Title: Multiplets in the Entanglement Spectrum

Date: Sep 06, 2012 01:00 PM

URL: http://pirsa.org/12090056

Abstract: What information can be determined about a state given
just the ground state wave function?

Quantum ground states, speaking intuitively, contain
fluctuations between many of the configurations one might want to understand.
The information about them can be organized by introducing an imaginary system,
dubbed the entanglement Hamiltonian.

What light does the dynamics of this Hamiltonian (a
precise version of the notion of "zero point motion") shed on the
actual system?

I will start my discussion near critical points, where
Lorentz invariance often emerges and the entanglement Hamiltonian becomes
tractable, revealing that it exists in one dimension less than the real system.
One application is to the fluctuations of angular momentum in a spin chain.

The entanglement Hamiltonian is especially successful at
clarifying the properties of quantum phases without order, such as topological
insulators. In particular, spin chains often have no long range order due to
quantum fluctuations. Nevertheless there can be phase transitions between two
of these disordered states, suggesting the existence of a hidden order.

My main aim in the talk will be to demonstrate that the
entanglement spectrum can serve as an order parameter for these unusual transitions; this observation leads to a
classification of one-dimensional phases.

One can understand the phases of the actual system simply
by looking at the spectrum of the entanglement Hamiltonian, just as one deduces
the properties of atoms from their spectra.

Multiplets in the

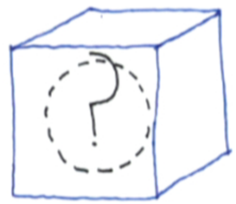
Ari Turner

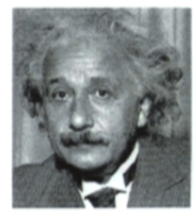
Work with
- Frank Pollmann
- Erez Berg
- Yi Zhang
- Masaki Oshikawa
- Ashvin Vishwanath

see Pollmann et al. PRB 81, 064439 (2010)
Turner et al. PRB 83, 075102 (2011)
Pollmann & Turner arXiv:1204.0704

Entanglement

B. Podolsky N. Rosen





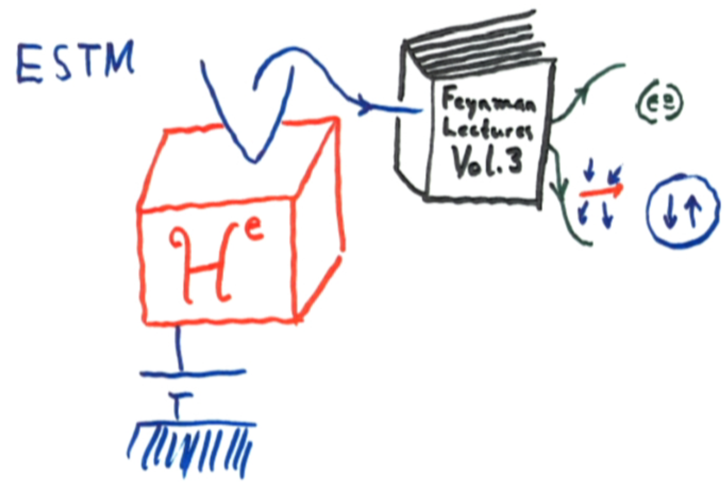
A. Einstein

$$\Psi = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_{PR}|\downarrow\rangle_{E} - |\downarrow\rangle_{PR}|\uparrow\rangle_{E} \right)$$

$$\rho_{\text{reduced}}(\text{Einstein}) = \mathrm{tr}_{PR}\,|\Psi\rangle\langle\Psi|$$

$$= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

What can one learn by studying the entanglement Hamiltonian as if it were physical?



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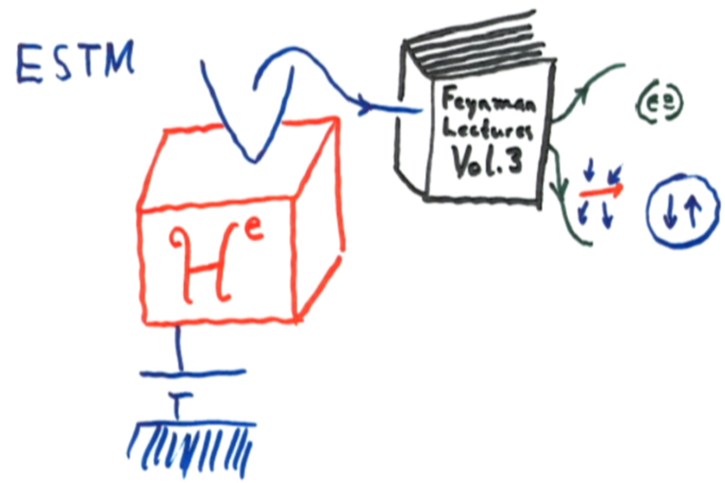
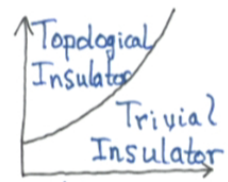


Topological Phases & Entanglement

Phases without order parameters

Typical Case



Hidden Order

Signature in Entanglement

$\Rightarrow$ A classification of 1D phases

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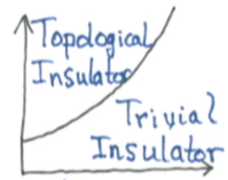

Hidden Order

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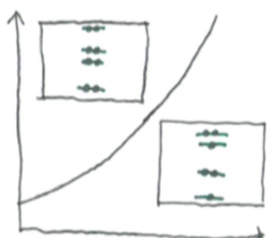


$\Rightarrow$ A classification of 1D phases

1. Is there a pseudohamiltonian for zero-point motion?
[E.g. I. Peschel J. Phys. A 2003]

2. Use it to describe interacting topological insulators [inspired by Li & Haldane 2008] especially in 1-D

What is the Entanglement Hamiltonian?

Example:

$$|\Psi\rangle = \sqrt{\tfrac{1}{3}}\, |\uparrow\rangle_A |\downarrow\rangle_B + \sqrt{\tfrac{2}{3}}\, |\downarrow\rangle_A |\uparrow\rangle_B$$

1st Step: Obtain a Mixed State

$$\rho_A = tr_B |\Psi\rangle\langle\Psi| = \begin{pmatrix} 2/3 & 0 \\ 0 & 1/3 \end{pmatrix}$$

"Quantum Fluctuations"

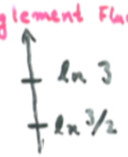
2nd Step: Define H^e by analogy

$H = -k_B T \ln \rho$ — "Thermal Fluctuations"

$H^e = -\ln \rho_A$ — "Entanglement Fluctu

$$\rho_A = \begin{pmatrix} 2/3 & 0 \\ 0 & 1/3 \end{pmatrix} = \begin{pmatrix} e^{-E_1} & 0 \\ 0 & e^{-E_2} \end{pmatrix}$$

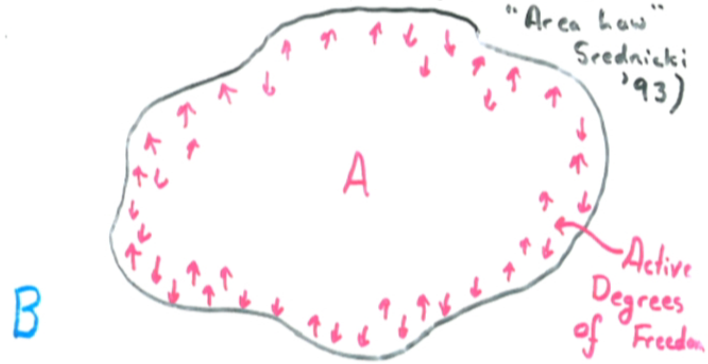
(I. Peschel; Li & Haldane)



Contrast the two Hamiltonians

in a gapped many-body system:

Entanglement Hamiltonian: Effectively d-1 dimensional (related to "Area Law" Srednicki '93)



Physical Hamiltonian: Extensive

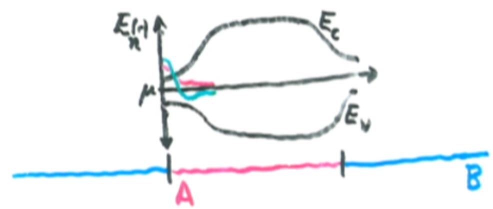
See also Osborne, Eisert, Verstraete PRL 105 (2010) on "Holographic Quantum States"

Guess for the Entanglement Hamiltonian

Frozen Inside:

$\mathcal{H}^{e}_{(r)} \approx D(r)\,\mathcal{H}(r)$

E.g: A band insulator



Can one actually calculate
the entanglement Hamiltonian?

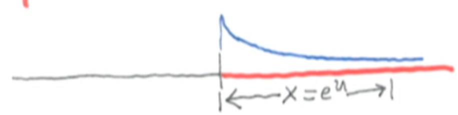
Finding Entanglement Hamiltonians

See Casini, Huerta & Myers
J. High Energy Phys. 2011

Assume Lorentz Invariant Theory

Can find $\mathcal{H}^e$ directly, without solving $\mathcal{H}$

Gapless case: $\mathcal{H}^e$ is 1-D



$$\mathcal{H} = \frac{1}{2}\int_{-\infty}^{\infty} (\Pi_\phi)^2 + \left(\frac{\partial\phi}{\partial x}\right)^2 dx$$

$$\mathcal{H}^{ent} = \pi \int_{-\infty}^{\infty} (\Pi_\phi)^2 + \left(\frac{\partial\phi}{\partial u}\right)^2 du$$

Can reconstruct physical
Hamiltonian from Ground State

Entanglement Hamiltonian ≈ Physical One
for CFTs

[Hinted from studies in Nienhuis, Campostrini & Calabrese J. Stat. Mech. 2009
Calabrese & Lefevre PR A 2008]

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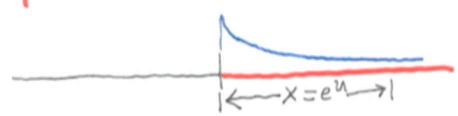

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Gapped Case: Zero Dimensional

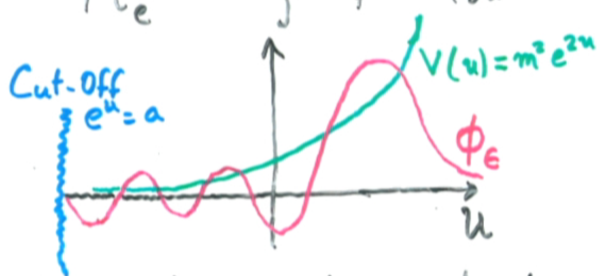
$$\mathcal{H} = \mathcal{H}_{gapless} + \frac{1}{2}\int m^2 \phi^2 \, dx$$

$Gap = m \Rightarrow$ Correlation length $= 1/m$

Convert from x to u

$$\xi_u = \frac{1/m}{dx/du} = e^{-u}/m$$

$$\mathcal{H}_e = \pi \int (\Pi_\phi)^2 + \left(\frac{\partial \phi}{\partial u}\right)^2 + m^2 e^{2u} \phi^2 \, du$$



Discrete spectrum due to confinement between $u \sim \ln 1/m$, $u \sim \ln a$.

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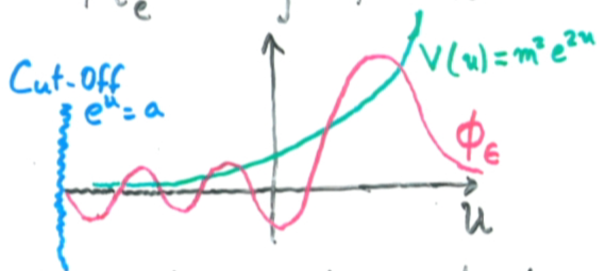

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Angular momentum fluctuations

Spin one atoms in
a nematic phase, e.g. Rb

Tune the correlation length
ξ.

What are the fluctuations
of spin in the segment
as a function of ξ, ℓ?

Is $S_{z,rms} \propto \sqrt{\ell}$?

[Use large N approximation
→ quadratic Hamiltonian]

(A project with Eugene Demler)

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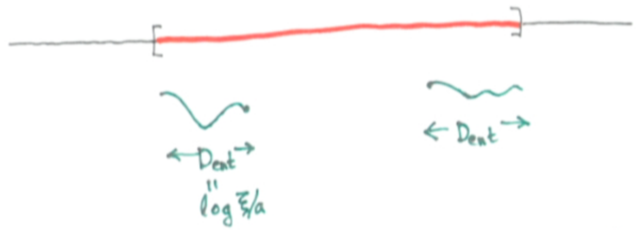
the Entanglement Hamiltonian



⇒ The fluctuations are independent of ℓ.

The entanglement modes are thermally excited ⇒ Planck distribution

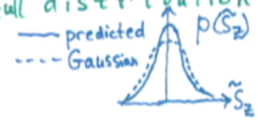
$$p(S_{z\nu}) \propto e^{-S_{z\nu}\omega_\nu}$$

in mode ν

$$\Rightarrow \langle S_z^2 \rangle \sim \frac{1}{\omega_{\nu=0}^2} \sim D_{ent}^2 \sim \left(\log \frac{\xi}{a}\right)^2$$

More precise: $S_{z,rms} = \frac{1}{\sqrt{2}\pi} \log \xi/a$

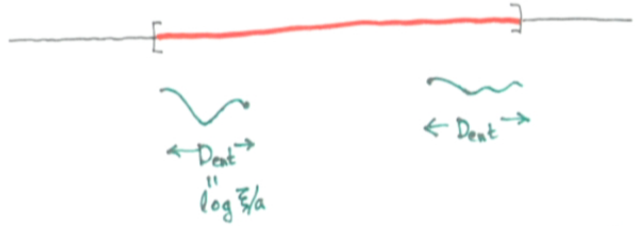
The full distribution is non-Gaussian



$$p(\tilde{S}_z) = \frac{\tilde{S}_z}{\sinh \tilde{S}_z / \pi\sqrt{2}}$$

What about $N<\infty$ where non-linearities are important?

the Entanglement Hamiltonian



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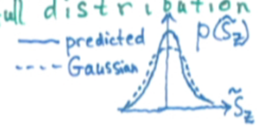

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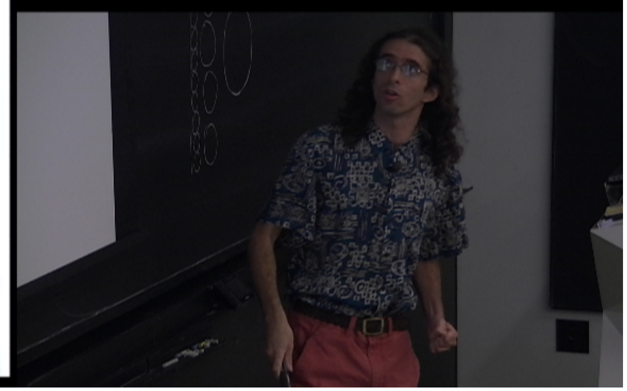


Summary of Entanglement Hamiltonians:

The entanglement Hamiltonian of a gapped d-dimensional system is d-1-dimensional

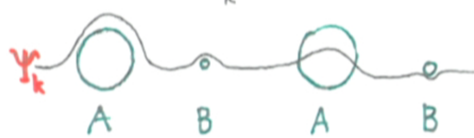
It can be calculated analytically for Lorentz-Invariant systems.

Topology and Conductivity:

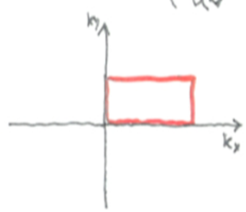
a 2D Example

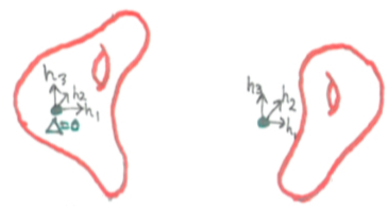
Bulk wave function

$$|G\rangle = \prod_{\vec{k}} c^{\dagger}_{\vec{k}} \, |\text{vacuum}\rangle$$



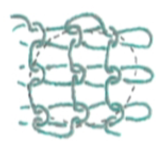
$$\epsilon(\vec{k})\begin{pmatrix} A(\vec{k}) \\ B(\vec{k}) \end{pmatrix} = \left(\epsilon_0 + \vec{h}(\vec{k}) \cdot \vec{\sigma}\right)\begin{pmatrix} A(\vec{k}) \\ B(\vec{k}) \end{pmatrix}$$





$\Downarrow$

$N_{Hall} = N_{Chern}$



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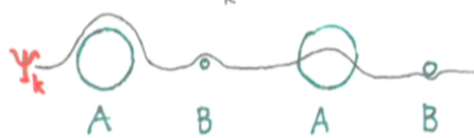

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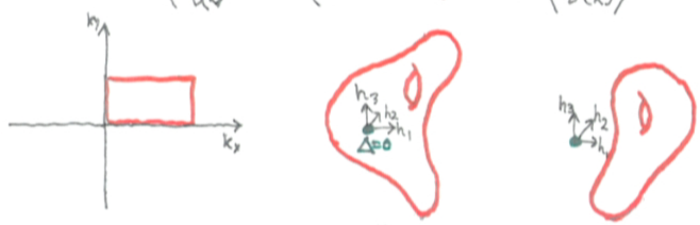

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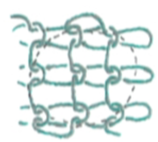

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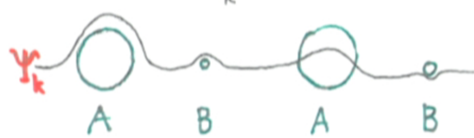

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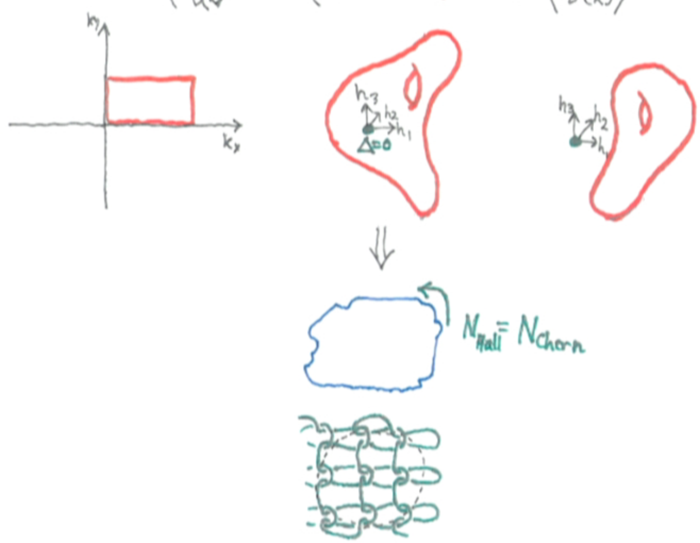
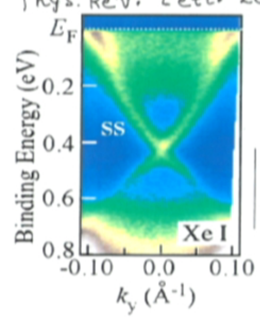


Topological Insulators
for Various Symmetries

	1D	2D	3D
Charge Conservation	–		–
Charge Conservation + Time Reversal	–	$\mathbb{Z}_2$	$\mathbb{Z}_2$ *
Superconductor + T ($T^2=1$)	$\mathbb{Z}$ Majorana	–	–

* Exists: For example, $TlBiSe_2$ Sato et al. Phys. Rev. Lett. 2010

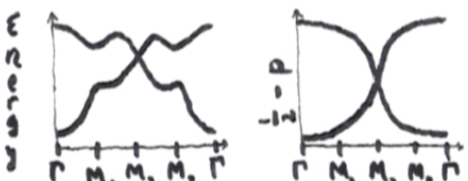


What are the effects of interactions?

First step:

Find a signature of topological phase in the wave function.

Reflection of Surface States in the Entanglement Spectrum?



Turner, Zhang & Vishwanath Phys. Rev. B 2010

Fidkowski Phys. Rev. Lett. 2010

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Try entanglement out in 1-D

A first test:

Spin-One Chain

$S = \frac{1}{2}$

Hagiwara et al.
Phys. Rev. Lett. 199

Classification of 1-D Phases

List symmetries $R_1, R_2, \ldots$

Landau classification:

Enumerate broken symmetries.

Refinement for topological phases:

Find how symmetries act on entanglement spectrum.

Characterizes phase

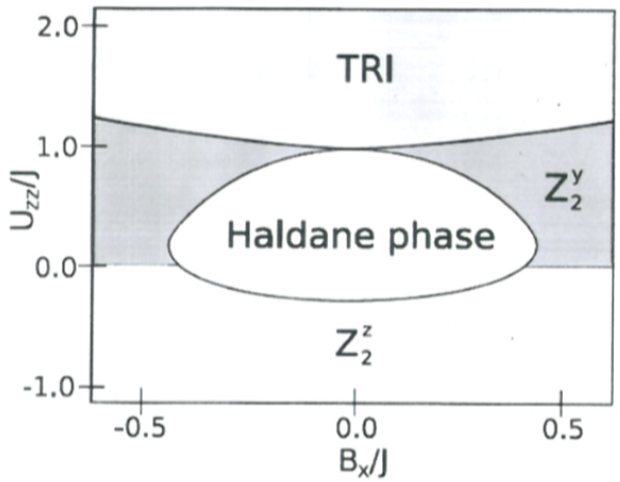
Extra phases $\boxed{\phi}$ Appear (w/ discrete Values)

Pollmann, Turner, Berg, and Oshikawa: Phys Rev. B (2010)
Chen, Gu and Wen: Phys. Rev. B (2011)
Schuch, Perez-Garcia & Cirac: Phys. Rev. B (2011)

This classification is complete.

Example: Spin One chain

$$\sum J S_i \cdot S_{i+1} + U_{zz} S_{iz}^2 + B_x S_{ix}$$



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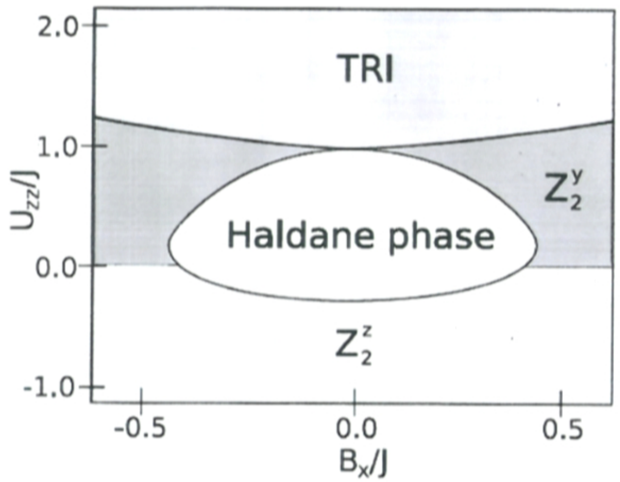

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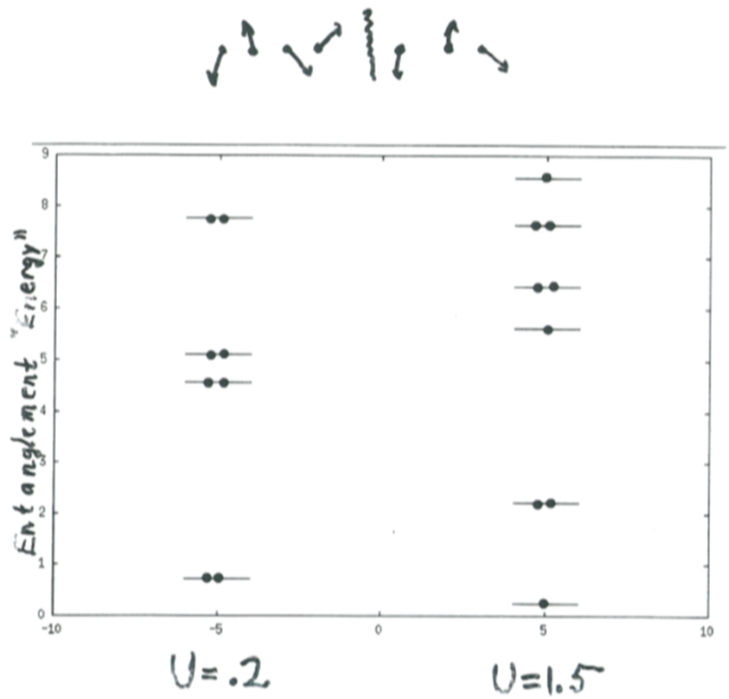
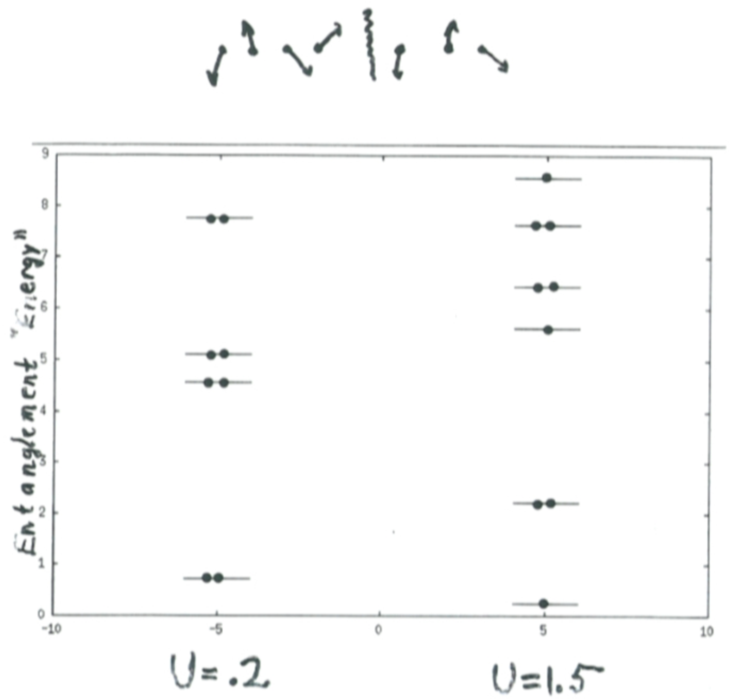


Entanglement "Energy"
U=.2
U=1.5



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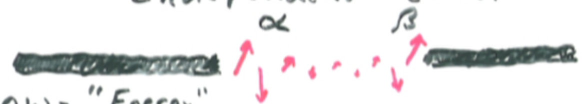
Action of Symmetry on Entanglement Spectrum

Entanglement Spec. of Finite Segment:
Independent ends.



Low-"Energy" States can be transformed into each other using operators on the ends

$$\Rightarrow \quad R_i |\alpha\rangle|\beta\rangle = V_i|\alpha\rangle\, U_i|\beta\rangle$$

Ambiguous Phase Factor!

$$R_i R_j = R_k \Rightarrow U_i U_j = e^{i\rho_{ij}} U_k$$

Schur Cycle

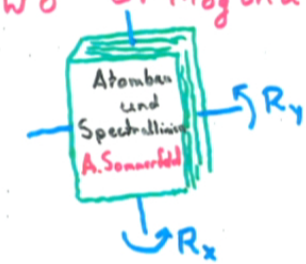
Examples of Symmetries:

Single axis: useless

$$R^n = \mathbb{1} \Rightarrow U^n = e^{i\beta}\mathbb{1}$$

Redefine $\tilde{U} = e^{-i\beta/n}U$

Two Orthogonal Axes:



$$R_x R_y = R_y R_x$$

but

$$U_x U_y = \underline{\underline{e^{i\phi}}}\, U_y U_x$$

$$\phi_{xy} = 0, \pi$$

Time Reversal Symmetry

$$\phi_T = 0, \pi$$

Inversion Symmetry

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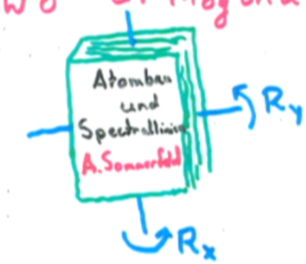

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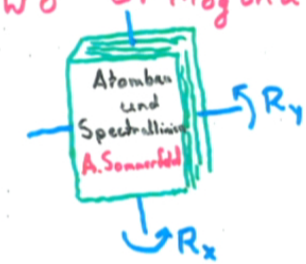

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$$U_x U_y = \underline{\underline{e^{i\phi}}} U_y U_x$$

$$\phi_{xy} = 0, \pi$$

Time Reversal Symmetry

$$\phi_T = 0, \pi$$

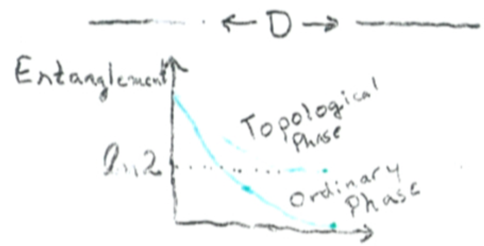
Inversion Symmetry

$$\phi_I = 0, \pi$$

Beyond Edge States:

Systems with mirror Symmetry:

Intuition about this characterization:



Correlations remain in separated chain

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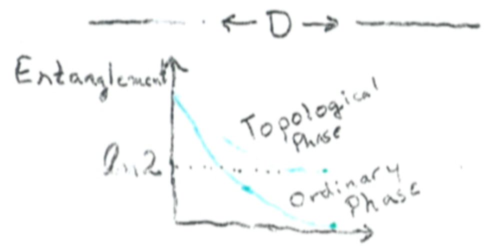

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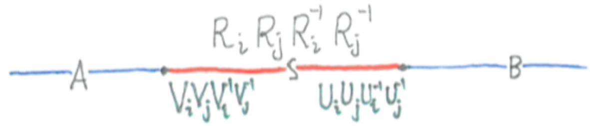
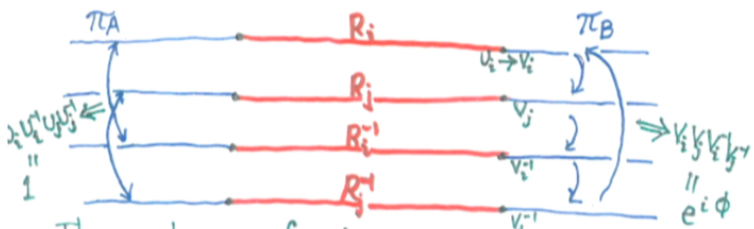
Measuring the Phase



Phase factors Cancel

A "string order" with multiple chains:

Measure $\langle \pi_A R_S \pi_B \rangle_{4\ \text{chains}} / \langle \pi_A \pi_B \rangle_{4\ \text{chains}}$

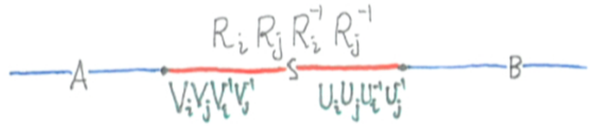


The phase factor can be determined by measuring correlations.

Jutho Haegeman et. al. Phys. Rev. Lett 2012

Frank Pollmann & Ari Turner arXiv:1204.0704

Measuring the Phase



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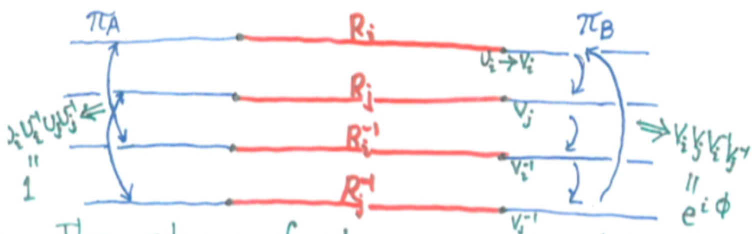

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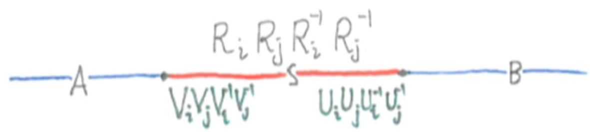

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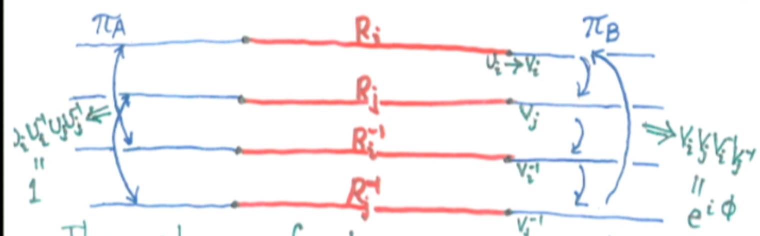

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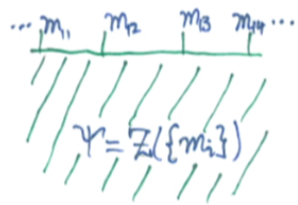
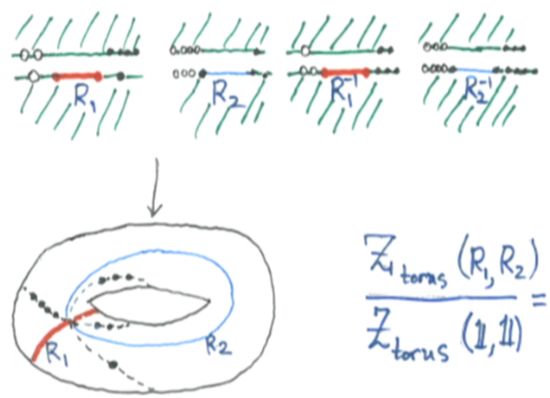
Topology & "Topological" Phases

The string order can be represented as a 2D sum via path integrals

1 copy



8 copies



$$\frac{Z_{torus}(R_1, R_2)}{Z_{torus}(\mathbb{1},\mathbb{1})} = e^{i\phi}$$

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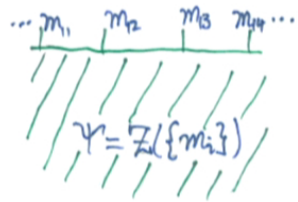

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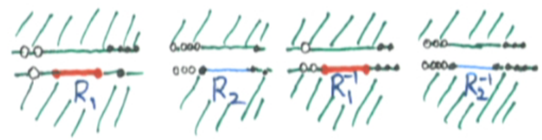

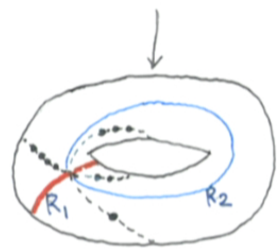

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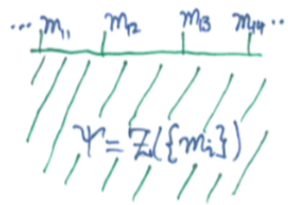

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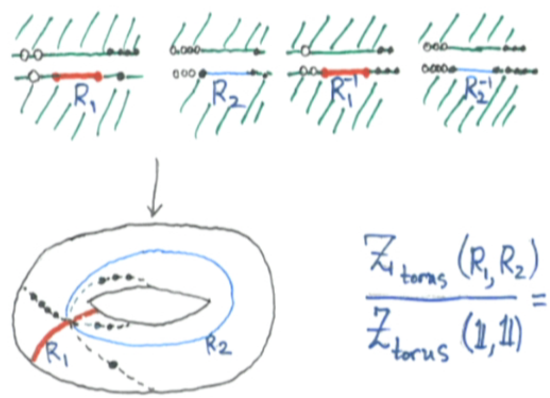

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Topological

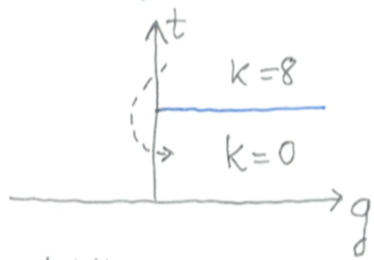
Insulators

Topological Insulators in 1D

If the particles are independent:

	1	2	3
BDI (Superconductors with T-symmetry $T^2=1$)	$\mathbb{Z}$	0	0

Infinitely many phases
↓
8



Fidkowski & Kitaev Phys Rev. B 2010

Topological Insulators in 1D

Topological insulators have fermions

$Q = (-1)^{N_F}$ is conserved automatically

Separated Operators can anticommute

Q alone $\Rightarrow$ 2 Phases

$$Q_A Q_B = \pm Q_B Q_A$$

Combining T and Q

gives 8

$$T_A Q_A = \pm Q_A T_A$$

$$T_A^2 = \pm \mathbb{1}$$

Turner, Berg & Pollmann Phys Rev.B 2011

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Topological Insulators in 1D

Summary 1D Phases

Topological insulators have fermions

1D Phases are distinguished by Schur phase factors

$Q = (-1)^{N_F}$ is conserved automatically

Degeneracies in the entanglement spectrum signal nontrivial phases.

Separated Operators can anticommute

Q alone $\Rightarrow$ 2 Phases

One-D superconductors with T-symmetry are classified by $\mathbb{Z}_8$ when interactions are included.

$Q_A \quad Q_B$

$Q_A Q_B = \pm Q_B Q_A$

Combining T and Q gives 8

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$T_A^2 = \pm 1$

What else can you use entanglement for?

Turner, Berg & Pollmann Phys Rev. B 2011

Fidkowski & Kitaev Phys Rev. B 2011

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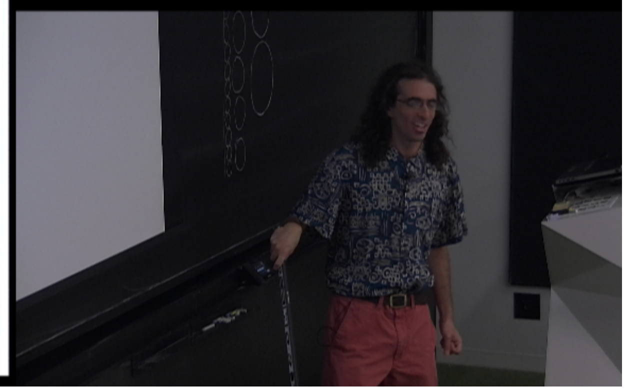
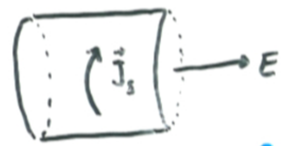


Does

Physical Response

= Entanglement Response

?



Magnetic/Electric Coupling θ:

a likely candidate.

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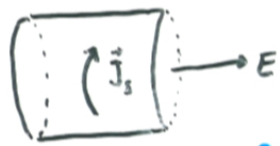

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Understand transitions between topological phases.

What are the low-energy modes?

Use Entanglement Spectrum as an order parameter.

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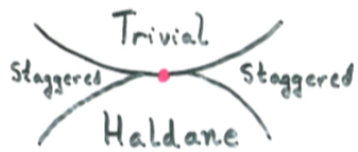
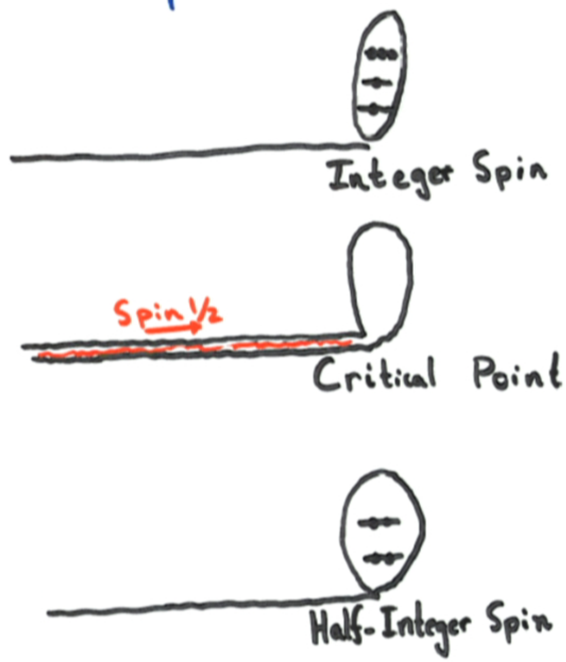


Use Entanglement Spectrum as an order parameter.

What happens at a phase transition?



Some other topics

Cold atoms: Geometry of Spinor Atoms & Vortices

Barnett, Turner, Demler
Phys. Rev. L. 2006

Turner, Vitelli & Nelson
Rev. Mod. Phys. 2010

Surface Properties of Materials

Weyl semimetals
Phys. Rev. B 83 (2011)

Magneto-Electric Coupling
Essin, Turner, Moore & Vanderbilt
PRB 81 (2010)

Fluctuations beyond the zero-point

Defects as a statistical measure
(for cosmic radiation?)
Beuman, Turner & Vitelli
arXiv /1207.3892

Jamming

?

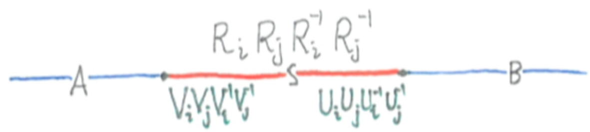
Summary

The Entanglement Hamiltonian of a gapped d-dimensional system is d-1-dimensional and can be determined analytically sometimes.

The entanglement Hamiltonian distinguishes among topological phases, perfectly in 1D.

Can it be used to understand more about topological insulators in higher dimensions and for other applications?

Measuring the Phase



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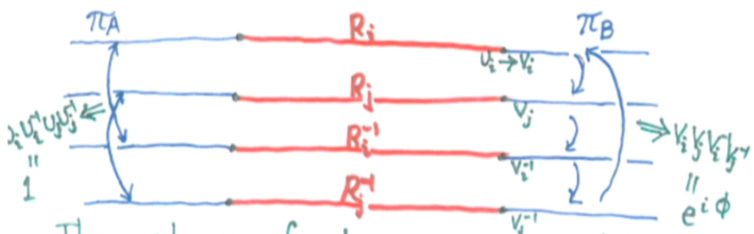

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