Title: The effect of initial correlations on the evolution of quantum states

Date: Oct 15, 2012 04:00 PM

URL: http://pirsa.org/12100045

Abstract: Until fairly recently, it was generally assumed that the initial state of a quantum system prepared for information processing was in a product state with its environment. If this is the case,
the evolution is described by a completely positive map. However, if the system and environment are initially correlated, or entangled, such that the so-called quantum discord is non-zero, then the
evolution is described by a map which is not completely positive. Maps that are not completely positive are not as well understood and the implications of having such a map are not completely known. I will discuss a few examples and a theorem (or two) which may help us understand the implications of having maps which are not completely positive.

The Effect of Prior Correlations on Open-System Evolution

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Perimeter Institute, Waterloo, ON, CA Oct. 15, 2012

QUNET WIKIBOOK: http://qunet.physics.siu.edu/wiki

Outline:

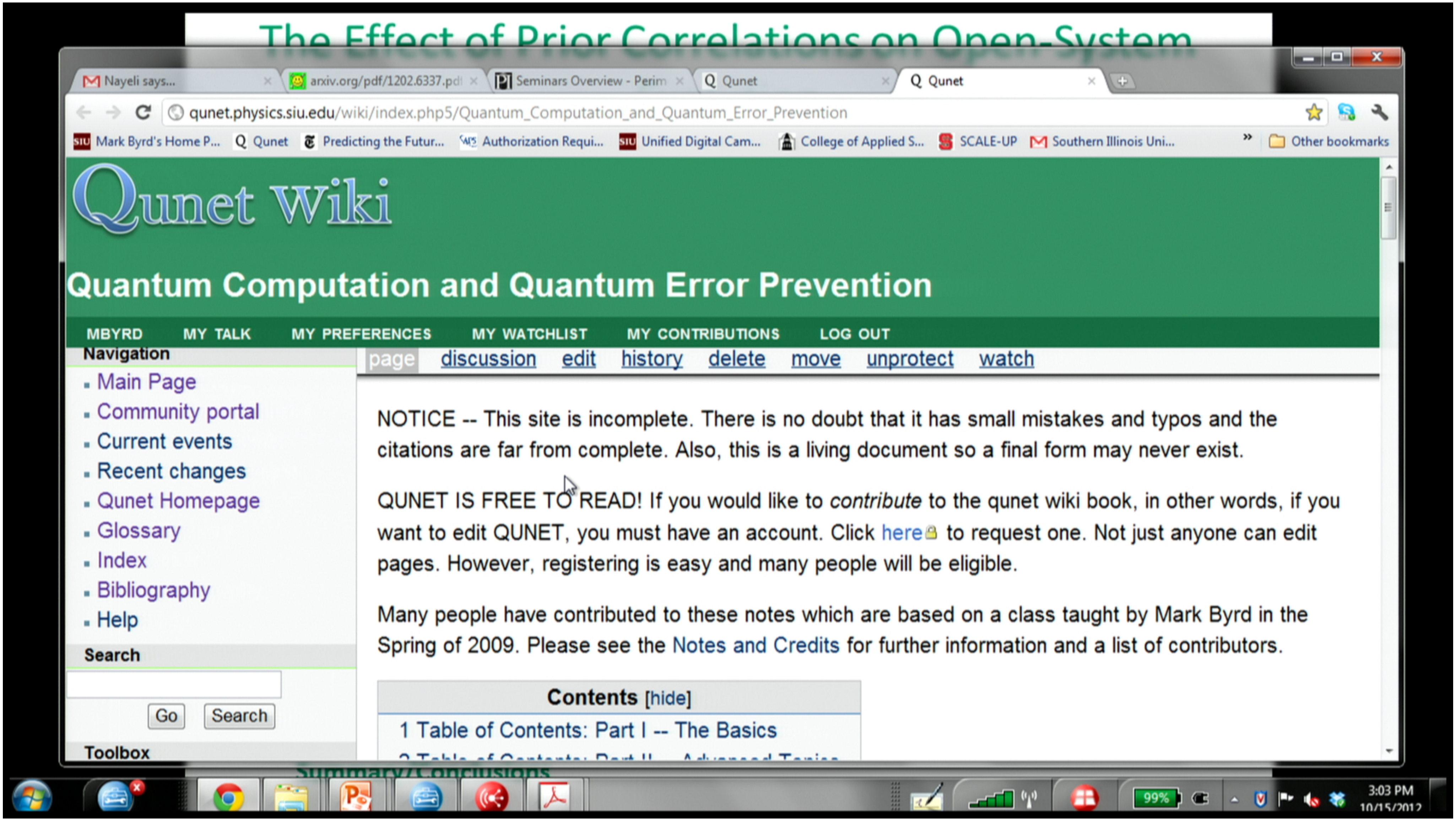
I. **Introduction:** How do we describe open system evolution?
Positivity, complete positivity, and discord

II. **Examples** of maps which are not completely positive

III. **Unitary/Pseudo-unitary freedom** in the OSR and reversibility conditions

IV. **Affine map** of the polarization vector (coherence or generalized Bloch vector)

Summary/Conclusions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Many people have contributed to these notes which are based on a class taught by Mark Byrd in the Spring of 2009. Please see the Notes and Credits for further information and a list of contributors.
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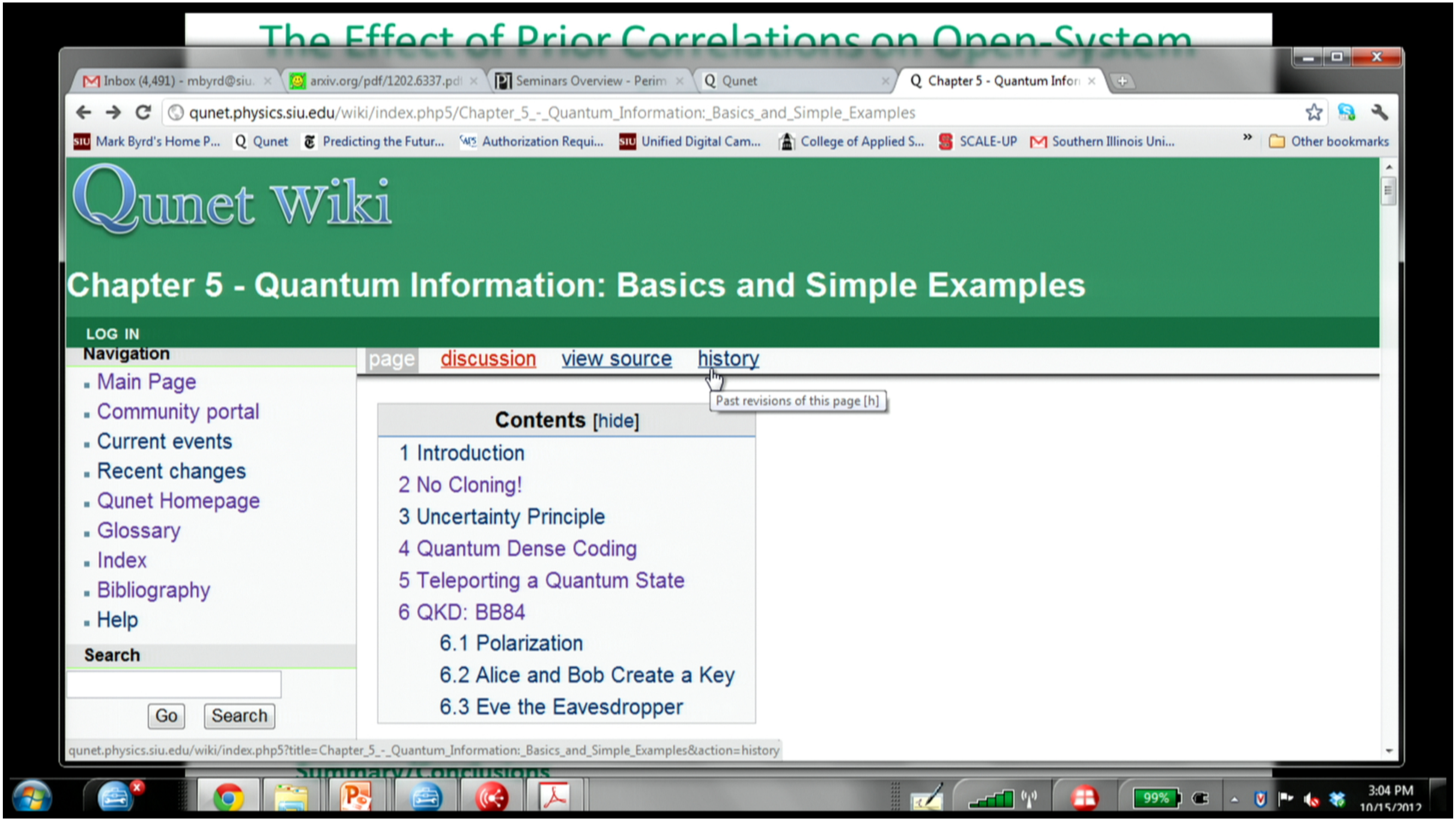
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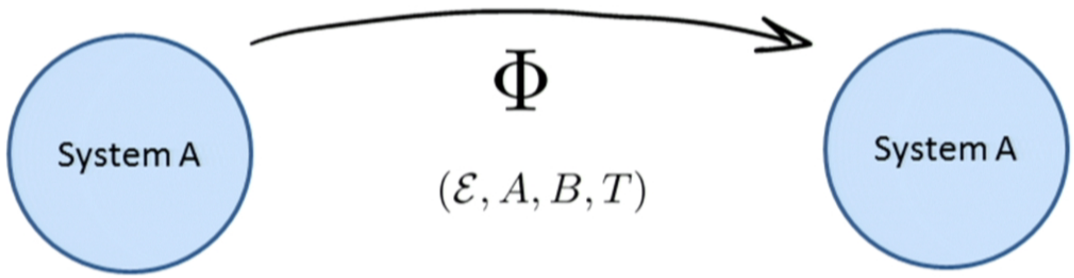

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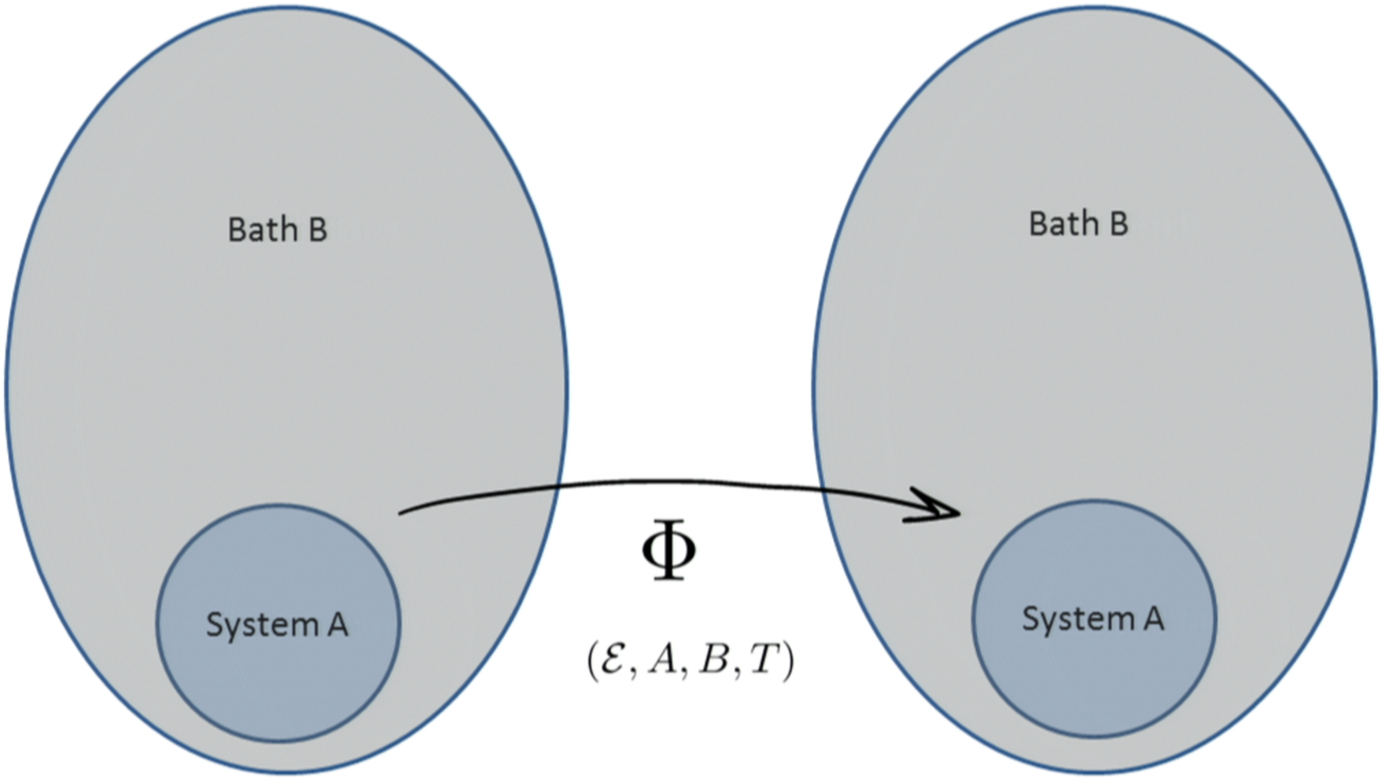

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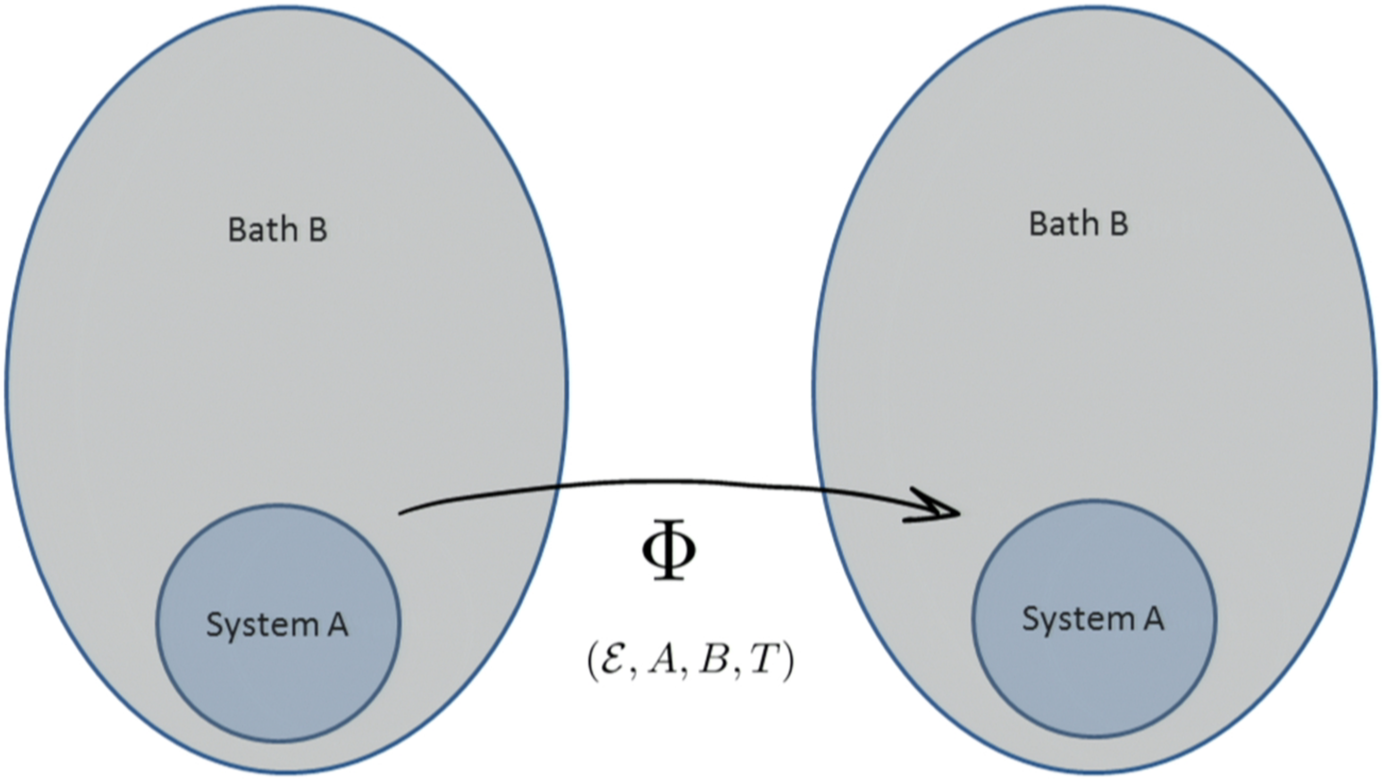

If we knew what we were doing, it wouldn't be called research.
--Albert Einstein

Open-system evolution, Sudarshan, et al. PR 61

A density operator must
(1) be Hermitian $\rho = \rho^\dagger,$

(2) be positive semi-definite $\rho \geq 0,$

(3) have trace one $\mathrm{Tr}(\rho) = 1.$

Consider a general map from one density operator to another (***Dynamical Map***)

$$\rho'_{r's'} = A_{r's',rs}\rho_{rs}.$$

ρ and ρ' density operators $\Rightarrow$

The map A must
(1) map Hermitian operators to Hermitian operators $A_{r's',rs} = A^*_{s'r',sr},$

(2) map positive operators to positive operators $x'_{r'}y'_{s'}A_{r's',rs}x_r y_s \geq 0.$

(3) preserve the trace $\sum_{r'} A_{r'r',rs} = \delta_{rs},$

Operator-Sum Representation (cont.)

From the map A, we can define a new matrix (called the "**B matrix**")

$$B_{r'r,s's} \equiv A_{r's',rs}.$$

The properties of B that follow from A are

(1) Hermiticity $\quad B_{r'r,s's} = B^*_{s's,r'r},$

(2) Positivity $\quad x'_{r'} y'_{s'} B_{r'r,s's} x_r y_s \geq 0.$

(3) Trace preservation $\quad \sum_{r'} B_{r'r,r's} = \delta_{rs},$

(1) Implies that B itself is Hermitian!

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Spectral Decomposition of B

Since B is Hermitian, it has an eigenvector/eigenvalue decomposition

$$B_{r'r,s's}\rho_{rs} = \sum_\alpha \gamma(\alpha) C^\alpha_{r'r}\rho_{rs}(C^\alpha_{s's})^*,$$

where C^α are the eigenvectors and $\gamma(\alpha)$ the eigenvalues of B.

We could also write

$$\Phi(\rho) = B\rho = \sum_\alpha \eta_\alpha A_\alpha \rho A^\dagger_\alpha \quad \left(= \sum_\alpha A_\alpha \rho A^\dagger_\alpha, \forall \eta_\alpha = 1\right)$$

where $A_\alpha \equiv \sqrt{|\gamma|} C^\alpha$ so that $\eta_\alpha = \pm 1.$

Note that the orthogonality of the eigenvectors corresponds to the orthogonality of the operators using the Hilbert-Schmidt norm

$$(A, B) = \mathrm{Tr}(A^\dagger B)$$

So this is a *minimal decomposition of the map*, meaning it cannot be written in terms of fewer operators.

Positive vs. Completely Positive

Any map that maps all positive density operators to positive density operators is called a **positive map**.

Consider a map

$$\Psi = \mathbb{I}_n \otimes \Phi$$

If this map is positive, then Φ is called n positive.

If it is positive for all n, then it is called a **completely positive map**.

The claim is that the extension should not change the positivity. However, when extending to another subsystem which is entangled with the first, one can find **physically relevant maps that are not completely positive.**

Completely Positive Maps

Theorem:
The map is completely positive if and only if all $\eta_\alpha = 1.$ The B-map has pos. eignvalues

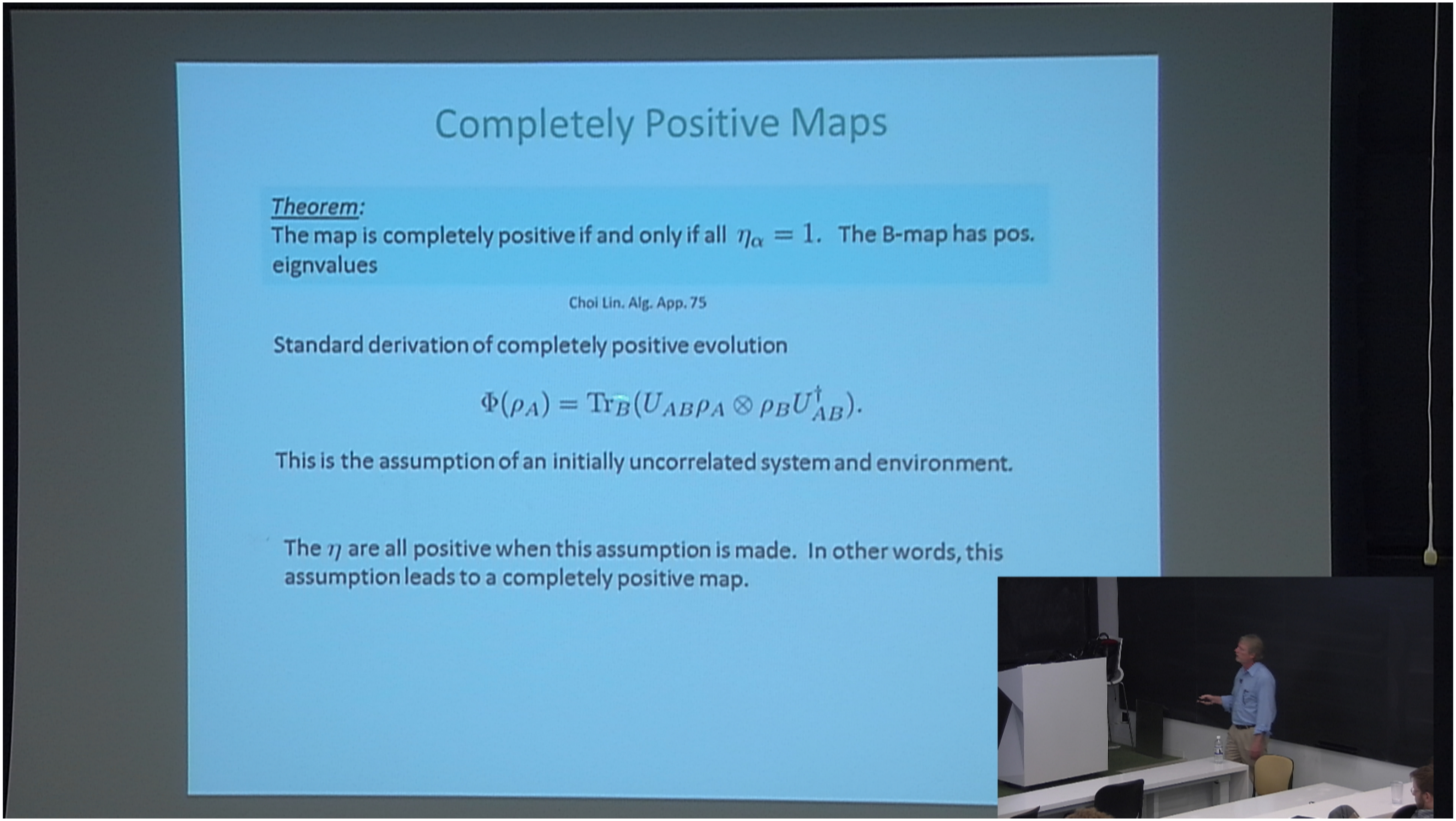
Choi Lin. Alg. App. 75

Standard derivation of completely positive evolution

$$\Phi(\rho_A) = \mathrm{Tr}_B(U_{AB}\rho_A \otimes \rho_B U^\dagger_{AB}).$$

This is the assumption of an initially uncorrelated system and environment.

The η are all positive when this assumption is made. In other words, this assumption leads to a completely positive map.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Quantum Discord

Let the density operator for two systems X and Y be

$$\rho^{XY} = \sum_j p_j \eta_j \otimes \tau_j ,$$

Ollivier/Zurek, PRL 01
Henderson/Vedral JPA 01

We could compute the mutual information

$$I(Y:X) = H(X) + H(Y) - H(X \cup Y),$$

or we could compute

$$J(Y:X) = H(Y) - H(Y|X),$$

where H is the entropy.

If X and Y are **classical**, (they are random probability distributions) then

$$J = I.$$

However, for quantum systems ...

Discord (cont.)

... for quantum systems, the amount of information obtained from the quantum system depends on the measurements made. We must choose a set of one-dimensional projectors $\{\Pi_j^X\}$ and calculate

$$H(Y|X) \to H(Y|\{\Pi_j^X\}) = \sum_j p_j H(\rho_{Y|\Pi_j^X}),$$

where $p_j = \mathrm{Tr}_{X,Y}(\Pi_j^X \rho^{XY})$ and $\rho_{Y|\Pi_j^Y} = \Pi_j^X \rho^{XY} \Pi_j^X / p_j$.

J=I is not always true for quantum systems. The **discord** *is the difference between these two.*

Theorem:
Zero quantum discord is necessary and sufficient to ensure completely positive evolution.

Rosario-Rodriguez, et al JPA 06, Shabani/Lidar PRL 09

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Part II. Examples

Example 1: NCP Map (Rodrigues-Rosario, et al.)

Consider the following initial density operator for two qubits

$$\rho = \frac{1}{4}(\mathbb{I} \otimes \mathbb{I} + \vec{a} \cdot \vec{\sigma} \otimes \mathbb{I} + c_{23}\sigma_2 \otimes \sigma_3).$$

Carteret, et al. 08
Rodriguez, et al 08

This is an unentangled state as can be seen using the partial transpose criterion.
One example is $a_1 = 1/\sqrt{2} = c_{23}, a_2 = 0 = a_3$.

$$U = \exp[it(\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3)]$$

The B map has eigenvalues

$$\frac{1}{2}\left(1 + \cos(2t)^2 - \sqrt{4\cos(2t)^4 + c_{23}^2 \cos(2t)^2 \sin(2t)^2}\right),$$

$$\frac{1}{2}\left(1 + \cos(2t)^2 + \sqrt{4\cos(2t)^4 + c_{23}^2 \cos(2t)^2 \sin(2t)^2}\right),$$

$$\frac{1}{4}(1 - \cos(4t) - c_{23}\sin(4t)), \quad \longleftarrow$$

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Example 2: NCP Maps

Consider the following model: a ring of fermions all coupled to a single particle (impurity)

$$H = \sum_{i=0}^{n} \varepsilon_{k_i} c^\dagger_{k_i} c_{k_i} + \epsilon b^\dagger b + V \sum_{i=0}^{n-1} (c^\dagger_{k_i} b + b^\dagger c_{k_i}) \delta_{k_i 0}.$$

For two particles, we can map the system via Jordan-Wigner transformation to the spin system to obtain (Ortiz, et al. PRA 01):

$$\bar{H} = \frac{\epsilon}{2}\sigma_z^1 + \frac{\varepsilon_{k_0}}{2}\sigma_z^2 + \frac{V}{2}(\sigma_x^1\sigma_x^2 + \sigma_y^1\sigma_y^2).$$

Consider an NMR device for the simulation. Ortiz, et al. 01

The NMR drift Hamiltonian:

$$\begin{aligned} H_d &= \frac{1}{2}\left(\frac{(\epsilon + \varepsilon_{k_0})}{2} - \sqrt{\left(\frac{\epsilon - \varepsilon_{k_0}}{2}\right)^2 + V^2}\right)\sigma_z^1 \\ &+ \frac{1}{2}\left(\frac{(\epsilon + \varepsilon_{k_0})}{2} + \sqrt{\left(\frac{\epsilon - \varepsilon_{k_0}}{2}\right)^2 + V^2}\right)\sigma_z^2. \end{aligned}$$

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Example 2 (cont.)

The control Hamiltonian for spins in the system is

$$H_c(t) = \sum_j [\alpha_{x_j}\sigma_x + \alpha_{y_j}\sigma_y] + \sum_{ij} \alpha_{i,j}\sigma_z^i \sigma_z^j,$$

where the α are controllable.

To obtain the representation of the Hamiltonian, the following control sequence must be applied:

$$\begin{aligned} U &= e^{i\frac{\pi}{4}\sigma_x^2} e^{-i\frac{\pi}{4}\sigma_y^1} e^{-i\frac{\theta}{2}\sigma_z^1\sigma_z^2} e^{i\frac{\pi}{4}\sigma_y^1} e^{i\frac{\pi}{4}\sigma_x^1} \\ &\quad \times e^{-i\frac{\pi}{4}\sigma_x^2} e^{-i\frac{\pi}{4}\sigma_y^2} e^{i\frac{\theta}{2}\sigma_z^1\sigma_z^2} e^{-i\frac{\pi}{4}\sigma_x^1} e^{i\frac{\pi}{4}\sigma_y^2}. \end{aligned}$$

This gives the model Hamiltonian which is being simulated on a NMR device.

Example 2 (cont.)

We add extra particles to represent a bath, which interacts (zz coupling) with both particles that represent the system of interest: the resonant impurity and the fermion site:

$$
\begin{aligned}
H_{\mathrm{NMR}} \quad = \quad & \frac{1}{2}\left(\frac{(\epsilon+\varepsilon_{k_0})}{2} - \sqrt{\left(\frac{(\epsilon-\varepsilon_{k_0})}{2}\right)^2 + V^2}\right)\sigma_z^1 \\
& +\frac{1}{2}\left(\frac{(\epsilon+\varepsilon_{k_0})}{2} + \sqrt{\left(\frac{(\epsilon-\varepsilon_{k_0})}{2}\right)^2 + V^2}\right)\sigma_z^2 \\
& +\frac{J_{zz}}{4}\sigma_z^1\sigma_z^3 + \frac{J_{zz}}{4}\sigma_z^2\sigma_z^3,
\end{aligned}
$$

Now we start with a variety of initial states which have non-zero discord

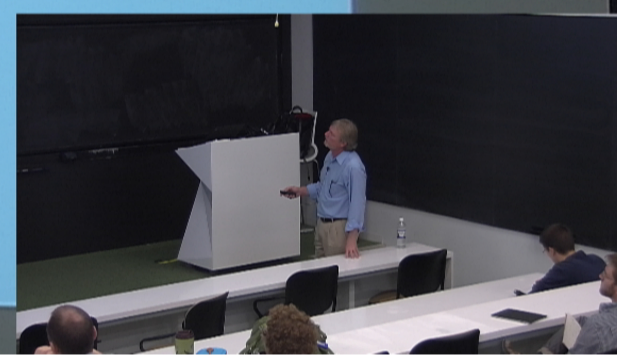
Examples Summary

What have we learned?

1) Maps do not need to be CP to be physically relevant. There are even examples of maps which are not positive which are physically relevant. (The map is positive only on a particular domain.)

2) Initial correlations can lead to NCP maps whether or not entanglement exists.

3) Not all correlated states lead to NCP maps. But, circumstances that lead to NCP maps do not seem rare.



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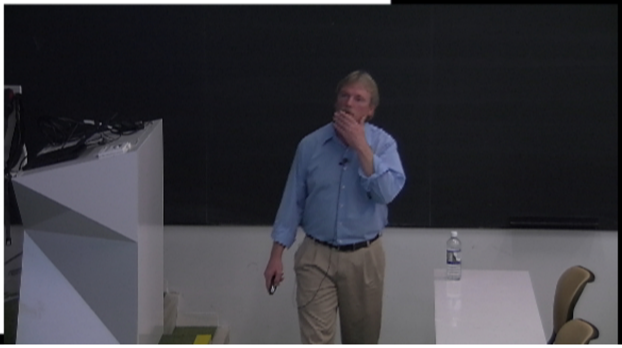
Part III: Unitary and Pseudo-Unitary Freedom in the OSR



Unitary Freedom in the OSR

Unitary Theorem:
The form of a completely positive Hermiticity-preserving map,

$$\Phi(\rho) = \sum_\alpha A_\alpha \rho A_\alpha^\dagger,$$

defined by operators $\{A_\alpha\}$, is not unique, but the operators $\{F_\beta\}$ give the same map, if and only if there is a unitary matrix with elements $u_{\alpha\beta}$ such that $F_\beta = u_{\beta\alpha} A_\alpha, \ \forall \beta.$ (Sum over α.)

Choi, Lin. Alg. App. 1975, also Nielsen and Chuang

To see sufficiency: $$\Phi(\rho) = \sum_\beta F_\beta \rho F_\beta^\dagger = \sum_{\alpha\beta\gamma} u_{\beta\alpha} A_\alpha \rho u_{\beta\gamma}^* A_\gamma^\dagger$$

Different physical circumstances can lead to the same evolution.

Application of Unitary Freedom: QECC Condition

Quantum error correcting condition (Knill/Laflamme PRA:95)

$$\langle i_L | A_\alpha^\dagger A_\beta | j_L \rangle = m_{\alpha\beta} \delta_{ij}$$

Theorem: Quantum Error Correcting Code Conditions (Nielsen and Chuang*)
Necessary and sufficient conditions for the existence of a quantum error correcting code are

$$P A_\alpha^\dagger A_\beta P = c_{\alpha\beta} P,$$

where P is a projector onto the code space.
(One can show that the above is true if and only if this is true.)

Step 1 of the proof (if) of the condition is to diagonalize $c_{\alpha\beta}$ using the unitary freedom in the OSR. It is also used for the (only if) part of the proof.

* Originally in Nielsen, et al. ITP conference proceedings

Pseudo-Unitary Degree of Freedom in the OSR

Pseudo-Unitary Theorem: The form of a Hermiticity-preserving map,

$$\Phi(\rho) = \sum_\alpha \eta_\alpha A_\alpha \rho A_\alpha^\dagger,$$

defined by $\{A_\alpha\}$, and $\{\eta_\alpha\}$ is not unique, but the operators $\{F_\beta\}$ give the same map, if and only if there is a pseudo-unitary matrix with elements $u_{\alpha\beta}$ such that $F_\alpha = u_{\alpha\beta} A_\beta,\ \forall\alpha.$ The signature of the matrix $U(p,q)$ is determined by the number of input and output sets $\{A_\alpha\}$ and $\{F_\beta\}$.

Ou/Byrd: 09

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$$U \text{ unitary} \Rightarrow U^\dagger \mathbb{I} U = \mathbb{I}, \quad U \text{ pseudo-unitary} \Rightarrow U^\dagger \eta U = \eta$$

$$\eta = \mathrm{diag}(1, 1, ..., 1, -1, ..., -1)$$

An example of a pseudo-Unitary group is the Lorentz group, SO(1,3) which fixes diag(1,-1,-1,-1)

Different physical circumstances can lead to the same evolution.

Pseudo-Unitary Degree of Freedom in the OSR

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QECC (Reversibility) Conditions

-- Problem: (Directly related to the composition of maps.)
Provide necessary and sufficient QECC conditions for maps which are not CP.

-- Shabani/Lidar (*PRA* 2009)
gave sufficient, but not necessary conditions.

Unitary Freedom in the OSR

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The form of a completely positive Hermiticity-preserving map,

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defined by operators $\{A_\alpha\}$, is not unique, but the operators $\{F_\beta\}$ give the same map, if and only if there is a unitary matrix with elements $u_{\alpha\beta}$ such that $F_\beta = u_{\beta\alpha} A_\alpha, \ \forall \beta.$ (Sum over α.)

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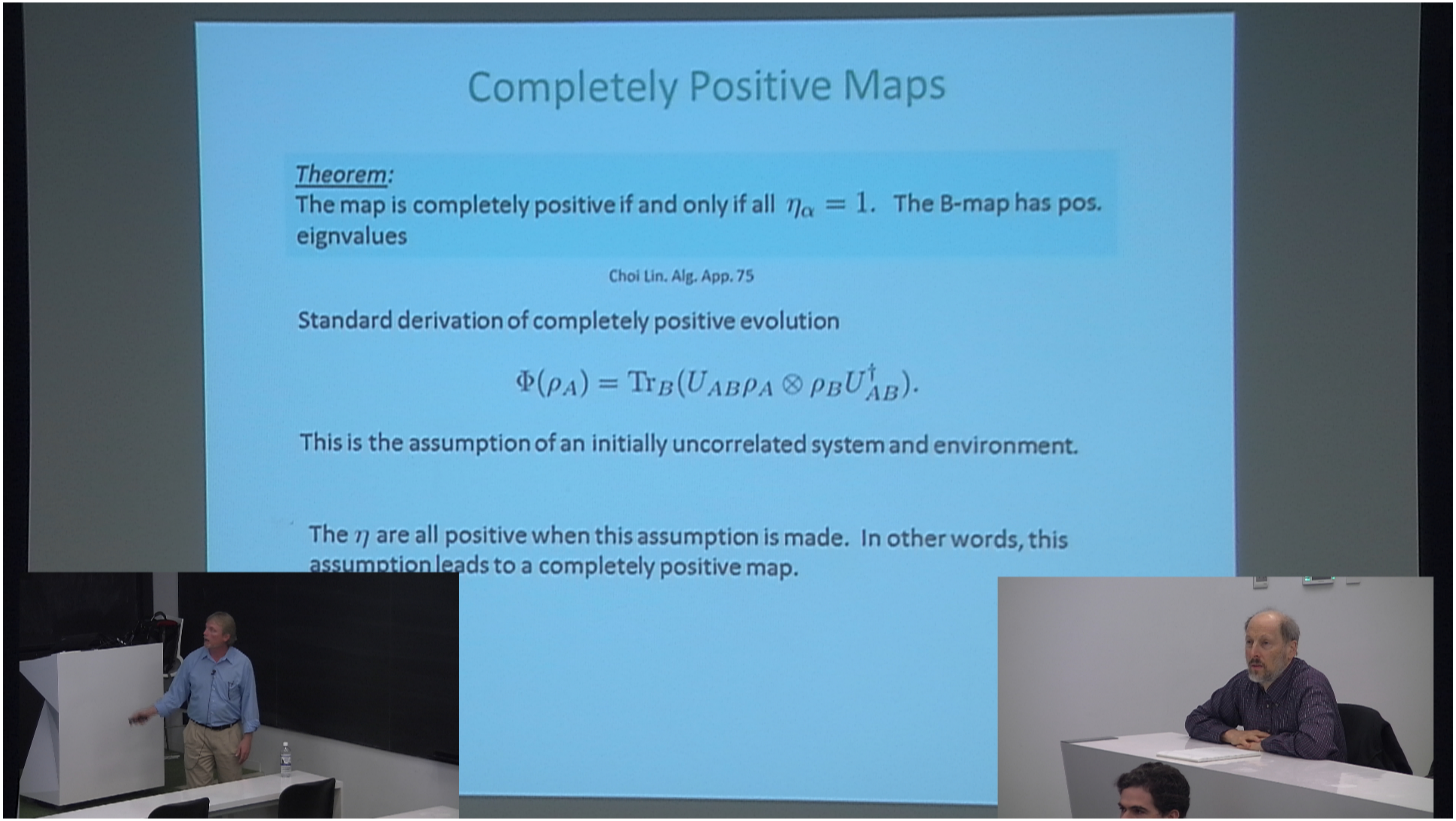
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$$\Phi(\rho_A) = \mathrm{Tr}_B(U_{AB}\rho_A \otimes \rho_B U_{AB}^\dagger).$$
This is the assumption of an initially uncorrelated system and environment.
The η are all positive when this assumption is made. In other words, this assumption leads to a completely positive map.

Pseudo-Unitary Degree of Freedom in the OSR

Pseudo-Unitary Theorem: The form of a Hermiticity-preserving map,

$$\Phi(\rho) = \sum_\alpha \eta_\alpha A_\alpha \rho A_\alpha^\dagger,$$

defined by $\{A_\alpha\}$, and $\{\eta_\alpha\}$ is not unique, but the operators $\{F_\beta\}$ give the same map, if and only if there is a pseudo-unitary matrix with elements $u_{\alpha\beta}$ such that $F_\alpha = u_{\alpha\beta} A_\beta,\ \forall \alpha$. The signature of the matrix $U(p,q)$ is determined by the number of input and output sets $\{A_\alpha\}$ and $\{F_\beta\}$.

Ou/Byrd: 09

To see sufficiency: $$\Phi(\rho) = \sum_\alpha \eta_\alpha F_\alpha \rho F_\alpha^\dagger = \sum_{\alpha\beta\gamma} \eta_\alpha u_{\alpha\beta} A_\beta \rho u_{\alpha\gamma}^* A_\gamma^\dagger$$

$$U \text{ unitary} \Rightarrow U^\dagger \mathbb{I} U = \mathbb{I}, \quad U \text{ pseudo-unitary} \Rightarrow U^\dagger \eta U = \eta$$

$$\eta = \mathrm{diag}(1, 1, ..., 1, -1, ..., -1)$$

An example of a pseudo-Unitary group is the Lorentz group, SO(1,3) which fixes diag(1,-1,-1,-1)

Different physical circumstances can lead to the same evolution.

Theorem: Reversibility (QECC) Conditions

Let C be a code and P the projector onto that code space. Suppose that

$$\mathcal{E}(\rho) = \sum_i \eta_i E_i \rho E_i^\dagger$$

is a map (NCP or CP). A necessary and sufficient condition for the existence of a quantum error correcting code C correcting $\mathcal{E}$ on C is

$$(\sqrt{\eta_i \eta_j}) P E_i^\dagger E_j P = m_{ij} P.$$

If such a code exists, we say that C is an error correcting code for the errors resulting from $\mathcal{E}$.

Byrd, et al: 12

Proof of Reversibility Conditions

Let

$$\mathcal{E}(\rho) = \sum_{i=1}^{p} E_i \rho E_i^\dagger - \sum_{i=p+1}^{p+q} \tilde{E}_i \rho \tilde{E}_i^\dagger$$

We can distinguish two cases, $(\sqrt{\eta_i \eta_j}) \in \mathbb{R}$ and $(\sqrt{\eta_i \eta_j}) \in \mathbb{C}$.

For $\eta_i \eta_j < 0$,

$$(\sqrt{\eta_i \eta_j}) P E_i^\dagger \tilde{E}_j P = b_{ij} P,$$

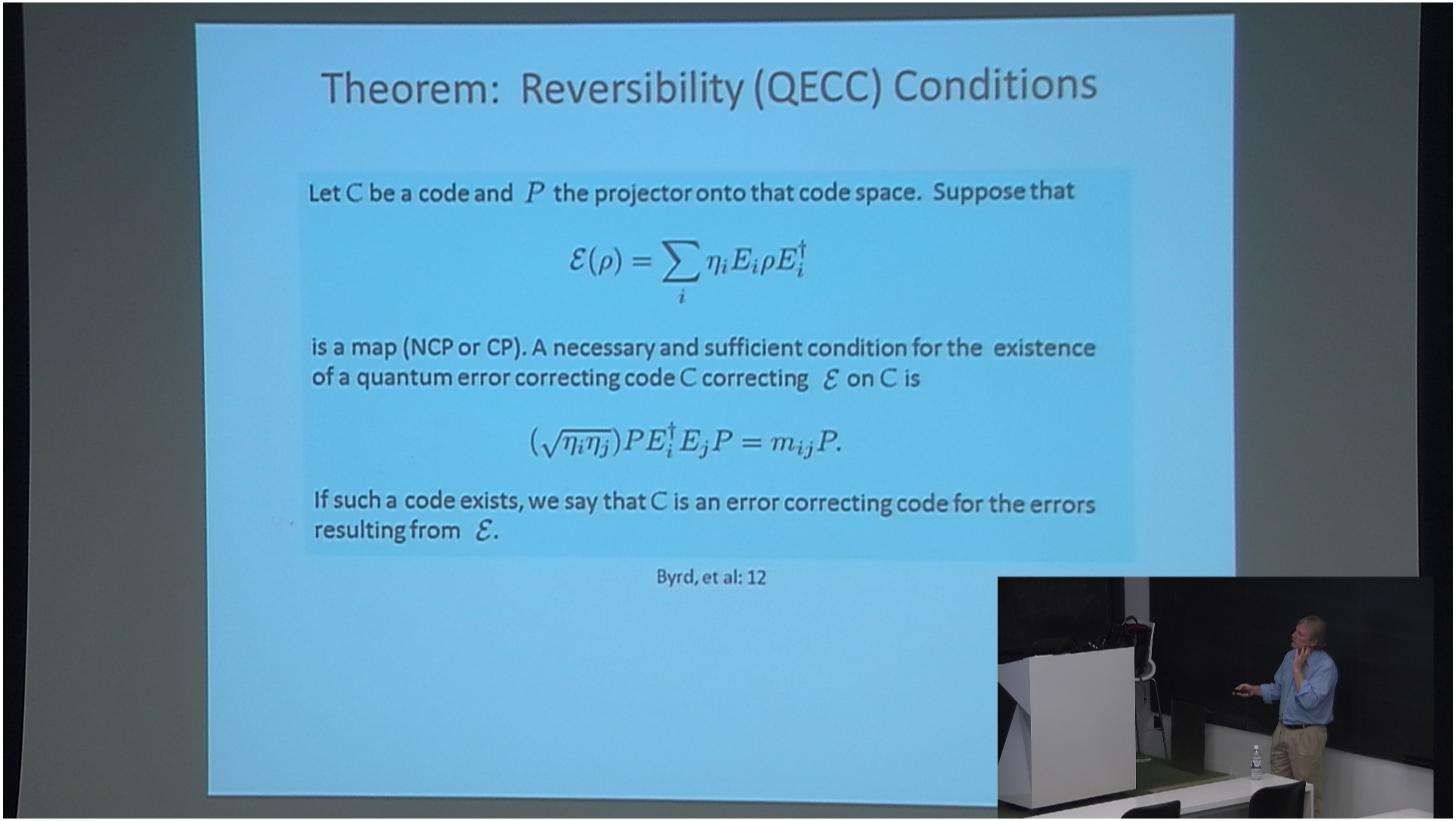
and

$$(\sqrt{\eta_i \eta_j}) P \tilde{E}_i^\dagger E_j P = c_{ij} P,$$

together with the positive case, imply the matrix m has the form

$$m = \begin{pmatrix} M & N \\ -N^\dagger & S \end{pmatrix}.$$

This matrix is pseudo-Hermitian and can be diagonalized with a pseudo-unitary matrix.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Theorem: Reversibility (QECC) Conditions

The proof now proceeds very similar to Nielsen and Chuang.

1) Diagonalize to get

$$(\sqrt{\eta_i \eta_j}) P F_i^\dagger F_j P = d_{ij} \delta_{ij} P.$$

2) Use the polar decomposition

$$F_j P = U_j \left[P F_j^\dagger F_j P \right]^{1/2} = (d_{jj})^{1/2} U_j P$$

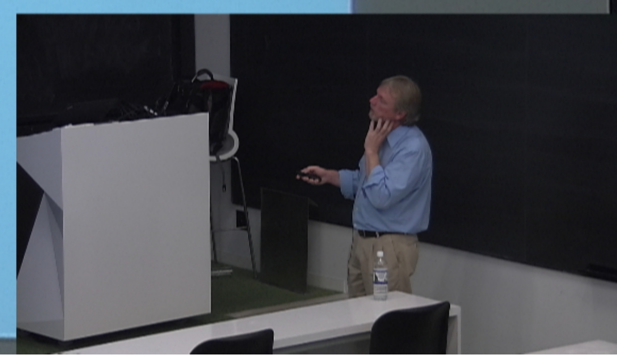
3) Define

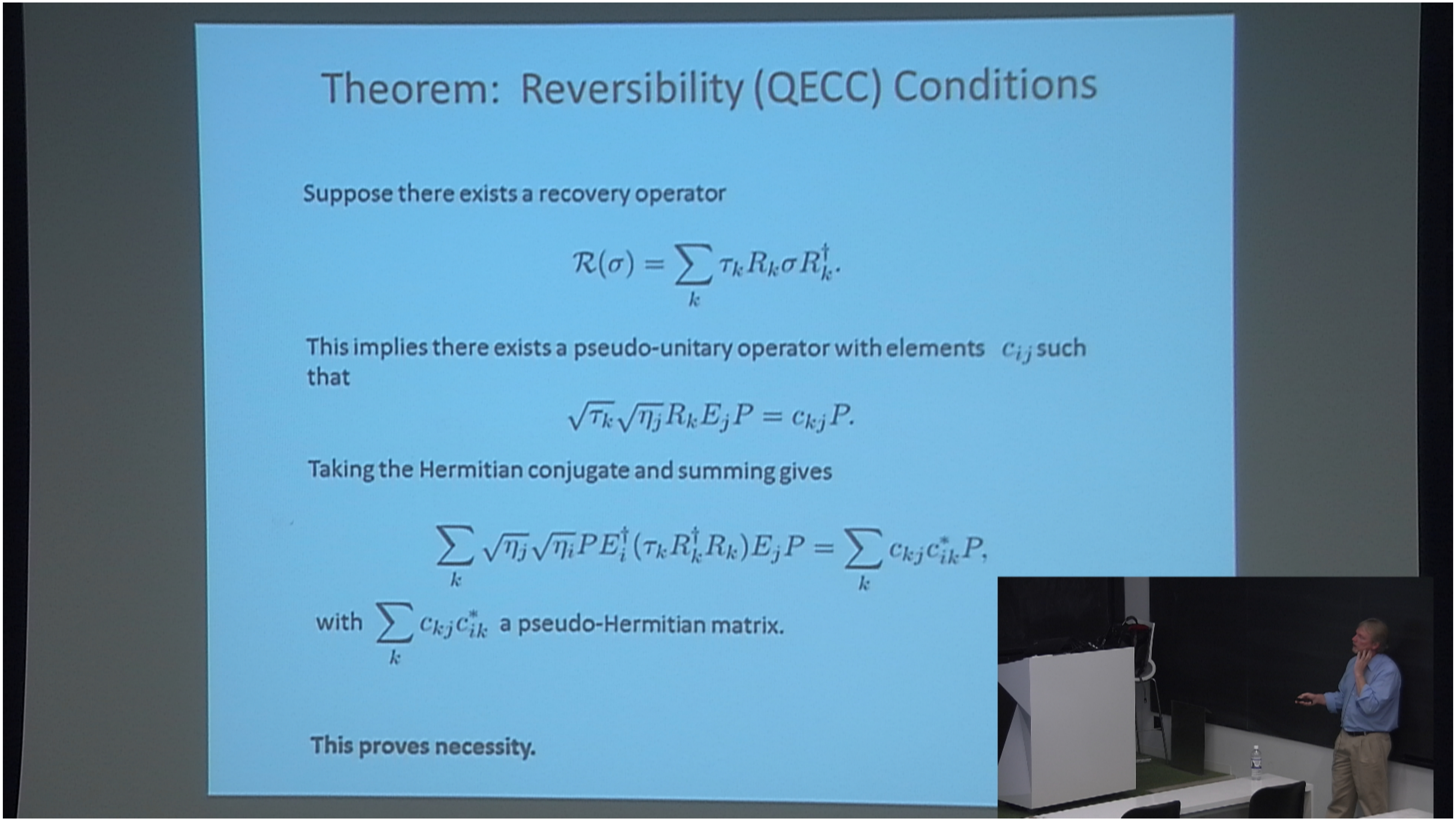
$$\mathcal{R}(\sigma) = \sum_l \eta_l U_l^\dagger P_l^\dagger \sigma P_l U_l$$

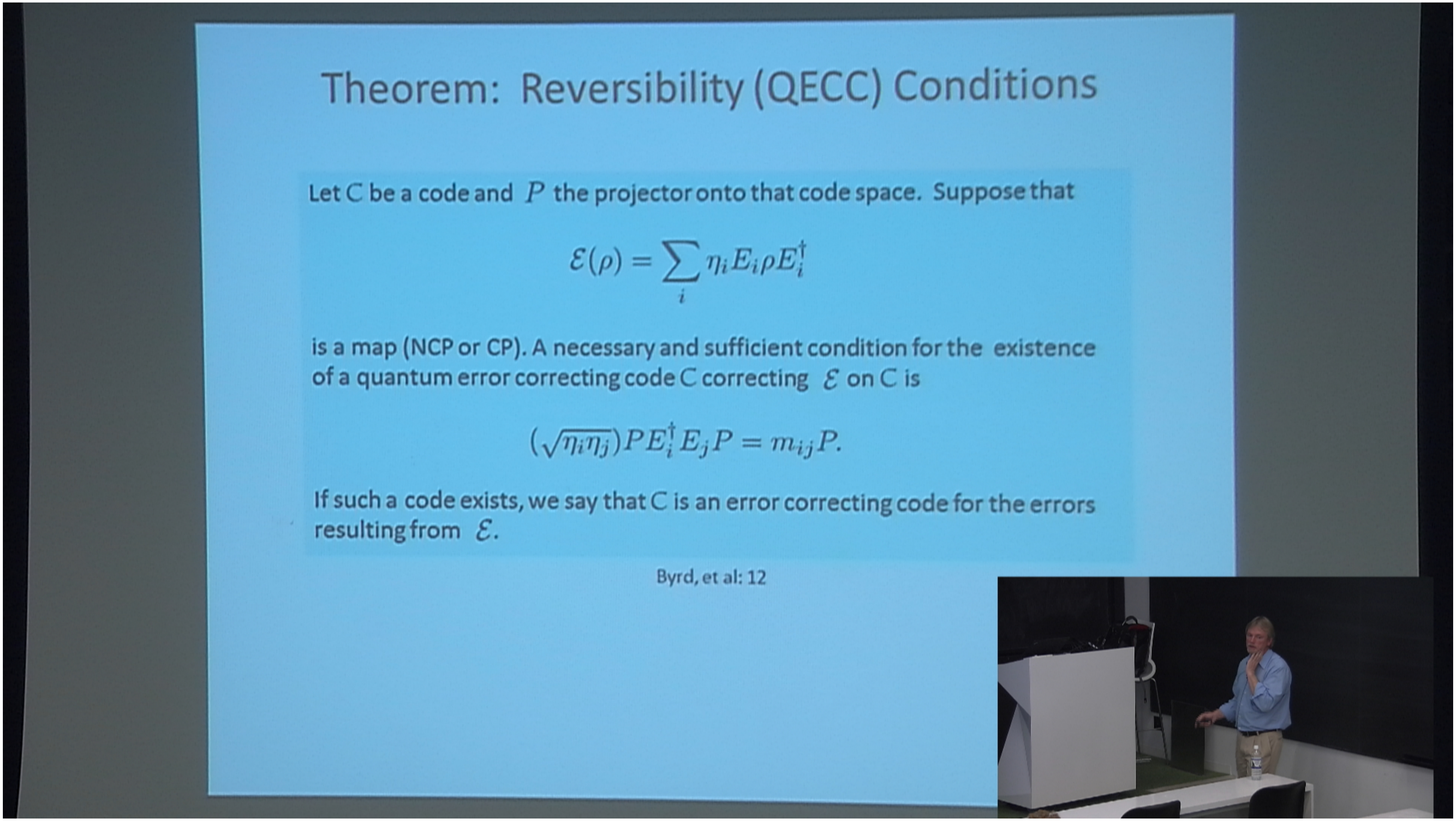
where

$$P_l \equiv U_l P U_l^\dagger = \frac{F_l P U_l^\dagger}{\sqrt{d_{ll}}}.$$

This proves sufficiency.




Theorem: Reversibility (QECC) Conditions
Suppose there exists a recovery operator
$$\mathcal{R}(\sigma) = \sum_k \tau_k R_k \sigma R_k^\dagger.$$
This implies there exists a pseudo-unitary operator with elements c_{ij} such that
$$\sqrt{\tau_k}\sqrt{\eta_j} R_k E_j P = c_{kj} P.$$
Taking the Hermitian conjugate and summing gives
$$\sum_k \sqrt{\eta_j}\sqrt{\eta_i} P E_i^\dagger (\tau_k R_k^\dagger R_k) E_j P = \sum_k c_{kj} c_{ik}^* P,$$
with $\sum_k c_{kj} c_{ik}^*$ a pseudo-Hermitian matrix.
This proves necessity.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This proves sufficiency.

Reversibility (QECC) Conditions

- The physical interpretation of the pseudo-unitary theorem is not well-understood.

- Since positivity and complete positivity are dependent upon the initial state of the system, the map cannot really be considered separately from the density operator:

$$\mathcal{E}(\rho) = \sum_i \eta_i E_i \rho E_i^\dagger$$

- We now have reversibility conditions for NCP maps.

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Part IV: Coherence or Polarization Vector Description of Open System Evolution

Summary/Conclusions

-- We know that initial correlations can lead to NCP maps.

-- If we begin with system-bath entangled initial states (similarly for non-zero discord states) and evolve, ***we cannot remove those errors using dynamical decoupling controls***.

Our simulations show that this ***is not necessarily a rare event*** … and we must know the difference if we are to treat the different physical circumstances correctly.

-- We proved this theorem, the Pseudo-Unitary Theorem, which provides equivalence classes of maps. For example, maps due to different physical circumstances, etc.

-- We provided reversibility conditions for maps which are not necessarily CP.

-- We have another picture for examining these maps in terms of the polarization vector.

-- More work needs to be done…

THE END

Thanks!

Example 1: NCP Map (Rodrigues-Rosario, et al.)

Consider the following initial density operator for two qubits

$$\rho = \frac{1}{4}(\mathbb{I} \otimes \mathbb{I} + \vec{a} \cdot \vec{\sigma} \otimes \mathbb{I} + c_{23}\sigma_2 \otimes \sigma_3).$$

Carteret, et al. 08
Rodriguez, et al 08

This is an unentangled state as can be seen using the partial transpose criterion.
One example is $a_1 = 1/\sqrt{2} = c_{23}, a_2 = 0 = a_3$.

$$U = \exp[it(\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3)]$$

The B map has eigenvalues

$$\frac{1}{2}\left(1 + \cos(2t)^2 - \sqrt{4\cos(2t)^4 + c_{23}^2\cos(2t)^2\sin(2t)^2}\right),$$

$$1 + \cos(2t)^2 + \sqrt{4\cos(2t)^4 + c_{23}^2\cos(2t)^2\sin(2t)^2}\Big),$$

$$- \cos(4t) - c_{23}\sin(4t)),$$

$$- \cos(4t) + c_{23}\sin(4t))$$