Title: A generalization of Kochen-Specker sets

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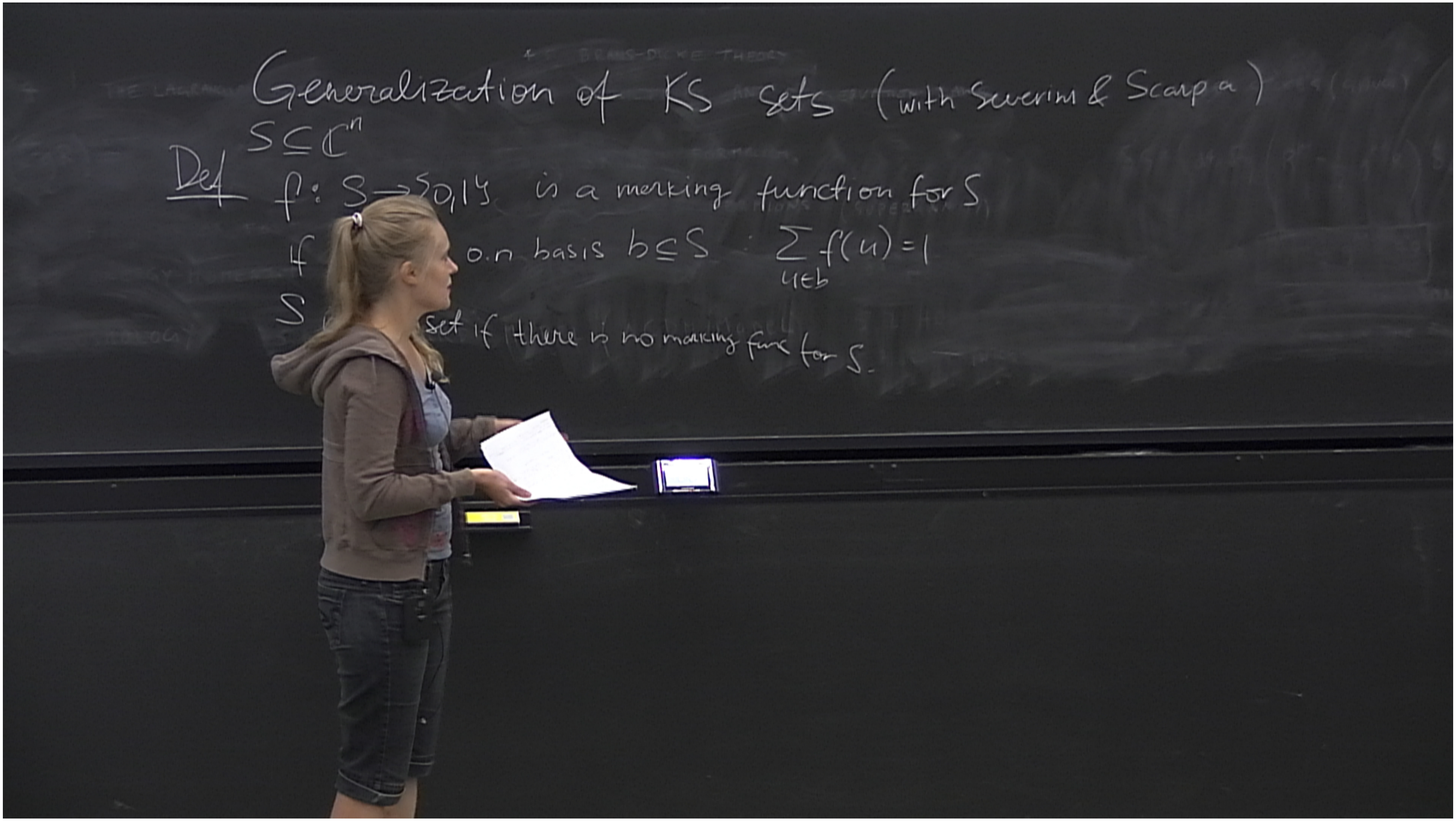
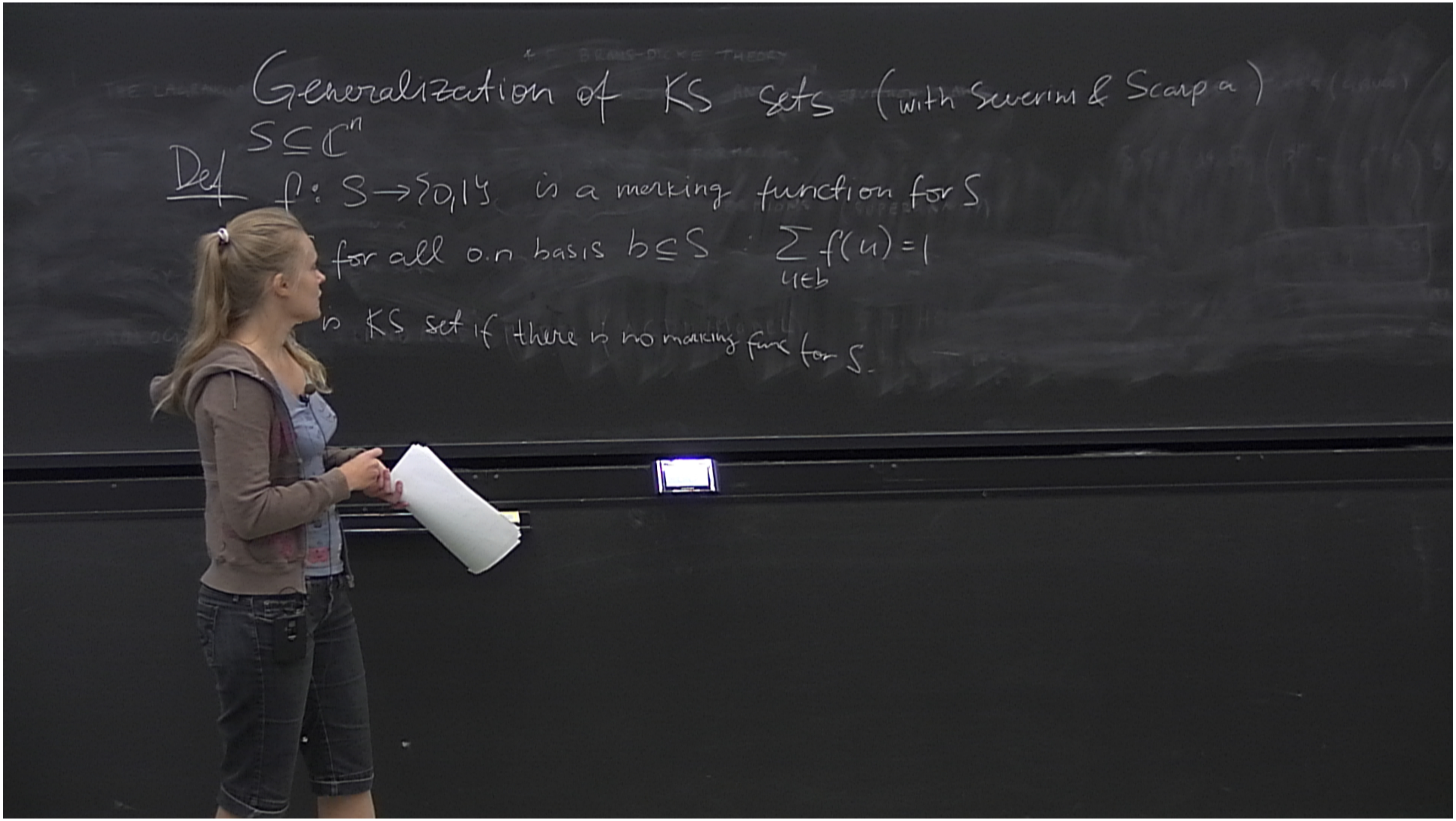
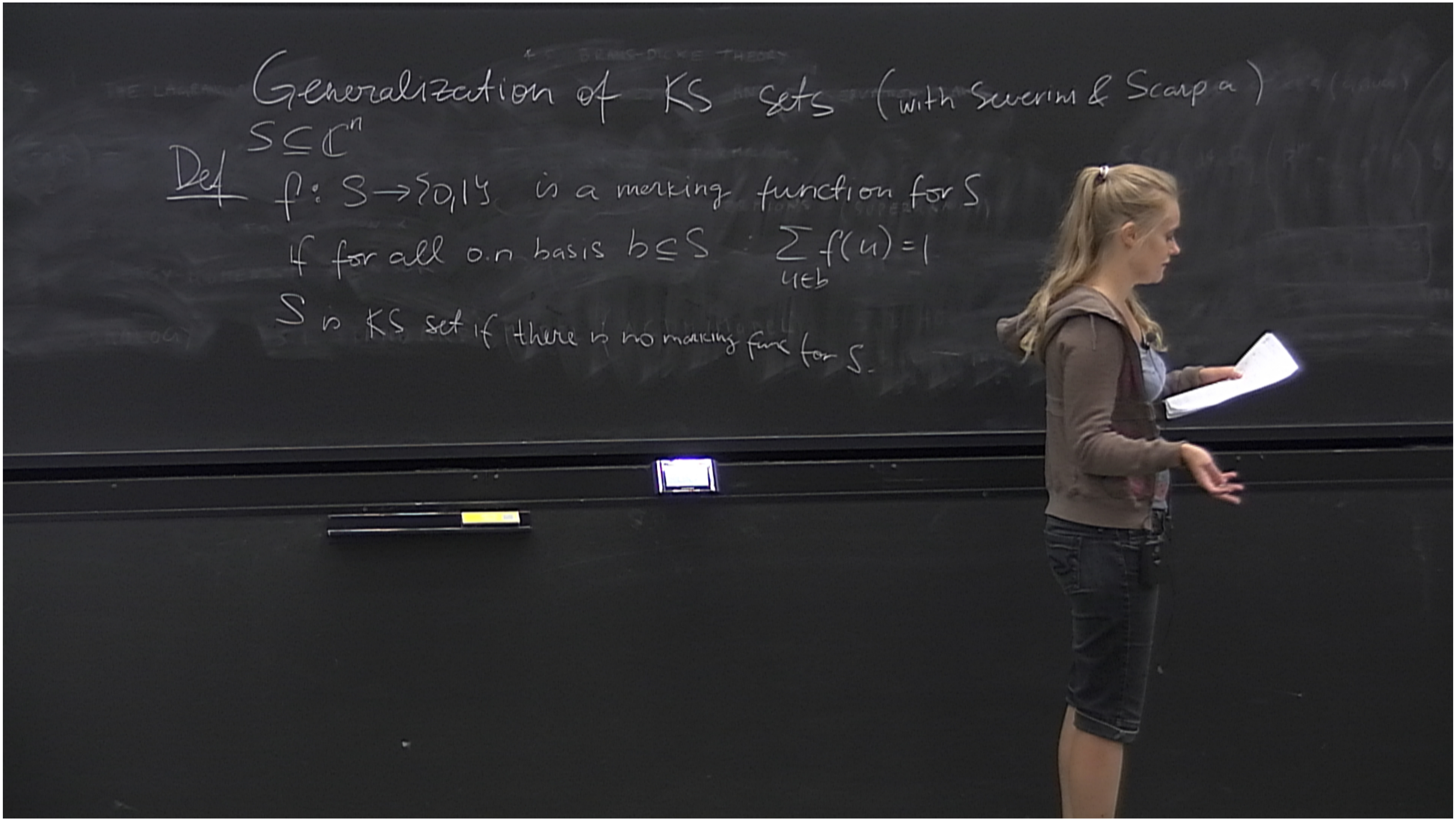
Abstract: I will discuss two generalizations of Kochen-Specker (KS) sets: projective KS sets and generalized KS sets. We will see that projective KS sets can be used to characterize all graphs for which the chromatic number is strictly larger than the quantum chromatic number. Here, the quantum chromatic number is defined via a nonlocal game based on graph coloring. We will further show that from any graph with separation between these two quantities, one can construct a classical channel for which entanglement assistance increases the one-shot zero-error capacity. As an example, we will consider a new family of classical channels with an exponential increase.


Generalization of KS sets (with Severini & Scarpa)
S ⊆ ℂ^n
Def
f: S → {0,1} is a marking function for S
if
o.n basis b ⊆ S : Σ_{u∈b} f(u) = 1
S
set if there is no marking func for S.


Generalization of KS sets (with Severini & Scarpa)
$S \subseteq \mathbb{C}^n$
Def $f: S \to \{0,1\}$ is a marking function for S
for all o.n basis $b \subseteq S$: $\sum_{u \in b} f(u) = 1$
KS set if there is no marking func for S.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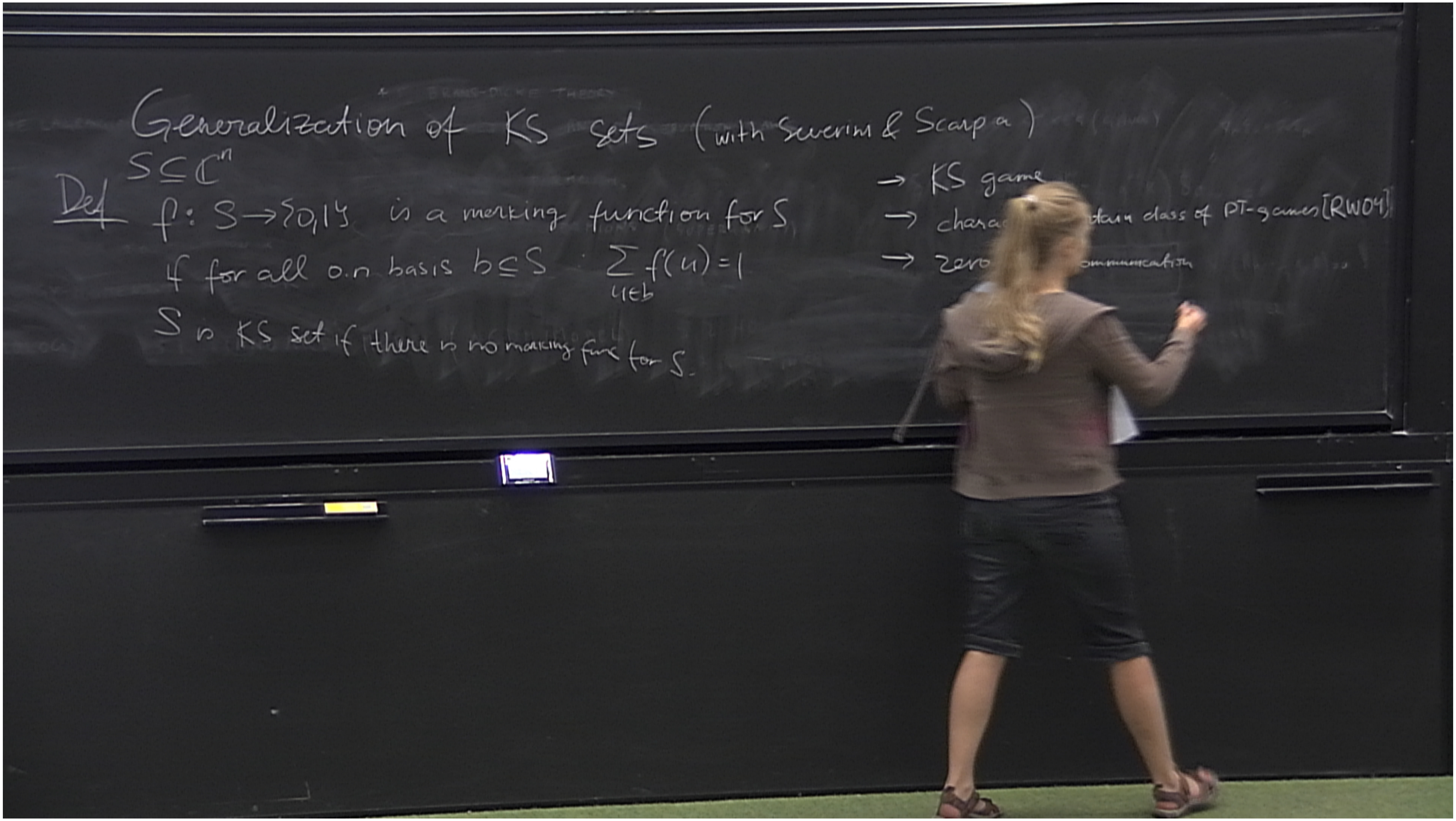
S is KS set if there is no marking func for S.


Generalization of KS sets (with Severini & Scarpa)
S ⊆ ℂ^n
→ KS game
Def f: S → {0,1} is a marking function for S
→ chara
class of PT-games [RW04]
if for all o.n basis b ⊆ S : Σ_{u∈b} f(u) = 1
→ zero
communication
S is KS set if there is no marking func for S.


Generalization of KS sets (with Severini & Scarpa)
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S is KS set if there is no marking func for
→ KS game
→ characterize certain class of PT-games [RW04]
→ zero-error communication [CLMW10]
→ graph colouring

Generalization of KS sets (with Severini & Scarpa)

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→ KS game

→ characterize certain class of PT-games [RW04]

→ zero-error communication [CLMW10]

→ graph colouring [SS12]


Def [RW04] S is a weak KS set if for
any marking func f, there exist u,v∈S
s.t u⊥v & f(u)=f(v)=1


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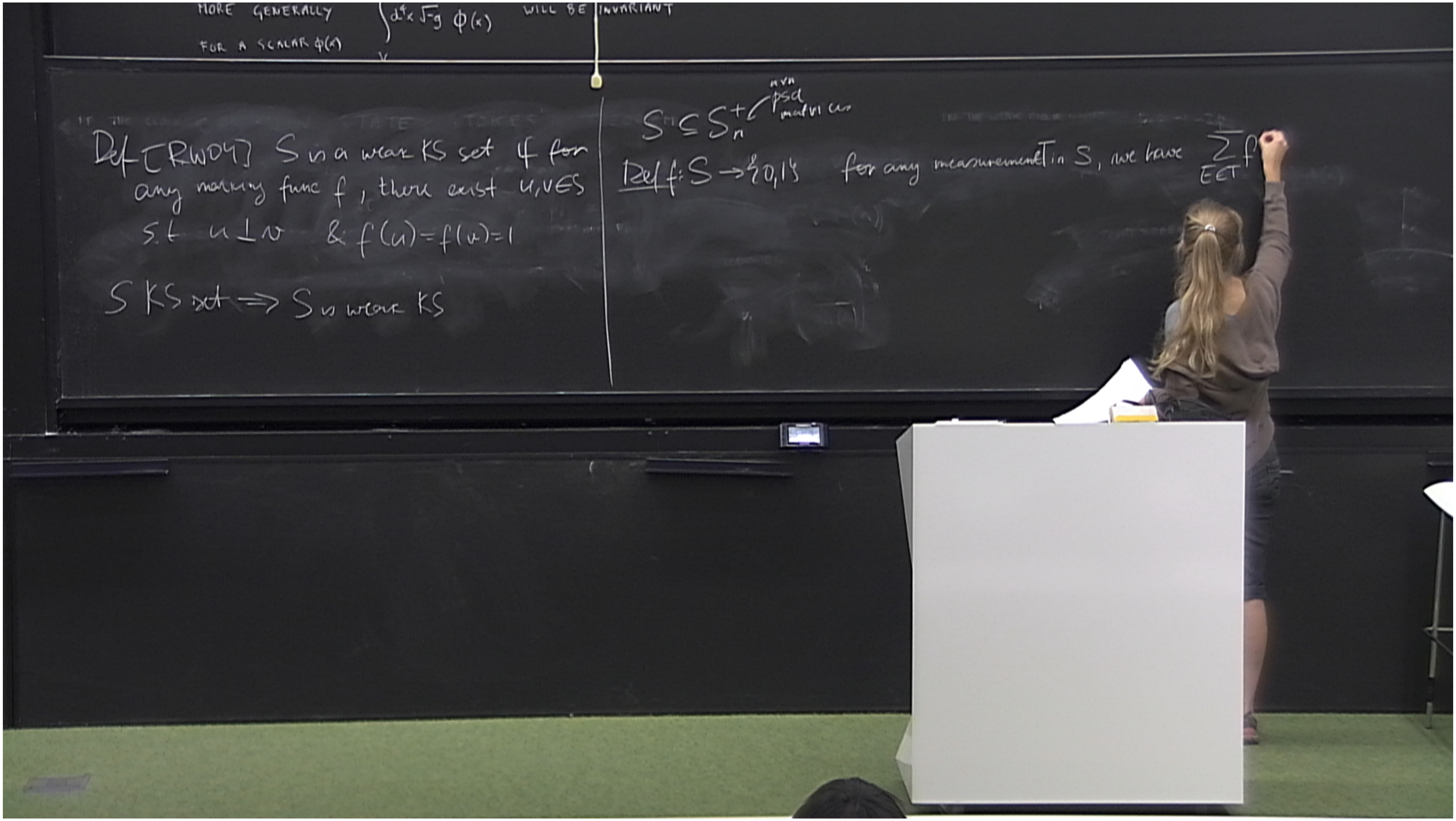
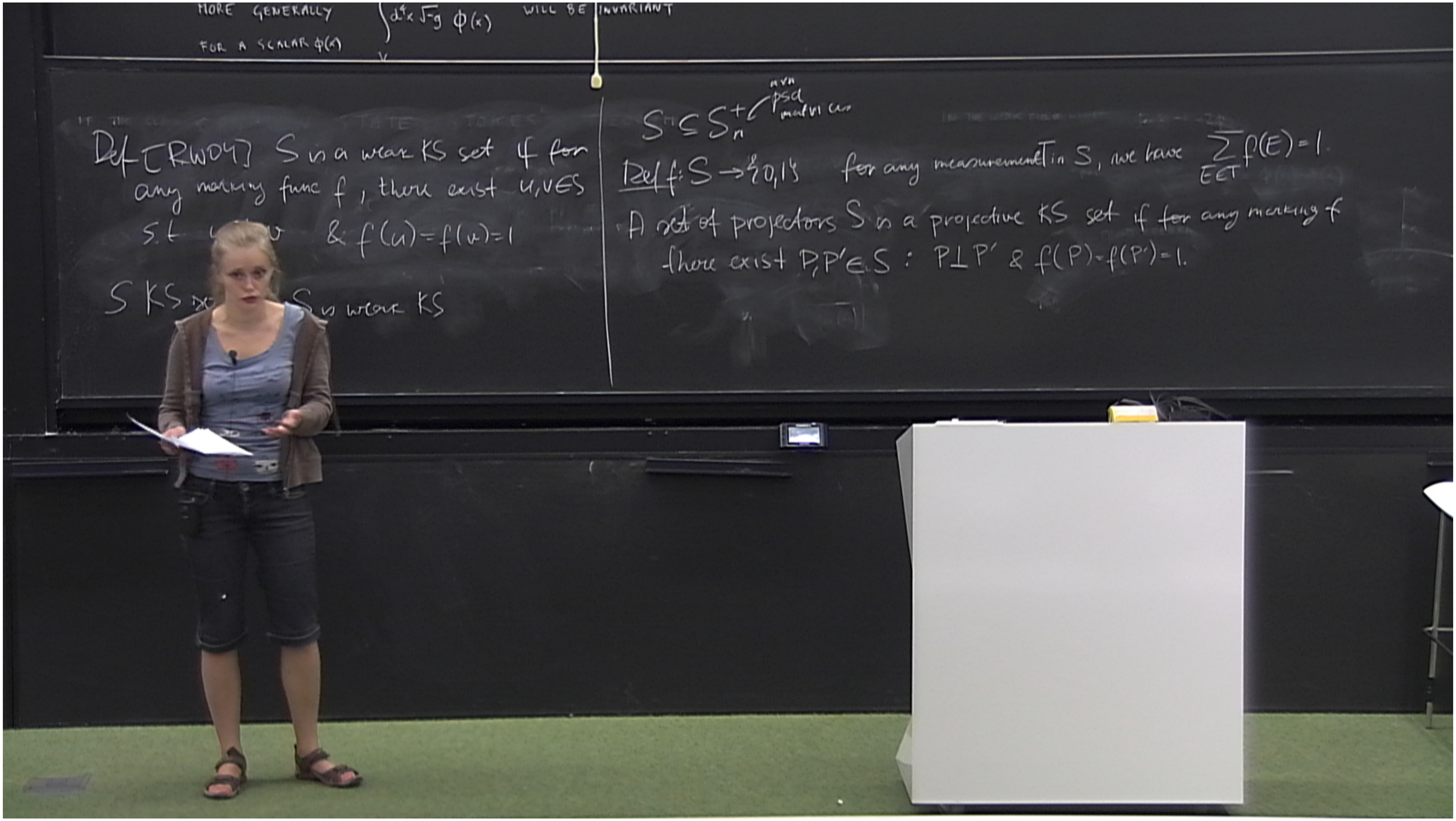
S KS set ⇒ S is weak KS


MORE GENERALLY ∫d⁴x √-g φ(x) WILL BE INVARIANT
FOR A SCALAR φ(x)
Def [RW04] S is a weak KS set if for any marking func f, there exist u,v ∈ S s.t u ⊥ v & f(u)=f(v)=1
S KS set ⇒ S is weak KS
S ⊆ S_n^+ nxn psd matrices
Def f: S → {0,1} for any measurement T in S, we have Σ_{E∈T} f



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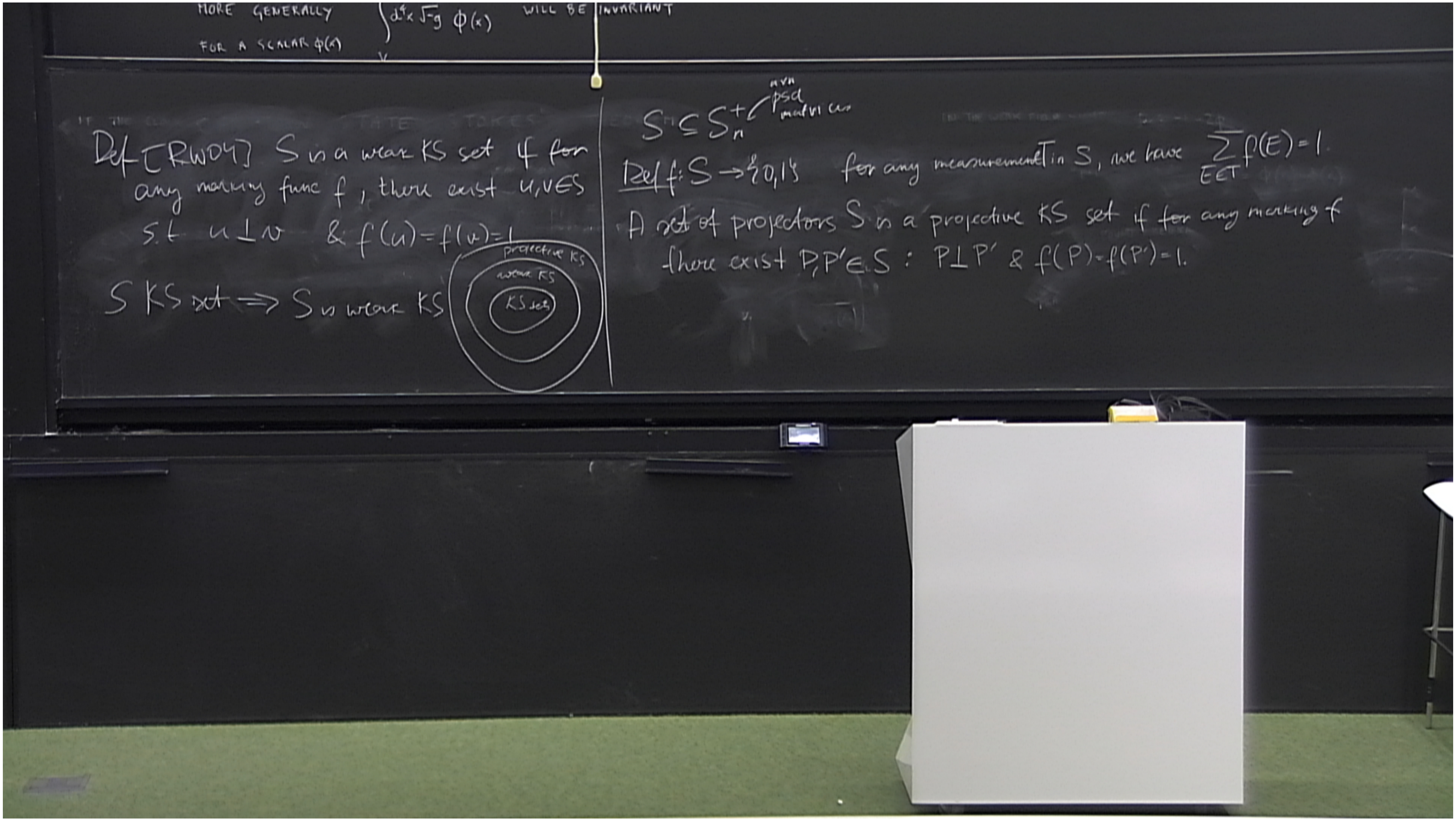
S KS ⇒ S is weak KS
S ⊆ S+n nxn psd matrices
Def f: S → {0,1} for any measurement T in S, we have Σ E∈T f(E)=1
A set of projectors S is a projective KS set if for any marking f
there exist P,P'∈S : P⊥P' & f(P)=f(P')=1.


MORE GENERALLY
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Def [RW04] S is a weak KS set if for any marking func f, there exist u,v∈S s.t. u⊥v & f(u)=f(v)=1
S KS set ⇒ S is weak KS
projective KS
weak KS
KS sets
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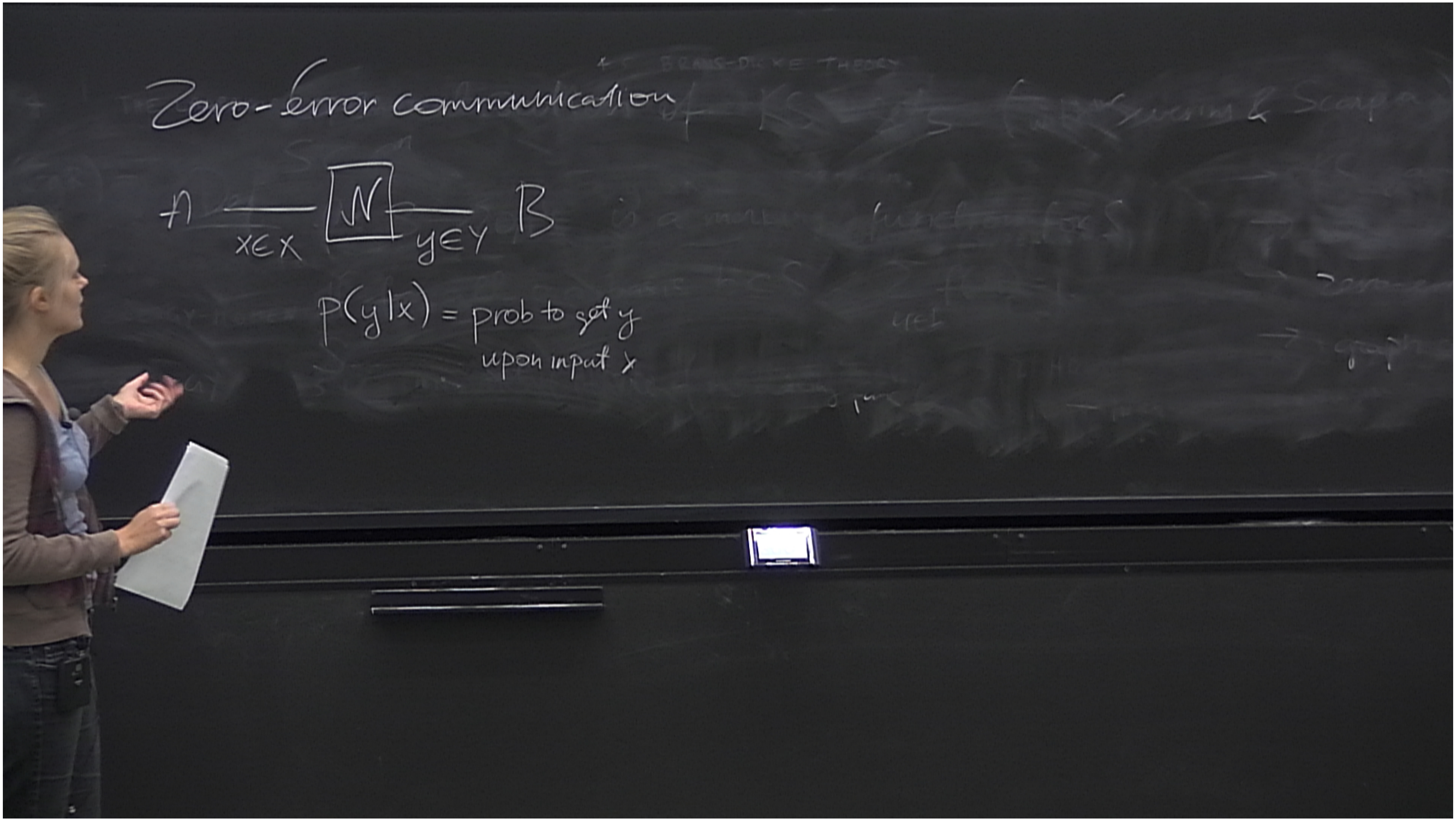
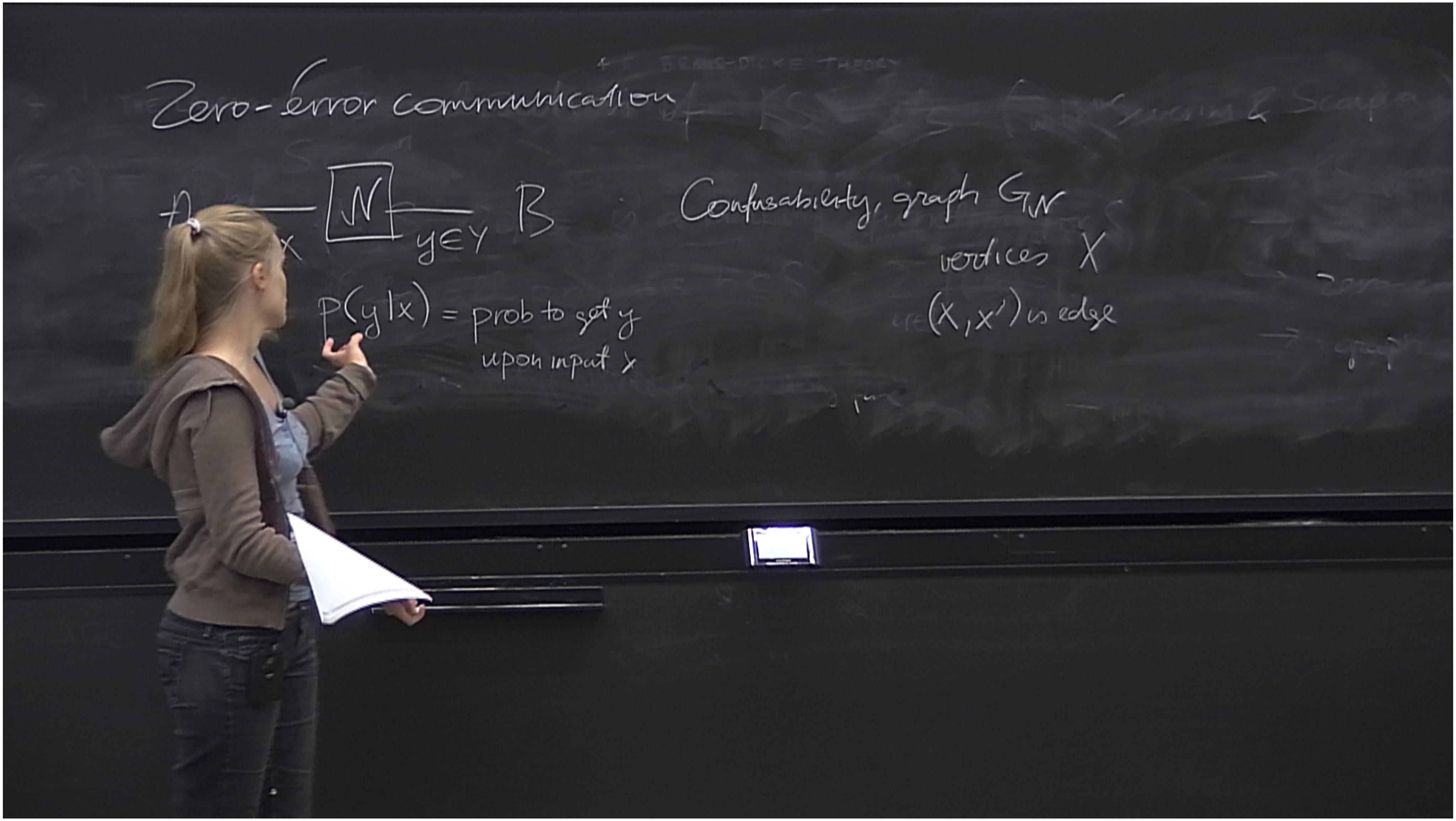
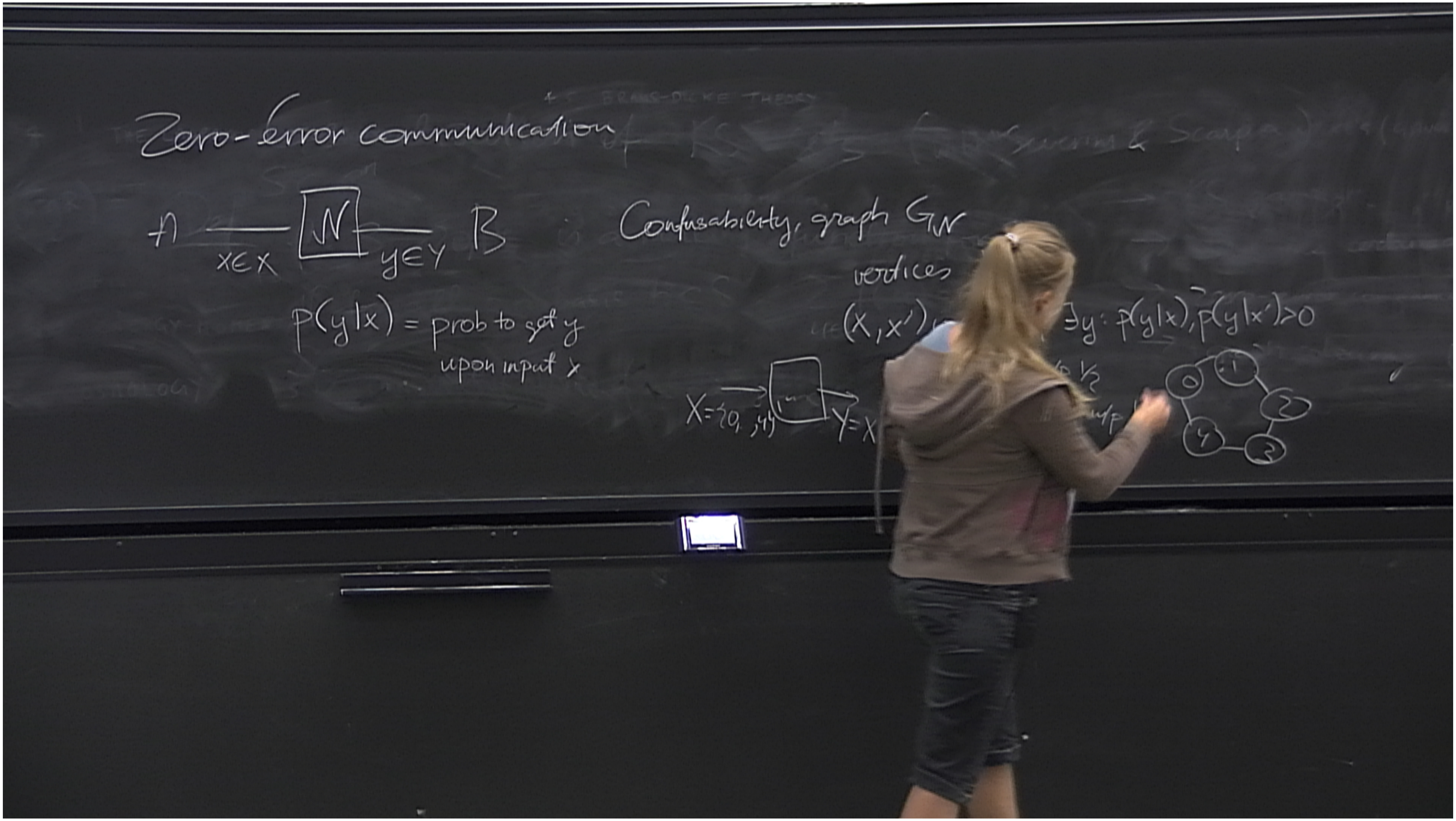

Zero-error communication
A
x∈X
N
y∈Y
B
p(y|x) = prob to get y
upon input x


Zero-error communication
A x N y∈Y B
p(y|x) = prob to get y
upon input x
Confusability graph G_N
vertices X
(x, x') is edge



Zero-error communication
A
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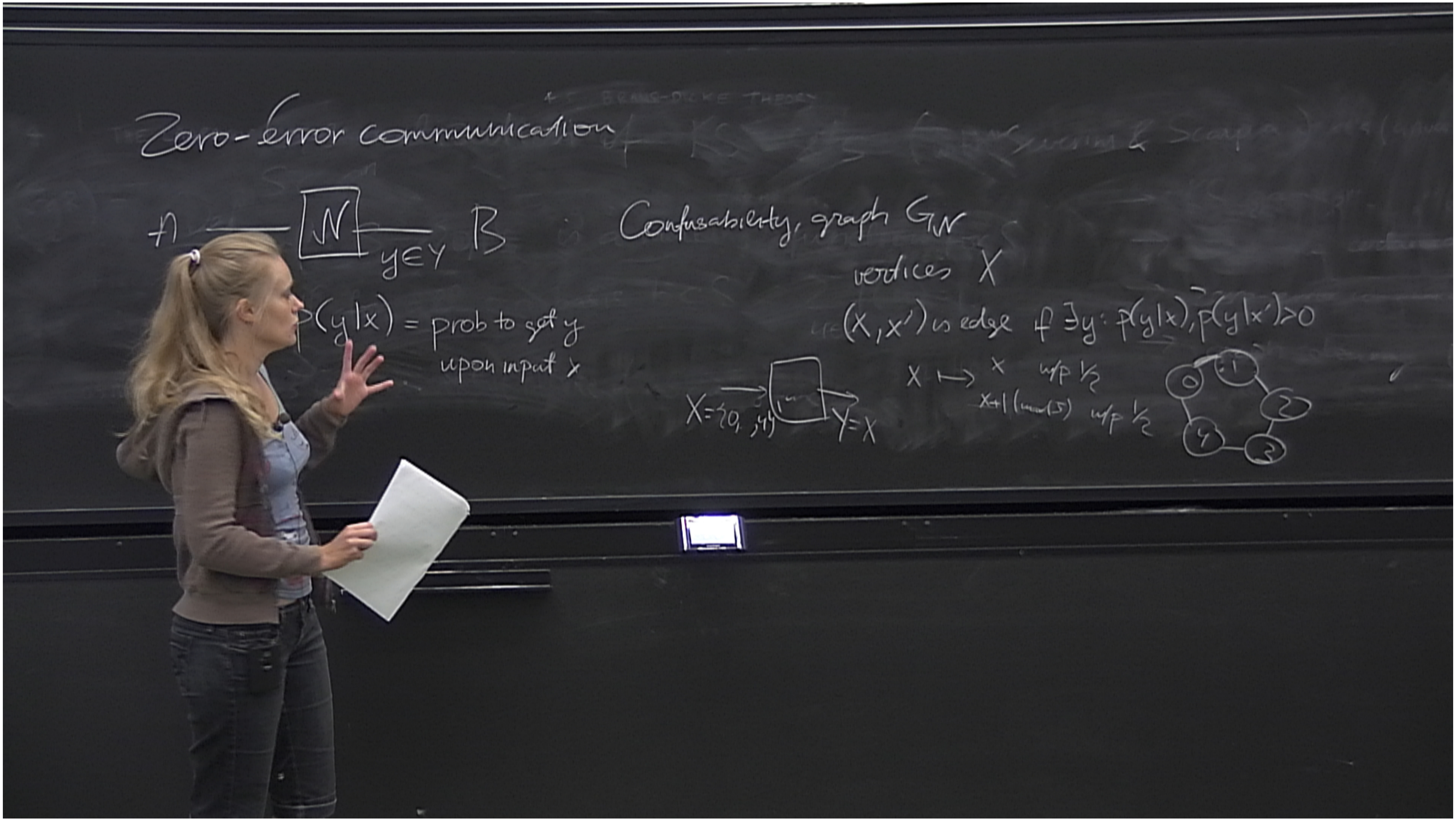
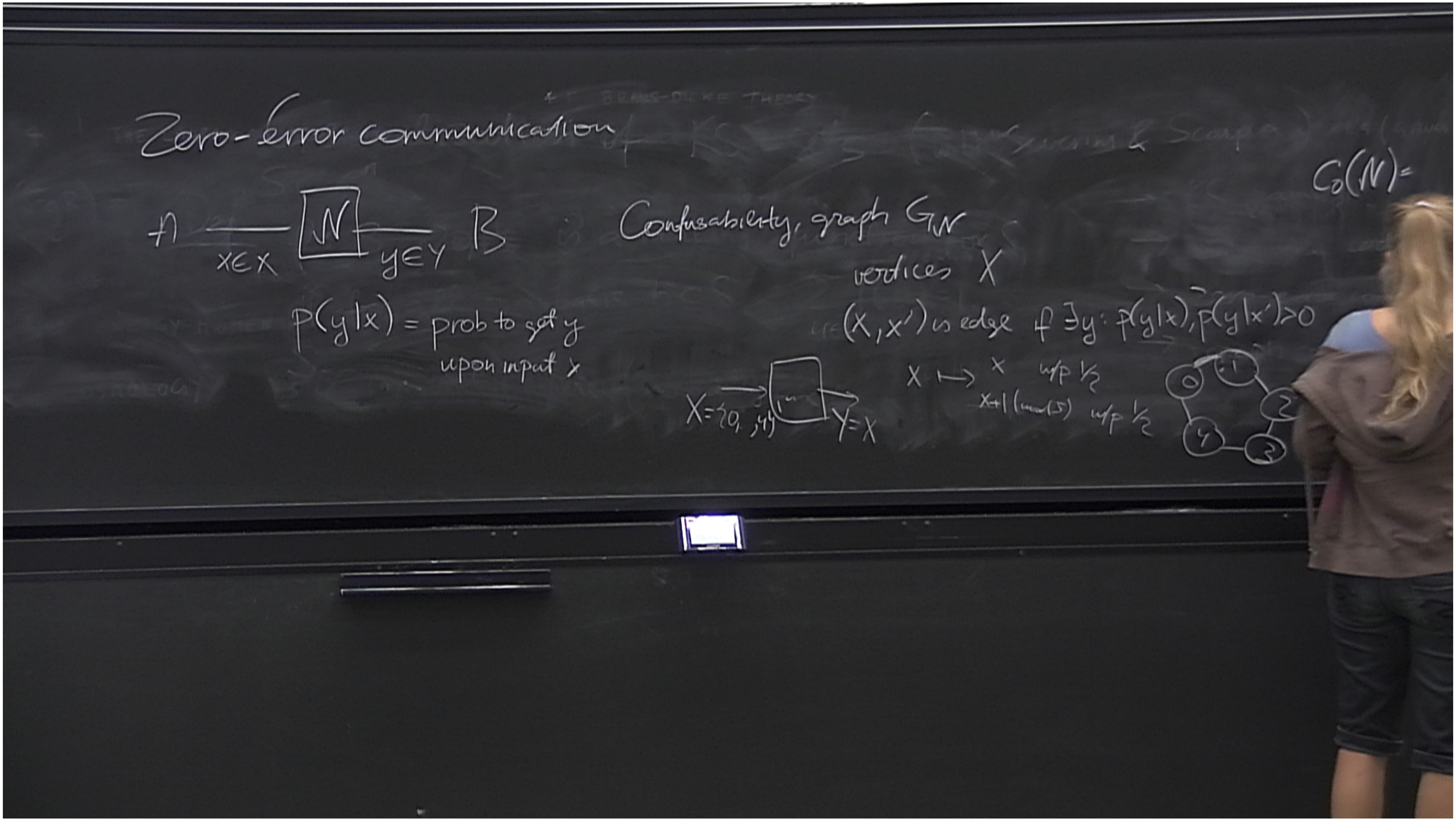
B
x∈X
y∈Y
p(y|x) = prob to get y
upon input x
Confusability graph G_N
vertices
(x,x')
∃y: p(y|x), p(y|x')>0


Zero-error communication
A
N
B
y∈Y
p(y|x) = prob to get y
upon input x
Confusability graph G_N
vertices X
(x, x') is edge if ∃y: p(y|x), p(y|x')>0
X={0,..,4}
Y=X
x ↦ x w/p 1/2
x+1 (mod 5) w/p 1/2
0
1
2
3
4


Zero-error communication
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Y=X
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x+1 (mod 5) w/p 1/2
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1
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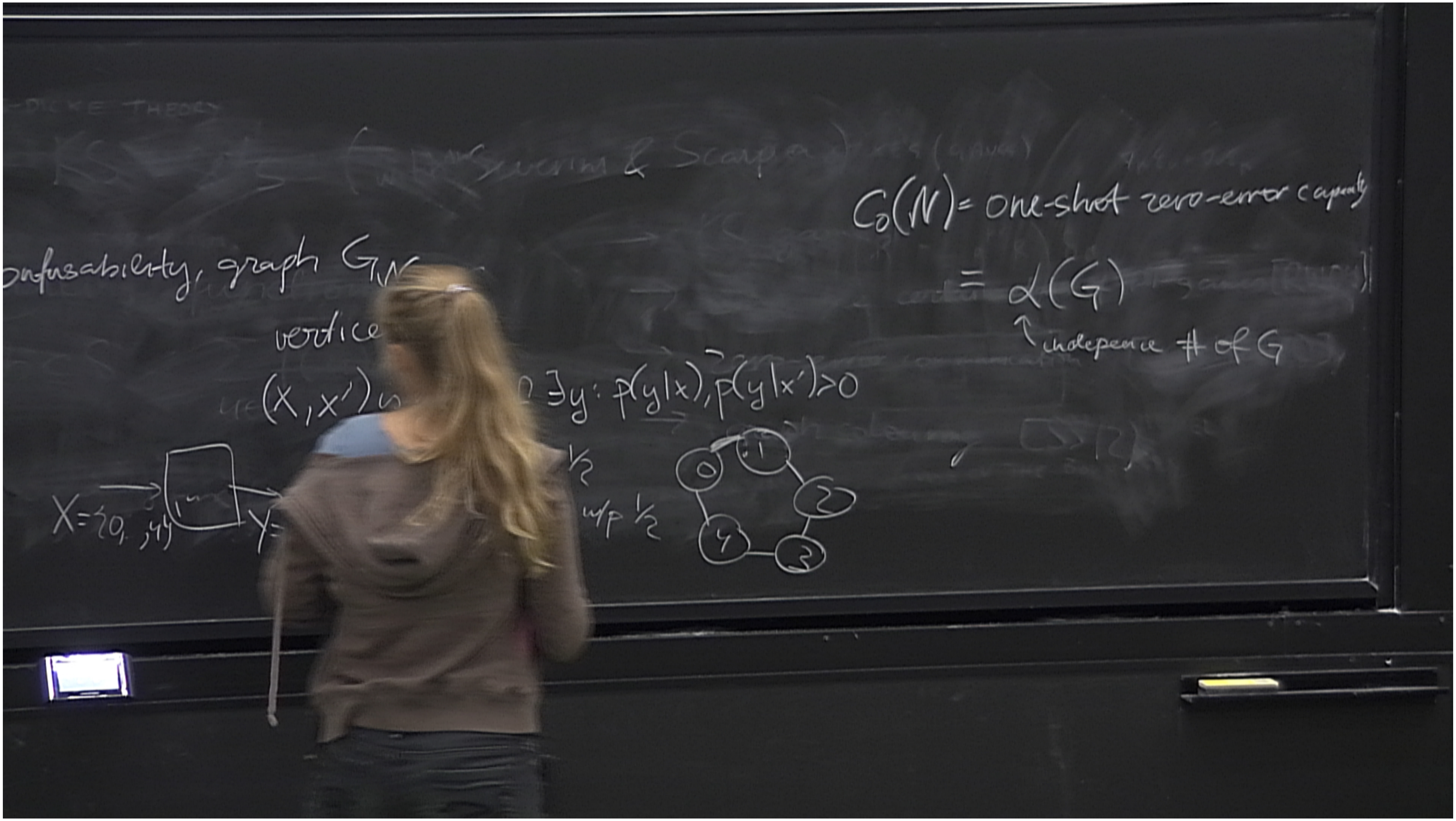
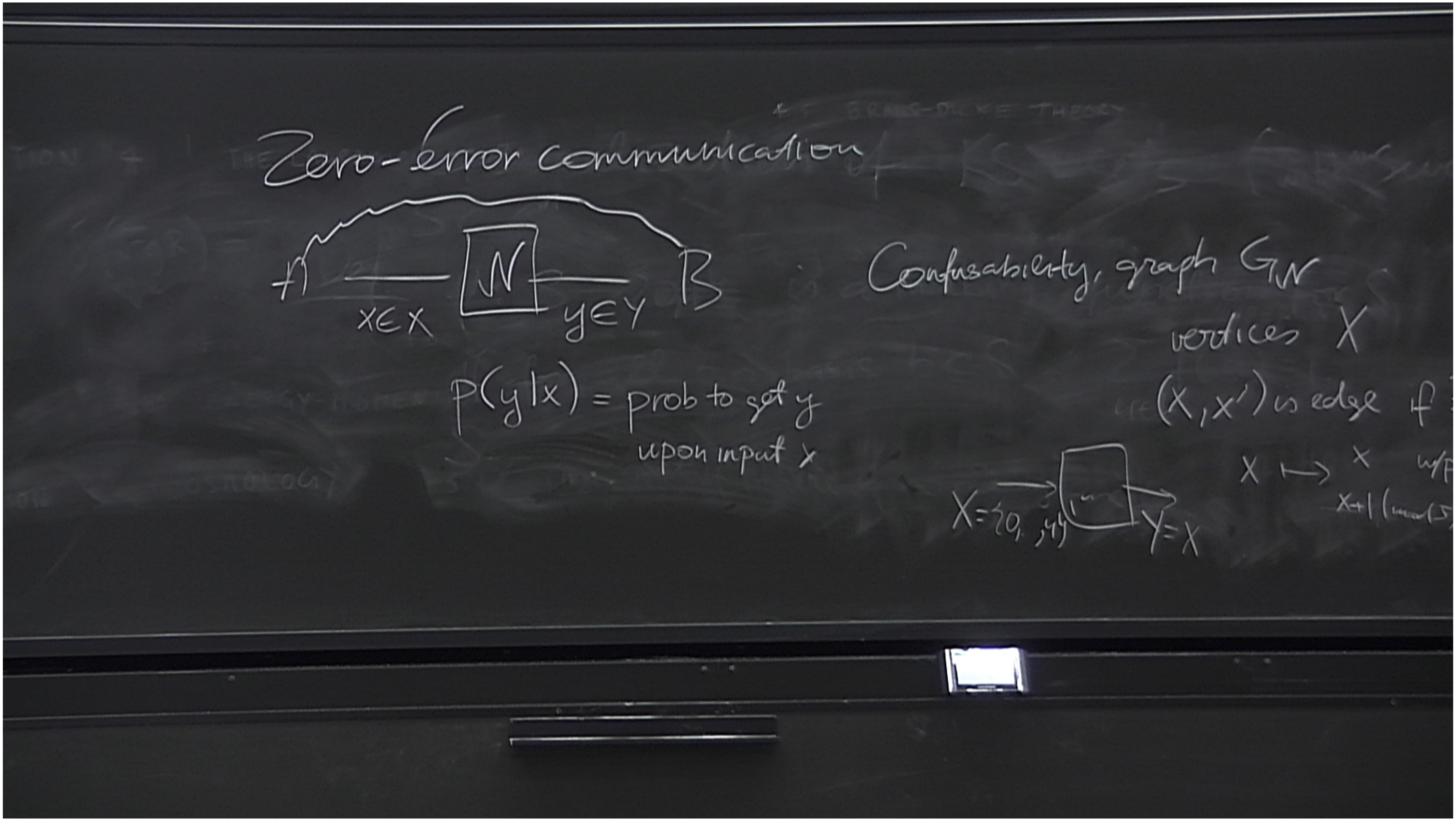
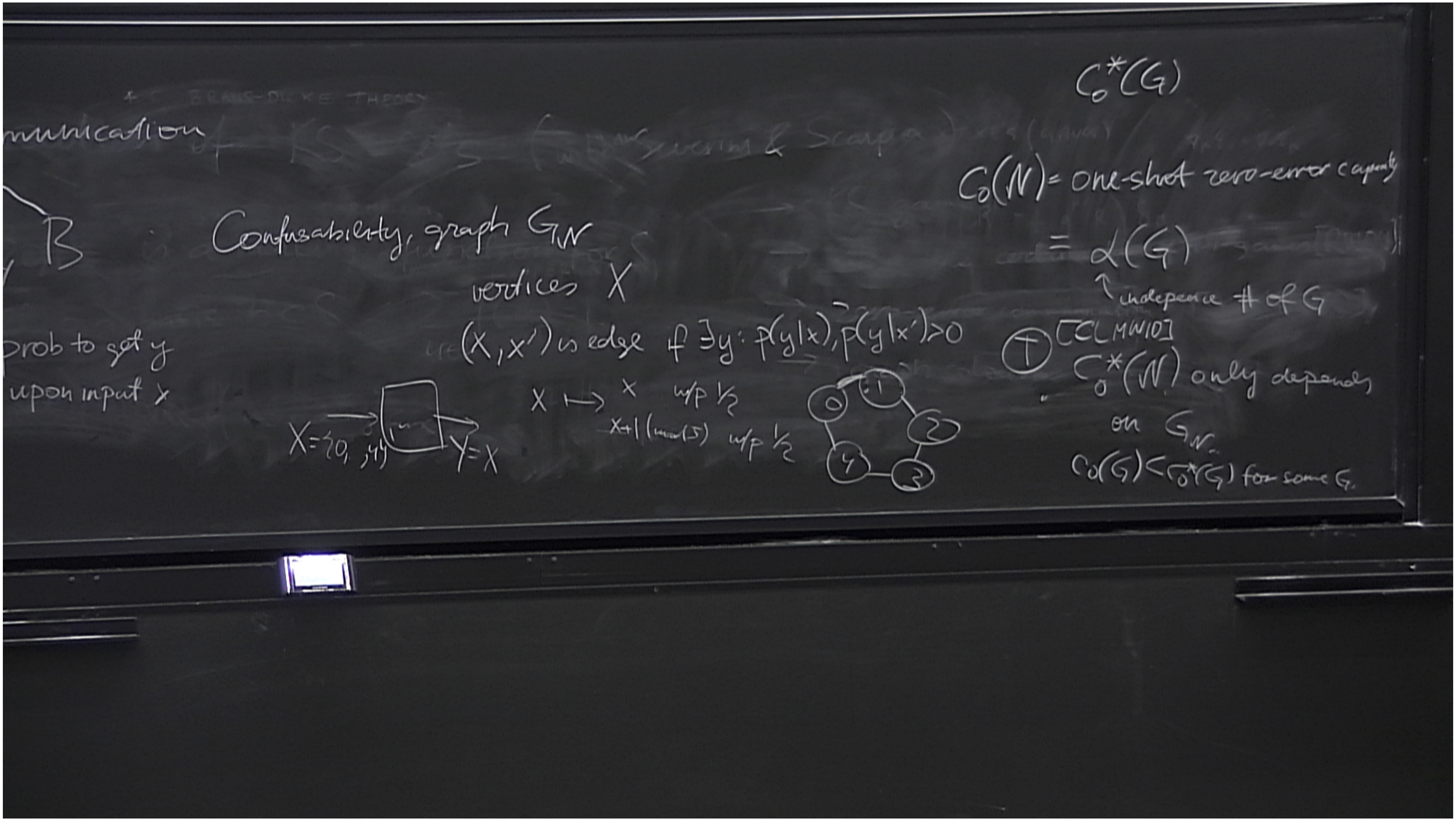
4
C_0(N)=


C0(N) = one-shot zero-error capacity
= α(G)
indepence # of G
onfusability graph G_N
vertice
(x, x') ∃y: p(y|x), p(y|x')>0
X={0, ,4}
w/p 1/2

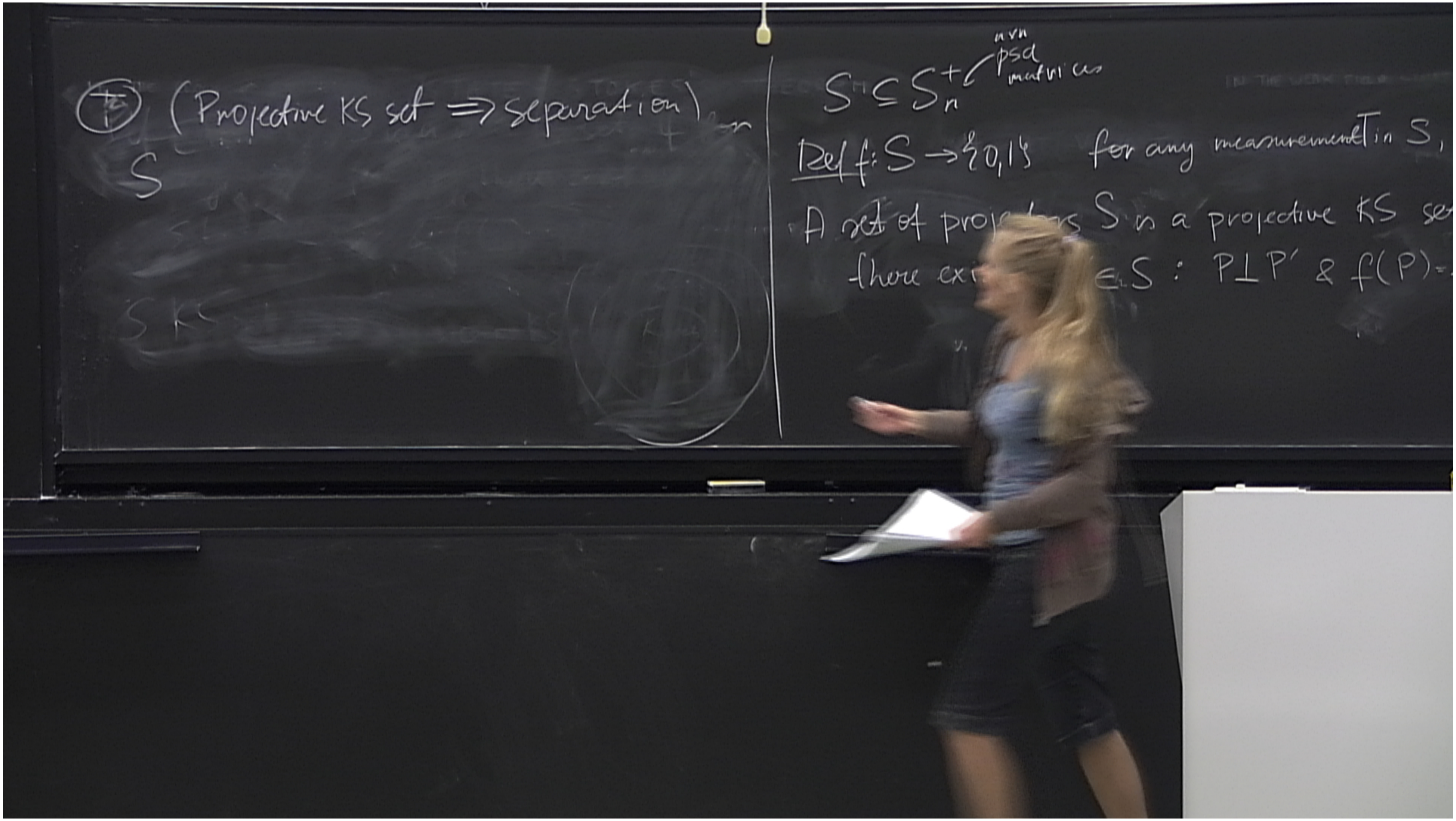


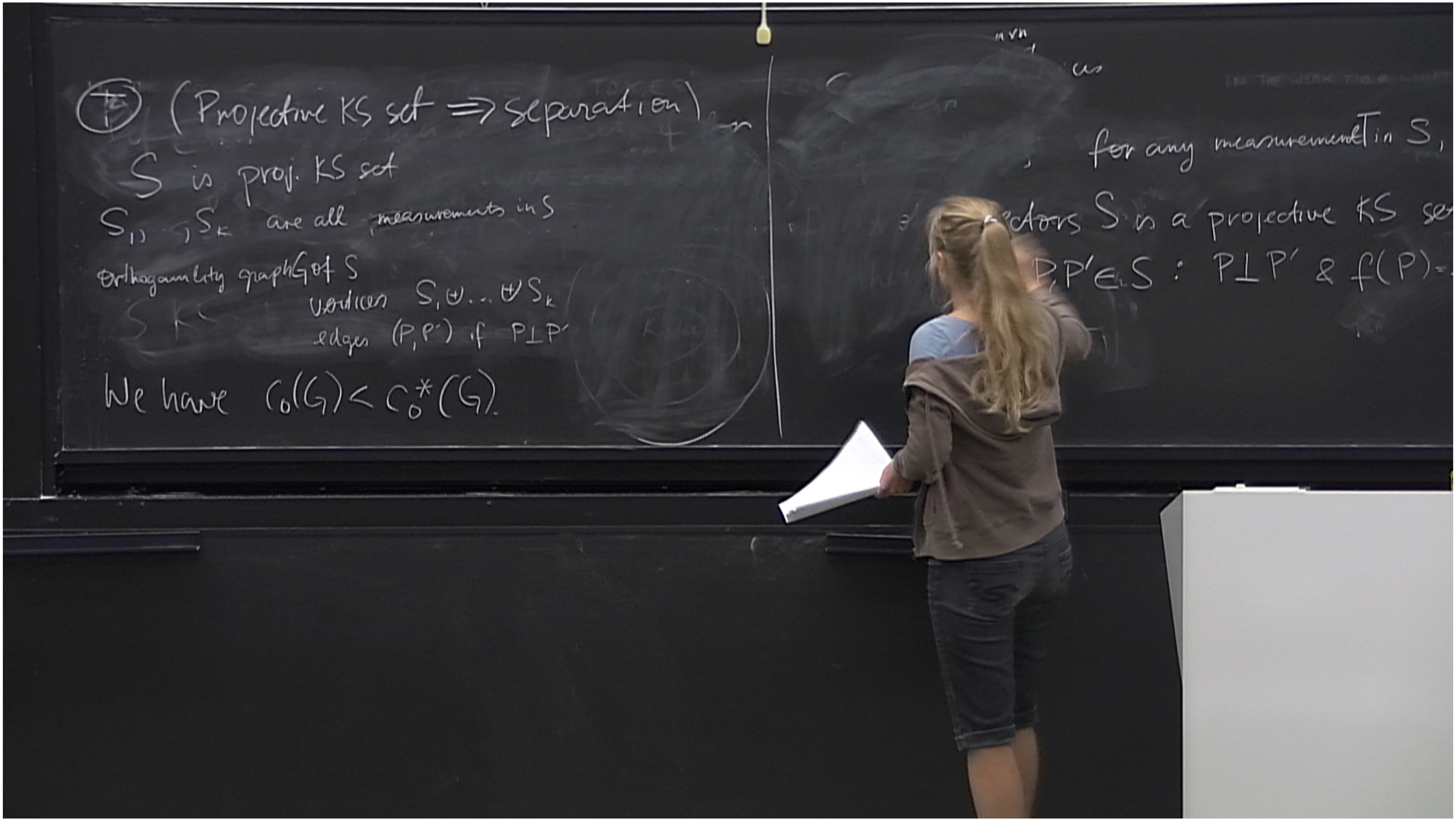
Zero-error communication
A
N
B
x∈X
y∈Y
p(y|x) = prob to get y
upon input x
Confusability graph G_N
vertices X
(x,x') is edge if
X = {0, ..., 4}
Y = X
x ↦ x
x+1 (mod 5)

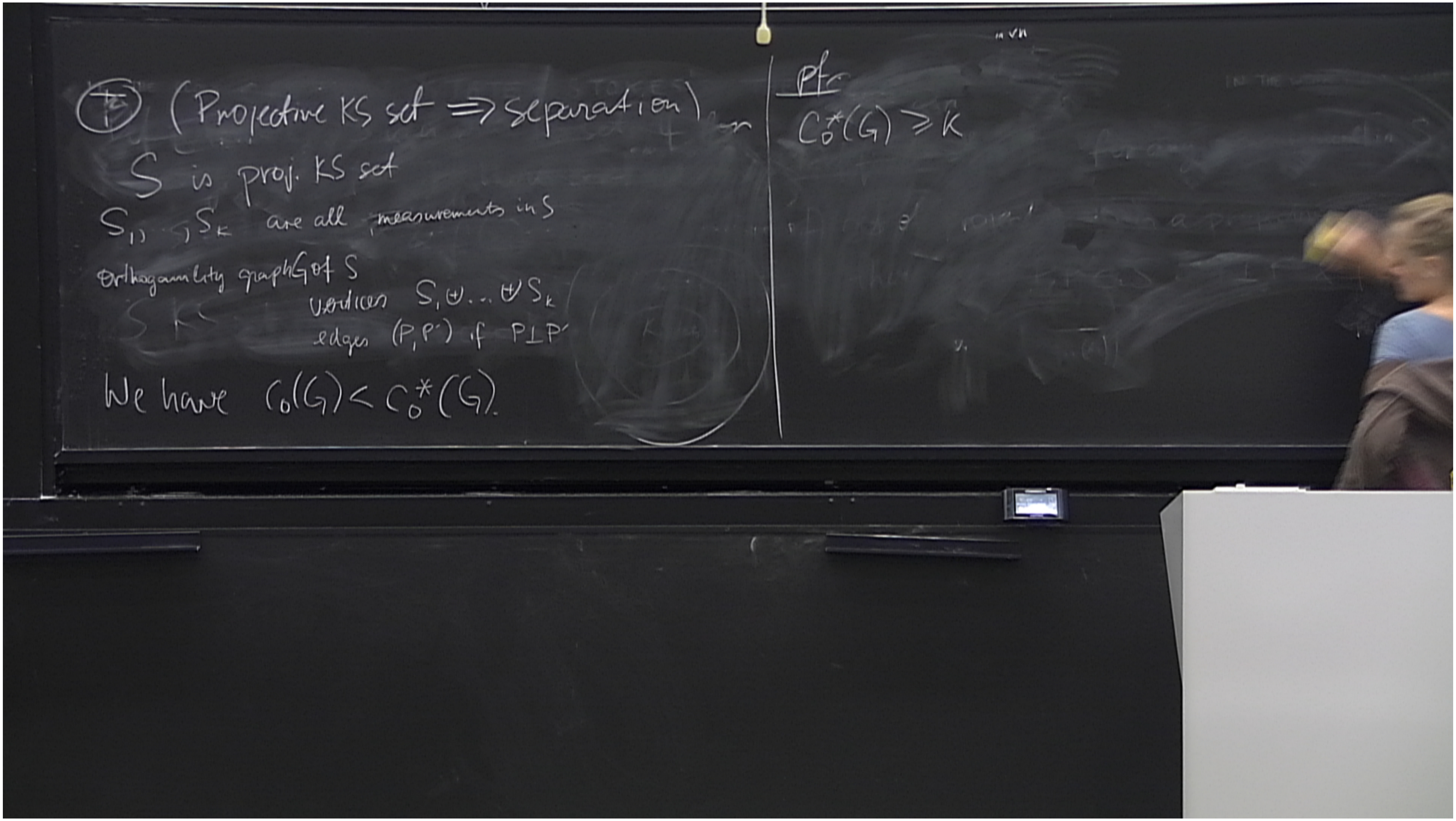
Confusability graph G_N
vertices X
(x, x') is edge
prob to get y upon input x
$C_0^*(G)$
$C_0(N)$ = one-shot zero-error capacity
$= \alpha(G)$
independence # of G
T [CLMW10]
$C_0^*(N)$ only depends on G_N.
$C_0(G) < C_0^*(G)$ for some G.

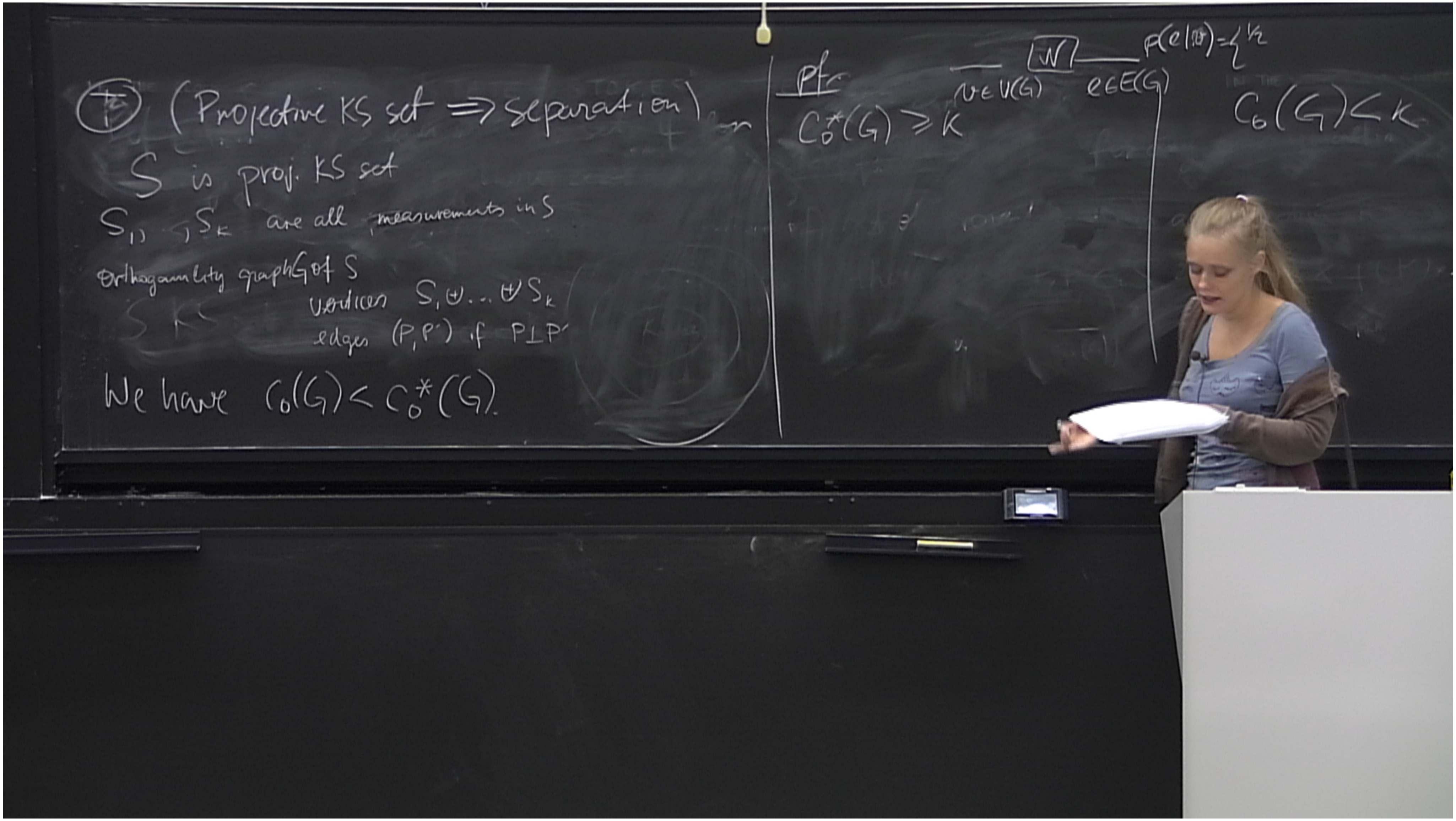


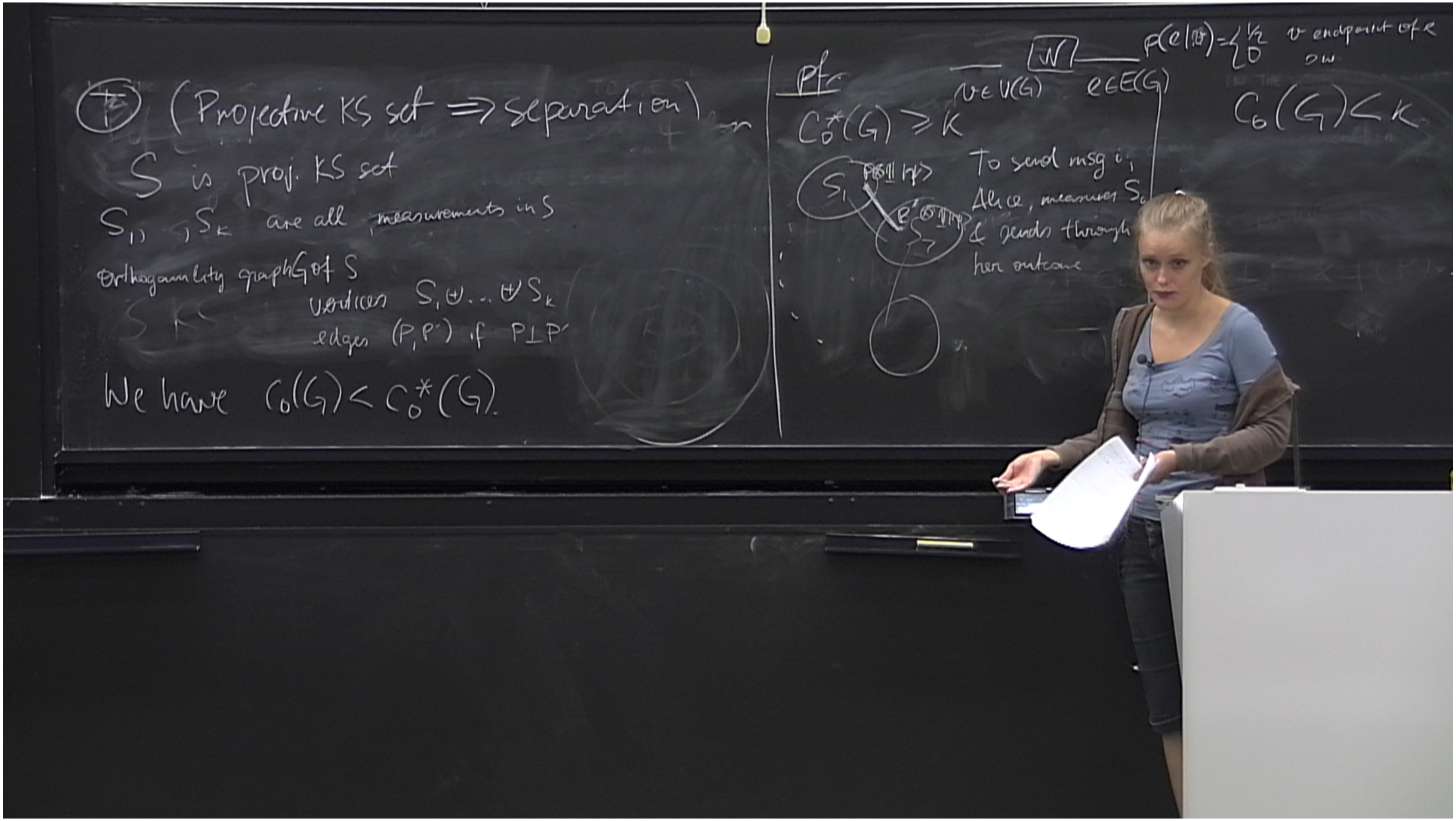
$C_0^*(G)$
mmunication
$C_0(N)$ = one-shot zero-error capacity
B
Confusability graph G_N
$= \alpha(G)$
independence # of G
vertices X
(x, x') is edge if $\exists y: p(y|x), p(y|x') > 0$
[CLMW10]
prob to get y
upon input x
$C_0^*(N)$ only depends
on G_N
$x \mapsto x$ w/p 1/2
$x+1 \pmod 5$ w/p 1/2
$X = \{0, \ldots, 4\}$
$Y = X$
0
1
2
3
4
$C_0(G) < C_0^*(G)$ for some G.


(Projective KS set ⇒ separation)
S ⊆ S+ n×n psd matrices
Def f: S → {0,1} for any measurement in S,
A set of proje s S is a projective KS se
there ex ∈ S : P⊥P' & f(P)=


(Projective KS set ⇒ separation)
S is proj. KS set
S₁, , Sₖ are all measurements in S
Orthogonality graph G of S
vertices S₁ ⊕ ... ⊕ Sₖ
edges (P, P') if P⊥P'
We have c₀(G) < c₀*(G).
for any measurement T in S,
ctors S is a projective KS se
P P' ∈ S : P⊥P' & f(P)=


T3 (Projective KS set ⇒ separation)
S is proj. KS set
$S_1, \ldots, S_k$ are all measurements in S
Orthogonality graph G of S
vertices $S_1 \oplus \ldots \oplus S_k$
edges (P, P') if $P \perp P'$
We have $c_0(G) < c_0^*(G)$.
Pf: $C_0^*(G) \geq k$


(Projective KS set ⇒ separation)
S is proj. KS set
S1, …, Sk are all measurements in S
Orthogonality graph G of S
vertices S1 ⊕ … ⊕ Sk
edges (P, P') if P⊥P'
We have C0(G) < C0*(G).
Pf:
C0*(G) ≥ K
v∈V(G)
e∈E(G)
C0(G) < K


(Projective KS set ⇒ separation)
S is proj. KS set
S1, ..., Sk are all measurements in S
Orthogonality graph G of S
vertices S1 ⊎ ... ⊎ Sk
edges (P, P') if P ⊥ P'
We have C0(G) < C0*(G).
Pf:
C0*(G) ≥ K
v ∈ V(G)
e ∈ E(G)
W
P(e|W) = 1/2 v endpoint of e, 0 o.w.
C0(G) < K
To send msg i, Alice measures Si & sends through her outcome

Thm (Projective KS set $\Rightarrow$ separation)

S is proj. KS set

$S_1, \dots, S_k$ are all measurements in S

Orthogonality graph G of S

vertices $S_1 \uplus \dots \uplus S_k$

edges (P, P') if $P \perp P'$

We have $c_0(G) < c_0^*(G)$.

Pf: $C_0^*(G) \geq k$

S_1 $P \otimes I|\psi\rangle$

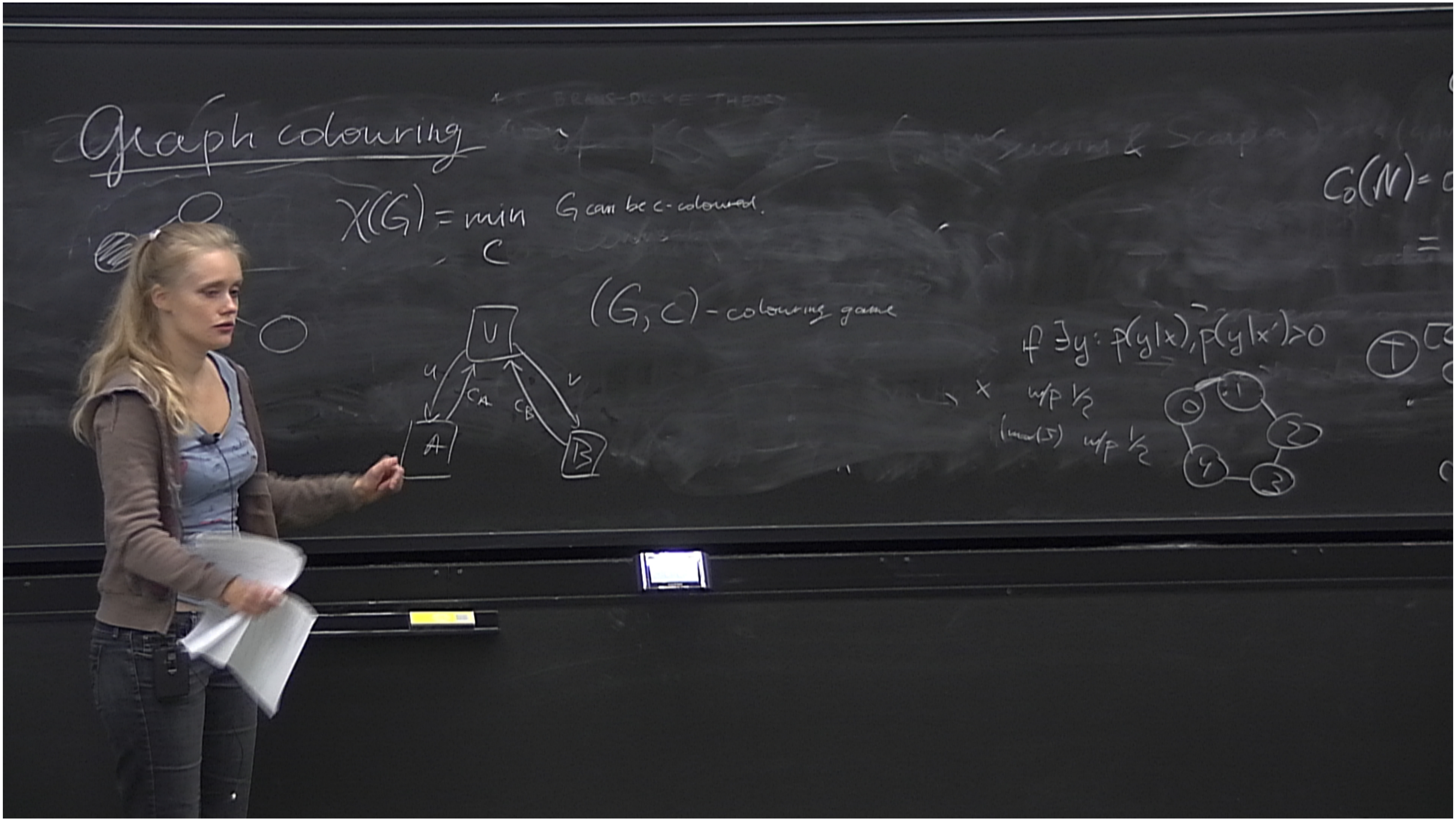
S_2 $P' \otimes I|\psi\rangle$

To send msg i, Alice measures S_i & sends through W her outcome

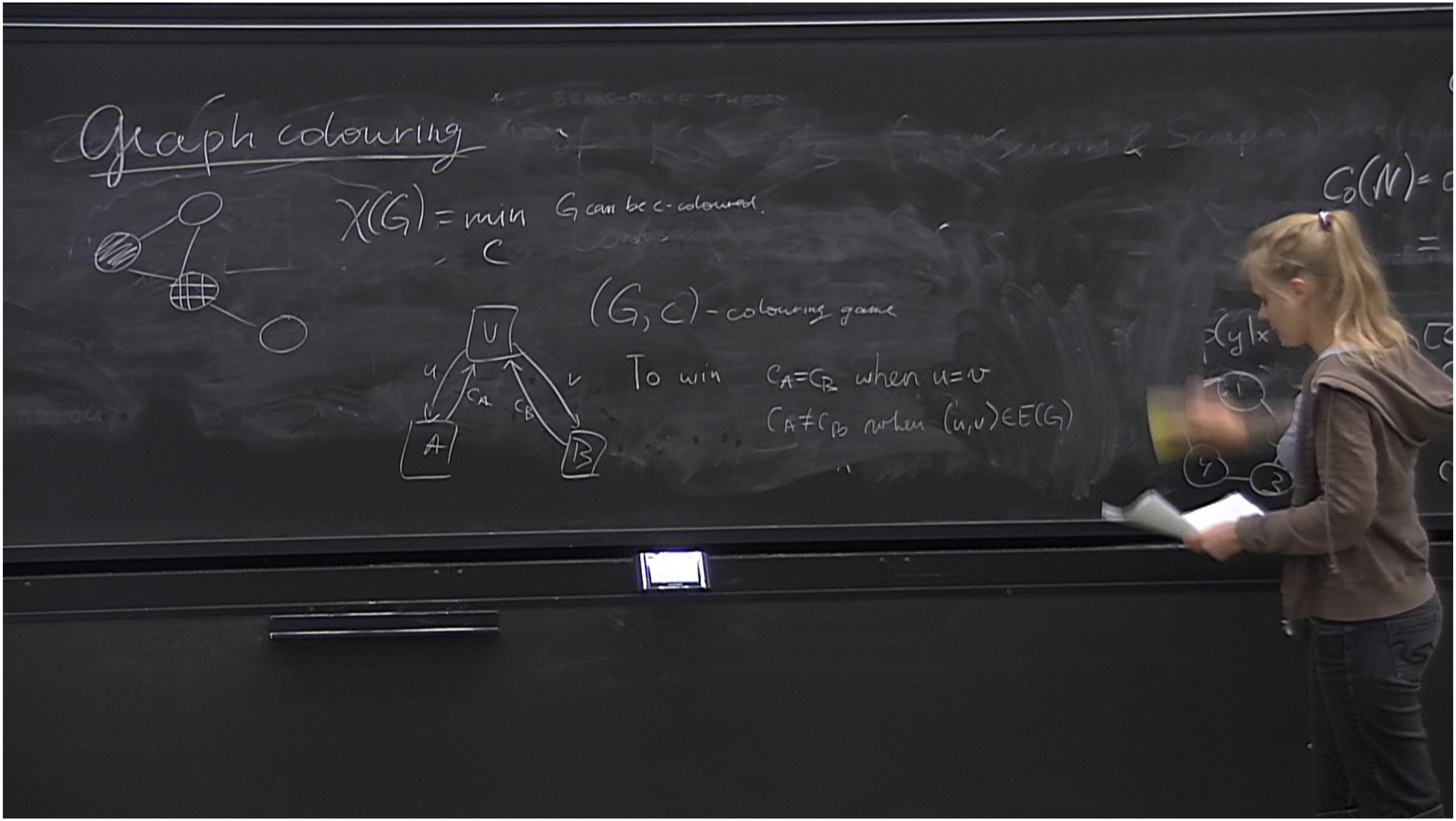
$v \in V(G)$ — $\boxed{W}$ — $e \in E(G)$

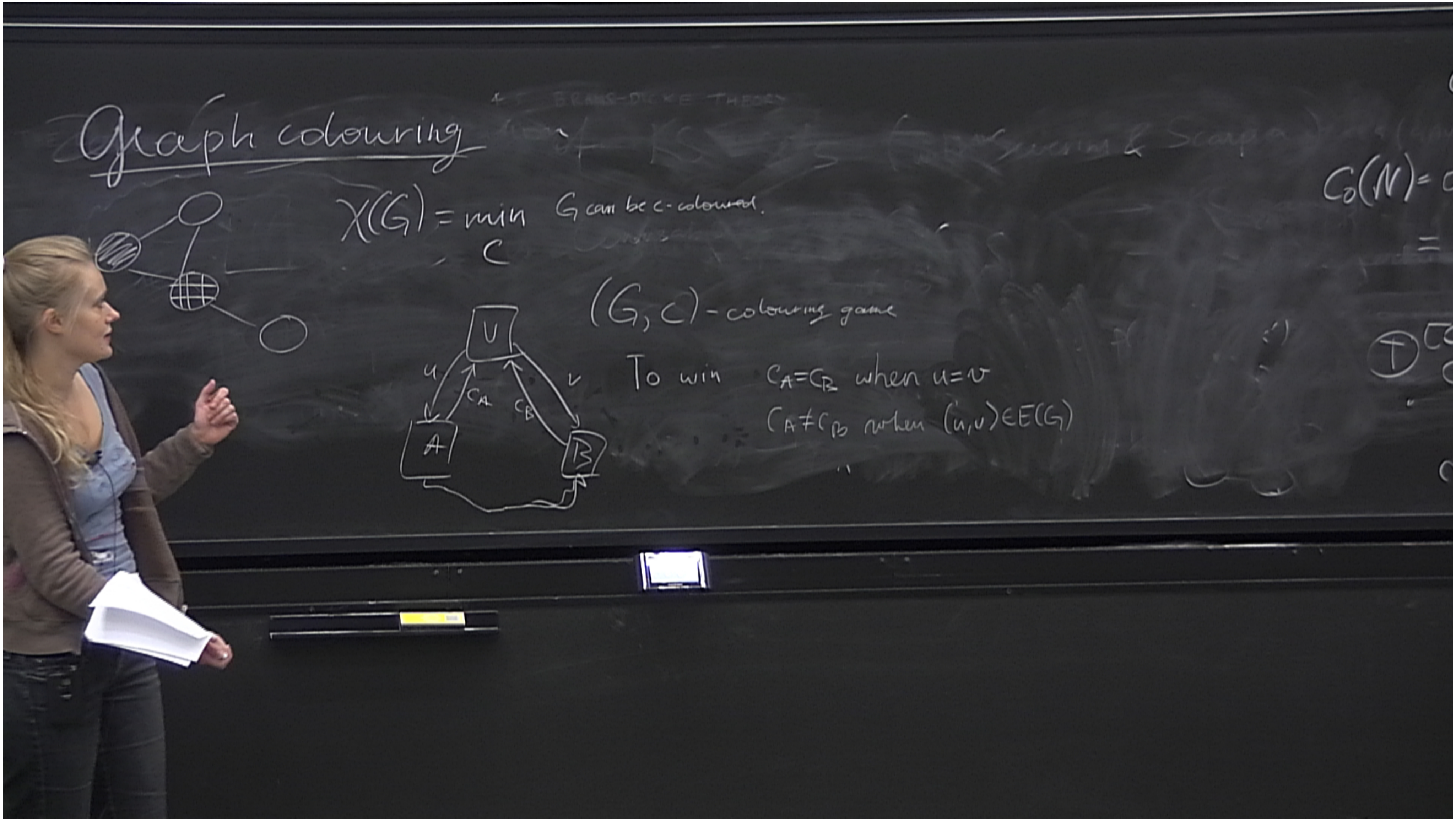
$p(e|v) = \begin{cases} 1/2 & v \text{ endpoint of } e \\ 0 & \text{o.w.} \end{cases}$

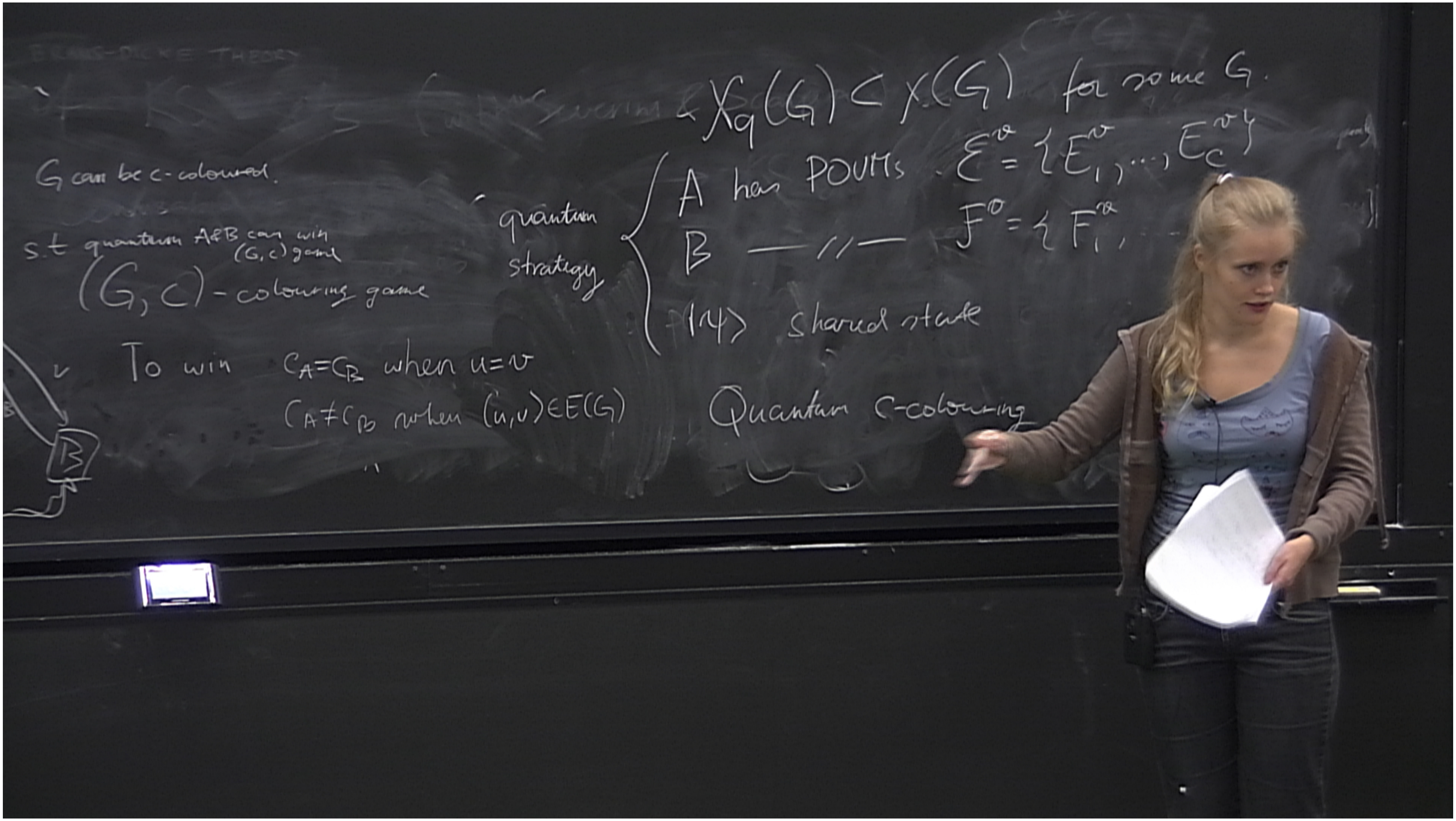
$C_0(G) < k$



Graph colouring
χ(G) = min c   G can be c-coloured.
(G, c) - colouring game
U
A
B
f ∃y: p(y|x), p(y|x')>0
w/p 1/2


Graph colouring
χ(G) = min G can be c-coloured.
c
(G, c) - colouring game
V
u
v
c_A
c_B
A
B
To win c_A = c_B when u = v
c_A ≠ c_B when (u,v) ∈ E(G)


Graph colouring
χ(G) = min G can be c-coloured.
c
(G, c) - colouring game
To win
c_A = c_B when u = v
c_A ≠ c_B when (u,v) ∈ E(G)
u
v
c_A
c_B
A
B


χ_q(G) < χ(G) for some G.
G can be c-coloured.
s.t quantum A&B can win (G,c) game
(G,c)-colouring game
To win c_A=c_B when u=v
c_A≠c_B when (u,v)∈E(G)
quantum strategy
A has POVMs ℰ^v = {E^v_1, ..., E^v_c}
B —— // ——
|ψ⟩ shared state
Quantum c-colouring

$\chi_q(G) < \chi(G)$ for some G.

G can be c-coloured.

s.t. quantum A&B can win (G,c) game

(G,c)-colouring game

To win $c_A = c_B$ when $u = v$

$c_A \neq c_B$ when $(u,v) \in E(G)$

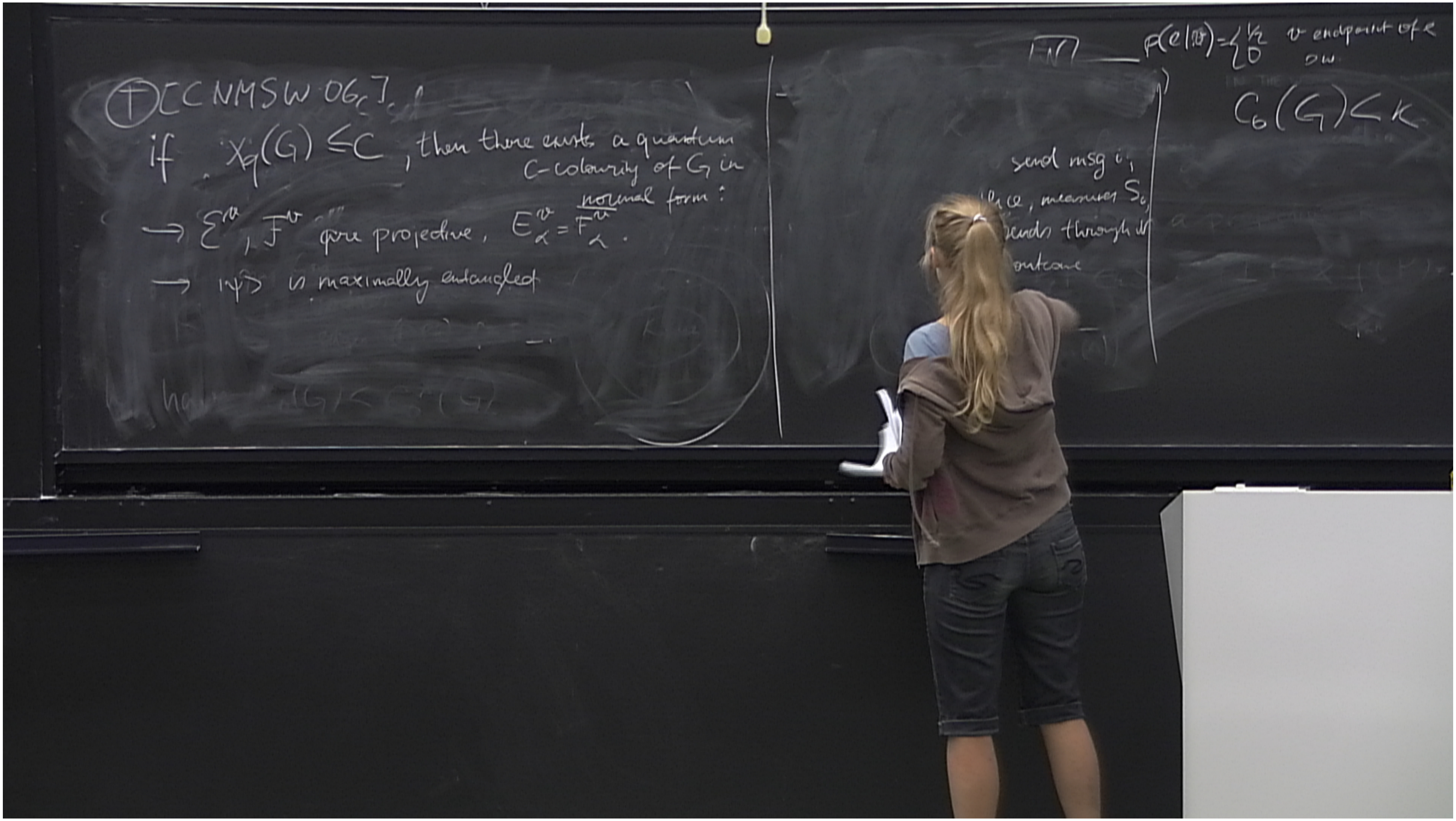
quantum strategy

A has POVMs $\mathcal{E}^v = \{E_1^v, \dots, E_c^v\}$

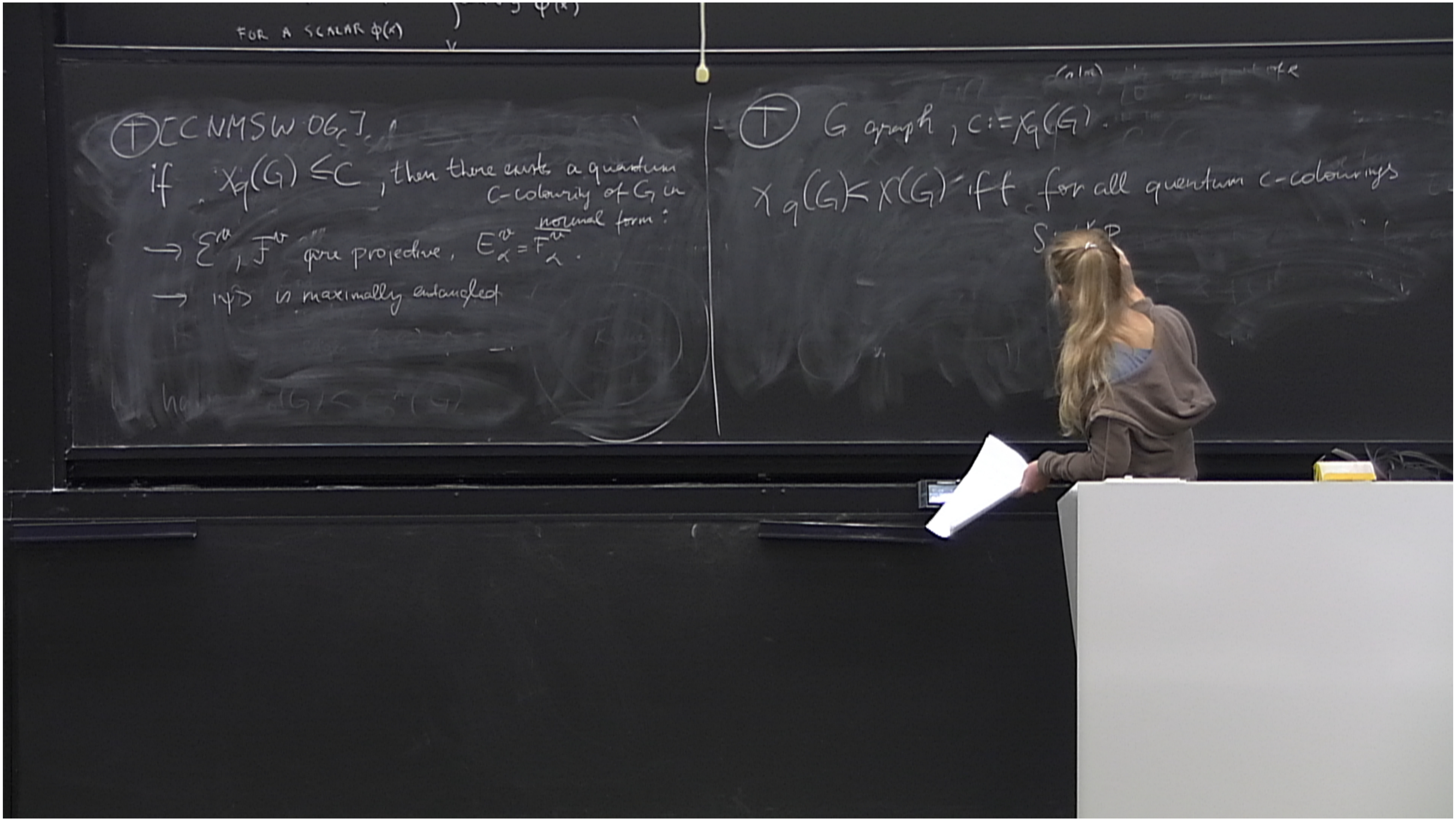
B —//— $\mathcal{F}^v = \{F_1^v, \dots, F_c^v\}$

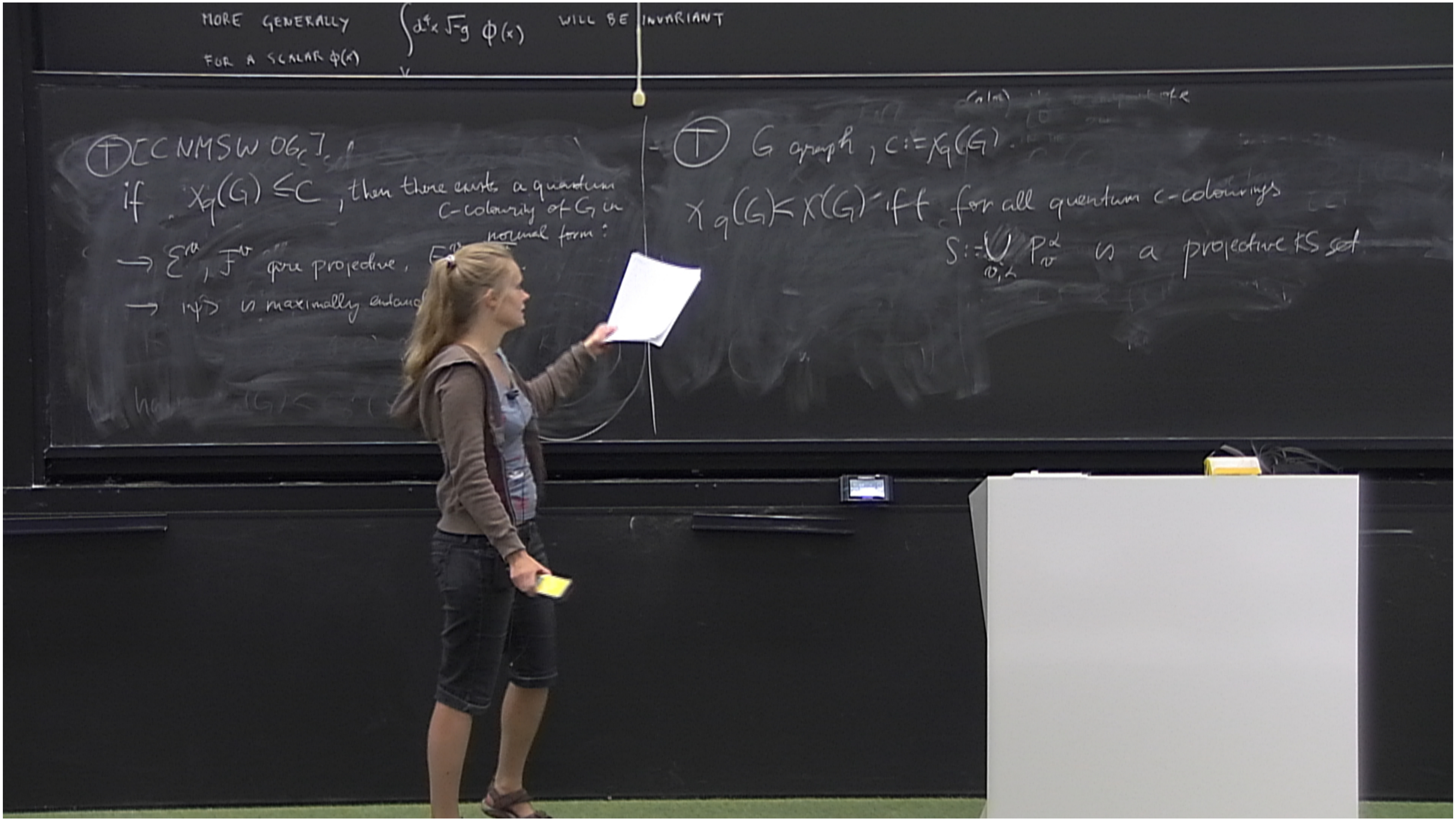
$|\psi\rangle$ shared state

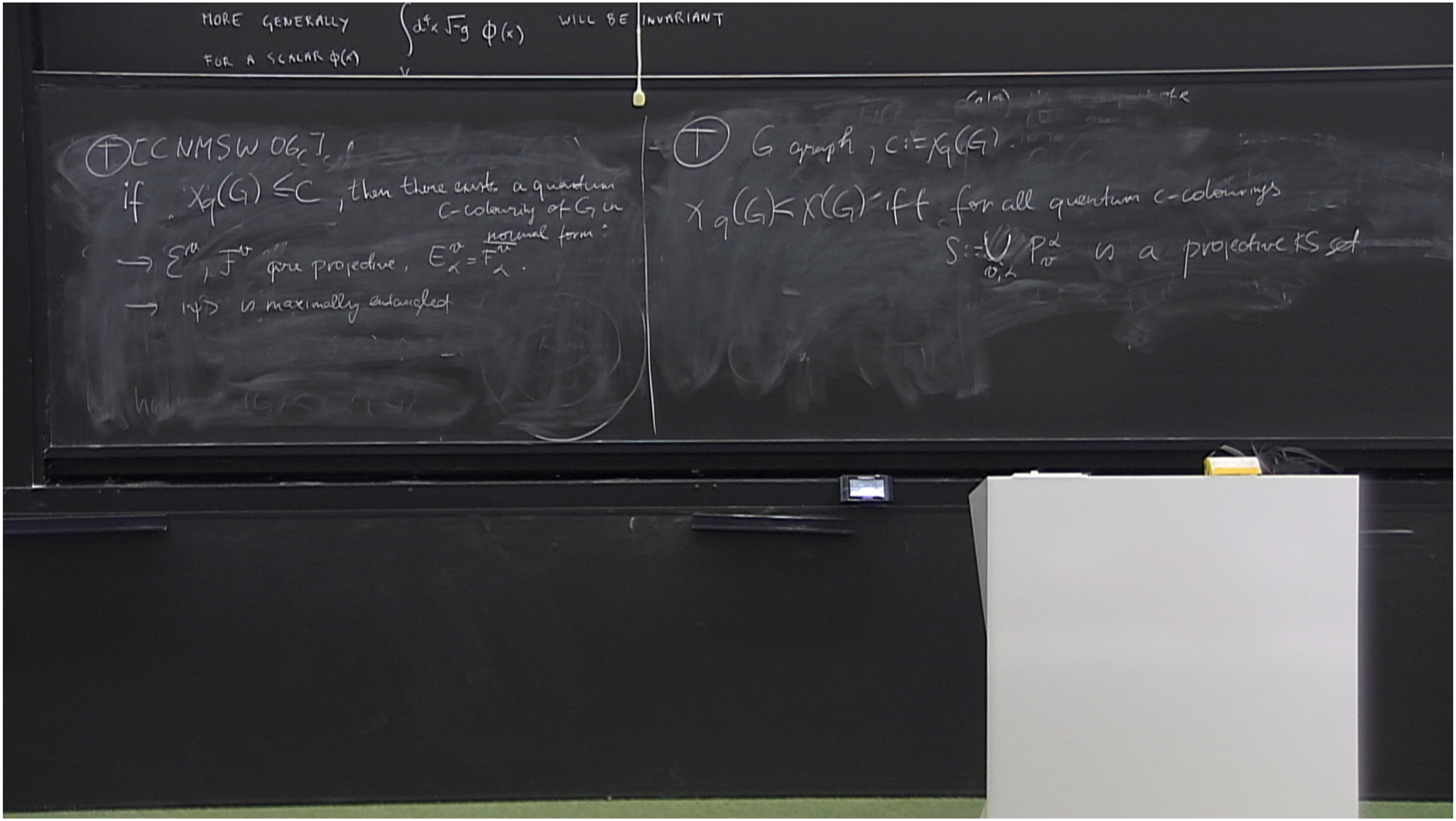
Quantum c-colouring is a strategy that convinces verifier w.p. 1

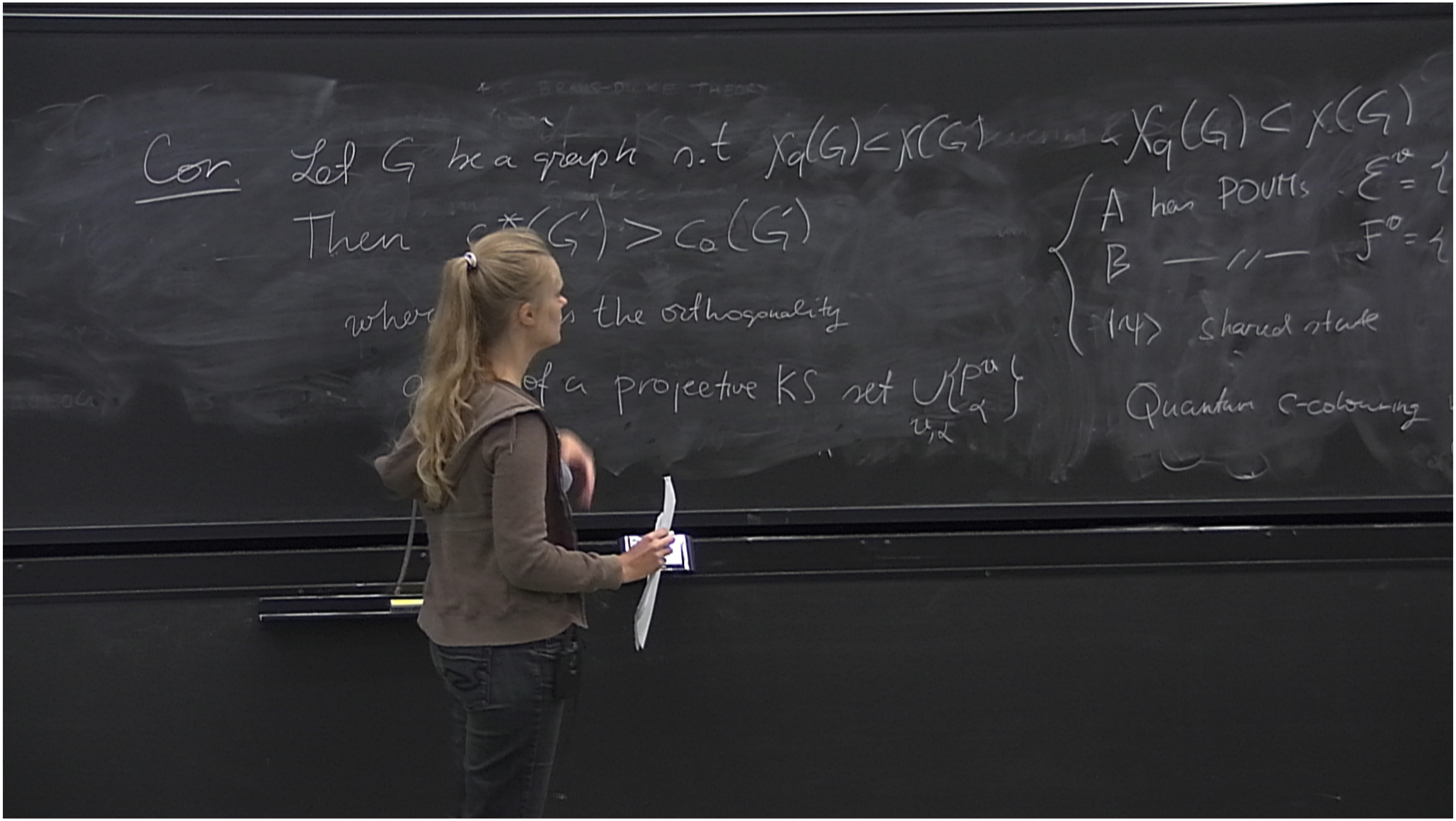


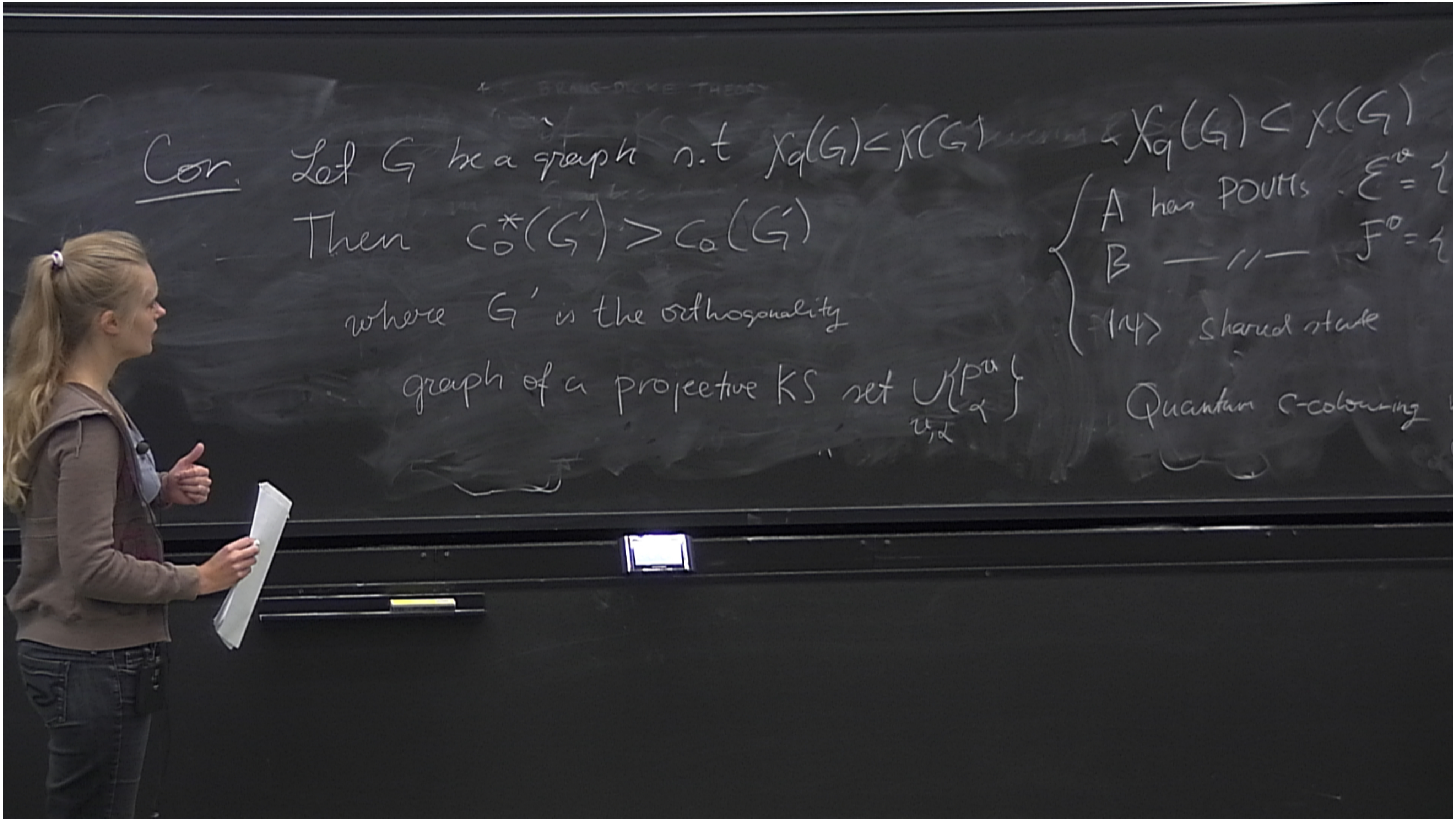
T [CNMSW 06]
if χq(G) ≤ C, then there exists a quantum
C-colouring of G in
normal form:
→ E^v, F^v are projective, E^v_α = F^v_α.
→ |ψ> is maximally entangled
p(e|v) = 1/2 v endpoint of e
0 o.w.
C_0(G) < K
send msg i,
outcome

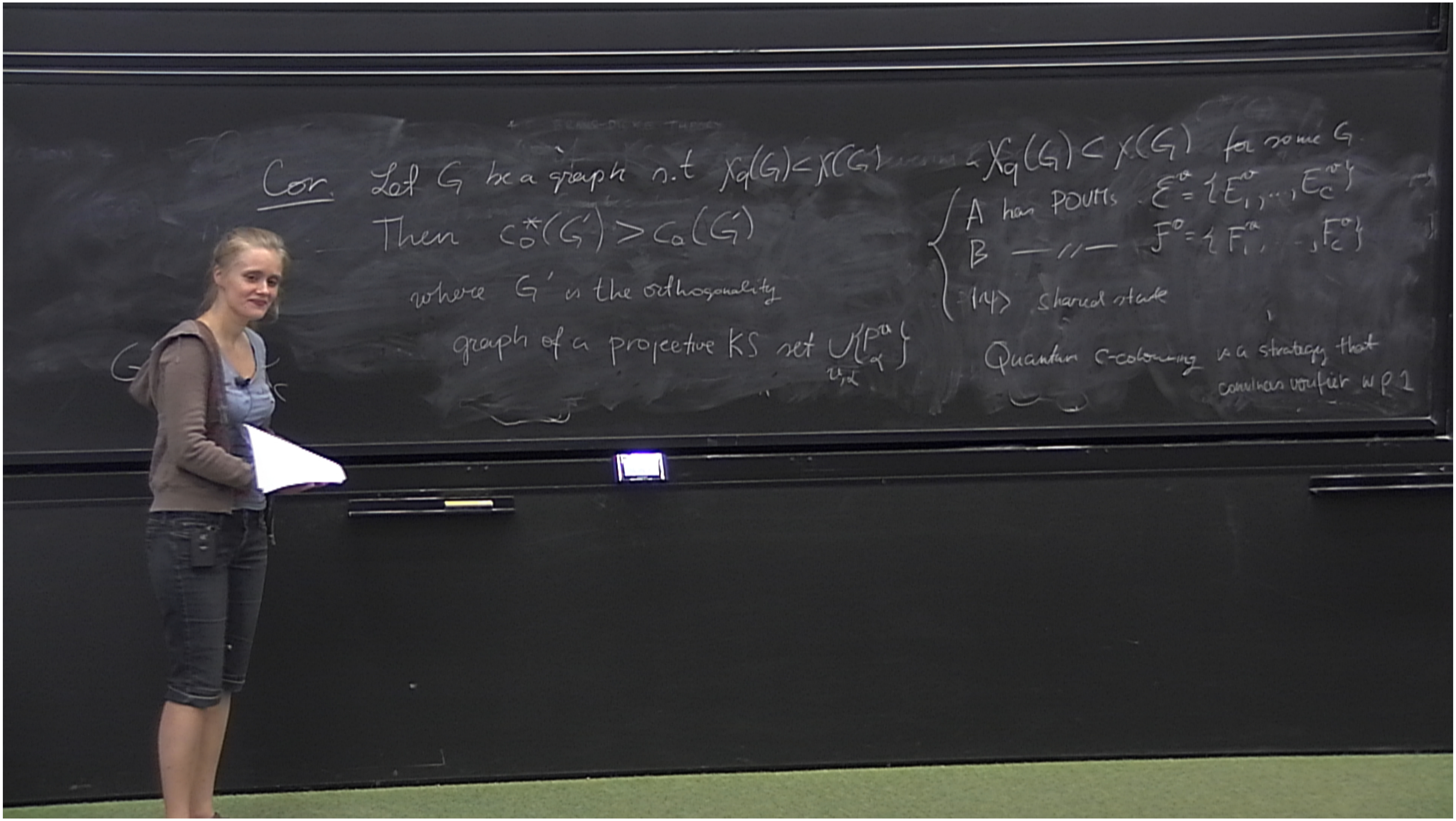

FOR A SCALAR φ(x)
T [CNMSW 06c]
if χq(G) ≤ c, then there exists a quantum c-colouring of G in normal form:
→ E^v, F^v are projective, E^v_α = F^v_α.
→ |ψ⟩ is maximally entangled
T G graph, c := χq(G).
χq(G) < χ(G) iff for all quantum c-colourings


MORE GENERALLY
FOR A SCALAR $\phi(x)$
$\int d^4x \sqrt{-g}\, \phi(x)$ WILL BE INVARIANT
Ⓣ [CNMSW 06]
if $\chi_q(G) \leq c$, then there exists a quantum c-colouring of G in normal form:
→ E^{α}, F^{v} are projective,
→ $|\psi\rangle$ is maximally entangled
Ⓣ G graph, $c := \chi_q(G)$
$\chi_q(G) < \chi(G)$ iff for all quantum c-colourings
$S := \bigcup_{v,\alpha} P_v^{\alpha}$ is a projective KS set


MORE GENERALLY $\int d^4x \sqrt{-g}\, \phi(x)$ WILL BE INVARIANT
FOR A SCALAR $\phi(x)$
Ⓣ [CNMSW 06]
if $\chi_q(G) \leq c$, then there exists a quantum c-colouring of G in normal form:
$\rightarrow E^v, F^v$ are projective, $E^v_\alpha = F^v_\alpha$.
$\rightarrow |\psi\rangle$ is maximally entangled
Ⓣ G graph, $c := \chi_q(G)$.
$\chi_q(G) < \chi(G)$ iff for all quantum c-colourings
$S := \bigcup_{v,\alpha} P^\alpha_v$ is a projective KS set


Cor. Let G be a graph s.t $\chi_q(G) < \chi(G)$
Then $c^*(G') > c_0(G')$
the orthogonality
of a projective KS set $\bigcup_{u,\alpha} \{P^u_\alpha\}$
$\chi_q(G) < \chi(G)$
A has POVMs
B
shared state
Quantum c-colouring


Cor. Let G be a graph s.t $\chi_q(G) < \chi(G)$
Then $c_0^*(G') > c_0(G')$
where G' is the orthogonality
graph of a projective KS set $\bigcup_{u,\alpha}\{P_\alpha^u\}$
$\chi_q(G) < \chi(G)$
A has POVMs $\mathcal{E}^u = \{$
B — // — $\mathcal{F}^v = \{$
$|\psi\rangle$ shared state
Quantum c-colouring


Cor. Let G be a graph s.t χq(G) < χ(G)
Then c0*(G') > c0(G')
where G' is the orthogonality
graph of a projective KS set ∪v,α {Pαv}
χq(G) < χ(G) for some G
A has POVMs ℰv = {E1v, ..., Ecv}
B —//— Fv = {F1v, ..., Fcv}
|ψ⟩ shared state
Quantum c-colouring is a strategy that
convinces verifier w.p. 1