

Title: Ising Model

Date: Sep 04, 2012 02:00 PM

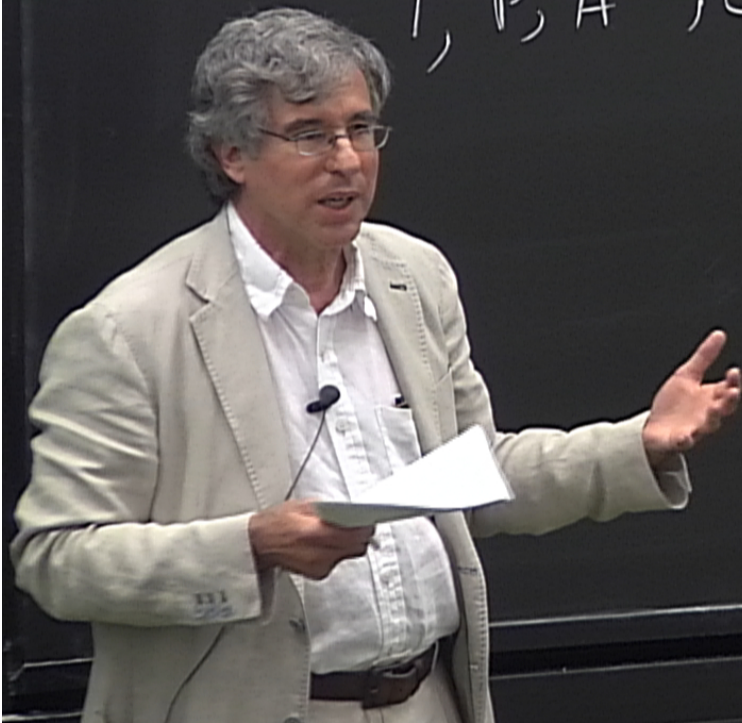
URL: <http://pirsa.org/12090054>

Abstract:

2. Thermodynamic Variable and Potential Functions

$$T, P, H; E, V, M$$

$$\rho = \frac{N}{V}$$



John Cardy

Things you should know.

$P_i =$

1. Foundations of Stat. Mech
e.g. Probability Distribution

a) E.G. For an isolated system with energy E , all states with that energy are equally probable.

b) For a system in e.g. with a thermal bath at temperature $T = \frac{1}{\beta}$ the prob that it is in state i with energy E_i is

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$$P_i = \frac{e^{-\beta E_i}}{Z} \quad \text{where } Z = \sum_{\substack{i=\text{all} \\ \text{state}}} e^{-\beta E_i}$$

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Free Energy (Helmholtz)

$$F(T, V, N) = -T \ln Z$$

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First derivs $\rightarrow$ average props
Second derivs $\rightarrow$ susceptibilities

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of Stat. Mech

Probability Distribut

For an isolated system with
energy E , all states with energy

are equally probable

System in contact with

temperature $T =$ in state

with energy

$$P_i = \frac{e^{-\beta E_i}}{Z} \quad \text{where } Z = \sum_{i=\text{all state}} e^{-\beta E_i}$$

Refs:

you should know.

Foundations of Stat. Mech
Probability Distribution

For a system with energy E , the probability of finding it in a state with that energy is proportional to $e^{-\beta E}$.
If the system is in contact with a thermal bath at temperature T , the probability that it is in state i is

$$P_i = \frac{e^{-\beta E_i}}{Z} \quad \text{where } Z = \sum_{i: \text{all state}} e^{-\beta E_i}$$

Refs: "States of Matter",
by David Goodstein

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Landau + Lifshitz Stat Phys.

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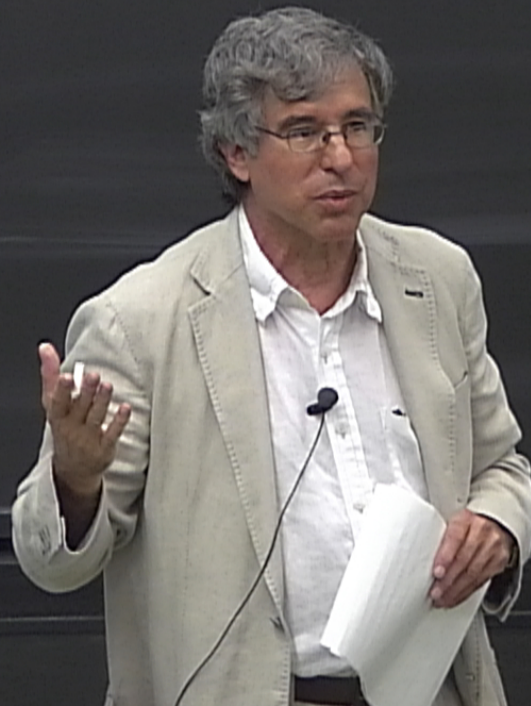
Berlinsky + A.B. Harris
on Wiki Chaps 1-6

The Ising Model

1

The Ising Model

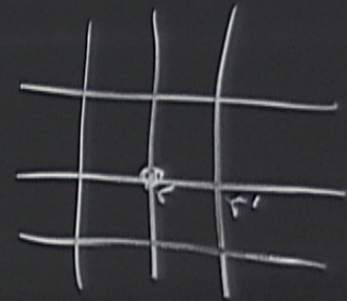
Interacting Spins on a lattice $S_r = \pm 1$



The Ising Model

Interacting Spins on a lattice

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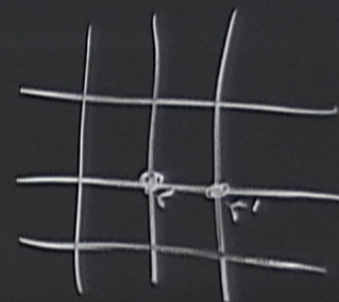


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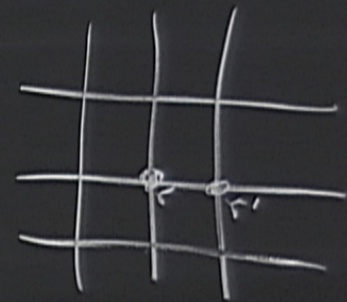
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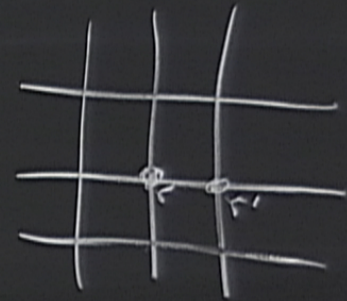
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State $\{S_r; r=1, \dots, N\}$

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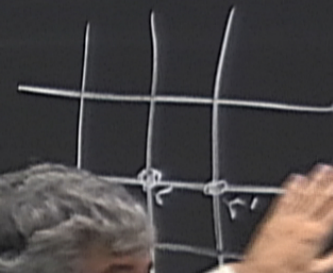
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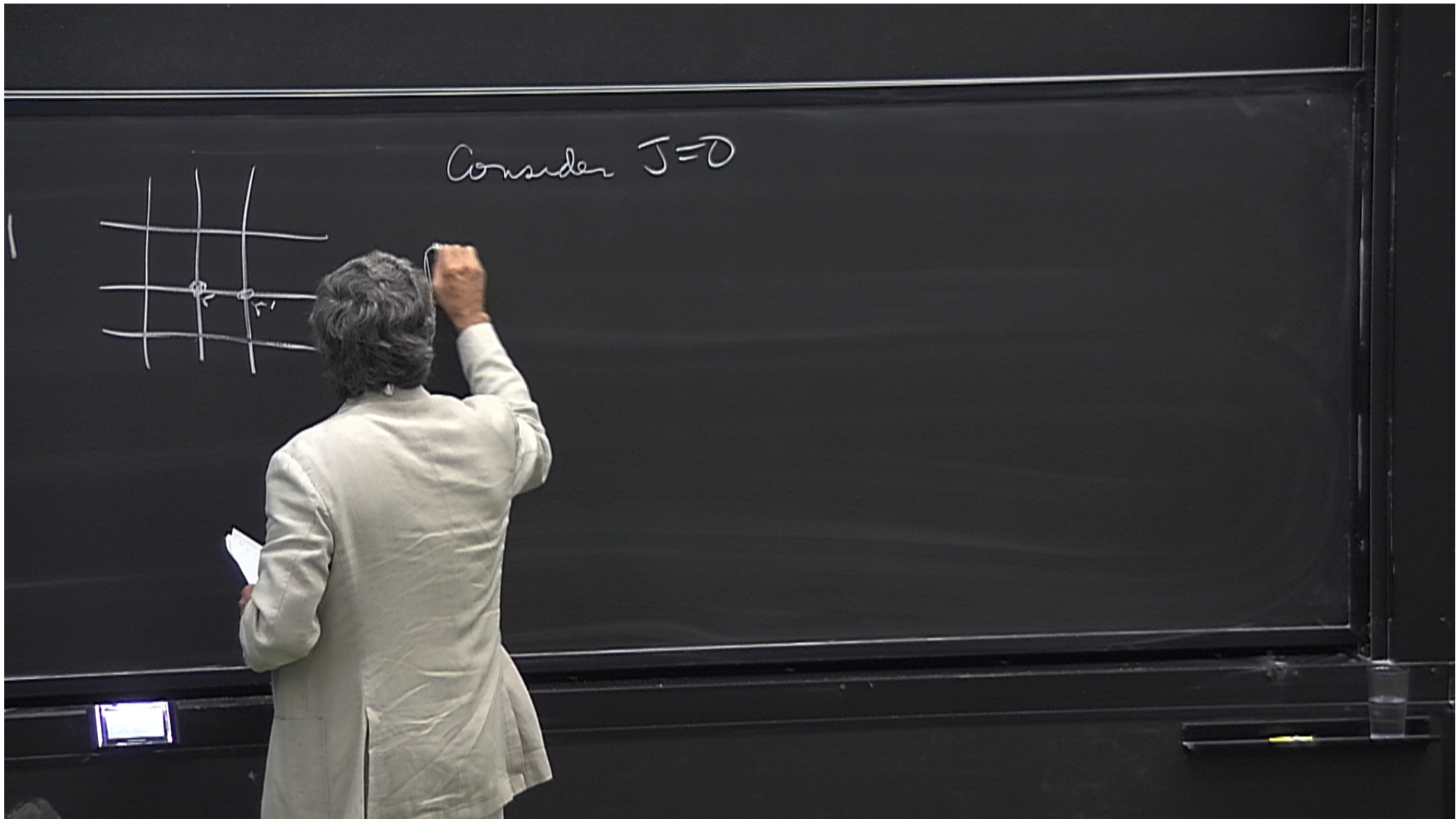
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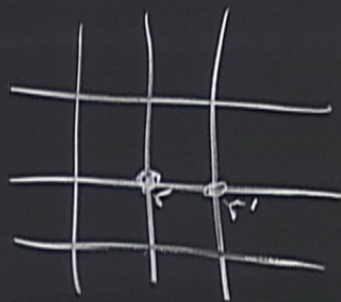
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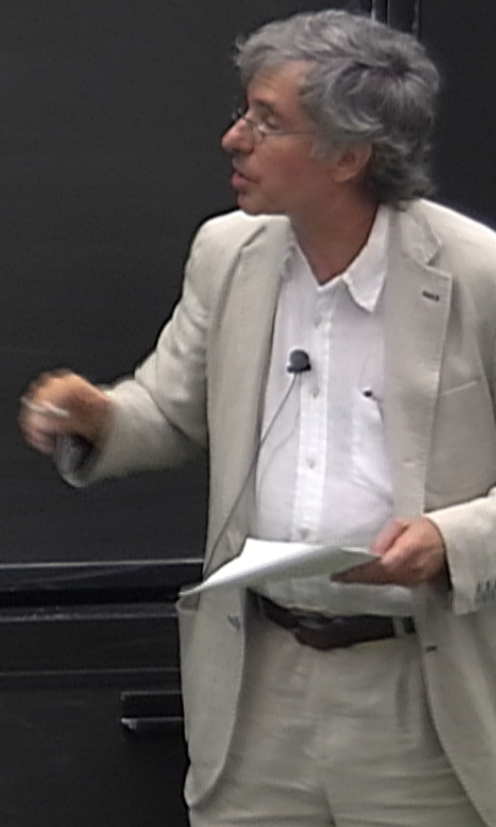


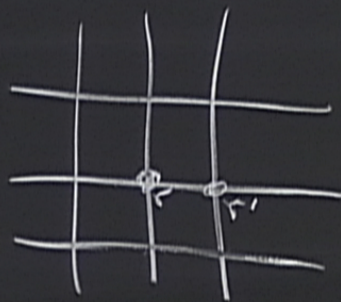




Consider $J=0$

$$P(\{s_r\}) = \frac{e^{\beta h \sum_r s_r}}{Z}$$

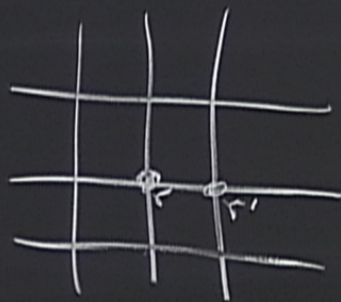




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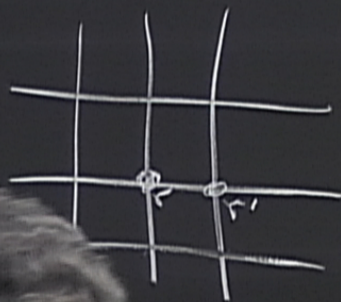
$$Z = \sum_{\{s_r = \pm 1\}} e^{\beta h \sum_r s_r} = \prod_r \sum_{s_r = \pm 1} e^{\beta h s_r}$$



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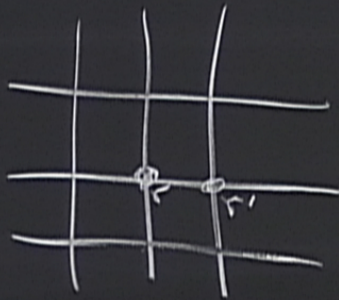


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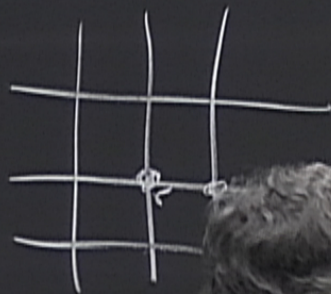


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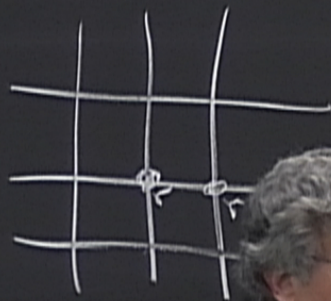


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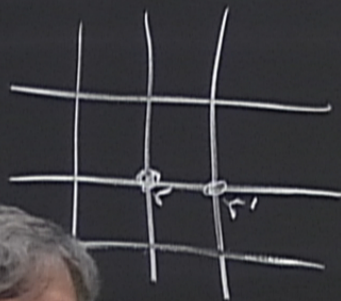
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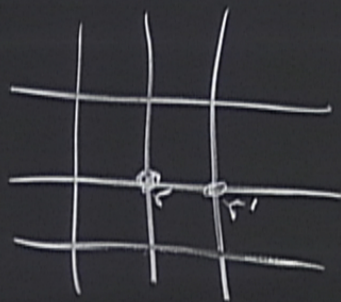
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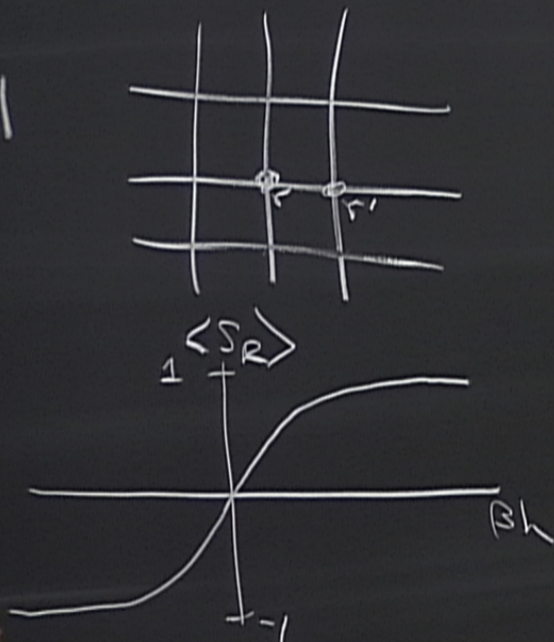
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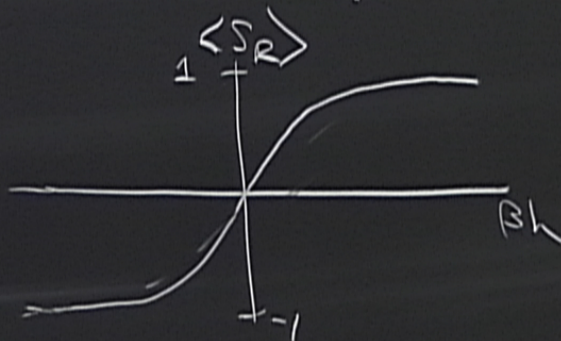
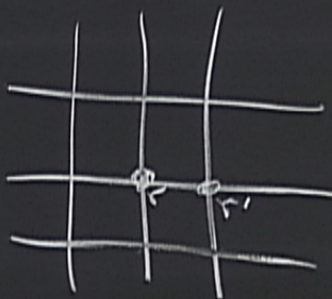
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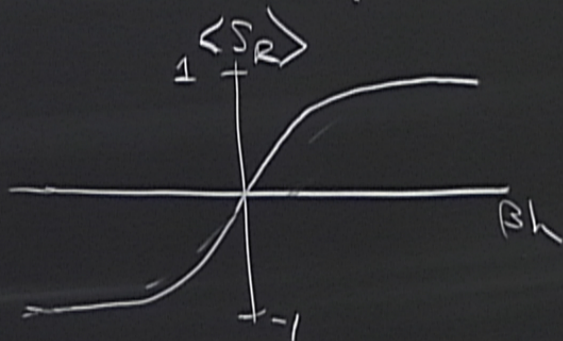
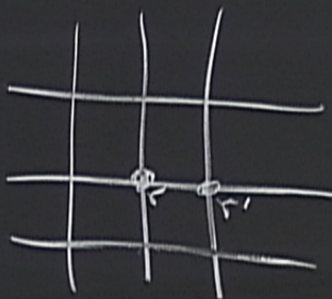
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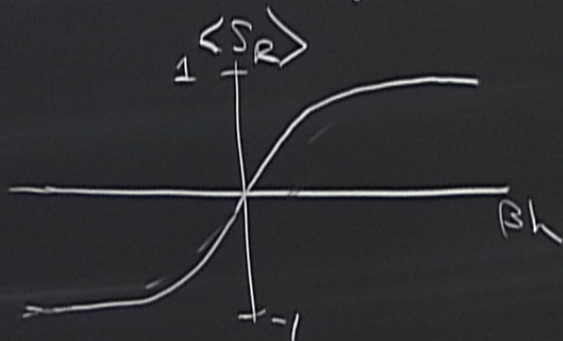
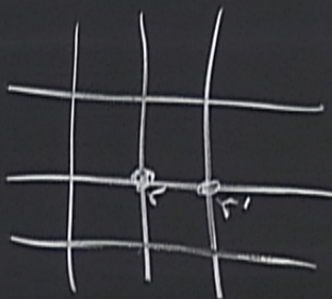
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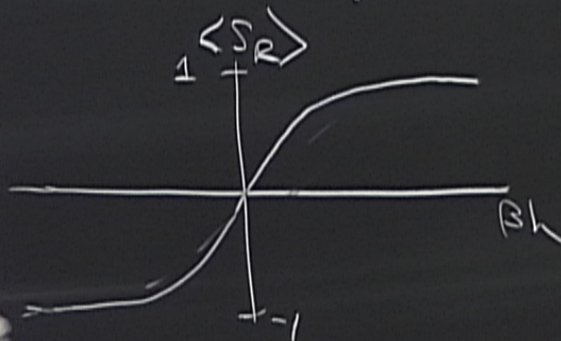
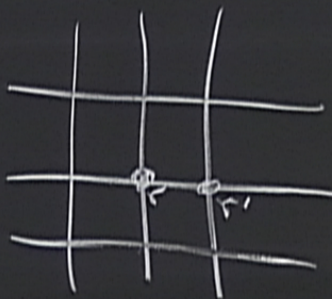
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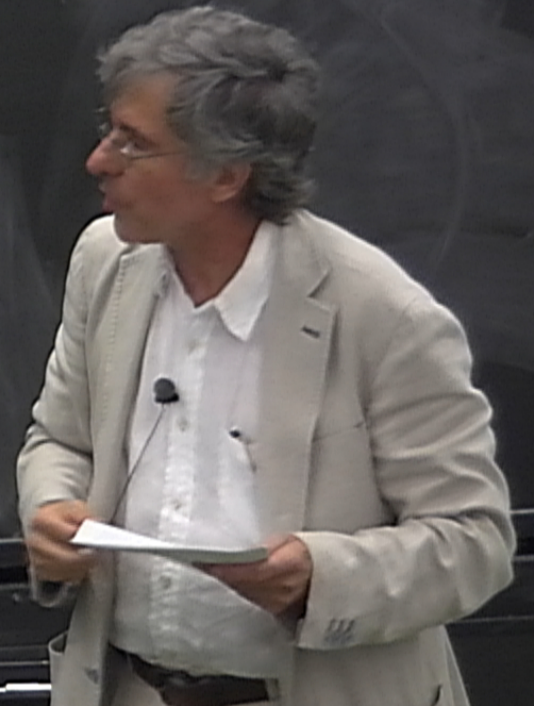
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- Mean Field Theory



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$$h_{\text{eff}} = h + \frac{z}{2} J \langle S \rangle$$

of nearest neighbours
↓

- Mean Field Theory

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$$\langle S \rangle = \tanh \beta h_{\text{eff}} = \tanh \beta (h + z J \langle S \rangle)$$

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"Self-consistent" equation for $\langle S \rangle$

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Mean Field Theory

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How to solve:

1.) $h = 0$

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$$\text{For } T=0 \langle S \rangle = \pm 1$$

For $T=0$ $\langle S \rangle = \pm 1$

If $\langle S \rangle$ is small

$$\langle S \rangle = \beta z J \left(\langle S \rangle - \frac{1}{3} (\beta z J)^2 \langle S \rangle^3 + \dots \right)$$

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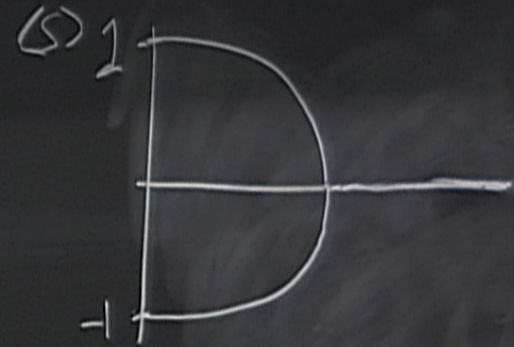
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$$T_c = zJ$$



$$\text{For } h=0 \quad \langle S \rangle = \pm 1$$

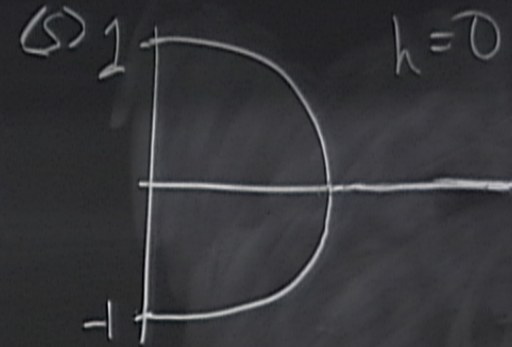
If $\langle S \rangle$ is small

$$\langle S \rangle = \beta z J \left(\langle S \rangle - \frac{1}{3} (\beta z J)^2 \langle S \rangle^3 + \dots \right)$$

$$\frac{T}{zJ} = 1 - \frac{1}{3} \left(\frac{zJ}{T} \right)^2 \langle S \rangle^2$$

$$\langle S \rangle = \pm \sqrt{3} \left(\frac{T}{zJ} \right)^{1/2} \sqrt{1 - \frac{T}{zJ}} \quad \text{for } T < zJ$$

$$T_c = zJ$$



$$\text{For } h=0 \quad \langle S \rangle = \pm 1$$

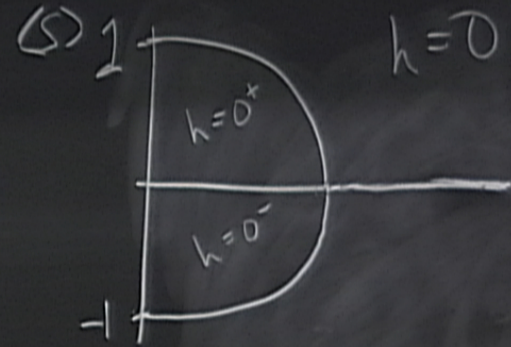
If $\langle S \rangle$ is small

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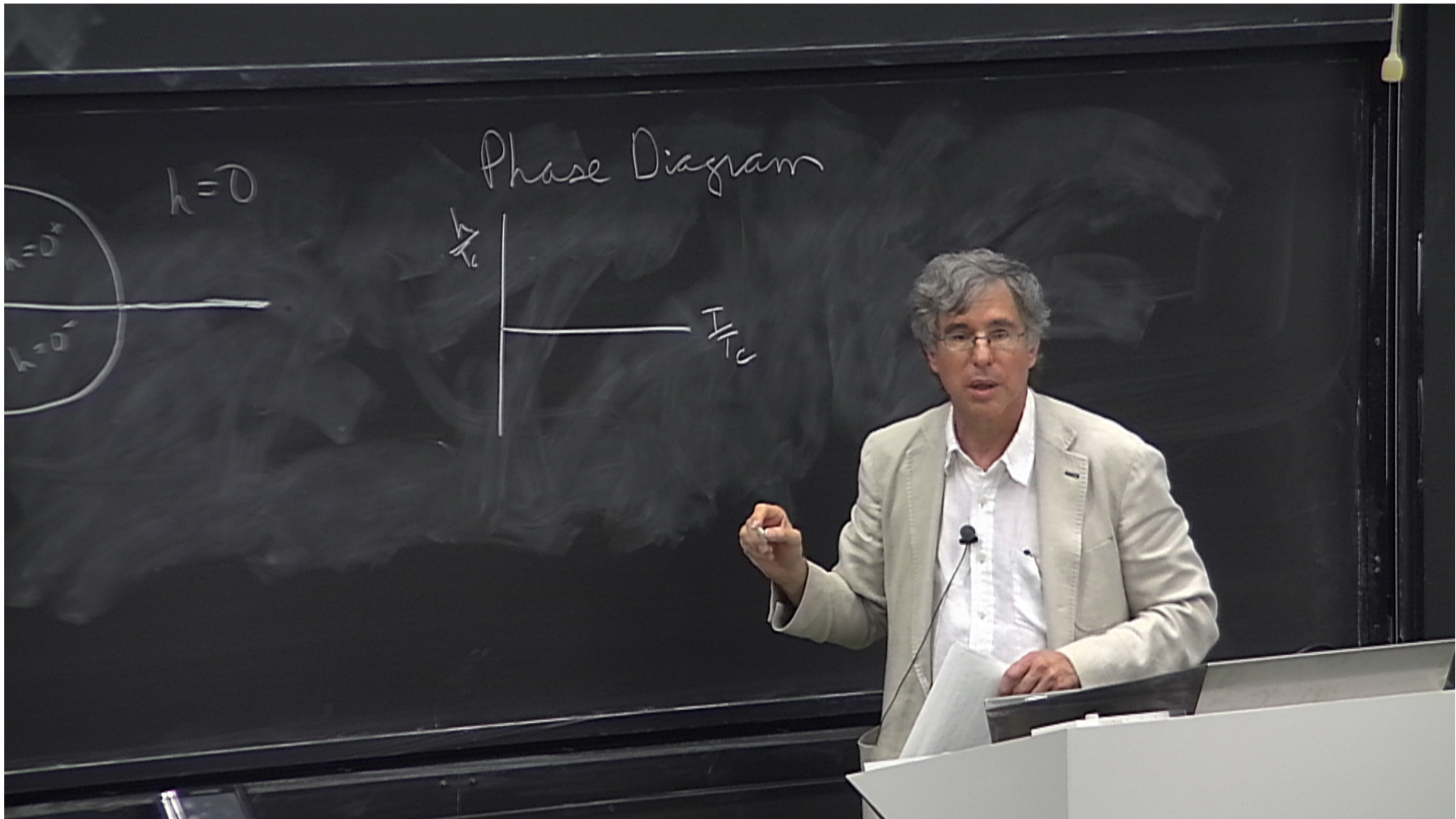
$$\frac{T}{zJ} = 1 - \frac{1}{3} \left(\frac{zJ}{T} \right)^2 \langle S \rangle^2$$

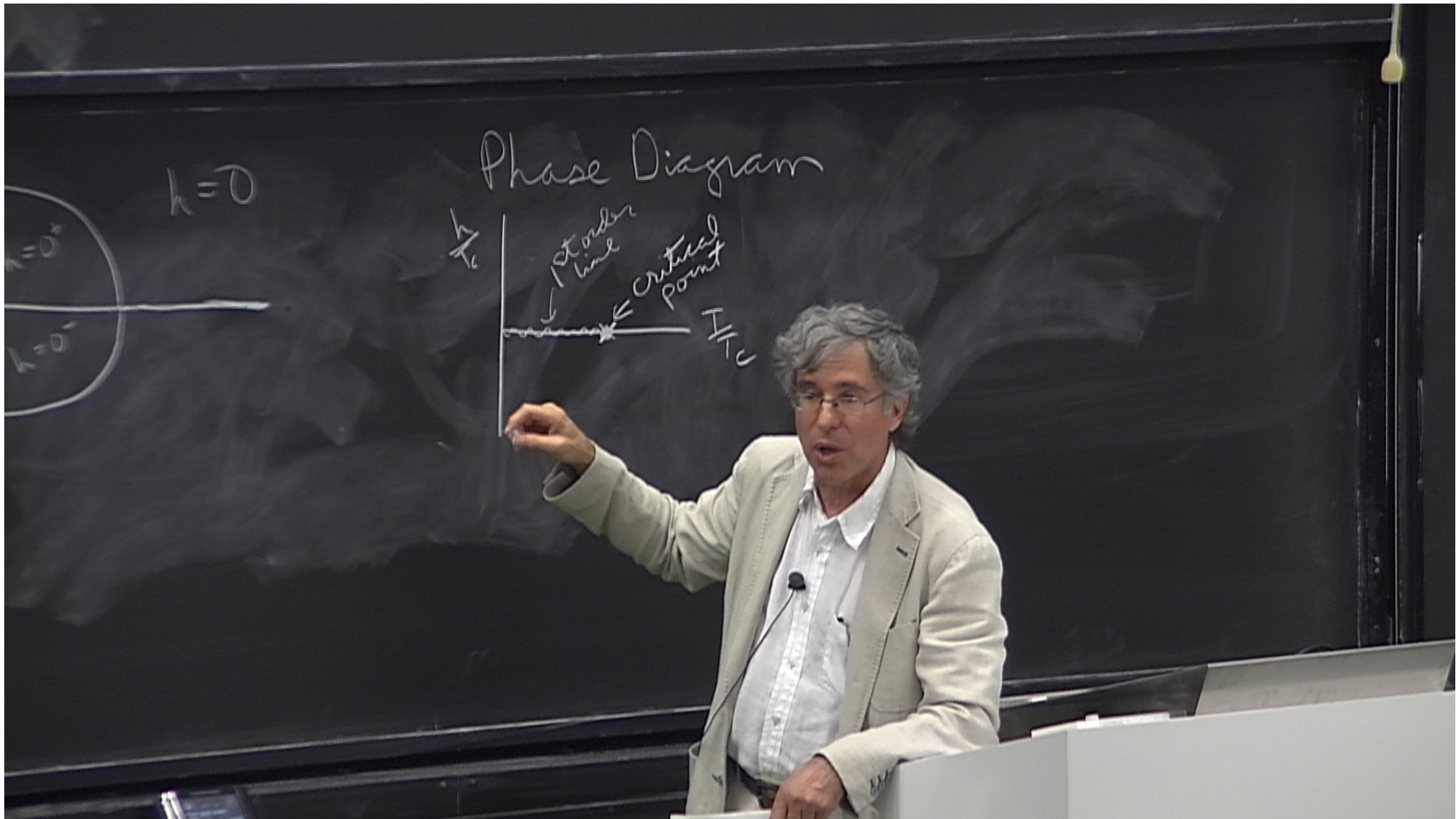
$$\langle S \rangle = \pm \sqrt{3} \left(\frac{T}{zJ} \right)^2 \sqrt{1 - \frac{T}{zJ}} \quad \text{for } T < zJ$$

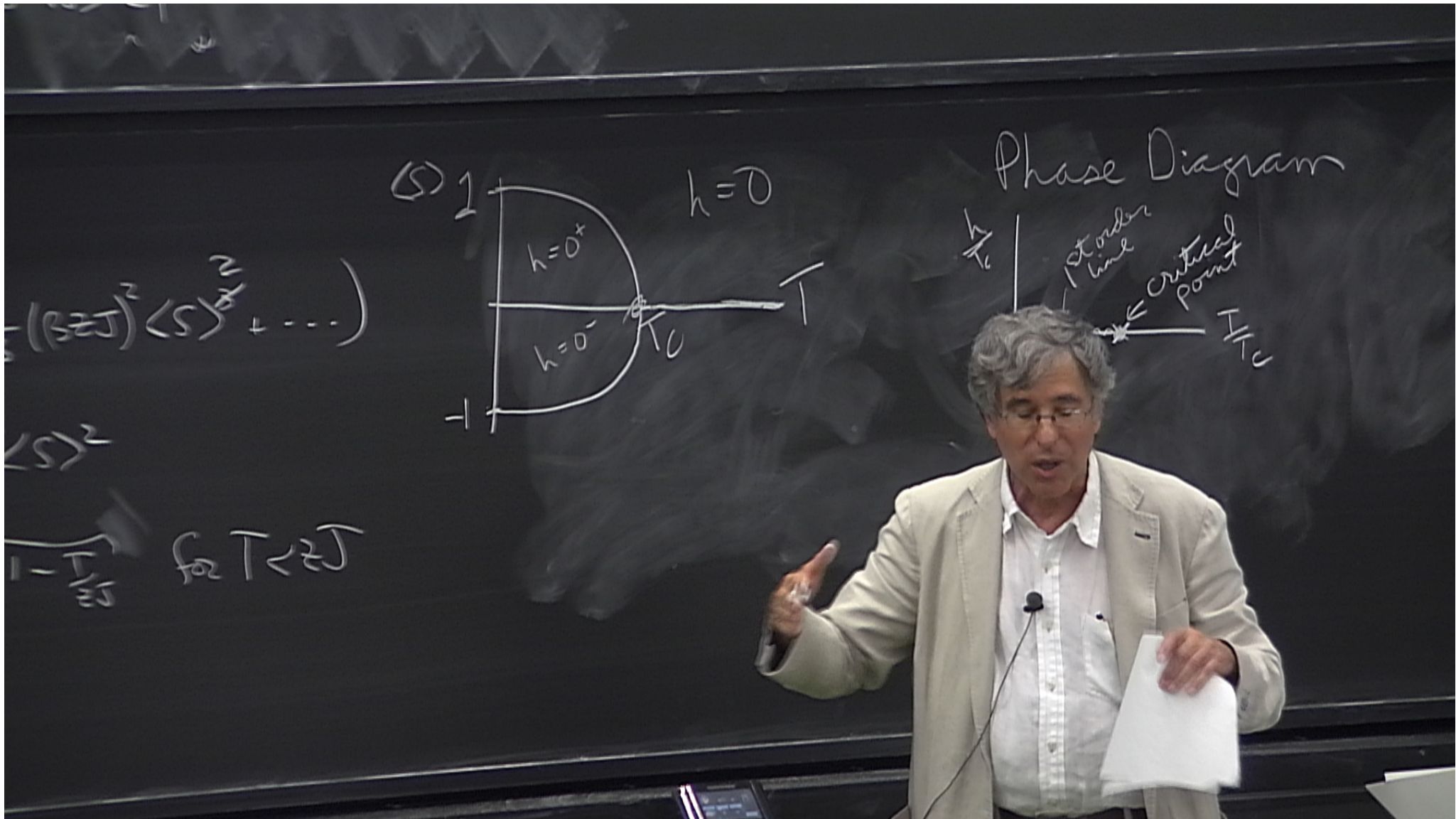
$$T_c = zJ$$

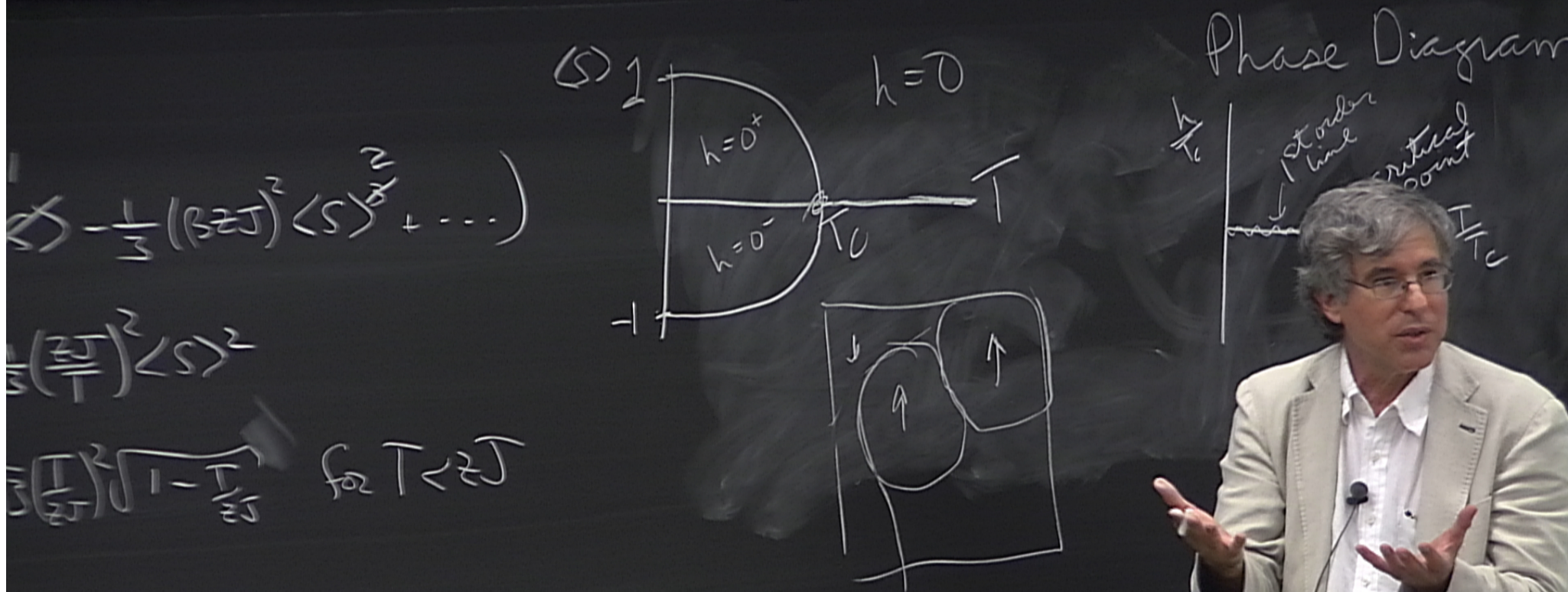


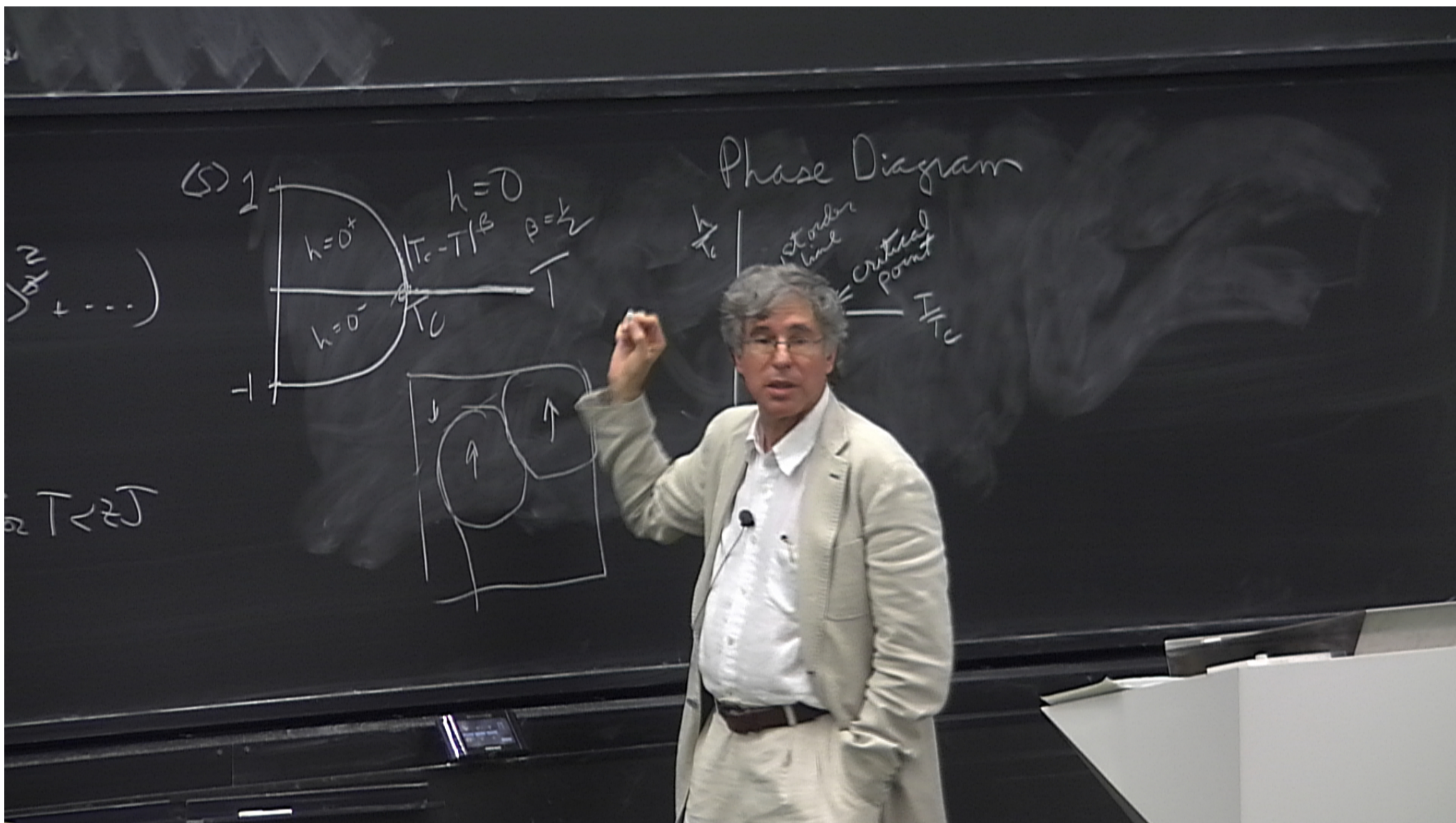








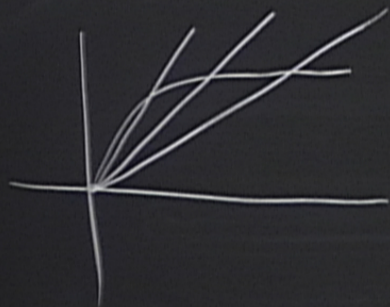




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