F-theorem relies on the renormalized entanglement entropy (Liu, Mezei)

$$\mathcal{F}(R) = -S(R) + R\,S'(R)\,.$$

They showed $\mathcal{F}'(R) \leq 0$, with equality only coming at fixed points where $\mathcal{F}(R) = F$.





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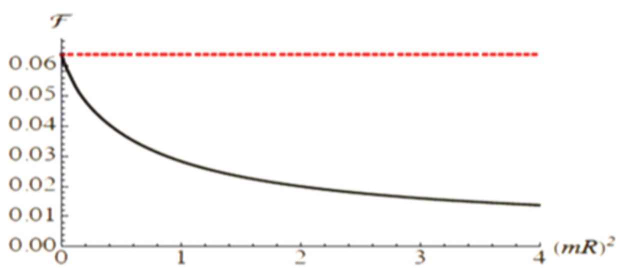

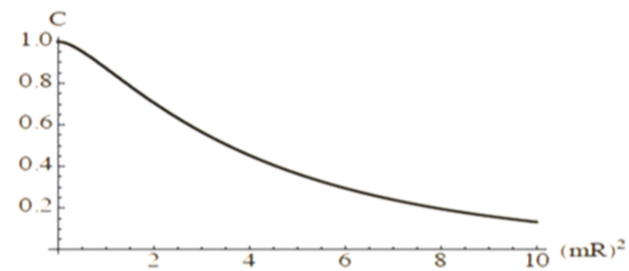

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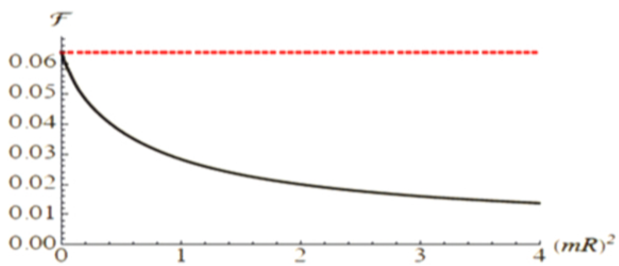

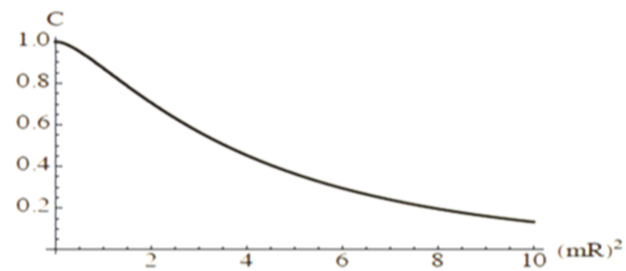

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If we perturb by an operator of dimension $\Delta = D - \delta$ and g is the renormalized, dimensionless coupling, then stationarity requires $c(g) = c_{\text{uv}} - g^2\delta + O(g^3)$.

2. Calculating F on the three-sphere

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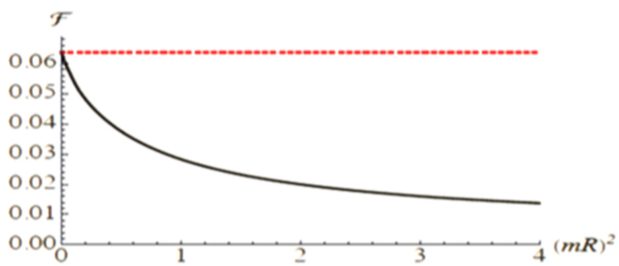

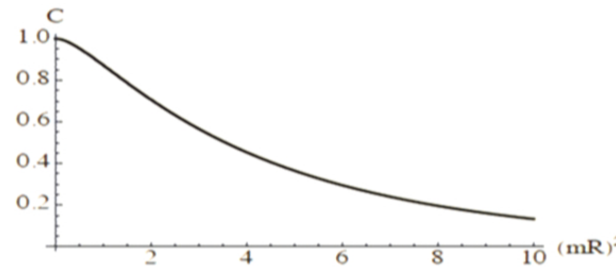

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Free fields

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- F_s is calculated from the partition function on the S^3 of radius a:

$$F_s = \log |Z_s| = \frac{1}{2} \log \det \left[\mu_0^{-2} \mathcal{O}_s\right] \ , \qquad \mathcal{O}_s = -\nabla^2 + \frac{3}{4a^2} \, .$$

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- The eigenvalues and degeneracies of $\mathcal{O}_s$:

$$F_s = \frac{1}{2} \sum_{n=0}^{\infty} m_n \log \left(\mu_0^{-2} \lambda_n \right) \, ,$$

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Free fields

The simplest F-value to calculate is that of the free conformal scalar.

- F_s is calculated from the partition function on the S^3 of radius a:

$$F_s = \log |Z_s| = \frac{1}{2} \log \det \left[\mu_0^{-2} \mathcal{O}_s \right] , \qquad \mathcal{O}_s = -\nabla^2 + \frac{3}{4a^2} .$$

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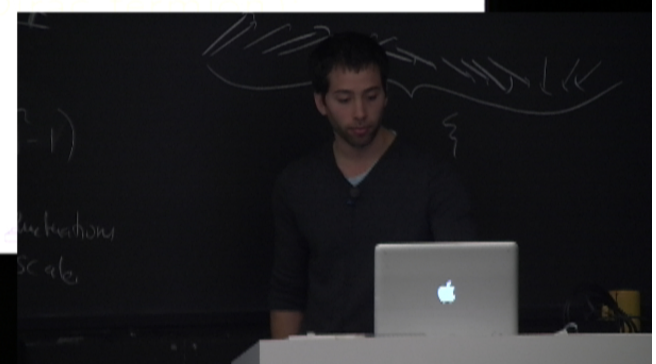

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- F for a chiral multiplet (complex scalar plus Dirac fermion):

$$F_{\text{chiral}} = 2(F_s + F_M) = \frac{\log 2}{2}\,.$$

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- **The action of the perturbed QFT on S^3**

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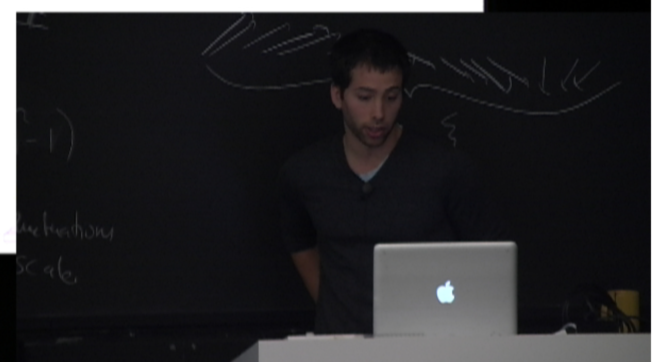
where S_0 is the action of the unperturbed CFT, $O(x)$ is a scalar operator of dimension $\Delta = 3 - \epsilon$ with $0 < \epsilon \ll 1$. λ_0 is the UV bare coupling defined at the UV scale μ_0.

Conformal invariance fixes the coefficients of the 2 and 3-point functions



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The theory flows:

$$\beta(g) = \mu \frac{dg}{d\mu} = -\epsilon g + 2\pi C g^2 + \mathcal{O}(g^3) \ ,$$

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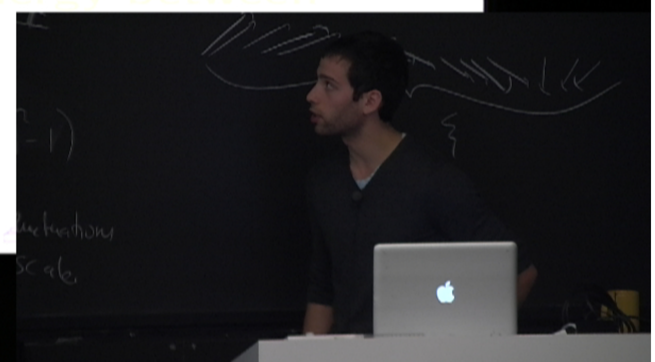

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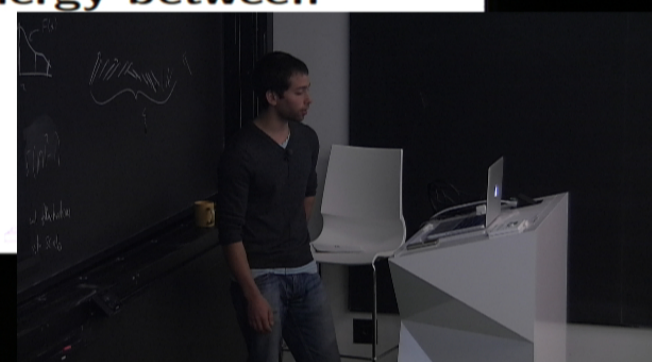
A short calculation gives the change in free energy between the UV and IR fixed points:

$$\delta F(g^*) = -\frac{\pi^2 \epsilon^3}{72 C^2} \, .$$



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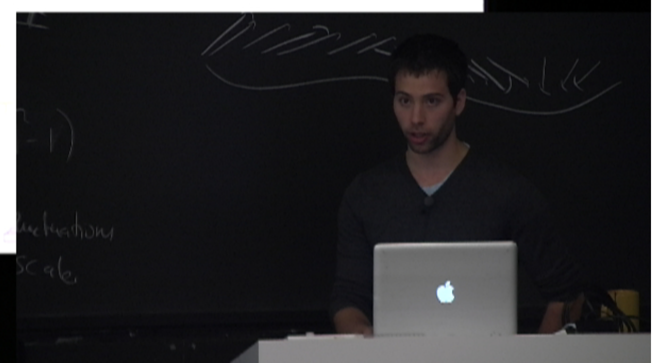
$$Z = \int D\phi \exp\left(-S_0 - \frac{\lambda_0}{2}\int d^D x\sqrt{G}\Phi^2\right) ,$$

where Φ is a single-trace operator of dimension $\Delta \in ((D-2)/2, D/2)$ and S_0 describes a large N CFT.



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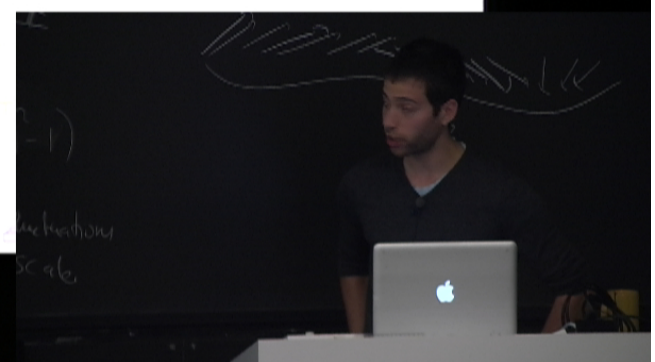

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$$\frac{Z}{Z_0} = \frac{\int D\sigma \left\langle \exp\left[\int d^D x\sqrt{G}\left(\frac{1}{2\lambda_0}\sigma^2 + \sigma\Phi\right)\right]\right\rangle_0}{\int D\sigma \exp(\frac{1}{2\lambda_0}\int d^D x\sqrt{G}\sigma^2)} .$$

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Higher point functions of Φ are suppressed relative to the two-point function by powers of $1/N$:

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The Gaussian integral over the auxiliary field $\sigma(x)$ then gives

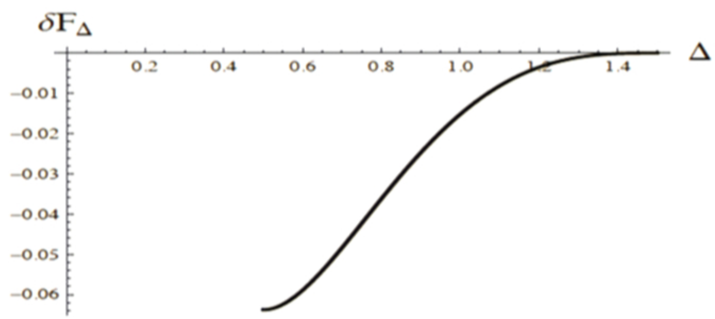
$$\delta F_\Delta = \frac{1}{2}\mathrm{tr}\log(K)\,,$$

$$K(x,y) = \frac{1}{\sqrt{G(x)}}\delta(x-y) + \lambda_0 a^D \langle\Phi(x)\Phi(y)\rangle_0 \,.$$

Double-trace deformations

- In the IR limit ($a \to \infty$) we find

$$\delta F_{\Delta} = -\frac{\pi}{6} \int_{\Delta}^{3/2} dx(x-1)(x-\frac{3}{2})(x-2)\cot(\pi x)\,.$$



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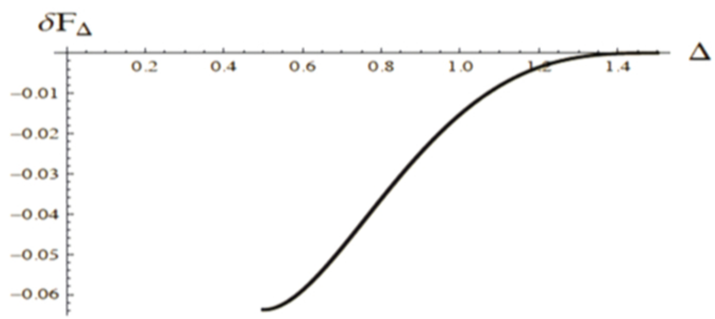

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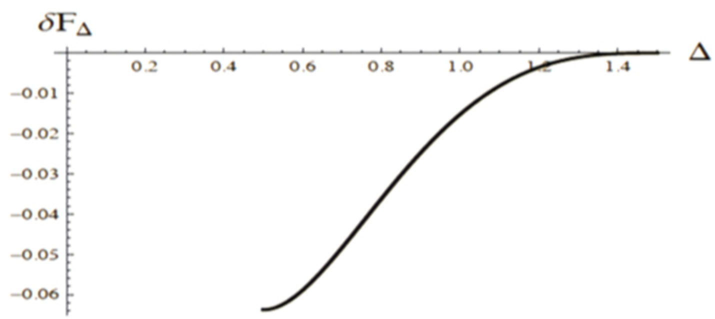


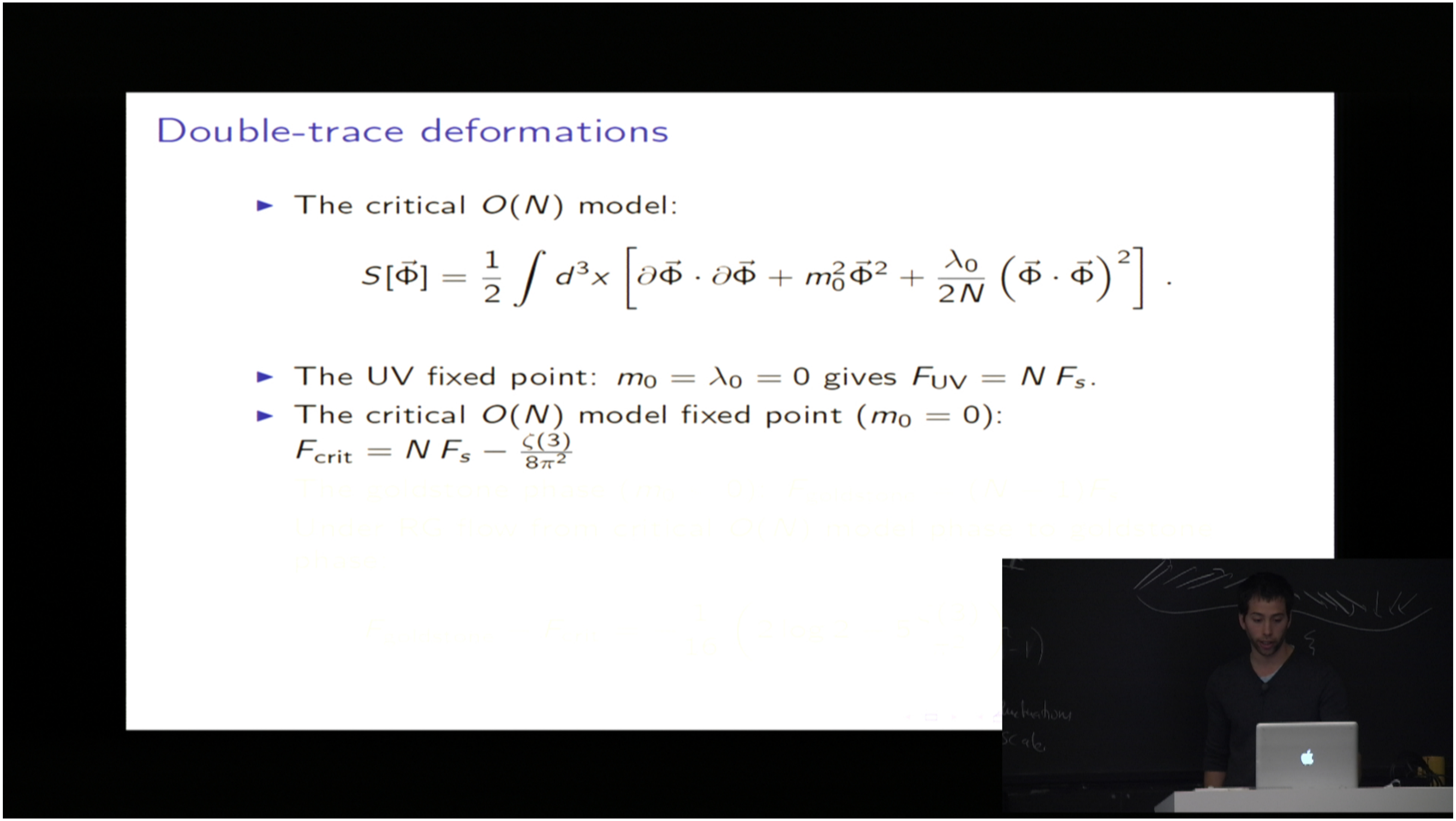
- When $\Delta = 1$ this describes the critical $O(N)$ model, and we find

$$\delta F_{\Delta=1} = -\frac{\zeta(3)}{8\pi^2} \approx -0.0152\,.$$

Double-trace deformations

- The critical $O(N)$ model:

$$S[\vec{\Phi}] = \frac{1}{2}\int d^3x \left[\partial\vec{\Phi}\cdot\partial\vec{\Phi} + m_0^2\vec{\Phi}^2 + \frac{\lambda_0}{2N}\left(\vec{\Phi}\cdot\vec{\Phi}\right)^2\right] .$$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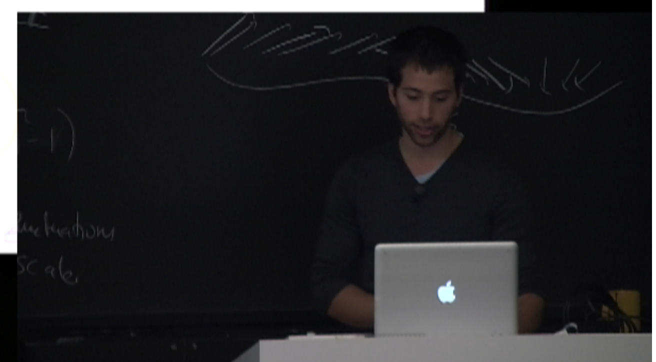

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- Under RG flow from critical $O(N)$ model phase to goldstone phase:

$$F_{\rm goldstone} - F_{\rm crit} = -\frac{1}{16}\left(2\log 2 - 5\frac{\zeta(3)}{\pi^2}\right) \approx -.0486$$

CS gauge theories with many flavors

- The gauge sector Lagrangian is

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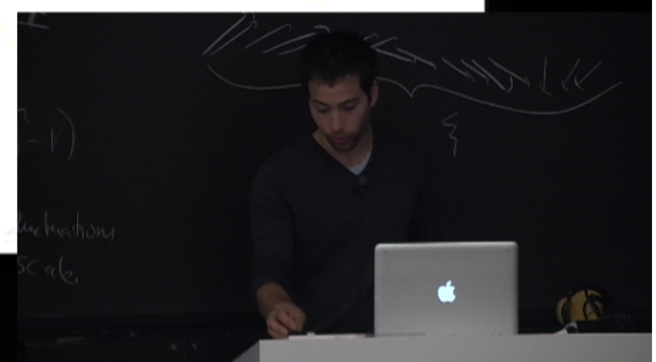
The gauge coupling e^2 has dimension of mass and flows to infinity in the IR.



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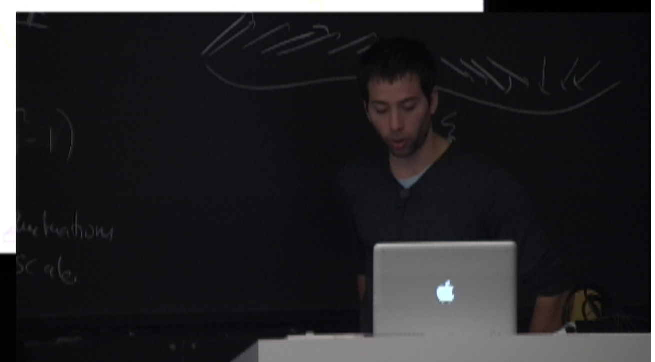

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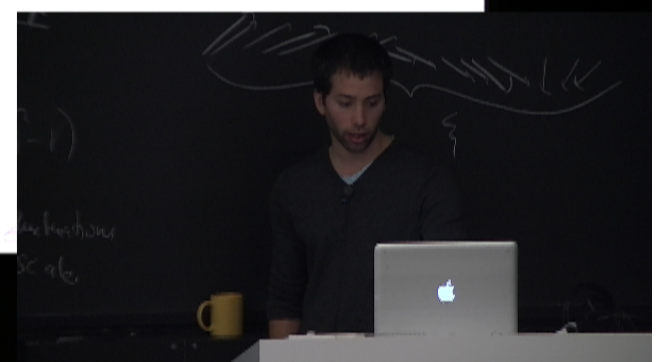

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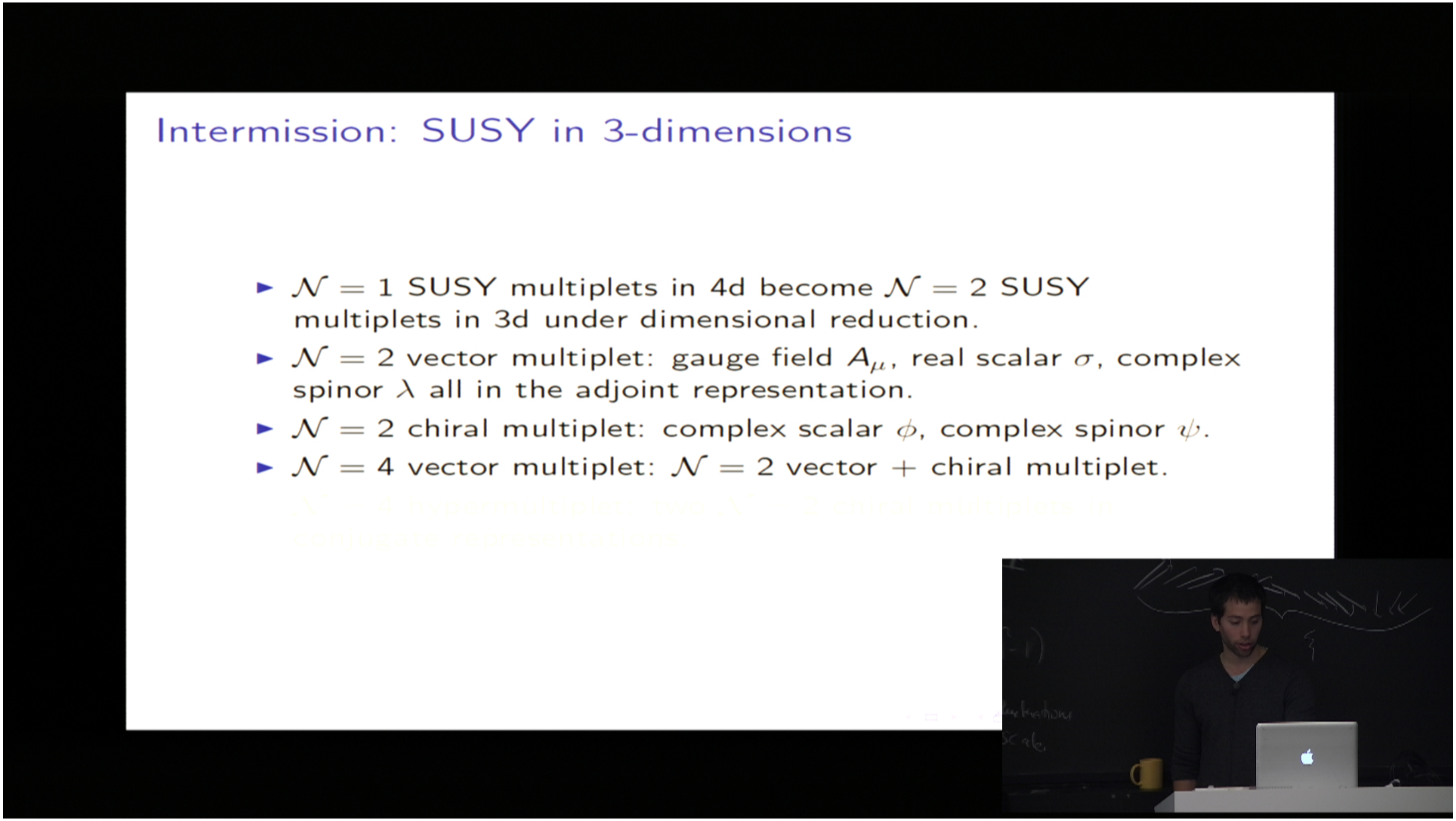
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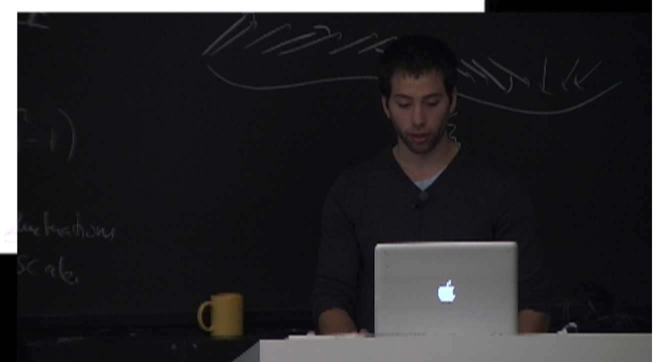

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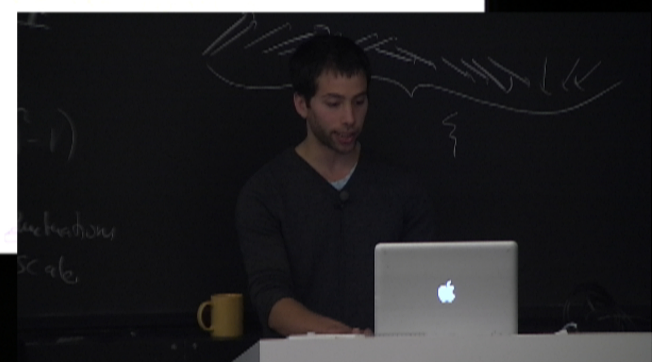

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    - In practice we can take $\mathcal{V} = Tr\left((Q\lambda)^\dagger\lambda\right)$ for vector multiplet and $\mathcal{V} = Tr\left[(Q\psi)^\dagger\psi + \psi^\dagger(Q\psi^\dagger)^\dagger\right]$ for chiral multiplet.

Example: $\mathcal{N}=4$ $U(1)$ gauge theory with many flavors

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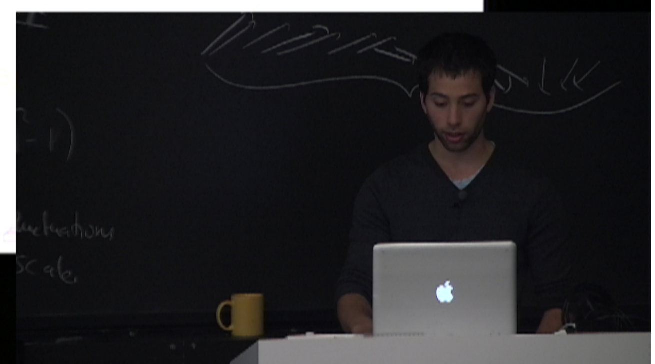

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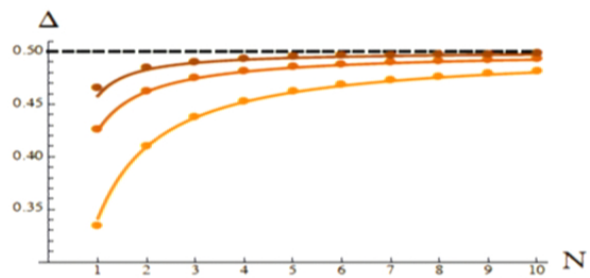

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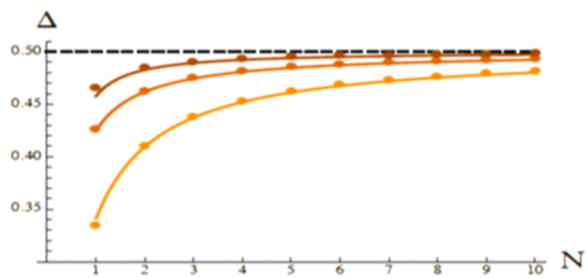

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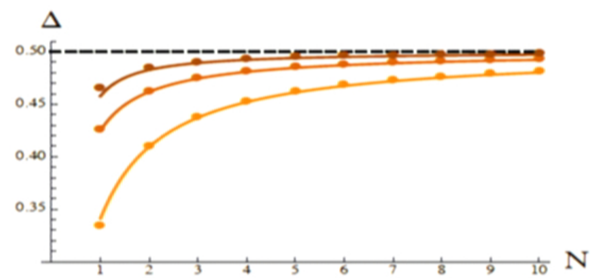

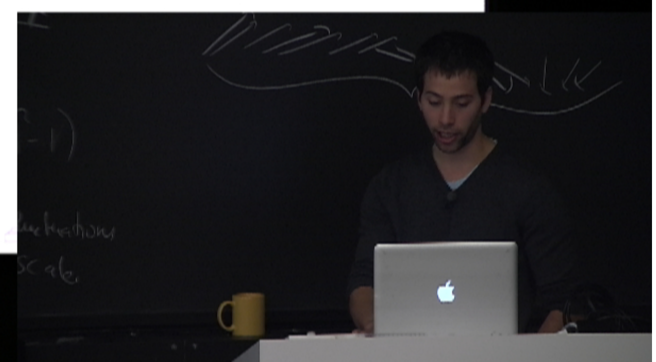

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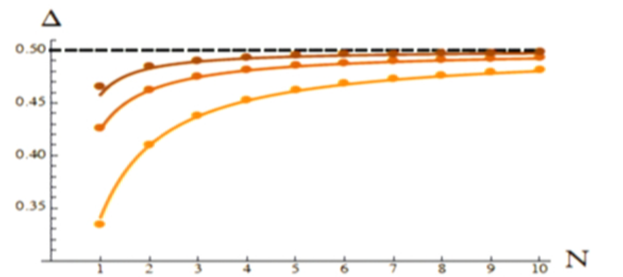
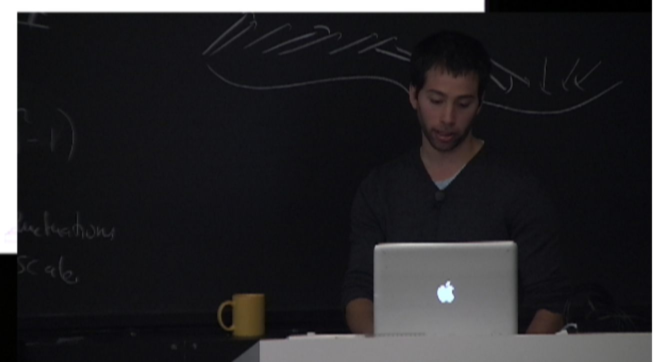
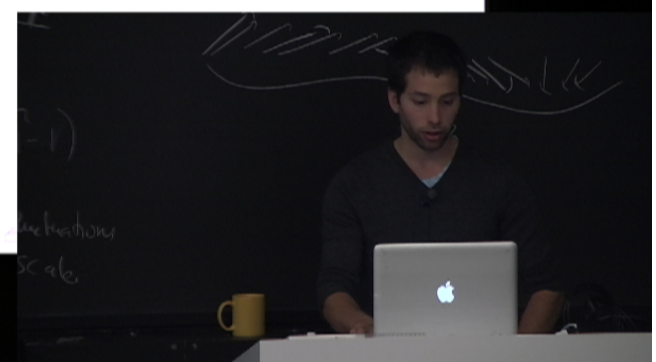
- Δ is the scaling dimension of the flavors! (Darker colors are increasing k from 0 to $4N$.)





Side-note: large N_c field theories with M-theory duals

- Let's consider $AdS_4 \times Y$ compactifications of M-theory



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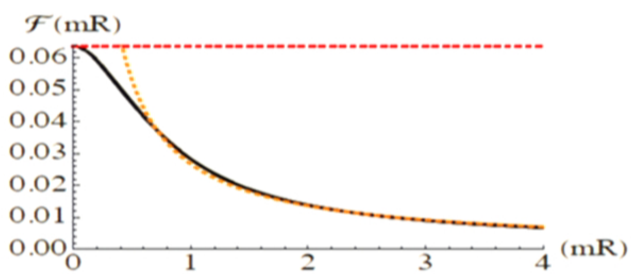
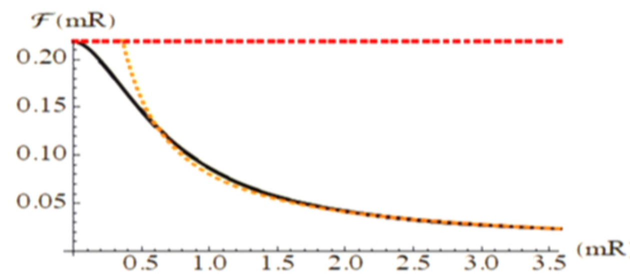
  - We can evaluate the on-shell supergravity action.
  - We can calculate the EE holographically (next section).
  - We can, in certain cases, calculate F directly in the field theory using localization.

Massive free fields

Recall the **renormalized entanglement entropy**, which is a monotonic interpolating function for the F-values along the RG flow:

$$\mathcal{F}(R) = -S(R) + R\,S'(R)\,.$$





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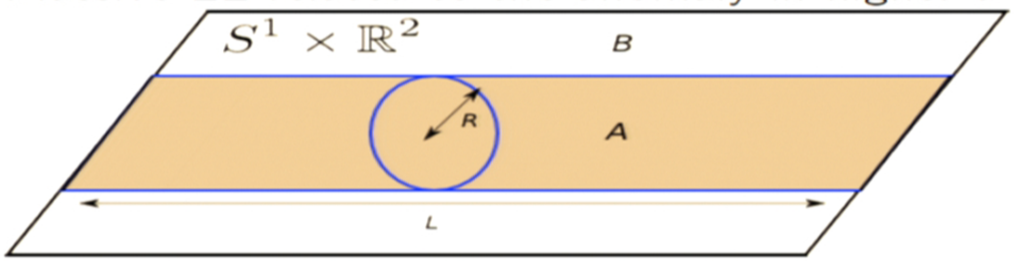
Massive EE related to the anomaly in higher d (Casini, Huerta):



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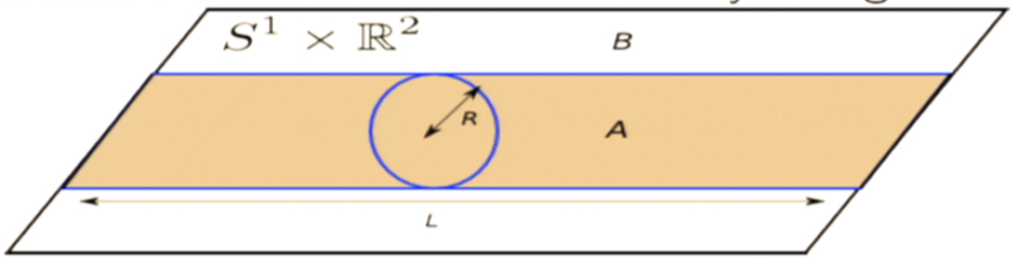


- Consider entanglement entropy of massless scalar field in 4-dimensions across $\Sigma_2 = \Sigma_1 \times S^1$ (Solodhukin):

$$S^{(4)}_{\Sigma_2} \supset \left(\frac{a}{720\pi} \int_{\Sigma_2} R_\Sigma + \frac{c}{240\pi} \int_{\Sigma_2} \left(k_a^{\mu\nu} k^a_{\nu\mu} - \frac{1}{2} k_a^{\mu\mu} k^a_{\nu\nu} \right) \right) \log \epsilon$$

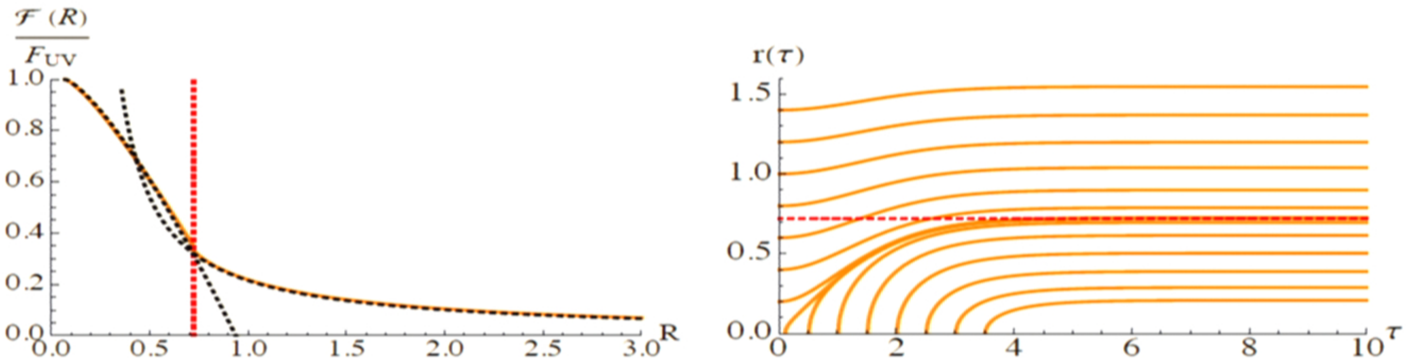
- KK reduce in direction of S^1 of length L: 3d masses $m_n^2 = (2\pi/L)^2 n^2$.
- Log term in 4d massless EE related to $1/m$ term in 3d massive EE: $S^{(4)}_{\Sigma_2} \propto \int dm\, S^{(3)}_{\Sigma_1}(m)$.
- (Huerta, B.R.S.) $\mathcal{F}(mR) = \frac{\pi}{24} \left(\frac{1}{mR} + \frac{3}{32} \frac{1}{(mR)^3} + \dots \right)$

A holographic example: the CGLP background

A few more details ...

- The warped product metric looks like
  $ds^2 = H^{-2/3}(-dt^2 + dr^2 + r^2 d\phi^2) + H^{1/3} ds_8^2$
- ds_8^2 is parameterized by radial coordinate $\tau \in [0, \infty)$ and 7 angles in $V_{5,2}$.
- At $\tau = 0$ an S^3 shrinks to zero size

The minimal surfaces and renormalized EE for this theory looks like ...



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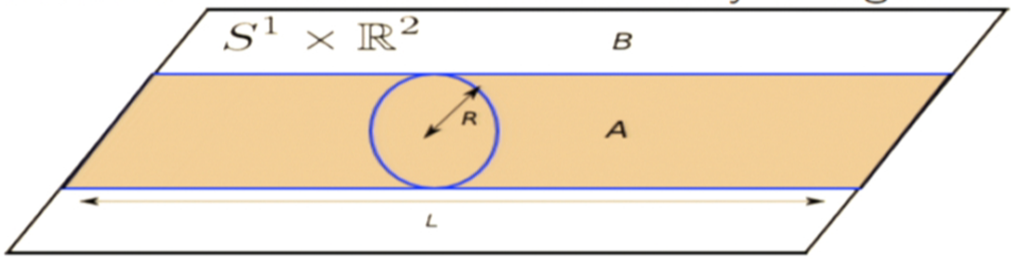

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- **Holographic prescription for calculating the entanglement entropy** (Ryu, Takayanagi, Klebanov, Kutasov, Murugan): find the minimal area codimension 2 hypersurface in the bulk which approaches entangling surface at the boundary: $S_\Sigma \propto \int_{\Sigma_{D-2}} d^{D-2}\sigma \sqrt{G_{\text{ind}}^{(D-2)}}$
- A nice example is the CGLP (Cvetic, Gibbons, Lu, Pope) background of M-theory. Supergravity background is a warped product of $\mathbb{R}^{2,1}$ and an eight dimensional Stenzel space (similar to KS background)

$$\sum_{i=1}^{5} z_i^2 = \epsilon^2 .$$

- This background is dual to a confining gauge theory, with UV fixed point dual to $AdS_4 \times V_{5,2}$.

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