

Title: The Exploration of Hot QCD Matter

Date: Sep 11, 2012 02:00 PM

URL: <http://pirsa.org/12090046>

Abstract: When nuclear matter is heated beyond a temperature of 2 trillion
degrees, it converts into a strongly coupled plasma of quarks and
gluons, the sQGP. Experiments using highly energetic collisions
between heavy nuclei have revealed that this new state of matter is a
nearly ideal, highly opaque liquid. A description based upon string
theory and black holes in five dimensions has made the quark- gluon
plasma an iconic example of a strongly coupled quantum system. In this
lecture I will survey the observed properties of the sQGP in the light
of the latest results from RHIC and LHC. On the theoretical side, I
will discuss the thermalization and entropy production problem and
origin and role of event-by-event fluctuations.



The Exploration of Hot QCD Matter

Flow, Fluctuations, Thermalization

Some possibly useful reviews:

General QGP (with B. Jacak):

Science 337, 310 (2012)

LHC (with J. Schukraft & B.

Wyslouch):

arXiv:1202.3233 (ARNPS)

Entropy (with A. Schäfer):

arXiv:1110.2378 (IJMPE 20, 2235)

Berndt Müller

Perimeter Institute

11 September 2012

Accelerators

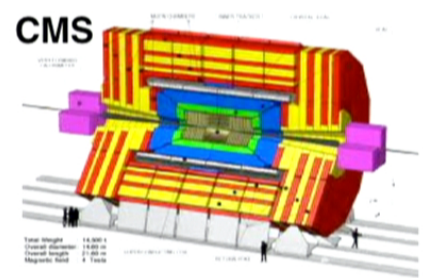
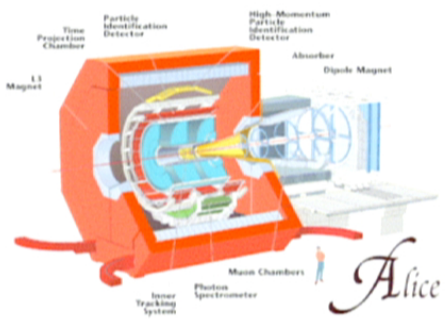
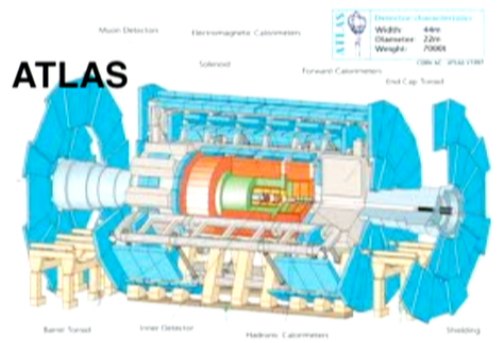
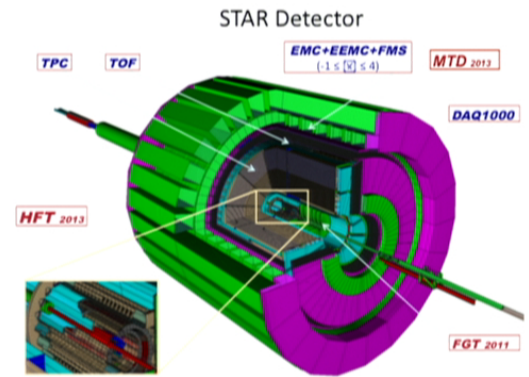
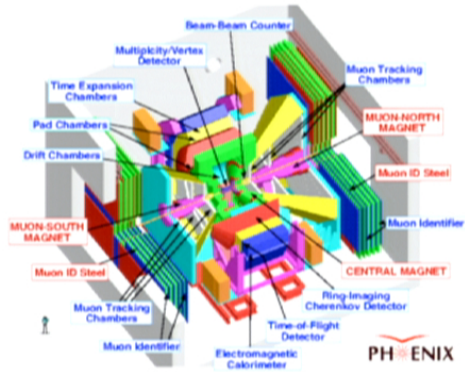


The Large Hadron Collider
27 km circumference
Energy: $E_{cm} = 2.76 \text{ TeV/NN}$



The Relativistic Heavy Ion Collider
3.8 km circumference
Top energy: $E_{cm} = 200 \text{ GeV/NN}$

Detectors



In the News

RHIC Scientists Serve Up "Perfect" Liquid

New state of matter more remarkable than predicted -- raising many new questions

April 18, 2005

TAMPA, FL -- The four detector groups conducting research at the [Relativistic Heavy Ion Collider](#) (RHIC) -- a giant atom "smasher" located at the U.S. Department of Energy's Brookhaven National Laboratory -- say they've created a new state of hot, dense matter out of the quarks and gluons that are the basic particles of atomic nuclei, but it is a state quite different and even more remarkable than had been predicted. In [peer-reviewed papers](#) summarizing the first three years of RHIC findings, the scientists say that instead of behaving like a gas of free quarks and gluons, as was expected, the matter created in RHIC's heavy ion collisions appears to be more like a *liquid*.

'Perfect' Liquid Hot Enough to be Quark Soup

Protons, neutrons melt to produce 'quark-gluon plasma' at RHIC

Monday, February 15, 2010

UPTON, NY — Recent analyses from the [Relativistic Heavy Ion Collider](#) (RHIC), a 2.4-mile-circumference "atom smasher" at the U.S. Department of Energy's (DOE) Brookhaven National Laboratory, establish that collisions of gold ions traveling at nearly the speed of light have created matter at a temperature of about 4 trillion degrees Celsius — the hottest temperature ever reached in a laboratory, about 250,000 times hotter than the center of the Sun. This temperature, based upon measurements by the PHENIX collaboration at RHIC, is higher than the temperature needed to melt protons and neutrons into a plasma of quarks and gluons. Details of the findings will be published in *Physical Review Letters*.

In the News

RHIC Scientists Serve Up "Perfect" Liquid

New state of matter more remarkable than predicted -- raising many new questions

April 18, 2005

TAMPA, FL -- The four detector groups conducting research at the [Relativistic Heavy Ion Collider](#) (RHIC) -- a giant atom "smasher" located at the U.S. Department of Energy's Brookhaven National Laboratory -- say they've created a new state of hot, dense matter out of the quarks and gluons that are the basic particles of atomic nuclei, but it is a state quite different and even more remarkable than had been predicted. In [peer-reviewed papers](#) summarizing the first three years of RHIC findings, the scientists say that instead of behaving like a gas of free quarks and gluons, as was expected, the matter created in RHIC's heavy ion collisions appears to be more like a *liquid*.

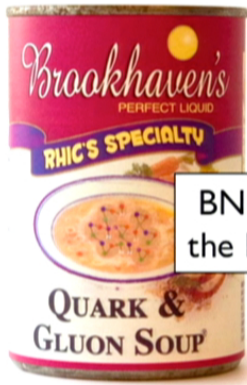
'Perfect' Liquid Hot Enough to be Quark Soup

Protons, neutrons melt to produce 'quark-gluon plasma' at RHIC

Monday, February 15, 2010

UPTON, NY — Recent analyses from the [Relativistic Heavy Ion Collider](#) (RHIC), a 2.4-mile-circumference "atom smasher" at the U.S. Department of Energy's (DOE) Brookhaven National Laboratory, establish that collisions of gold ions traveling at nearly the speed of light have created matter at a temperature of about 4 trillion degrees Celsius — the hottest temperature ever reached in a laboratory, about 250,000 times hotter than the center of the Sun. This temperature, based upon measurements by the PHENIX collaboration at RHIC, is higher than the temperature needed to melt protons and neutrons into a plasma of quarks and gluons. Details of the findings will be published in *Physical Review Letters*.

The Big Questions



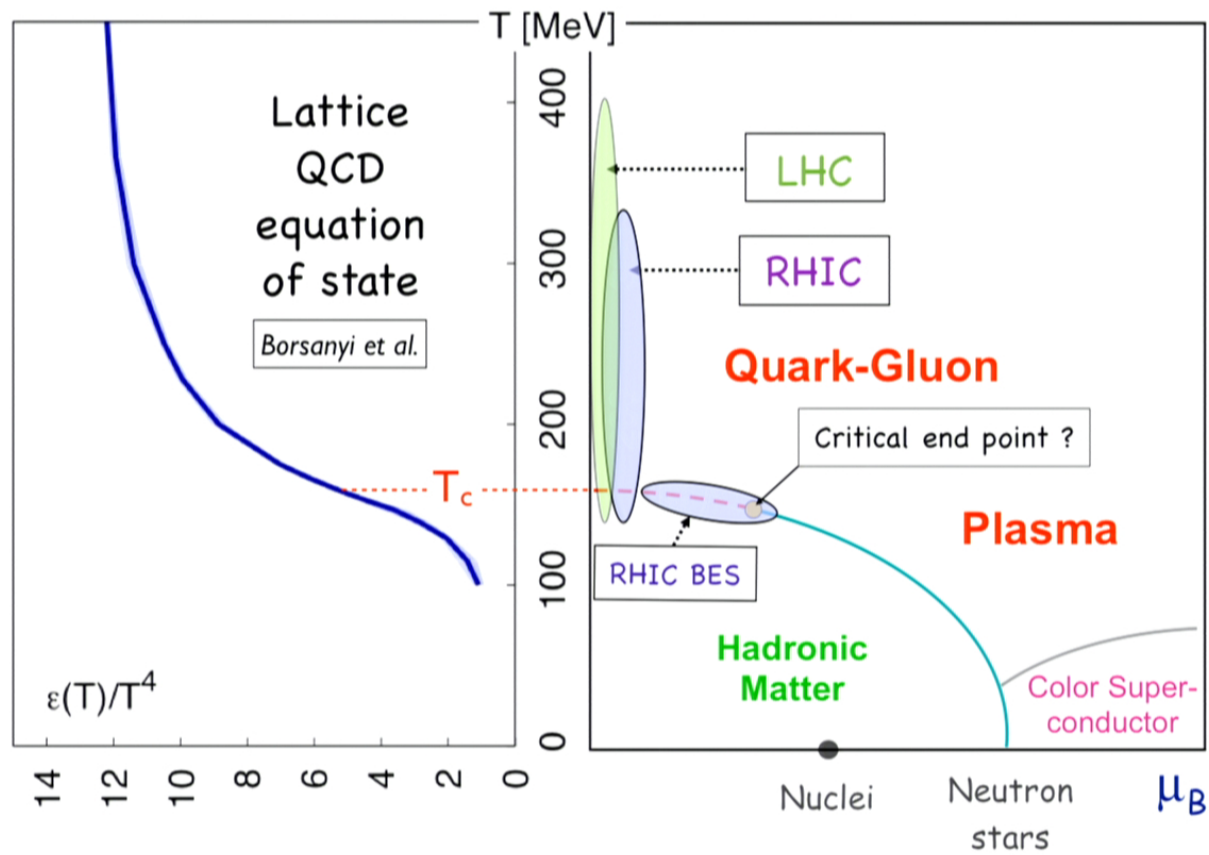
BNL's version of the Perfect Liquid



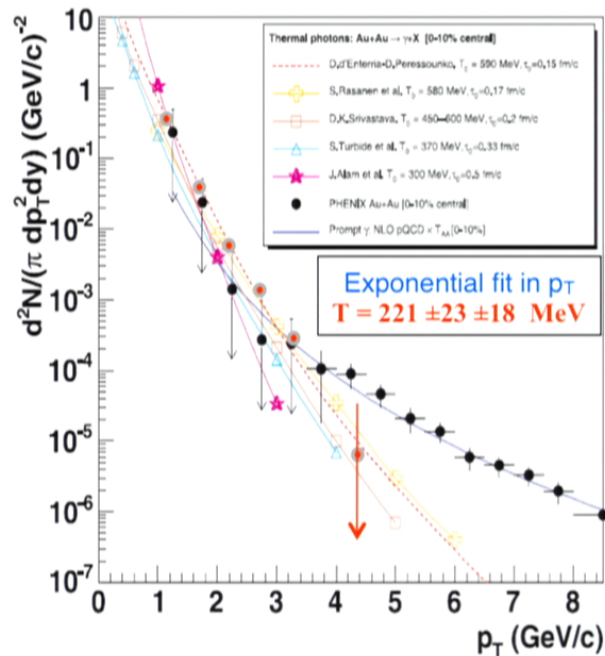
Shiseido's version of the Perfect Liquid

- What is a “Perfect” Liquid ?
- What is the structure of the QGP produced at RHIC ?
- Is the QGP produced at LHC also a “Perfect” Liquid ?
- At which scale is the QGP strongly coupled ?
- How does the QGP thermalize ?
- How does the structure of colliding nuclei manifest itself in the QGP ?

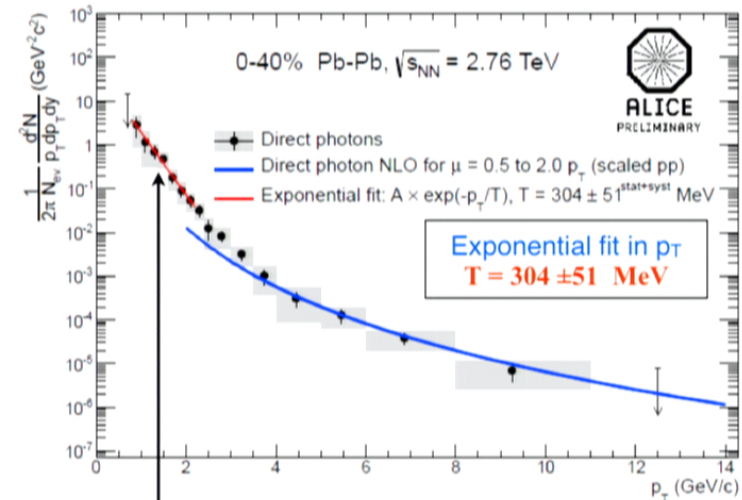
QCD Phase Diagram



Direct photons



Hydro fits
 $T_{init} \geq 300$ MeV

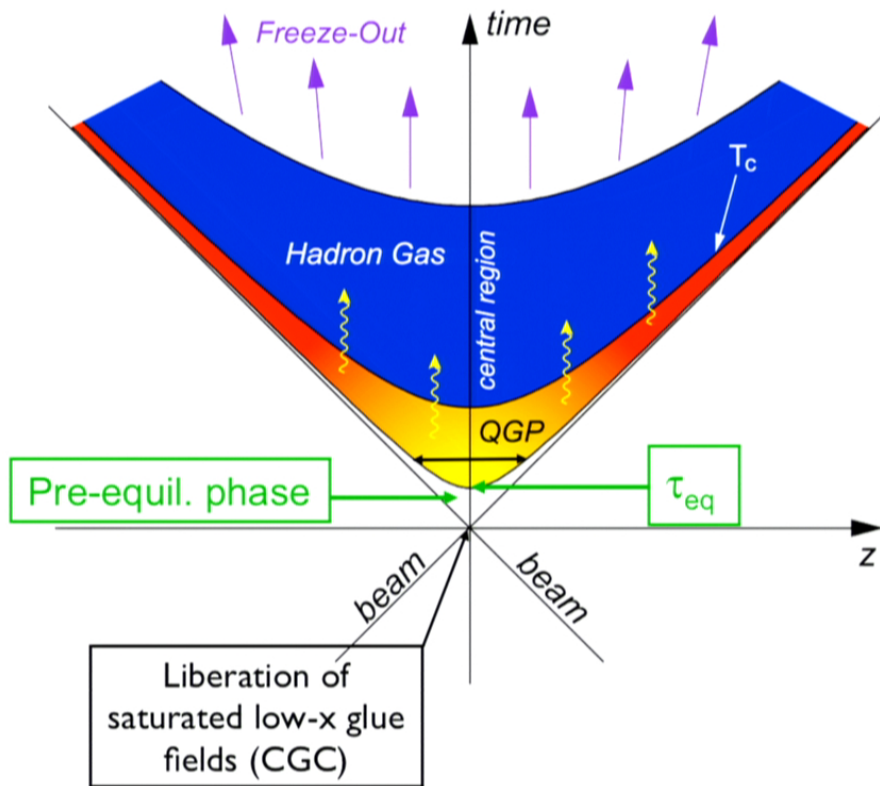


New **record "temperature"**
 measured in Pb+Pb at LHC:

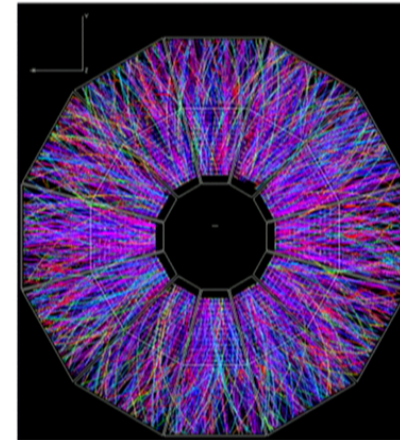
$$T_{LHC} = 1.37 T_{RHIC}.$$

Reflects larger initial temperature T_{in} ,
 but not to be identified with T_{in} .

Space-time picture

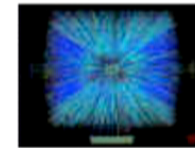


along the beam

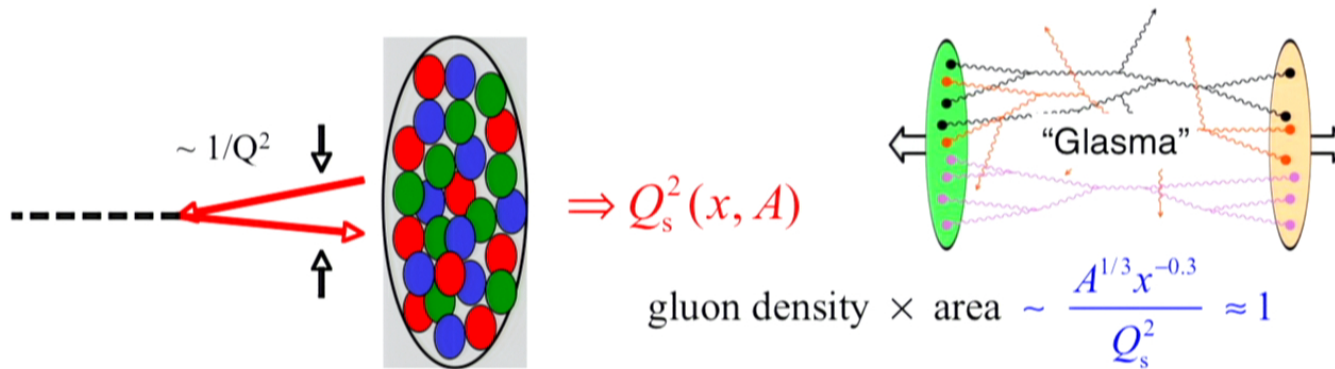


Au-Au collision as seen by STAR

from the side

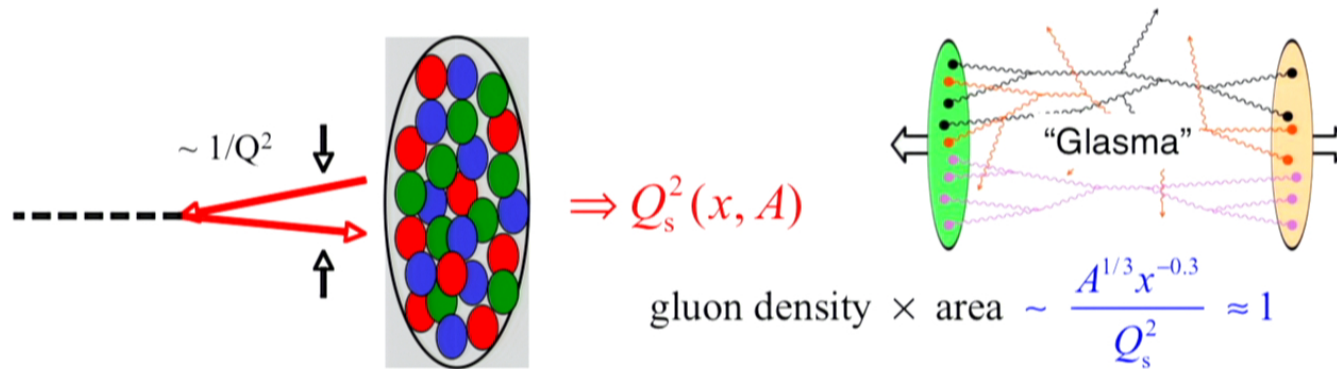


GGC and glasma

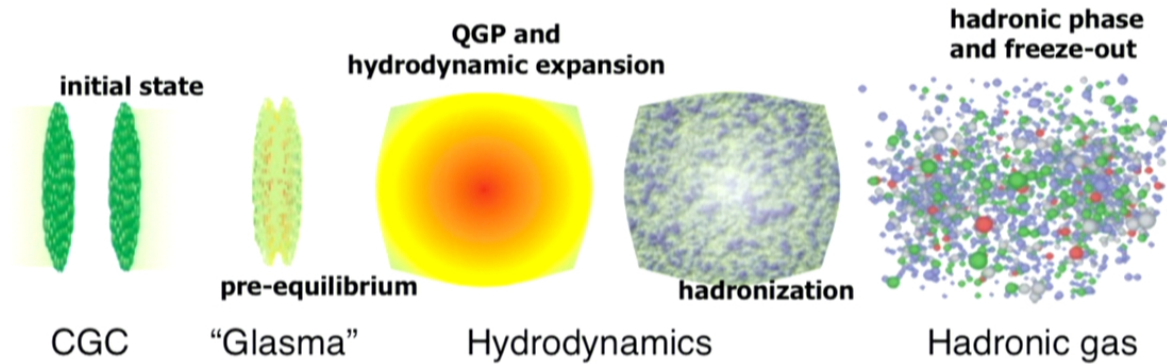


Universal state at small x : **Color glass condensate (CGC)**

GGC and glasma



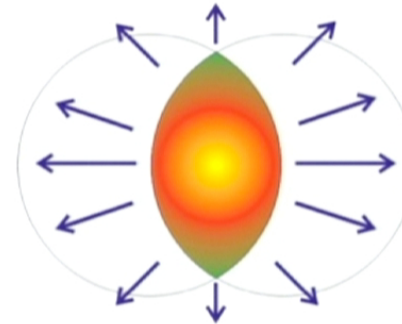
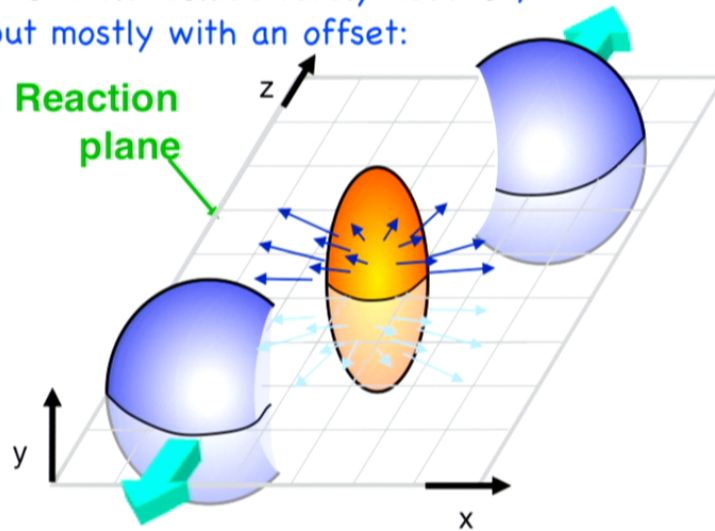
Universal state at small x : **Color glass condensate (CGC)**



The fluid QGP

Elliptic flow

- two nuclei collide rarely head-on, but mostly with an offset:



only matter in the overlap area gets compressed and heated:
Expansion is anisotropic

$$2\pi \frac{dN}{d\phi} = N_0 \left(1 + 2 \sum_n v_n(p_T, \eta) \cos n(\phi - \psi_n(p_T, \eta)) \right)$$

anisotropic flow coefficients

event plane angle

Viscous hydrodynamics

Hydrodynamics = effective theory of energy and momentum conservation

energy-momentum tensor = **ideal fluid** + **dissipation**

$$\partial_{\mu} T^{\mu\nu} = 0 \quad \text{with} \quad T^{\mu\nu} = (\varepsilon + P)u^{\mu}u^{\nu} - Pg^{\mu\nu} + \Pi^{\mu\nu}$$

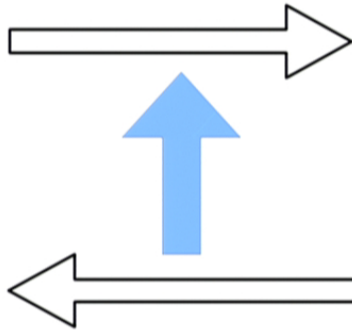
$$\tau_{\Pi} \left[\frac{d\Pi^{\mu\nu}}{d\tau} + (u^{\mu}\Pi^{\nu\lambda} + u^{\nu}\Pi^{\mu\lambda}) \frac{du^{\lambda}}{d\tau} \right] = \eta (\partial^{\mu}u^{\nu} + \partial^{\nu}u^{\mu} - \text{trace}) - \Pi^{\mu\nu}$$

Input: Equation of state $P(\varepsilon)$, shear viscosity, initial conditions $\varepsilon(x,0)$, $u^{\mu}(x,0)$

Shear viscosity is normalized by density: kinematic viscosity η/ρ .

Relativistically, the appropriate normalization factor is the entropy density $s = (\varepsilon+P)/T$, because the particle density is not conserved: η/s .

Shear viscosity



Shear viscosity describes ability to transport momentum across flow gradients! Kinetic theory:

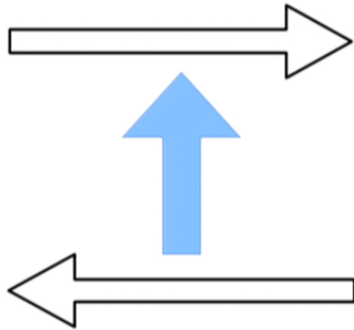
$$\eta \approx \frac{1}{3} n \bar{p} \lambda_f \quad \lambda_f = \frac{1}{n\sigma} \rightarrow \eta \approx \frac{\bar{p}}{3\sigma}$$

$$\sigma \leq \frac{4\pi}{\bar{p}^2} \rightarrow \eta \geq \frac{\bar{p}^3}{12\pi}$$

Relativistic system of massless particles: $\bar{p} \sim T \rightarrow \bar{p}^3 \sim T^3 \sim s$

$$\Rightarrow \frac{\eta}{s} \geq \text{some lower bound} = \# \cdot \left[\frac{\hbar}{k_B} \right]$$

Shear viscosity



Shear viscosity describes ability to transport momentum across flow gradients! Kinetic theory:

$$\eta \approx \frac{1}{3} n \bar{p} \lambda_f \quad \lambda_f = \frac{1}{n\sigma} \rightarrow \eta \approx \frac{\bar{p}}{3\sigma}$$

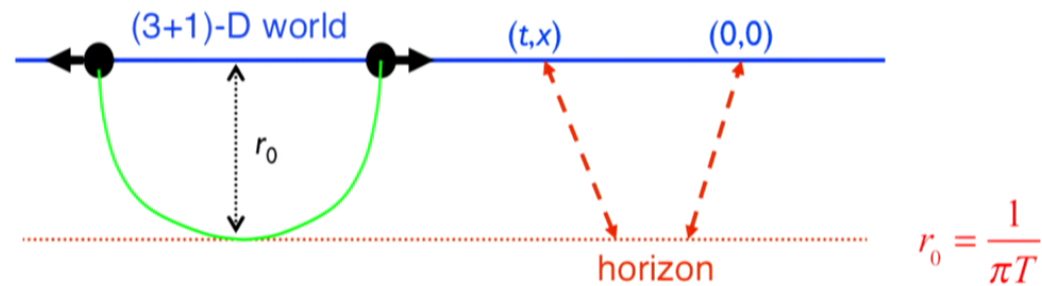
$$\sigma \leq \frac{4\pi}{\bar{p}^2} \rightarrow \eta \geq \frac{\bar{p}^3}{12\pi}$$

Relativistic system of massless particles: $\bar{p} \sim T \rightarrow \bar{p}^3 \sim T^3 \sim s$

$$\Rightarrow \frac{\eta}{s} \geq \text{some lower bound} = \# \cdot \left[\frac{\hbar}{k_B} \right]$$

String theory weighs in

General argument [Kovtun, Son & Starinets, PRL 94 (2005) 111601] based on duality between thermal QFT and string theory in 5-dimensional curved space with a “black-brane” metric (Maldacena 1998).



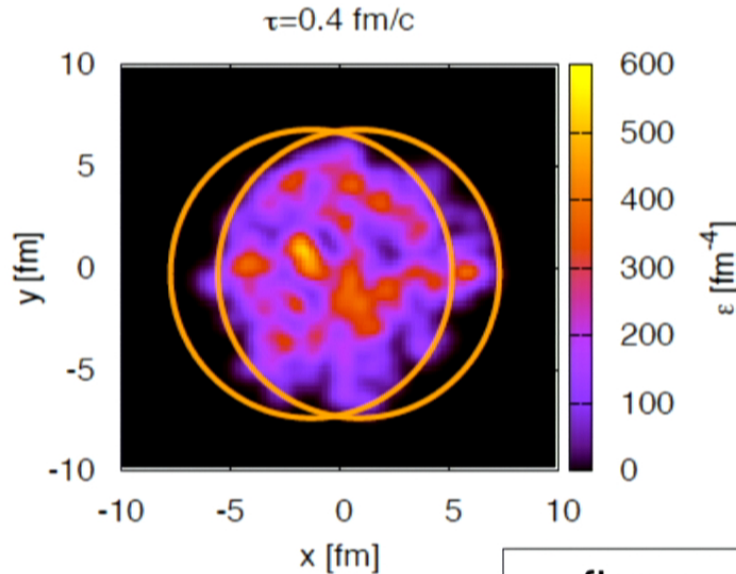
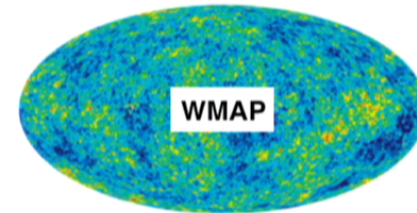
Dissipation in QFT is dual to absorption of gravitons by the black hole:

$$\sigma_{\text{abs}}(\omega) = \frac{8\pi G}{\omega} \int dt d^3x e^{i\omega t} \left\langle \left[T_{xy}(t, \vec{x}), T_{xy}(0, 0) \right] \right\rangle \xrightarrow{\omega \rightarrow 0} a \quad (\text{horizon area})$$

Thus: $\eta = \frac{\sigma_{\text{abs}}(0)}{16\pi G} = \frac{a}{16\pi G} = \frac{s}{4\pi}$ because $s = \frac{a}{4G} \rightarrow \boxed{\frac{\eta}{s} = \frac{1}{4\pi}}$

Event-by-event fluctuations

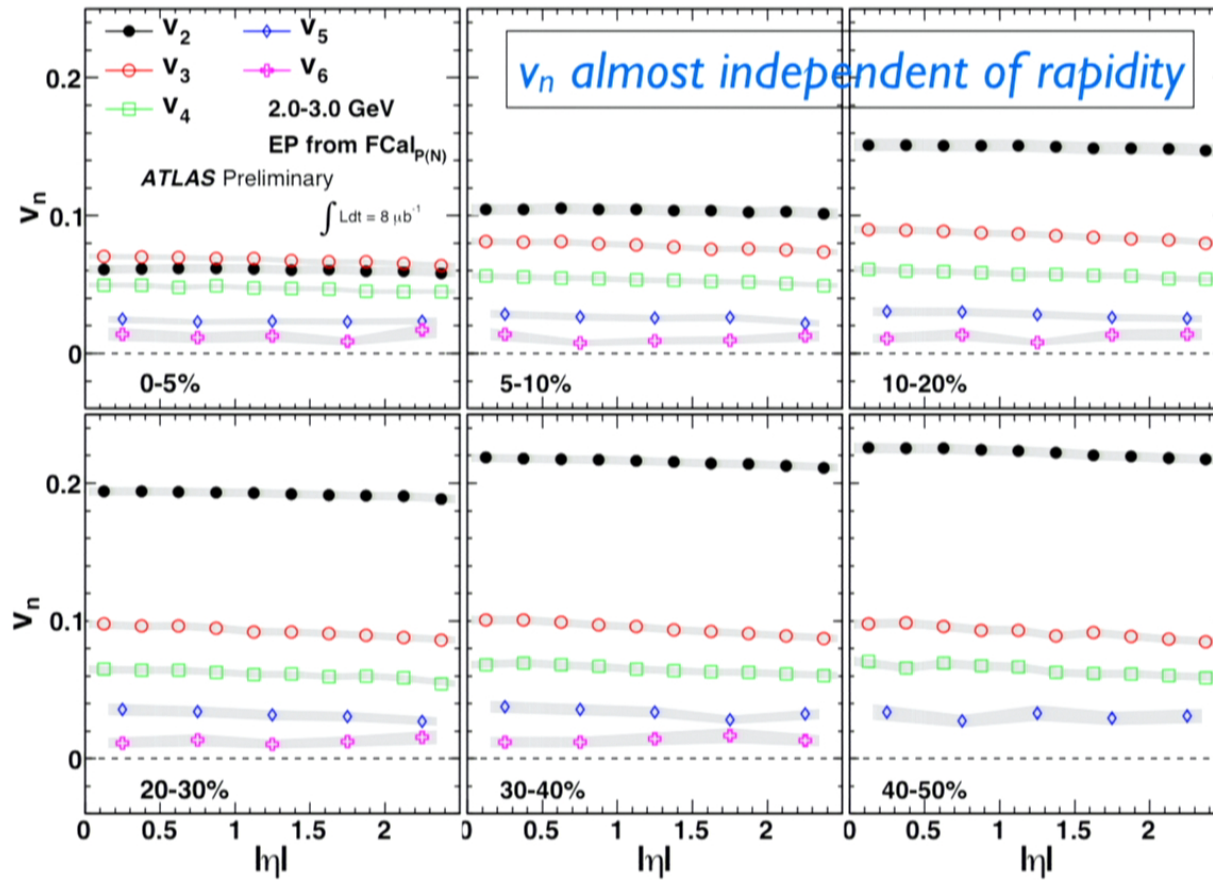
Initial state generated in A+A collision is grainy
 event plane \neq reaction plane
 \Rightarrow eccentricities $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \text{etc.} \neq 0$



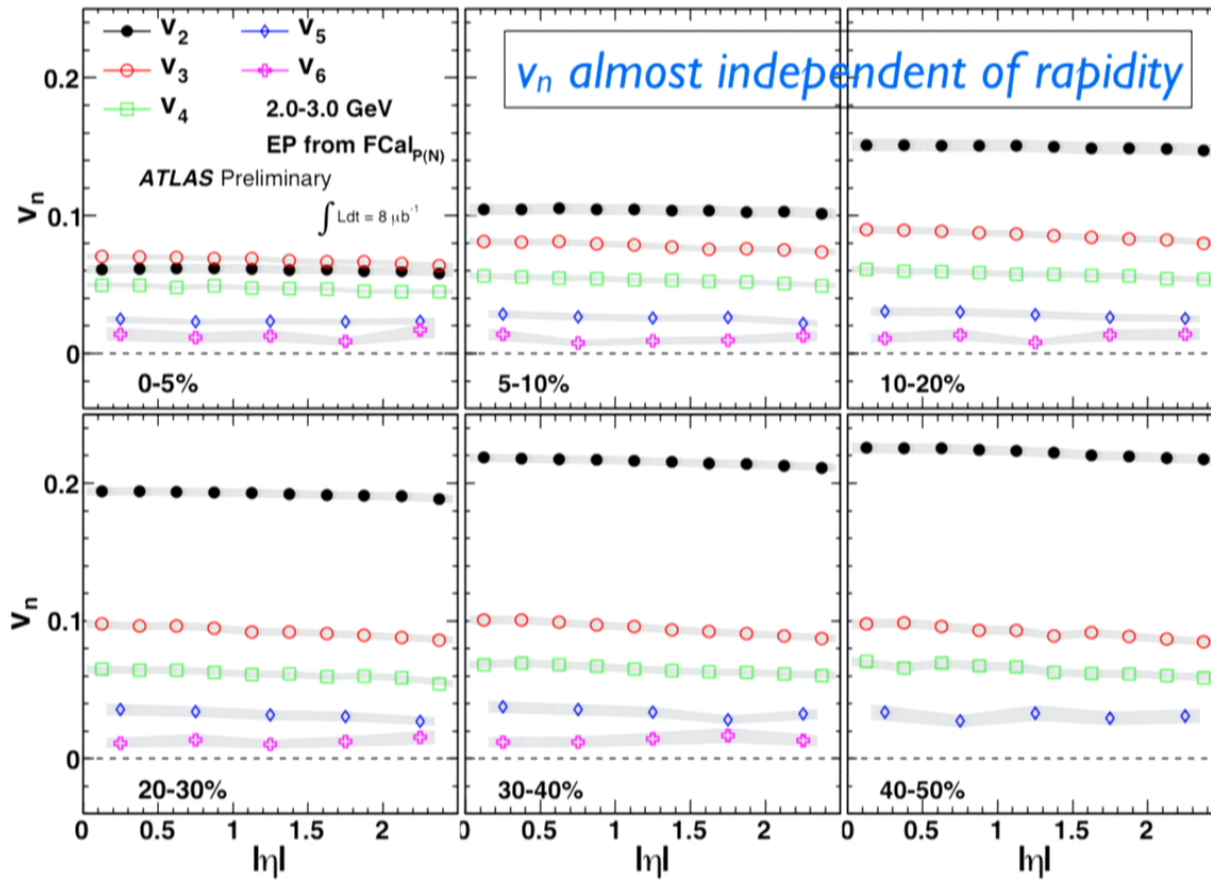
Idea: Energy density fluctuations in transverse plane from initial state quantum fluctuations. These thermalize to different temperatures locally and then propagate hydrodynamically to generate angular flow velocity fluctuations in the final state.

\Rightarrow flows $v_1, v_2, v_3, v_4, \dots$

$v_n (n = 2, \dots, 6)$

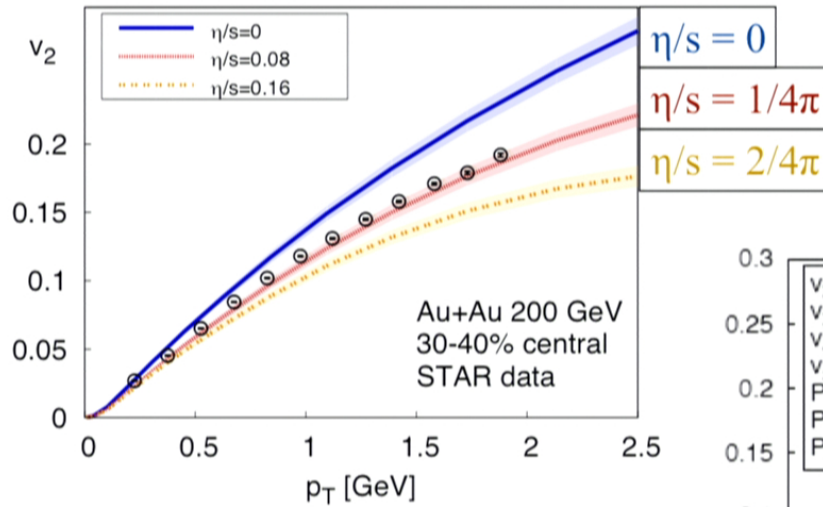


$v_n (n = 2, \dots, 6)$



Elliptic flow “measures” η_{QGP}

Schenke, Jeon, Gale, PRL 106 (2011) 042301

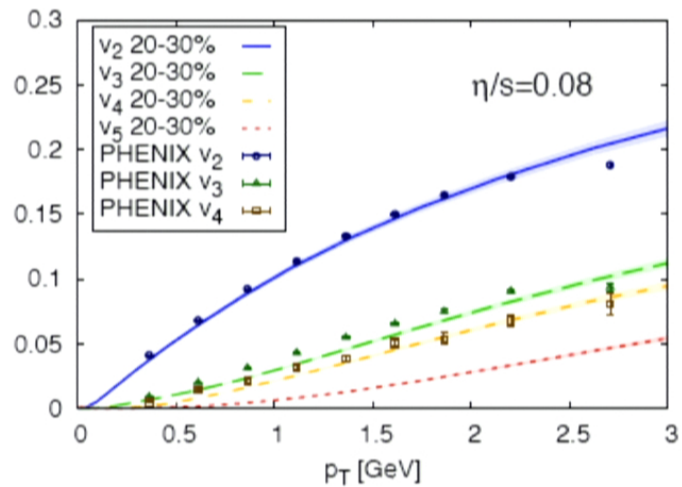


Schenke, Jeon, Gale, PRC 85 (2012) 024901

Universal strong coupling limit of non-abelian gauge theories with a gravity dual:

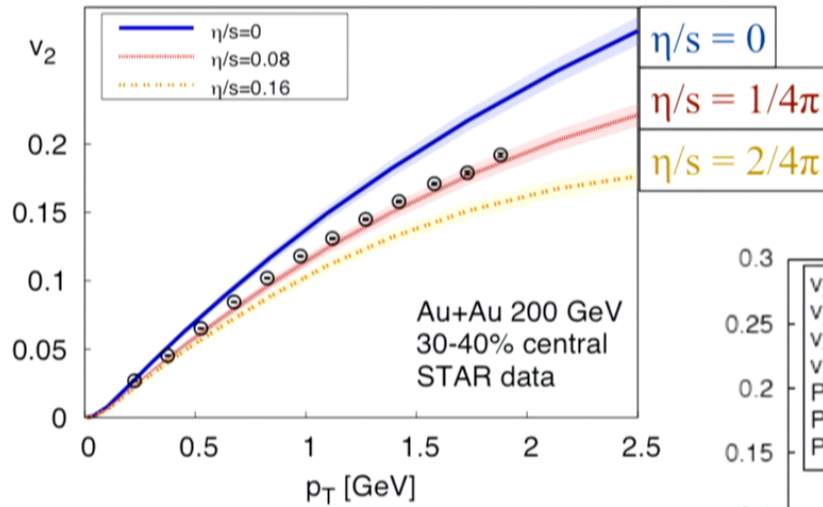
$$\eta/s \rightarrow 1/4\pi$$

aka: the “perfect” liquid



Elliptic flow “measures” η_{QGP}

Schenke, Jeon, Gale, PRL 106 (2011) 042301

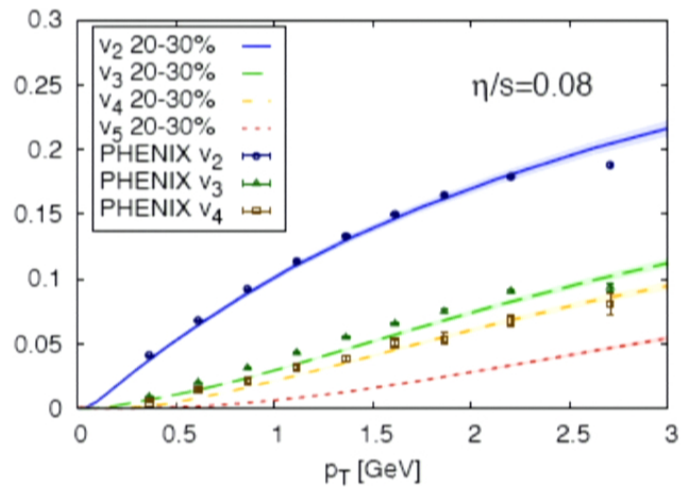


Schenke, Jeon, Gale, PRC 85 (2012) 024901

Universal strong coupling limit of non-abelian gauge theories with a gravity dual:

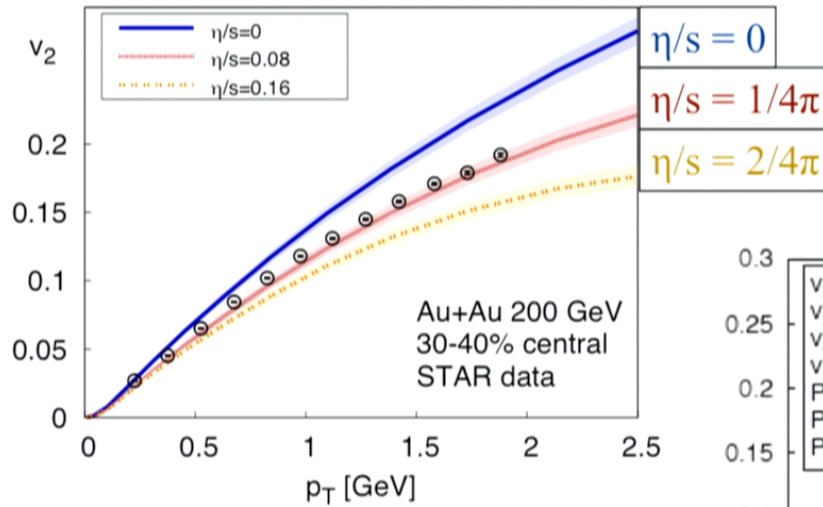
$$\eta/s \rightarrow 1/4\pi$$

aka: the “perfect” liquid



Elliptic flow “measures” η_{QGP}

Schenke, Jeon, Gale, PRL 106 (2011) 042301

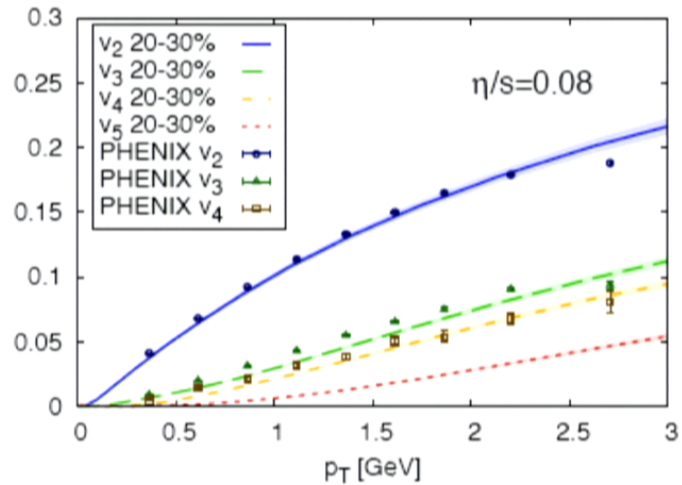


Schenke, Jeon, Gale, PRC 85 (2012) 024901

Universal strong coupling limit of non-abelian gauge theories with a gravity dual:

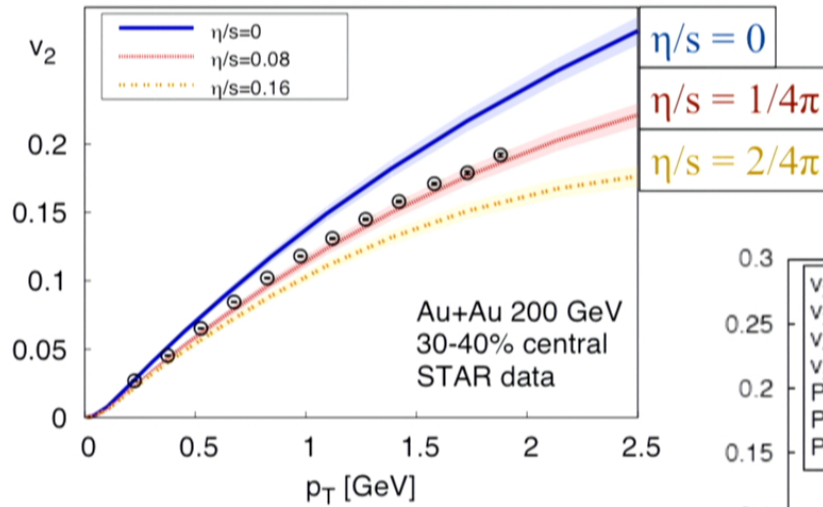
$$\eta/s \rightarrow 1/4\pi$$

aka: the “perfect” liquid



Elliptic flow “measures” η_{QGP}

Schenke, Jeon, Gale, PRL 106 (2011) 042301

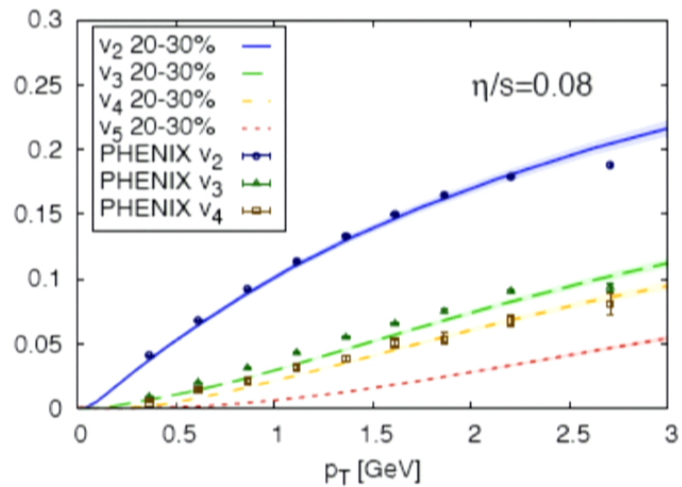


Schenke, Jeon, Gale, PRC 85 (2012) 024901

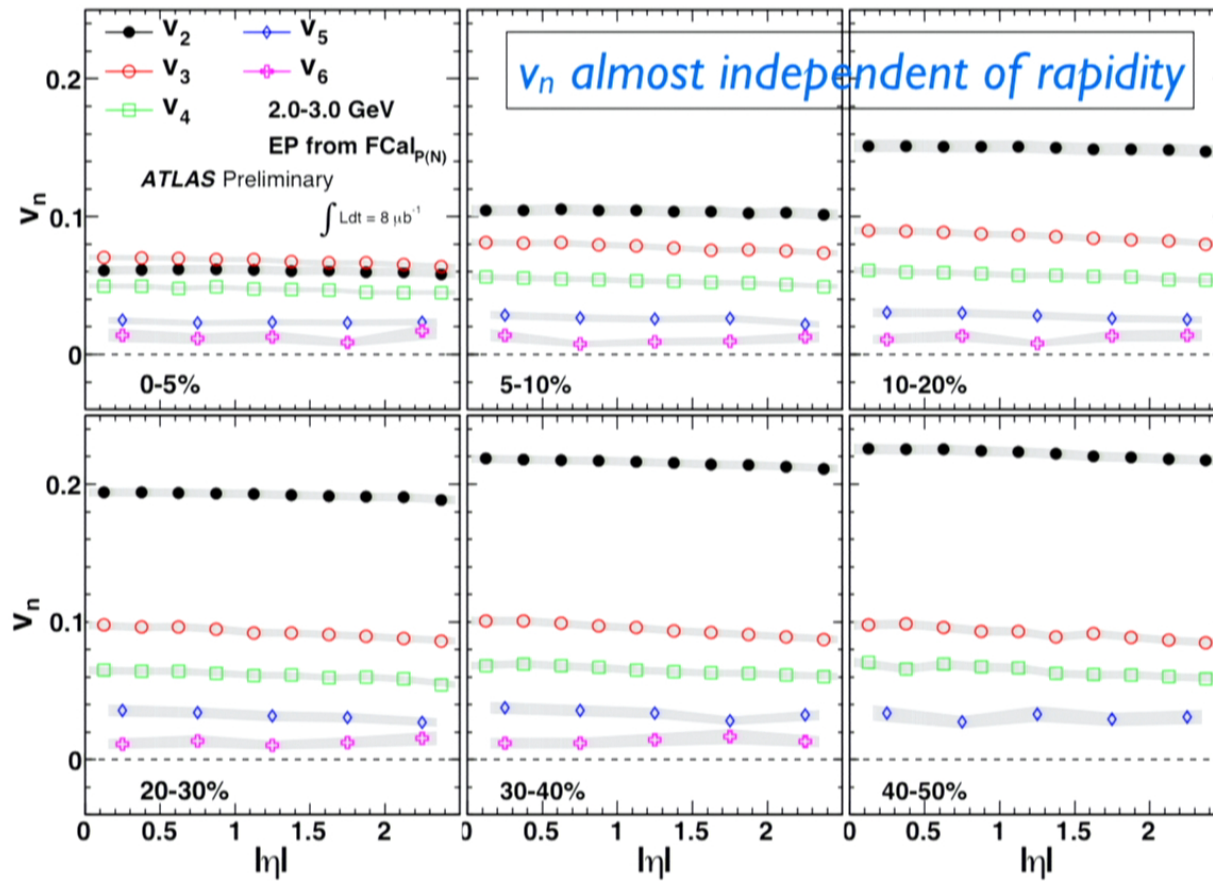
Universal strong coupling limit of non-abelian gauge theories with a gravity dual:

$$\eta/s \rightarrow 1/4\pi$$

aka: the “perfect” liquid

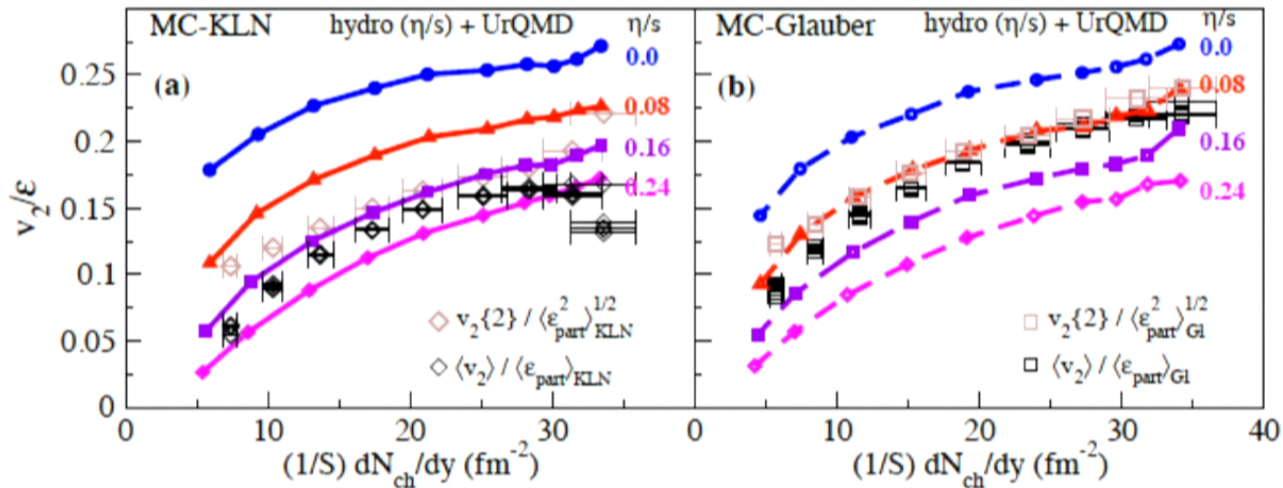


$v_n (n = 2, \dots, 6)$



Shear viscosity

Song, Bass, Heinz, Hirano, Shen, PRL 106 (2011) 192301

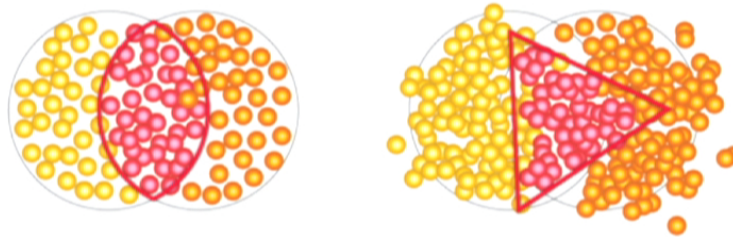


Conclusion: $1 \leq 4\pi\eta/s \leq 2.5$

Remaining uncertainty mainly due to initial density profile

How far can we reduce the uncertainty ?

Color charge fluctuations



Quantum fluctuations in the positions of the colliding nucleons give rise to a position dependent density of valence partons and other hard partons: $\mu^2(x)$.

For given μ , color charges of the partons combine in a random walk in SU(3). This generates an approximately Gaussian distribution of color charges $\rho^a(x)$.

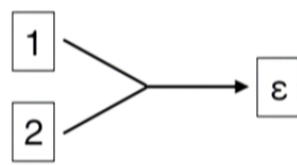
$$P[\rho] \propto \exp\left(-\frac{1}{2g^2\mu^2} \int d^2x \rho^a(x)\rho^a(x)\right)$$

Neglected: transverse correlations among color charges, x-dependence of μ , confinement related effects, etc.

Energy density fluctuations

Quantity to calculate: $\langle \varepsilon(\mathbf{x})\varepsilon(\mathbf{y}) \rangle - \langle \varepsilon(\mathbf{x}) \rangle \langle \varepsilon(\mathbf{y}) \rangle$

Energy density deposited by two colliding sheets of CGC:



$$\varepsilon(\mathbf{x}) = \frac{1}{4} F_{ij}^c(\mathbf{x}) F_{ij}^c(\mathbf{x}) + 2A^{\eta c}(\mathbf{x}) A^{\eta c}(\mathbf{x})$$

$$F_{ij}^c(\mathbf{x}) = g f_{abc} \left(A_i^a(1; \mathbf{x}) A_j^b(2; \mathbf{x}) + A_i^a(2; \mathbf{x}) A_j^b(1; \mathbf{x}) \right)$$

$$A^{\eta c}(\mathbf{x}) A^{\eta c}(\mathbf{x}) = \frac{g^2}{4} f_{abc} f_{a'b'c} A_i^a(1; \mathbf{x}) A_i^b(2; \mathbf{x}) A_j^{a'}(1; \mathbf{x}) A_j^{b'}(2; \mathbf{x})$$

CGC field correlator (light-cone gauge):

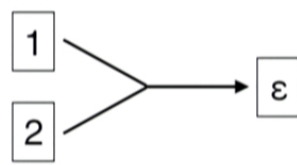
$$\langle A_i^a(n; \mathbf{x}) A_j^b(m; \mathbf{y}) \rangle = \delta_{mn} \delta_{ab} \int \frac{d^2 p}{(2\pi)^2} \cos[\mathbf{p} \cdot (\mathbf{x} - \mathbf{y})] \frac{p_i p_j}{p^2} G(|\mathbf{p}|)$$

$$G(|\mathbf{x}|) = \frac{4}{g^2 N |\mathbf{x}|^2} \left[1 - \exp \left(\frac{g^2 N}{8\pi} g^2 \mu^2 |\mathbf{x}|^2 \ln(\Lambda |\mathbf{x}|) \right) \right] \Theta(1 - \Lambda |\mathbf{x}|)$$

Energy density fluctuations

Quantity to calculate: $\langle \varepsilon(\mathbf{x})\varepsilon(\mathbf{y}) \rangle - \langle \varepsilon(\mathbf{x}) \rangle \langle \varepsilon(\mathbf{y}) \rangle$

Energy density deposited by two colliding sheets of CGC:



$$\varepsilon(\mathbf{x}) = \frac{1}{4} F_{ij}^c(\mathbf{x}) F_{ij}^c(\mathbf{x}) + 2A^{\eta c}(\mathbf{x}) A^{\eta c}(\mathbf{x})$$

$$F_{ij}^c(\mathbf{x}) = g f_{abc} \left(A_i^a(1; \mathbf{x}) A_j^b(2; \mathbf{x}) + A_i^a(2; \mathbf{x}) A_j^b(1; \mathbf{x}) \right)$$

$$A^{\eta c}(\mathbf{x}) A^{\eta c}(\mathbf{x}) = \frac{g^2}{4} f_{abc} f_{a'b'c} A_i^a(1; \mathbf{x}) A_i^b(2; \mathbf{x}) A_j^{a'}(1; \mathbf{x}) A_j^{b'}(2; \mathbf{x})$$

CGC field correlator (light-cone gauge):

$$\langle A_i^a(n; \mathbf{x}) A_j^b(m; \mathbf{y}) \rangle = \delta_{mn} \delta_{ab} \int \frac{d^2 p}{(2\pi)^2} \cos[\mathbf{p} \cdot (\mathbf{x} - \mathbf{y})] \frac{p_i p_j}{p^2} G(|\mathbf{p}|)$$

$$G(|\mathbf{x}|) = \frac{4}{g^2 N |\mathbf{x}|^2} \left[1 - \exp \left(\frac{g^2 N}{8\pi} g^2 \mu^2 |\mathbf{x}|^2 \ln(\Lambda |\mathbf{x}|) \right) \right] \Theta(1 - \Lambda |\mathbf{x}|)$$

$\Delta\epsilon/\epsilon$ is large

$$G(|x|) = G_0 \phi(|x|^2 / \xi^2)$$

with

$$G_0 = \frac{4}{9} \pi \mu^2$$

$$1 / \xi^2 = \frac{1}{9} N \pi (g\mu)^2$$

$$\phi_{MV}(u) = (1 - e^{-u}) / u$$

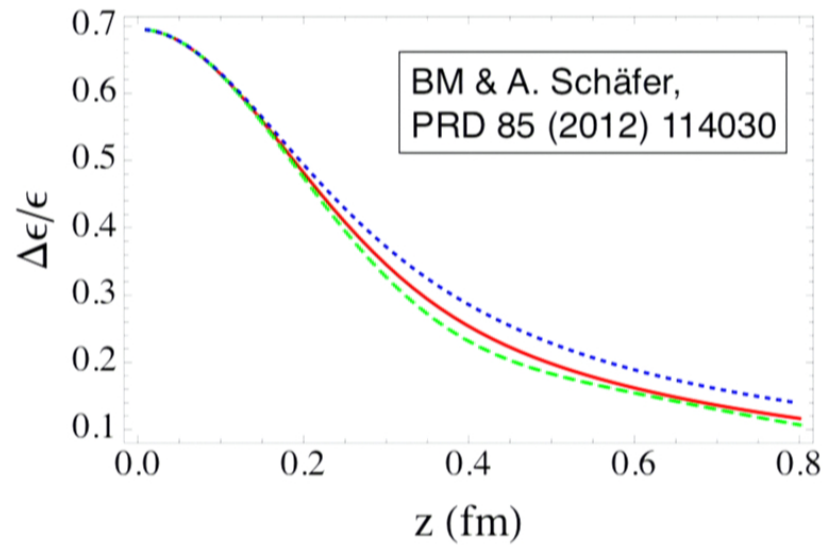
$$\phi_1(u) = e^{-u/2}$$

$$\phi_2(u) = \left(1 + \frac{u}{2}\right)^{-1}$$

$$Q_s^2 = (g^2 \mu)^2 = 2 \text{ GeV}^2;$$

$$g^2(\mu^2) = 3.785;$$

$$g^2(1/x^2) = \frac{16\pi^2}{9 \ln(1/(\Lambda^2 x^2))}.$$



$\Delta\epsilon/\epsilon$ is large

$$G(|x|) = G_0 \phi(|x|^2 / \xi^2)$$

with

$$G_0 = \frac{4}{9} \pi \mu^2$$

$$1 / \xi^2 = \frac{1}{9} N \pi (g\mu)^2$$

$$\phi_{MV}(u) = (1 - e^{-u}) / u$$

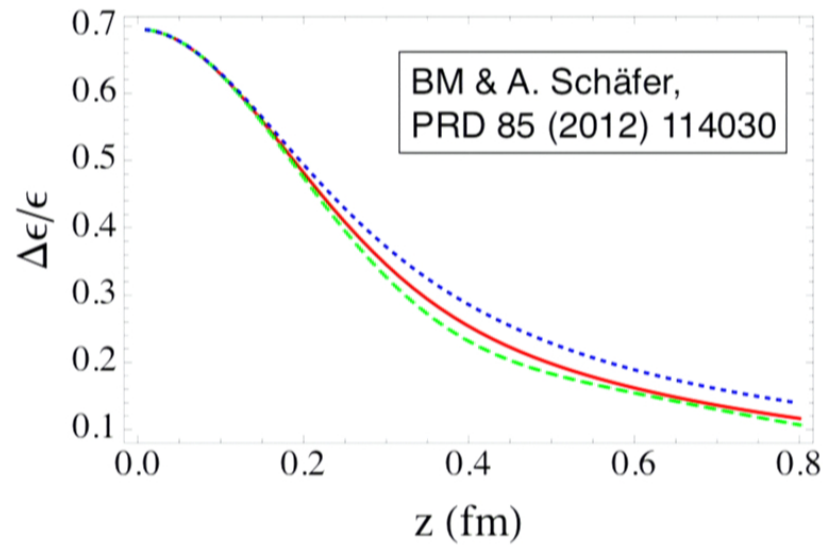
$$\phi_1(u) = e^{-u/2}$$

$$\phi_2(u) = \left(1 + \frac{u}{2}\right)^{-1}$$

$$Q_s^2 = (g^2 \mu)^2 = 2 \text{ GeV}^2;$$

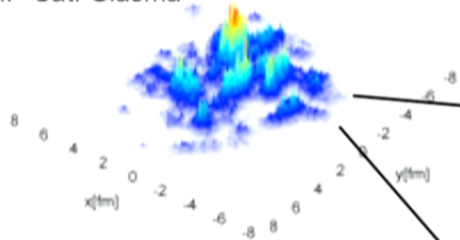
$$g^2(\mu^2) = 3.785;$$

$$g^2(1/x^2) = \frac{16\pi^2}{9 \ln(1/(\Lambda^2 x^2))}.$$

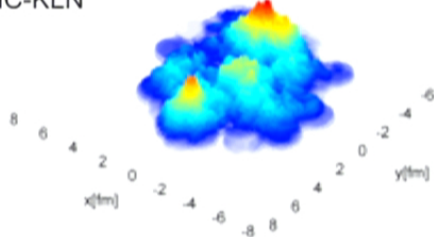


Glasma fluctuations

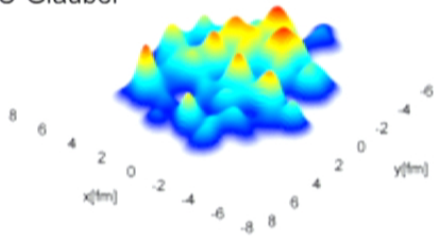
IP-Sat. Glasma



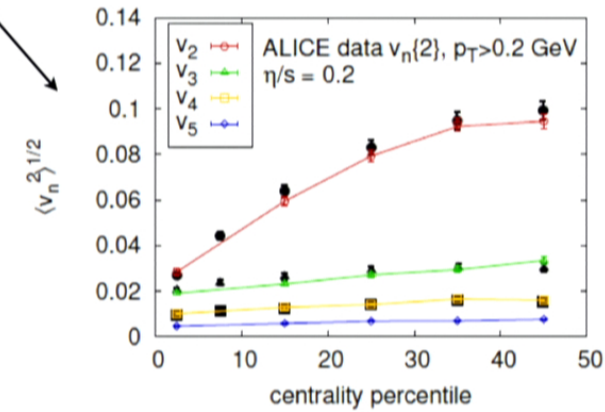
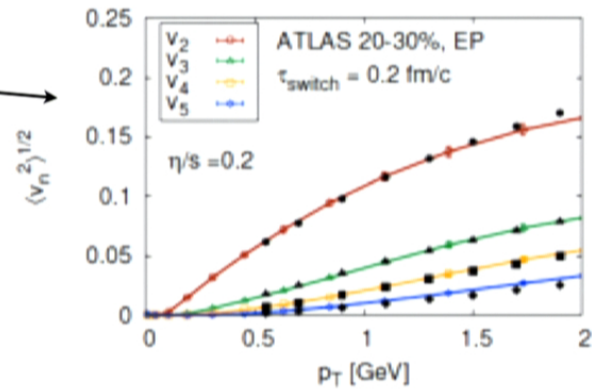
MC-KLN



MC-Glauber



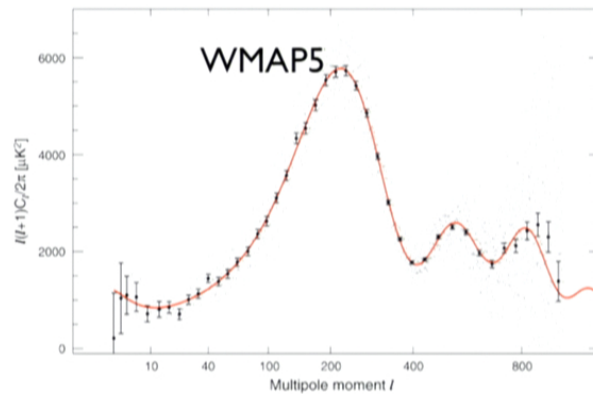
Schenke, Tribedy, Venugopalan (QM 2012)



Fluctuation spectrum

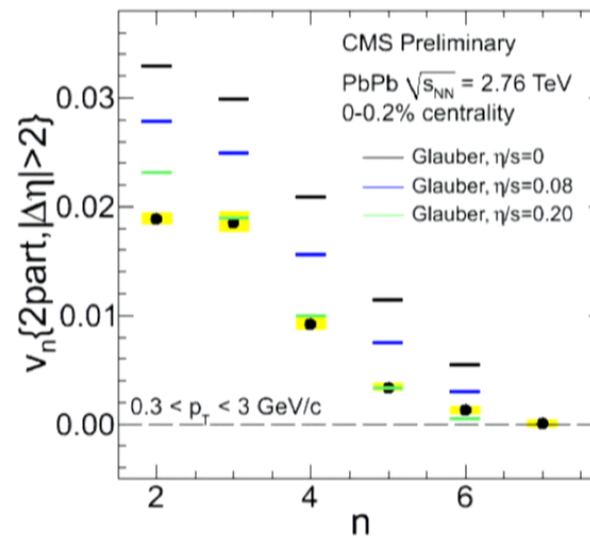
Can different distributions of various eccentricities in different collision systems be used to discriminate between energy deposition models / theories?

Can the power spectrum of v_n be used to determine η/s and v_{sound} ?



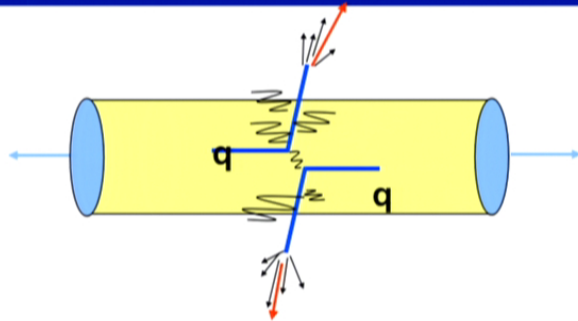
The RHIC/LHC advantage:
There are many knobs to turn, not just a single universe to observe.

M. Luzum et al.



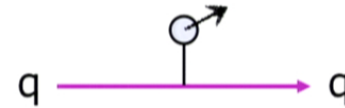
The opaque QGP

Parton energy loss

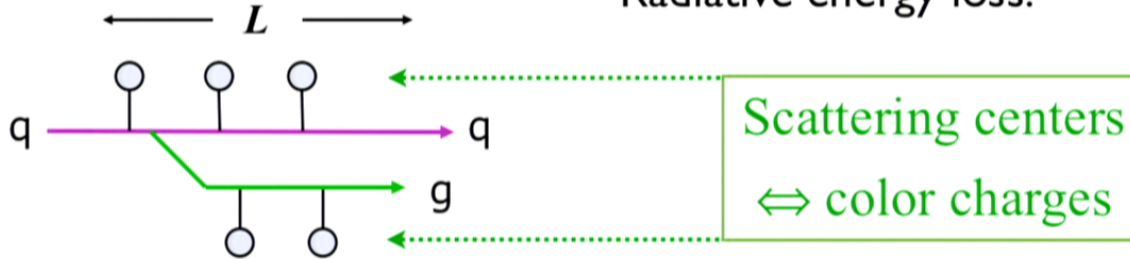


Elastic energy loss:

$$\frac{dE}{dx} = -C_2 \hat{e}$$



Radiative energy loss:



$$\frac{dE}{dx} = -C_2 \hat{q} L$$

$$\hat{q} = \rho \int q^2 dq^2 \frac{d\sigma}{dq^2} = \int dx^- \langle F_i^+(x^-) F^{+i}(0) \rangle$$

Connecting jets with the medium

Hard partons probe the medium via the density of colored scattering centers:

$$\hat{q} = \rho \int q^2 dq^2 \left(d\sigma / dq^2 \right) \sim \int dx^- \left\langle F^{++}(x^-) F_{\perp}^+(0) \right\rangle$$

If kinetic theory applies, thermal gluons are quasi-particles that experience the same medium. Then the shear viscosity is:

$$\eta \approx \frac{1}{3} n \langle p \lambda_f(p) \rangle = \frac{1}{3} \left\langle \frac{p}{\sigma_{rr}(p)} \right\rangle$$

In QCD, small angle scattering dominates: $\sigma_{rr}(p) \approx \frac{2\hat{q}}{\langle p \rangle^2 n}$

With $\langle p \rangle \sim 3T$ and $s \approx 4n$
(for massless particles):

$$\frac{\eta}{s} \approx O(1) \times \frac{T^3}{\hat{q}}$$

A. Majumder, BM, X-N. Wang,
PRL 99 (2007) 192301

LHS is bounded below: $\eta/s \geq 1/(4\pi)$

RHS is unbounded: $\sim 1/\sqrt{\lambda}$

Connecting jets with the medium

Hard partons probe the medium via the density of colored scattering centers:

$$\hat{q} = \rho \int q^2 dq^2 \left(d\sigma / dq^2 \right) \sim \int dx^- \left\langle F^{++}(x^-) F_{\perp}^+(0) \right\rangle$$

If kinetic theory applies, thermal gluons are quasi-particles that experience the same medium. Then the shear viscosity is:

$$\eta \approx \frac{1}{3} n \langle p \lambda_f(p) \rangle = \frac{1}{3} \left\langle \frac{p}{\sigma_{rr}(p)} \right\rangle$$

In QCD, small angle scattering dominates: $\sigma_{rr}(p) \approx \frac{2\hat{q}}{\langle p \rangle^2 n}$

With $\langle p \rangle \sim 3T$ and $s \approx 4n$
(for massless particles):

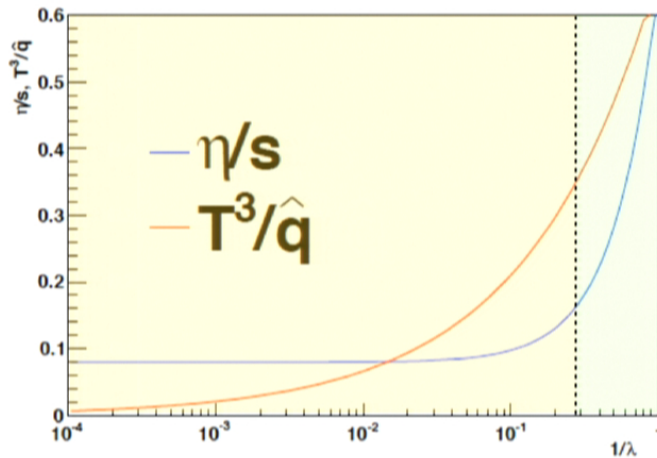
$$\frac{\eta}{s} \approx O(1) \times \frac{T^3}{\hat{q}}$$

A. Majumder, BM, X-N. Wang,
PRL 99 (2007) 192301

LHS is bounded below: $\eta/s \geq 1/(4\pi)$

RHS is unbounded: $\sim 1/\sqrt{\lambda}$

Why \hat{q} is important

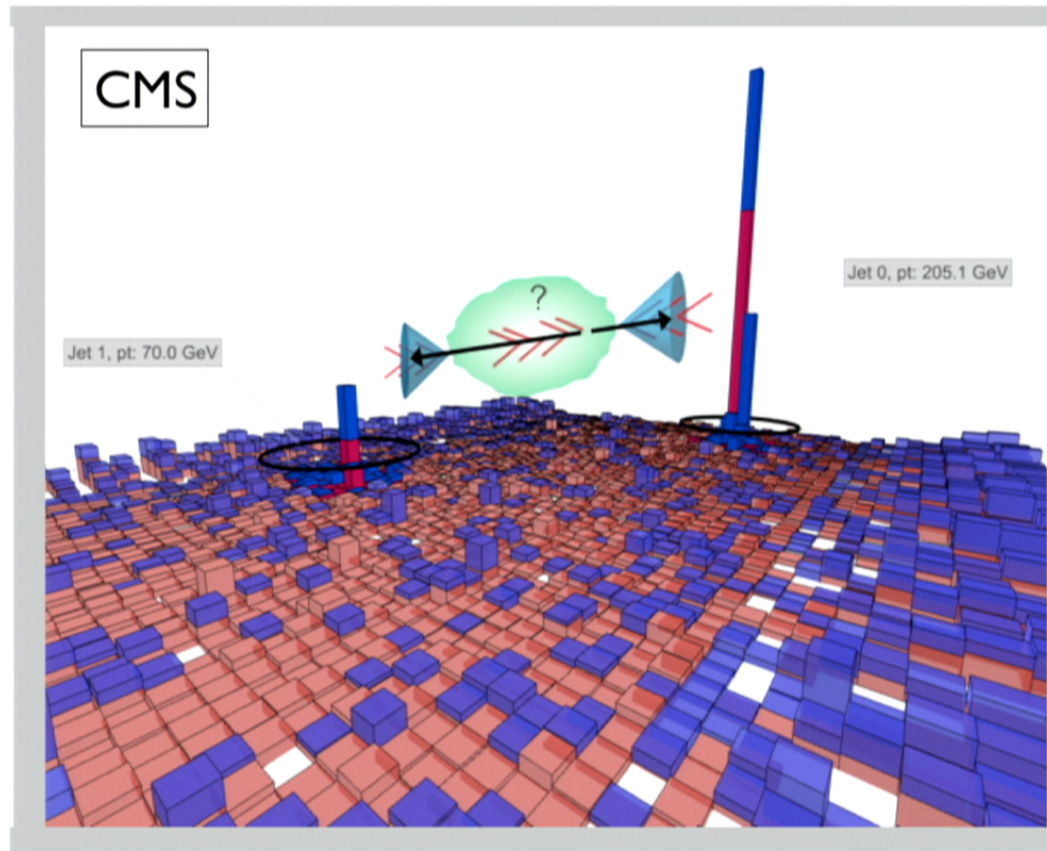


Majumder, BM, Wang argued that η/s and \hat{q} are related at weak coupling in gauge theories [PRL 99, 192301 (2007)]:

$$\eta / s = \text{const} \times T^3 / \hat{q}$$

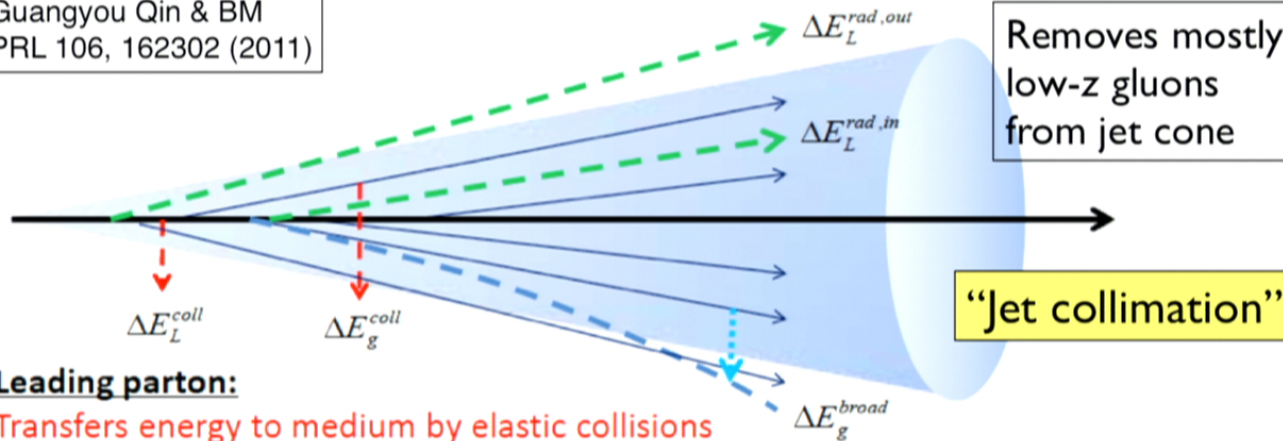
At strong coupling, η/s saturates at $1/4\pi$, but \hat{q} increases without limit. Unambiguous criterion for weak vs. strong coupling?

Di-jet asymmetry



Parton shower in matter

Guangyou Qin & BM
PRL 106, 162302 (2011)



Removes mostly low-z gluons from jet cone

“Jet collimation”

Leading parton:

Transfers energy to medium by elastic collisions

Radiates gluons scattering in the medium (*inside* and *outside* jet cone)

$$E_L(t) = E_L(t_1) - \int \hat{e}_L dt - \int \omega d\omega dk_{\perp}^2 dt \frac{dN_g^{med}}{d\omega dk_{\perp}^2 dt}$$

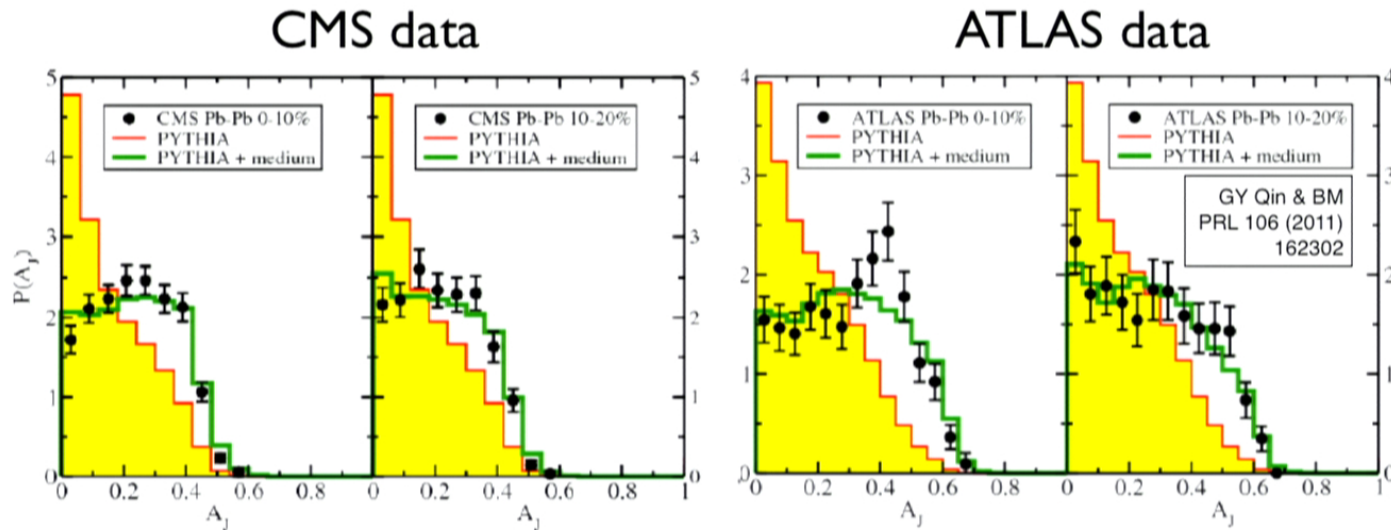
Radiated gluons (*vacuum & medium-induced*):

Transfer energy to medium by elastic collisions

Be kicked out of the jet cone by multiple scatterings after emission

$$\frac{df_g(\omega, k_{\perp}^2, t)}{dt} = \hat{e} \frac{\partial f_g}{\partial \omega} + \frac{1}{4} \hat{q} \nabla_{k_{\perp}}^2 f_g + \frac{dN_g^{med}}{d\omega dk_{\perp}^2 dt}$$

Di-jet asymmetry



ATLAS and CMS data differ in cuts on jet energy, cone angle, etc; results depend somewhat on precise cuts and background corrections. Fits of CMS and ATLAS data require $\sim 20\%$ different parameters. Several other calculations using pQCD physics input also fit the data.

General conclusion: *pQCD jet quenching can explain these data.*

$\Delta\epsilon/\epsilon$ is large

$$G(|x|) = G_0 \phi(|x|^2 / \xi^2)$$

with

$$G_0 = \frac{4}{9} \pi \mu^2$$

$$1 / \xi^2 = \frac{1}{9} N \pi (g\mu)^2$$

$$\phi_{MV}(u) = (1 - e^{-u}) / u$$

$$\phi_1(u) = e^{-u/2}$$

$$\phi_2(u) = \left(1 + \frac{u}{2}\right)^{-1}$$

$$Q_s^2 = (g^2 \mu)^2 = 2 \text{ GeV}^2;$$

$$g^2(\mu^2) = 3.785;$$

$$g^2(1/x^2) = \frac{16\pi^2}{9 \ln(1/(\Lambda^2 x^2))}.$$

