Title: Combinatorics and the thermodynamic limit for black holes in Loop Quantum Gravity

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Abstract: The
combinatorial problems associated with the counting of black hole states in loop quantum gravity can be analyzed by using suitable generating functions. These not only provide very useful tools for exact computations, but can also be used to perform an statistical analysis of the black hole degeneracy spectrum, study the interesting substructure found in the entropy of microscopic black holes and its asymptotic behavior for large horizon areas. The methods that will be described are relevant for the discussion of the thermodynamic limit for black holes in the area canonical ensemble. This is an important issue in order to understand sub-leading corrections to the Bekenstein-Hawking law.


Combinatorics and the thermodynamic limit for black holes in LQG
J. Fernando Barbero G.
Instituto de Estructura de la Materia, CSIC.
Perimeter Institute, September 13, 2012
Work in collaboration with I. Agulló, J. Díaz-Polo, E. F.-Borja, and E.J.S. Villaseñor.
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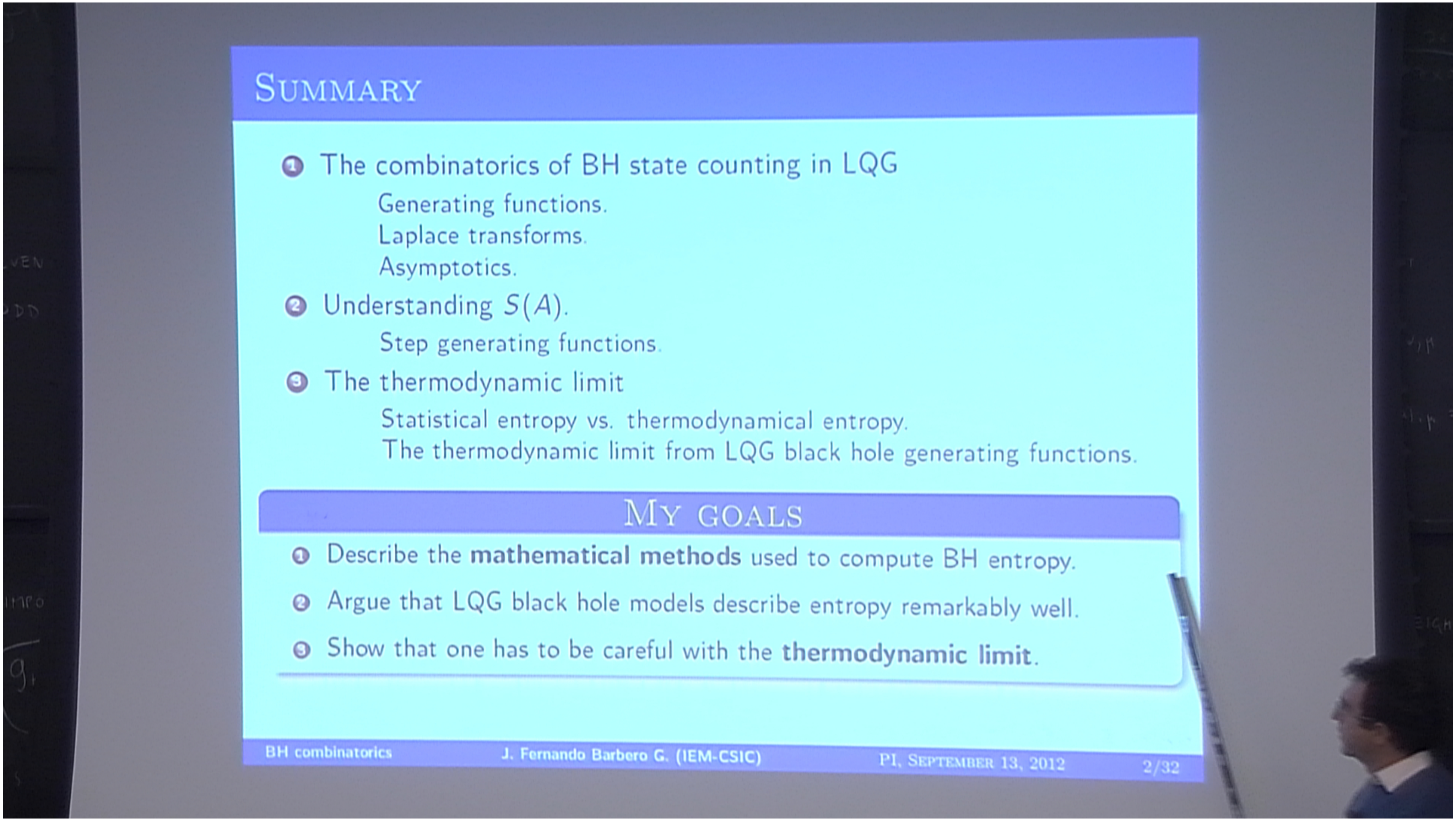
Summary

1. The combinatorics of BH state counting in LQG
   - Generating functions.
   - Laplace transforms.
   - Asymptotics.


SUMMARY
1. The combinatorics of BH state counting in LQG
Generating functions.
Laplace transforms.
Asymptotics.
2. Understanding $S(A)$.
Step generating functions.
3. The thermodynamic limit
Statistical entropy vs. thermodynamical entropy.
The thermodynamic limit from LQG black hole generating functions.
MY GOALS
1. Describe the **mathematical methods** used to compute BH entropy.
2. Argue that LQG black hole models describe entropy remarkably well.
3. Show that one has to be careful with the **thermodynamic limit**.
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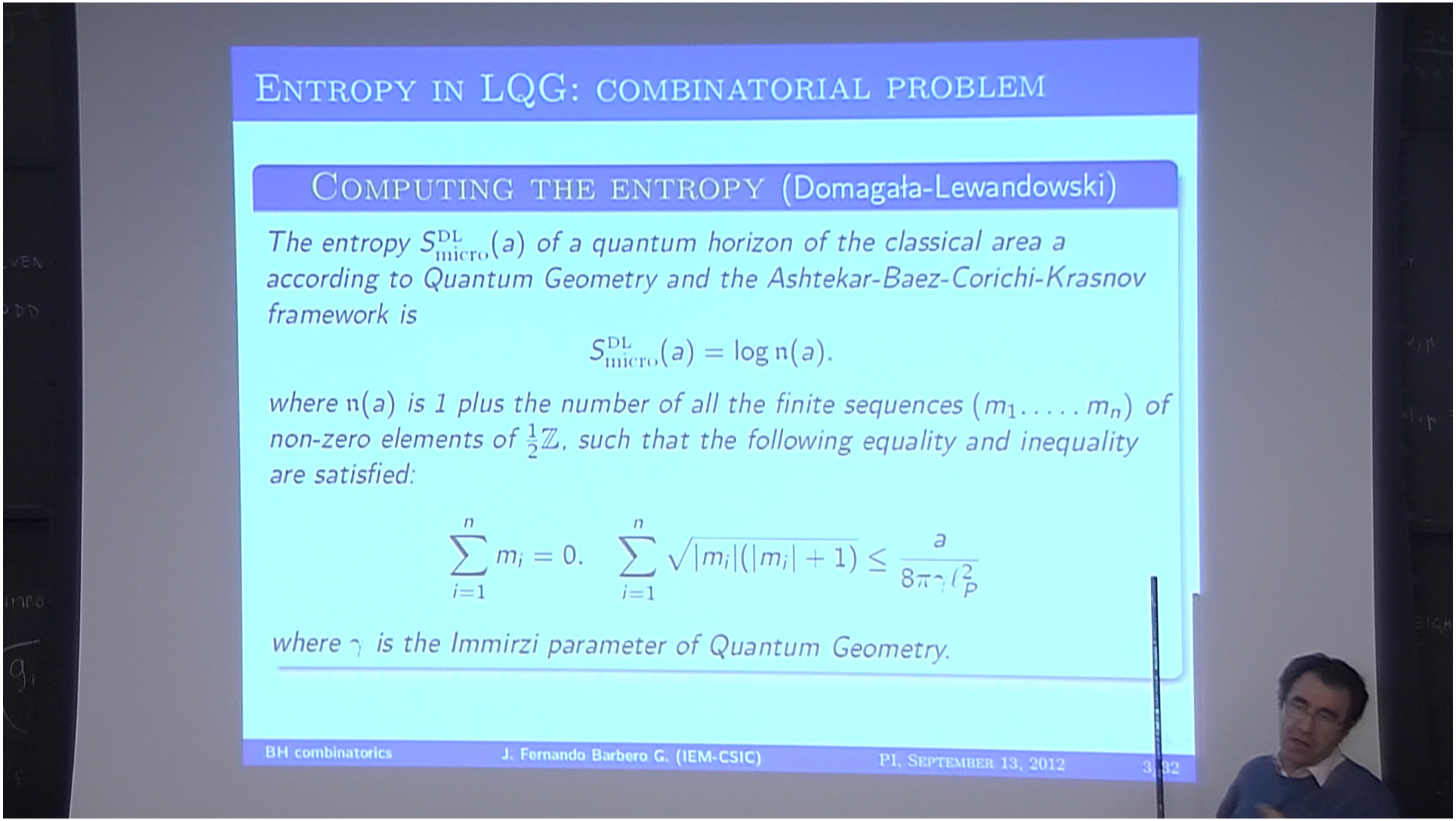

Entropy in LQG: combinatorial problem
Computing the entropy (Domagała-Lewandowski)
The entropy $S^{\mathrm{DL}}_{\mathrm{micro}}(a)$ of a quantum horizon of the classical area a according to Quantum Geometry and the Ashtekar-Baez-Corichi-Krasnov framework is
$S^{\mathrm{DL}}_{\mathrm{micro}}(a) = \log \mathfrak{n}(a).$
where $\mathfrak{n}(a)$ is 1 plus the number of all the finite sequences $(m_1, \ldots, m_n)$ of non-zero elements of $\frac{1}{2}\mathbb{Z}$, such that the following equality and inequality are satisfied:
$\sum_{i=1}^{n} m_i = 0, \quad \sum_{i=1}^{n} \sqrt{|m_i|(|m_i|+1)} \leq \frac{a}{8\pi\gamma\ell_P^2}$
where γ is the Immirzi parameter of Quantum Geometry.
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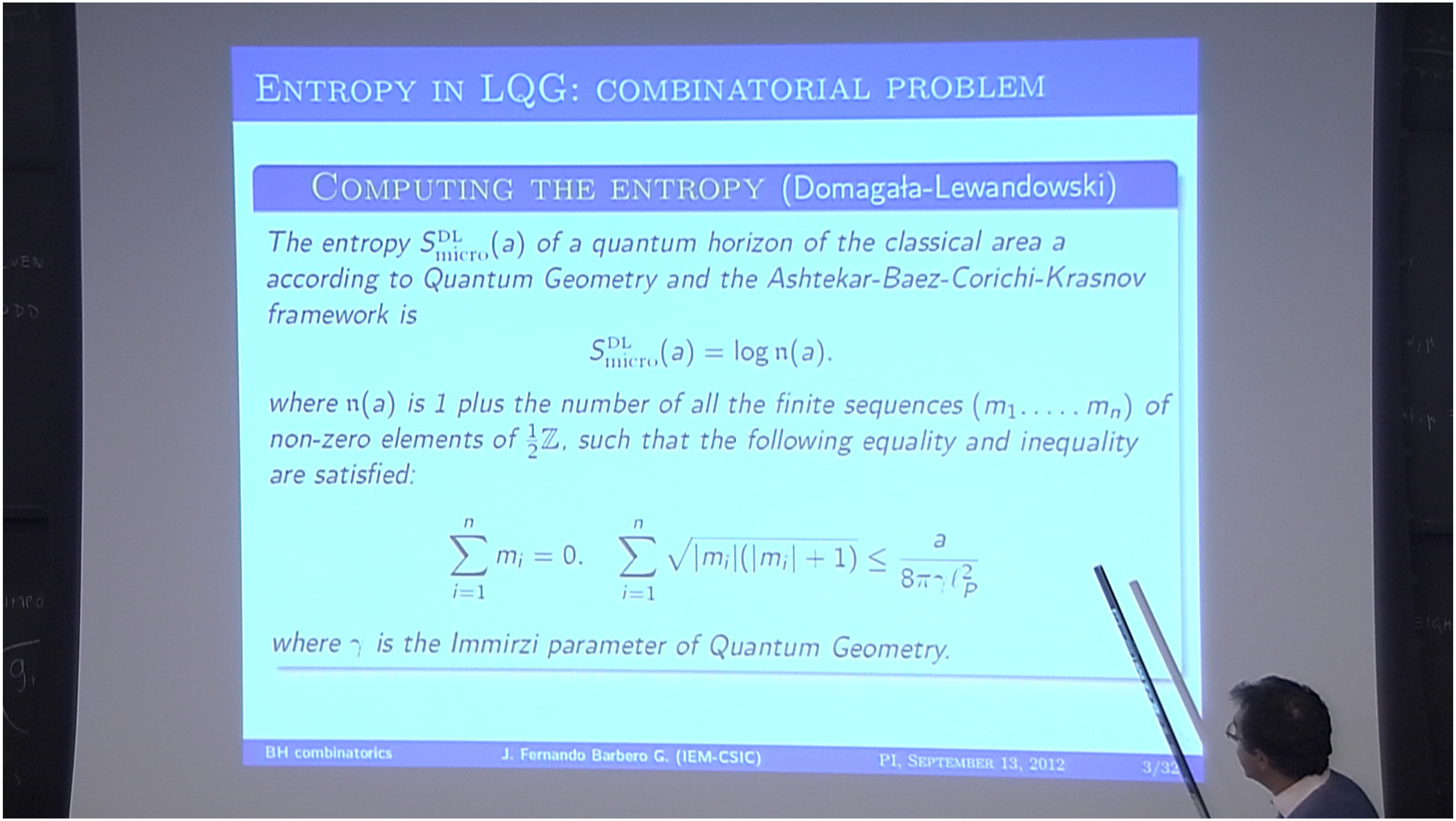
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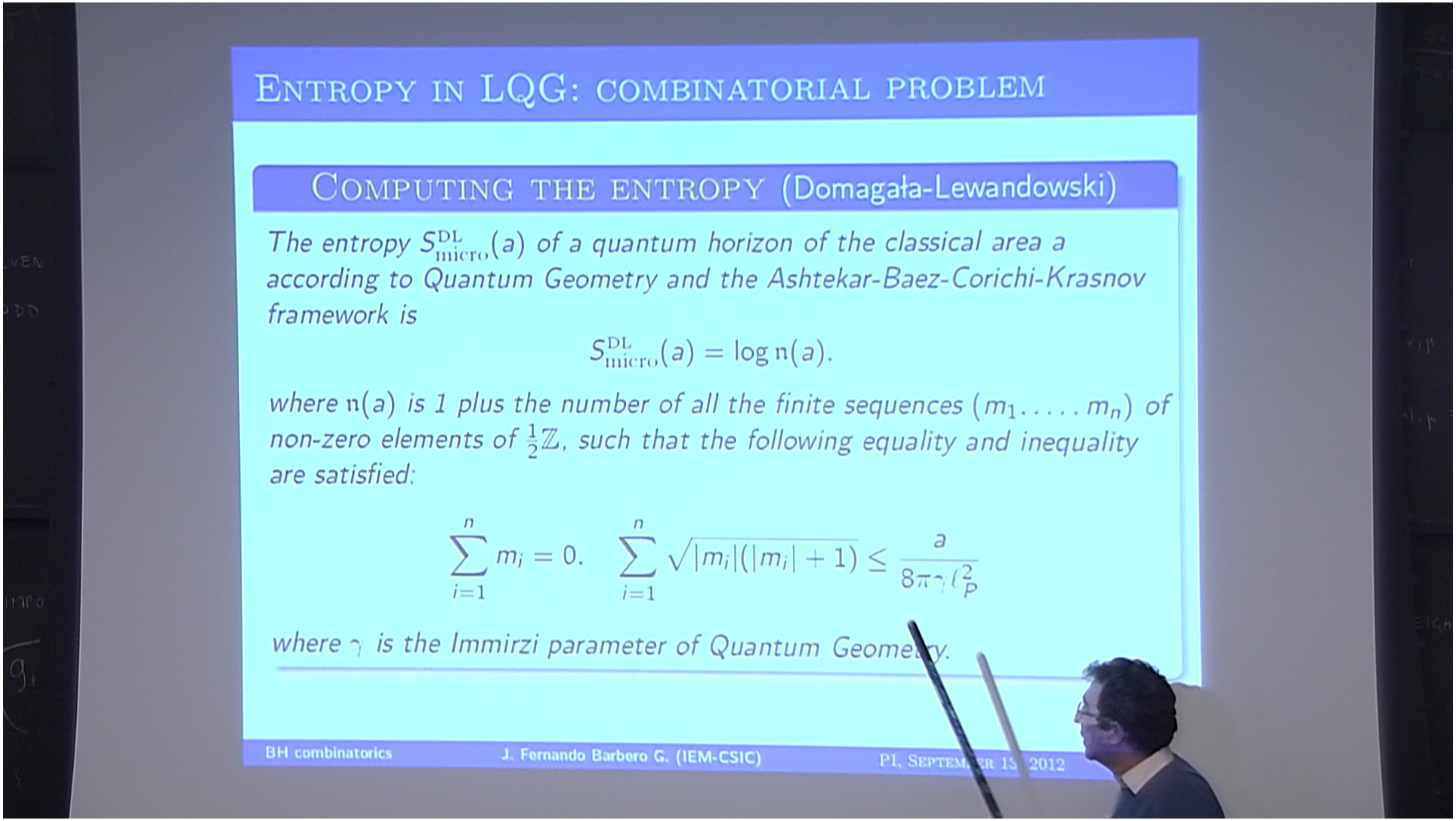
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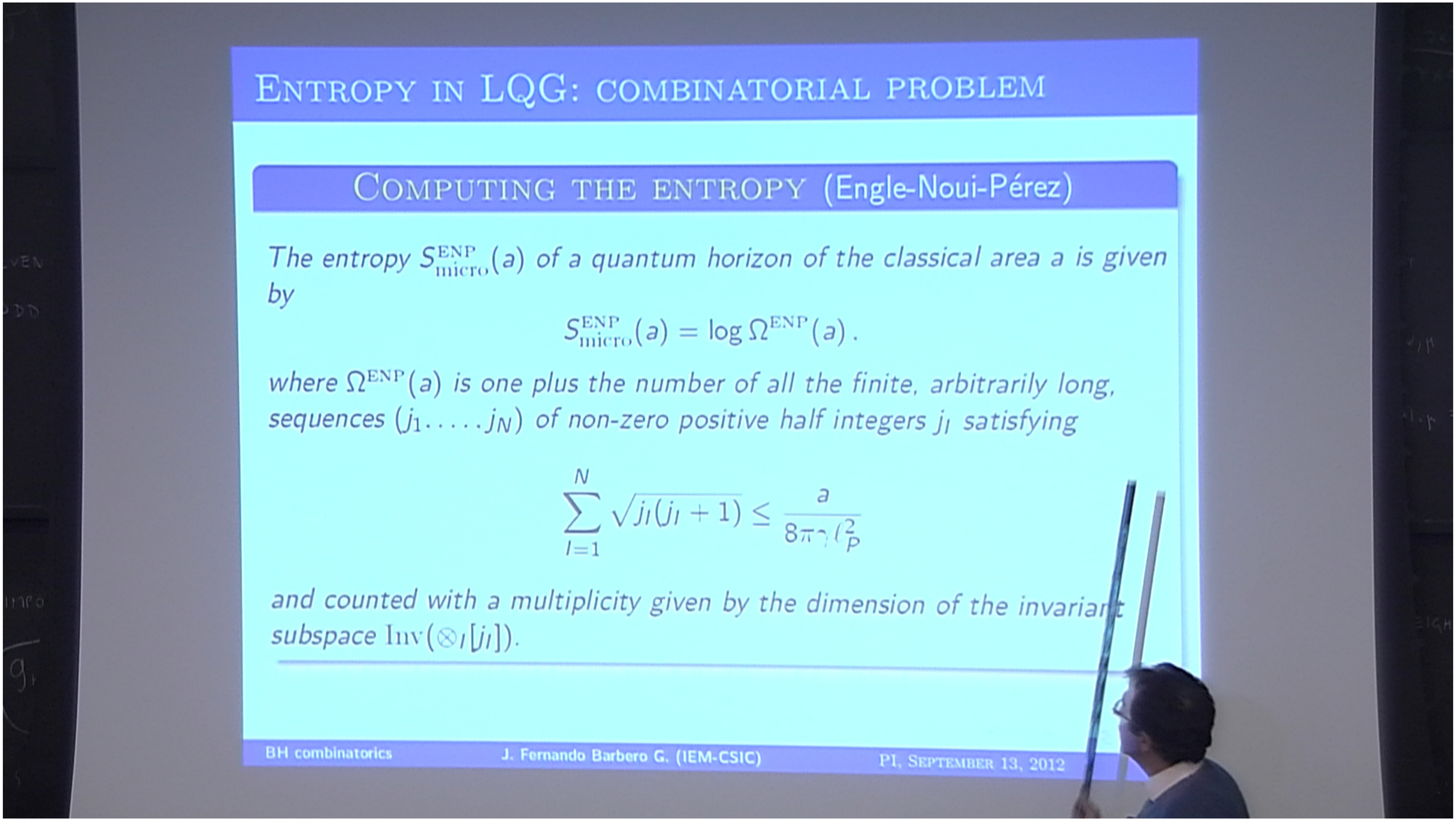
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$\sum_{I=1}^{N} \sqrt{j_I(j_I+1)} \leq \frac{a}{8\pi\gamma \ell_P^2}$
and counted with a multiplicity given by the dimension of the invariant subspace $\mathrm{Inv}(\otimes_I [j_I])$.
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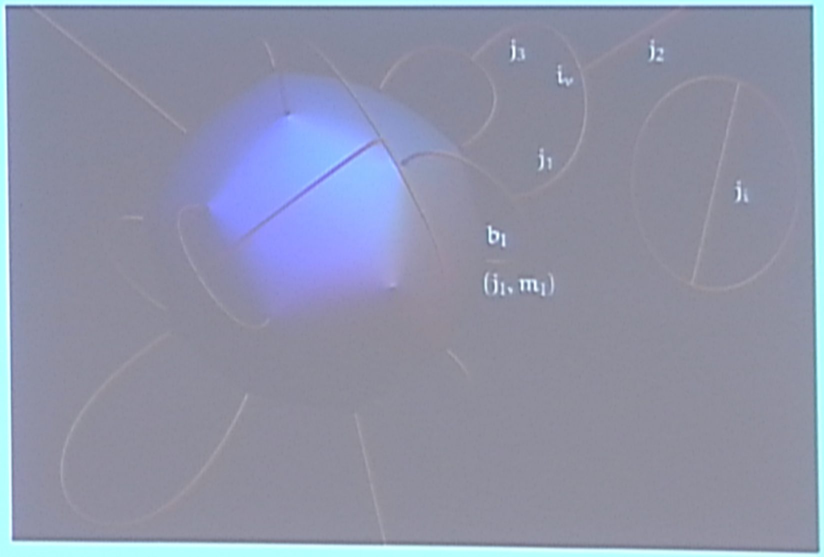
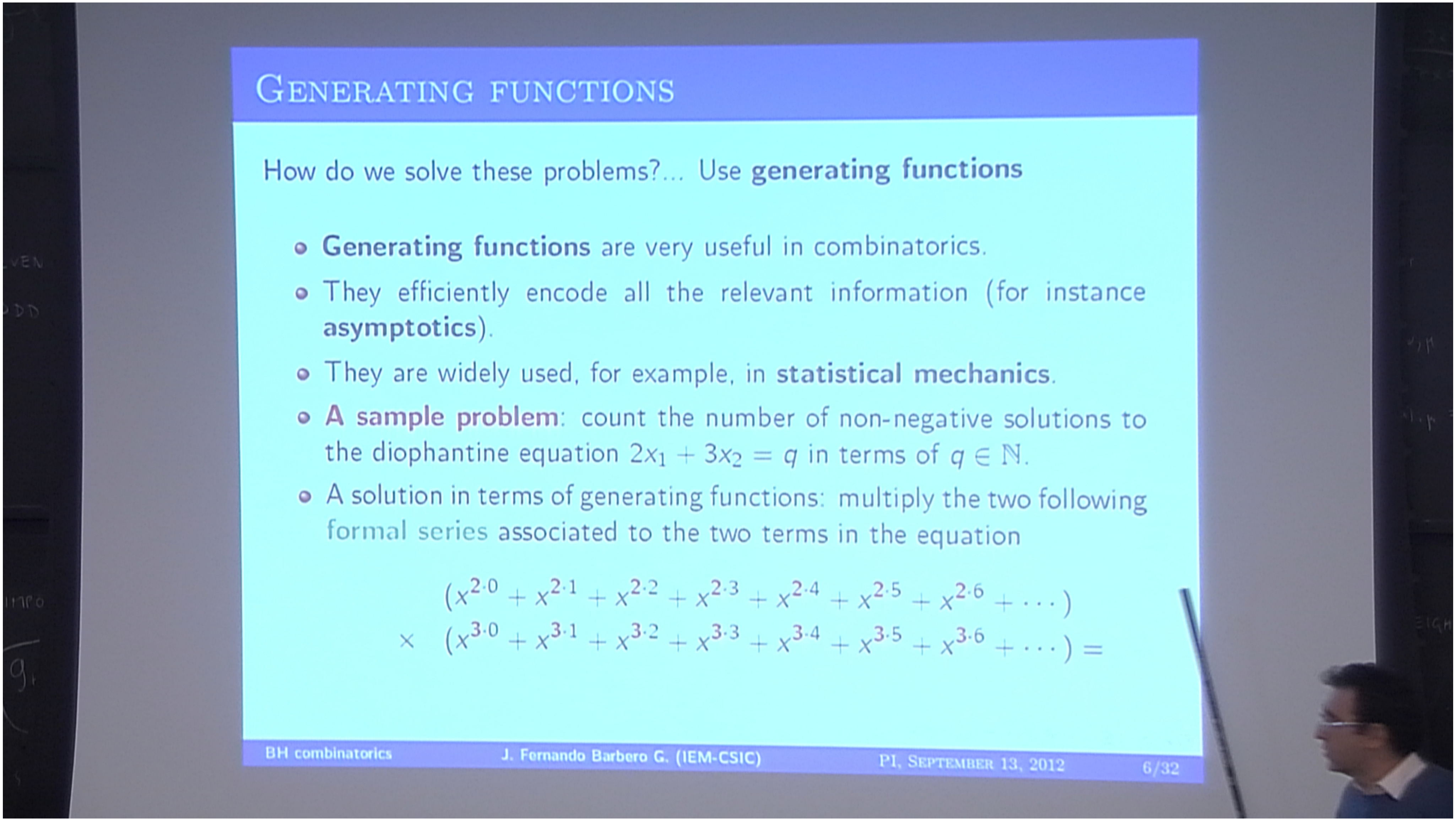
Comments

- The **area operator** has a **discrete spectrum** of the form: $\sum_I \sqrt{j_I(j_I+1)}$ where the j_I correspond to the edges of the graph labeling a quantum state that pierce the horizon. The entropy is computed by solving combinatorial problems involving these labels.




Generating functions
How do we solve these problems?... Use generating functions
Generating functions are very useful in combinatorics.
They efficiently encode all the relevant information (for instance asymptotics).
They are widely used, for example, in statistical mechanics.
A sample problem: count the number of non-negative solutions to the diophantine equation $2x_1 + 3x_2 = q$ in terms of $q \in \mathbb{N}$.
A solution in terms of generating functions: multiply the two following formal series associated to the two terms in the equation
$(x^{2\cdot 0} + x^{2\cdot 1} + x^{2\cdot 2} + x^{2\cdot 3} + x^{2\cdot 4} + x^{2\cdot 5} + x^{2\cdot 6} + \cdots)$
$\times \quad (x^{3\cdot 0} + x^{3\cdot 1} + x^{3\cdot 2} + x^{3\cdot 3} + x^{3\cdot 4} + x^{3\cdot 5} + x^{3\cdot 6} + \cdots) =$
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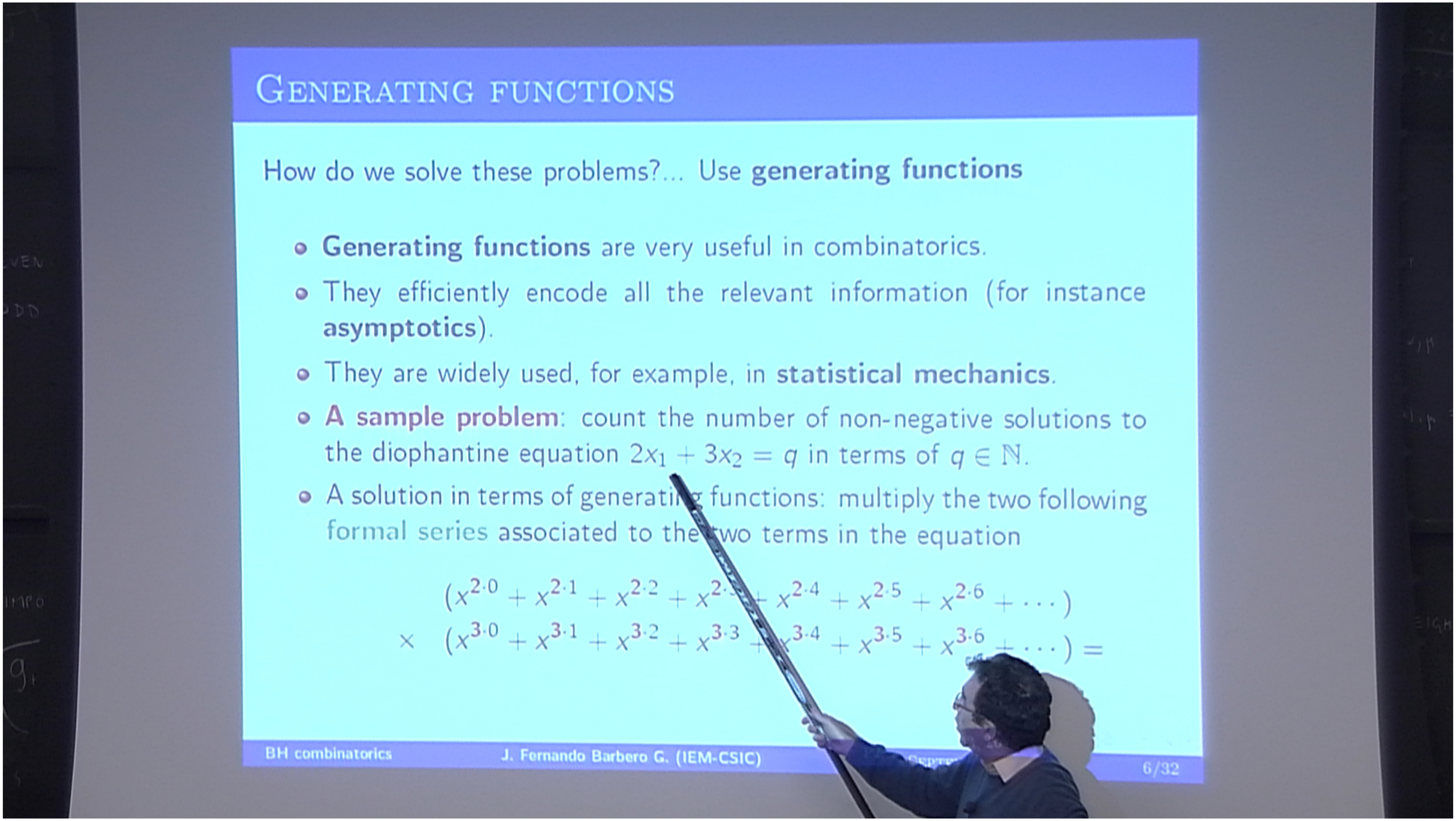
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Generating functions

$$= \underbrace{x^{(2\cdot 0+3\cdot 0)}}_{1} + \underbrace{x^{(2\cdot 1+3\cdot 0)}}_{x^2} + \underbrace{x^{(2\cdot 0+3\cdot 1)}}_{x^3} + \underbrace{x^{(2\cdot 2+3\cdot 0)}}_{x^4} + \underbrace{x^{(2\cdot 1+3\cdot 1)}}_{x^5} +$$

$$\underbrace{x^{(2\cdot 0+3\cdot 2)} + x^{(2\cdot 3+3\cdot 0)}}_{2x^6} + \underbrace{x^{(2\cdot 2+3\cdot 1)}}_{x^7} + \underbrace{x^{(2\cdot 4+3\cdot 0)} + x^{(2\cdot 1+3\cdot 2)}}_{2x^8} + \cdots$$

- The coefficient of the term x^q gives the number of solutions to the diophantine equation $2x_1 + 3x_2 = q$ for the chosen value of q.
- The formal series given above actually correspond to meromorphic functions of a complex variable $x \in \mathbb{C}$. In this case

$$(x^{2\cdot 0} + x^{2\cdot 1} + x^{2\cdot 2} + x^{2\cdot 3} + x^{2\cdot 4} + x^{2\cdot 5} + x^{2\cdot 6} + \cdots) = \frac{1}{1-x^2}$$

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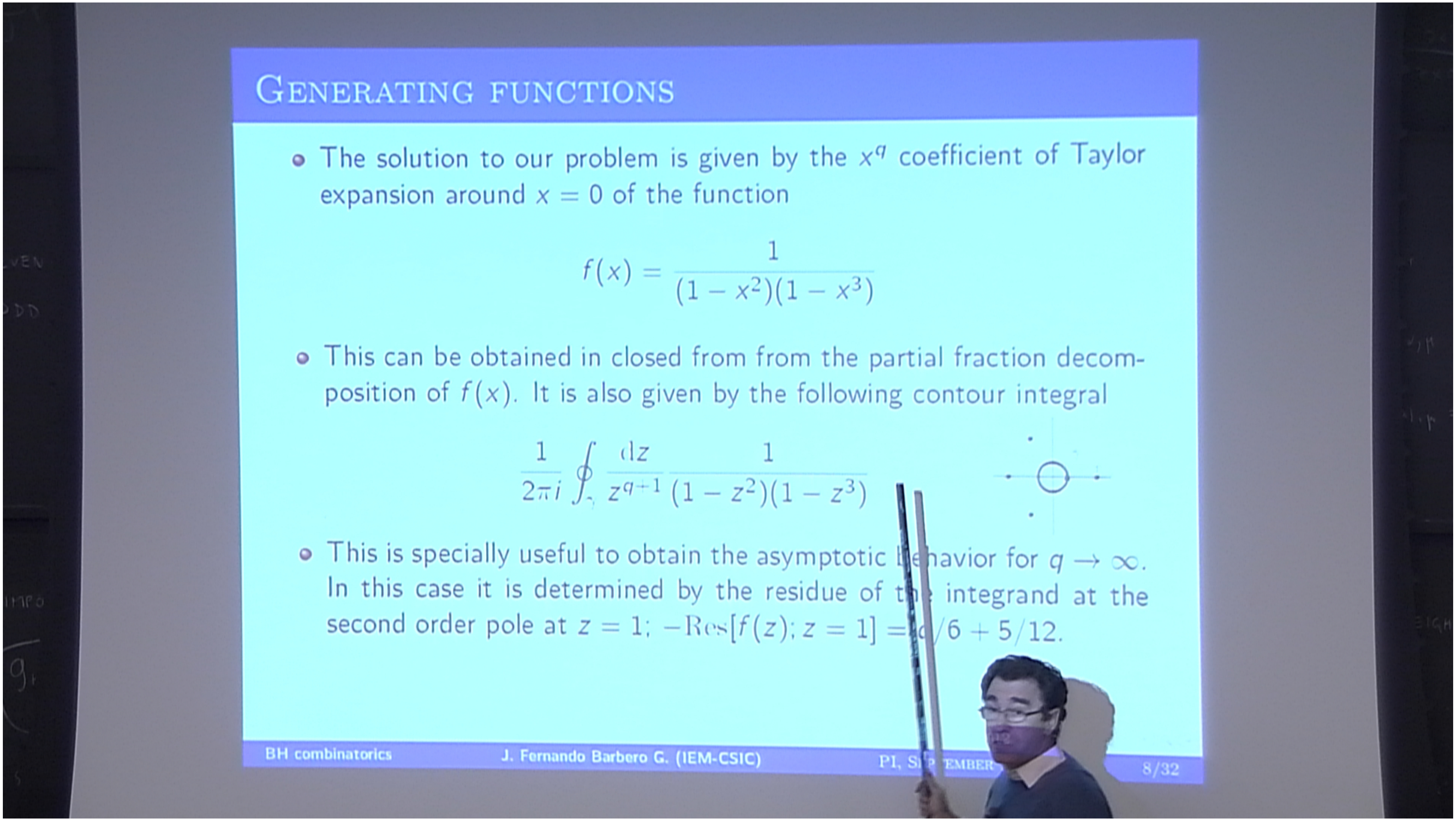
Generating functions
The solution to our problem is given by the x^q coefficient of Taylor expansion around $x = 0$ of the function
$f(x) = \frac{1}{(1-x^2)(1-x^3)}$
This can be obtained in closed from from the partial fraction decomposition of $f(x)$. It is also given by the following contour integral
$\frac{1}{2\pi i}\oint \frac{dz}{z^{q+1}}\frac{1}{(1-z^2)(1-z^3)}$
This is specially useful to obtain the asymptotic behavior for $q \to \infty$. In this case it is determined by the residue of the integrand at the second order pole at $z = 1$; $-\mathrm{Res}[f(z); z = 1] =$ /6 + 5/12.
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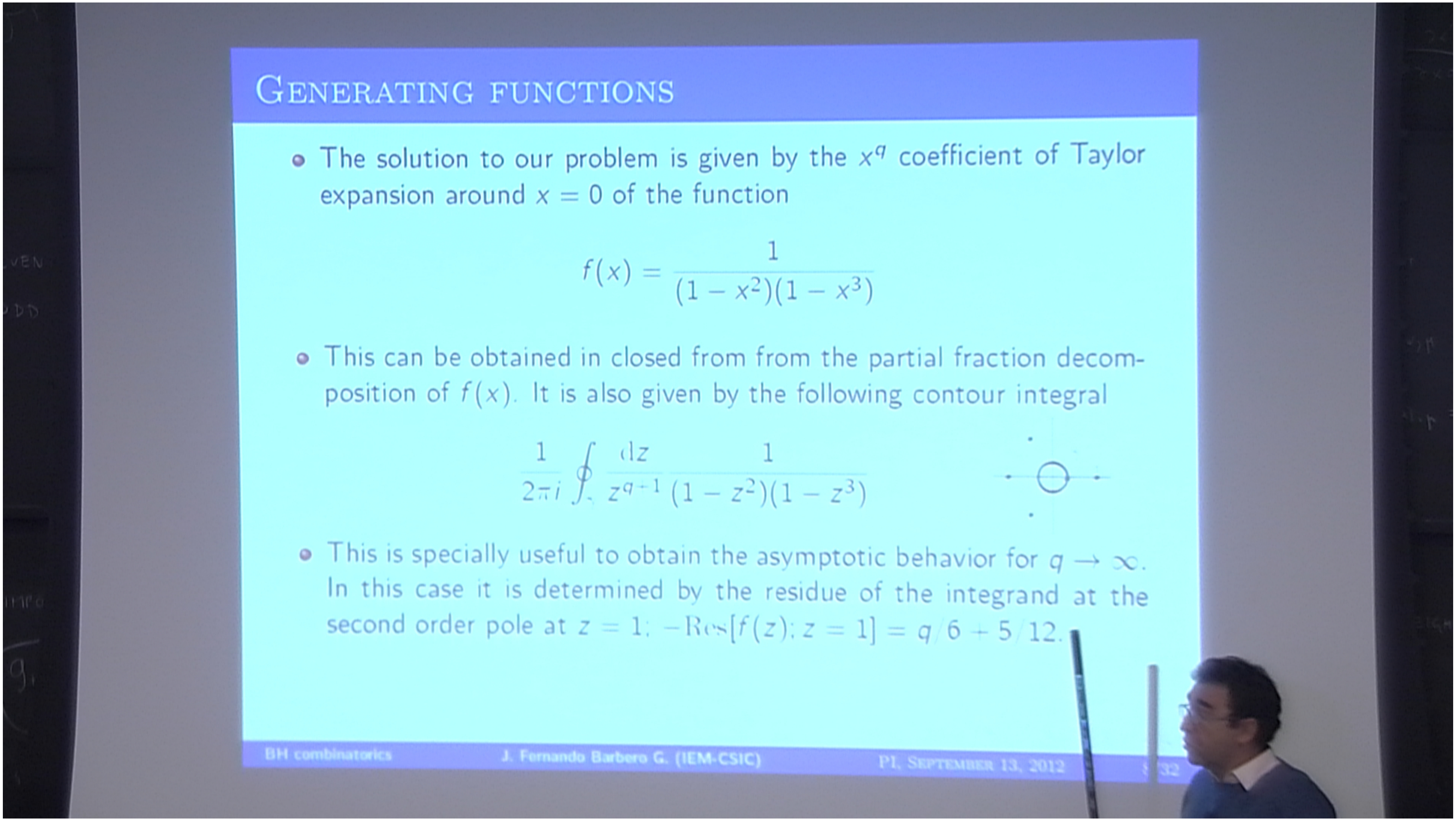
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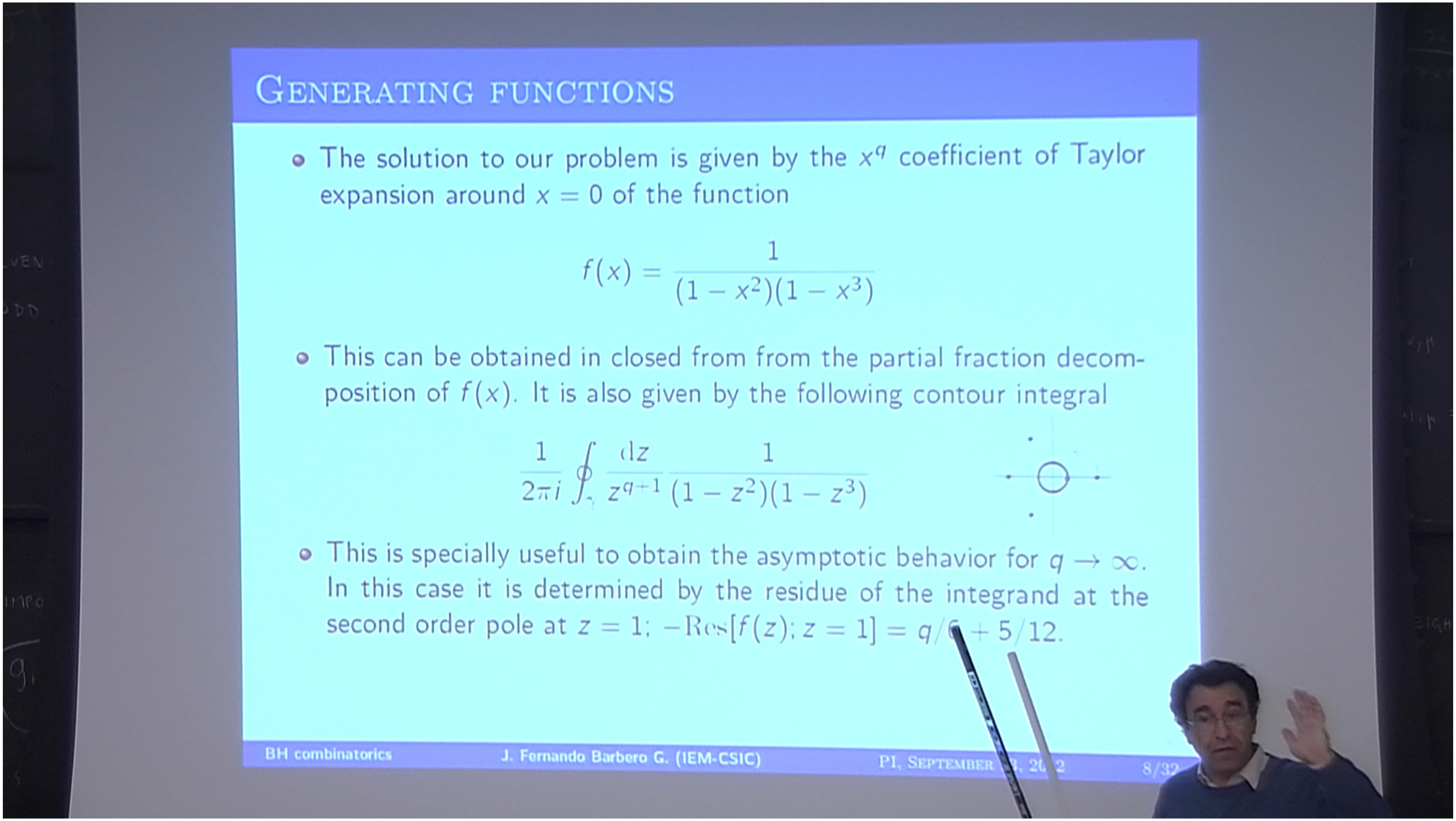
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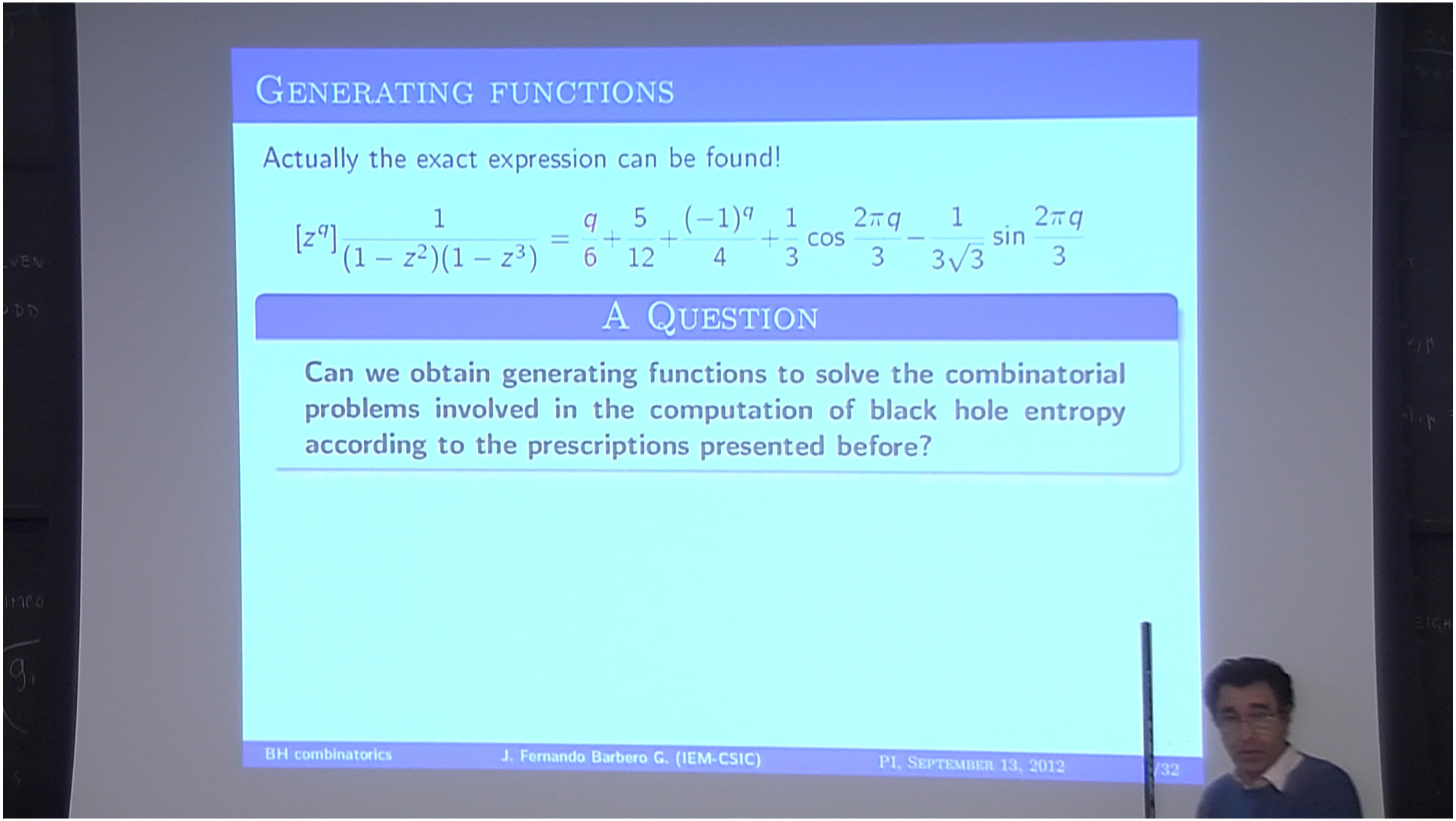
GENERATING FUNCTIONS
Actually the exact expression can be found!
$[z^q]\frac{1}{(1-z^2)(1-z^3)} = \frac{q}{6}+\frac{5}{12}+\frac{(-1)^q}{4}+\frac{1}{3}\cos\frac{2\pi q}{3}-\frac{1}{3\sqrt{3}}\sin\frac{2\pi q}{3}$
A QUESTION
Can we obtain generating functions to solve the combinatorial problems involved in the computation of black hole entropy according to the prescriptions presented before?
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YES!

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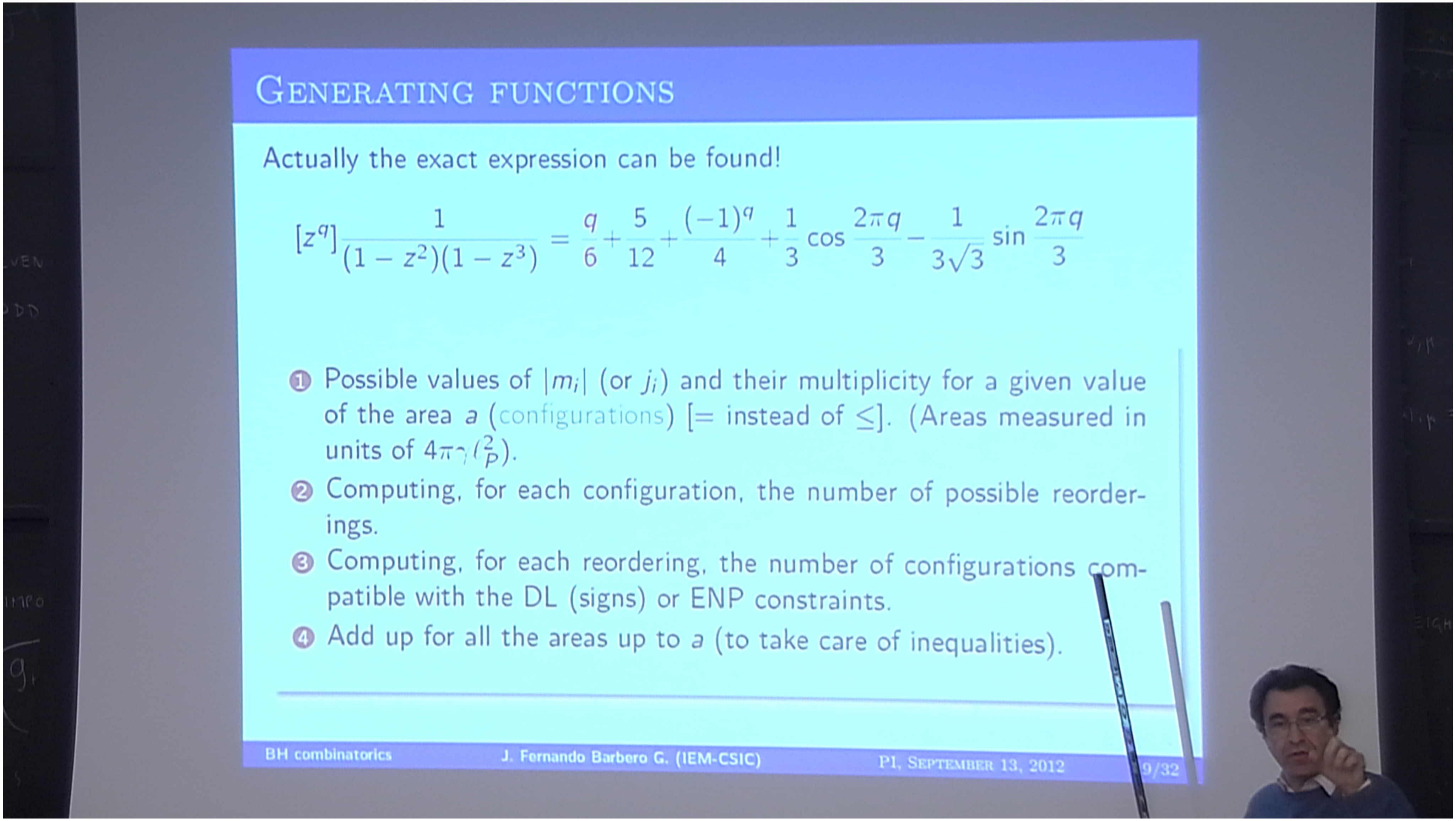
1. Possible values of $|m_i|$ (or j_i) and their multiplicity for a given value of the area a (configurations) [= instead of $\leq$]. (Areas measured in units of $4\pi\gamma\ell_P^2$).
2. Computing, for each configuration, the number of possible reorderings.
3. Computing, for each reordering, the number of configurations compatible with the DL (signs) or ENP constraints.
4. Add up for all the areas up to a (to take care of inequalities).



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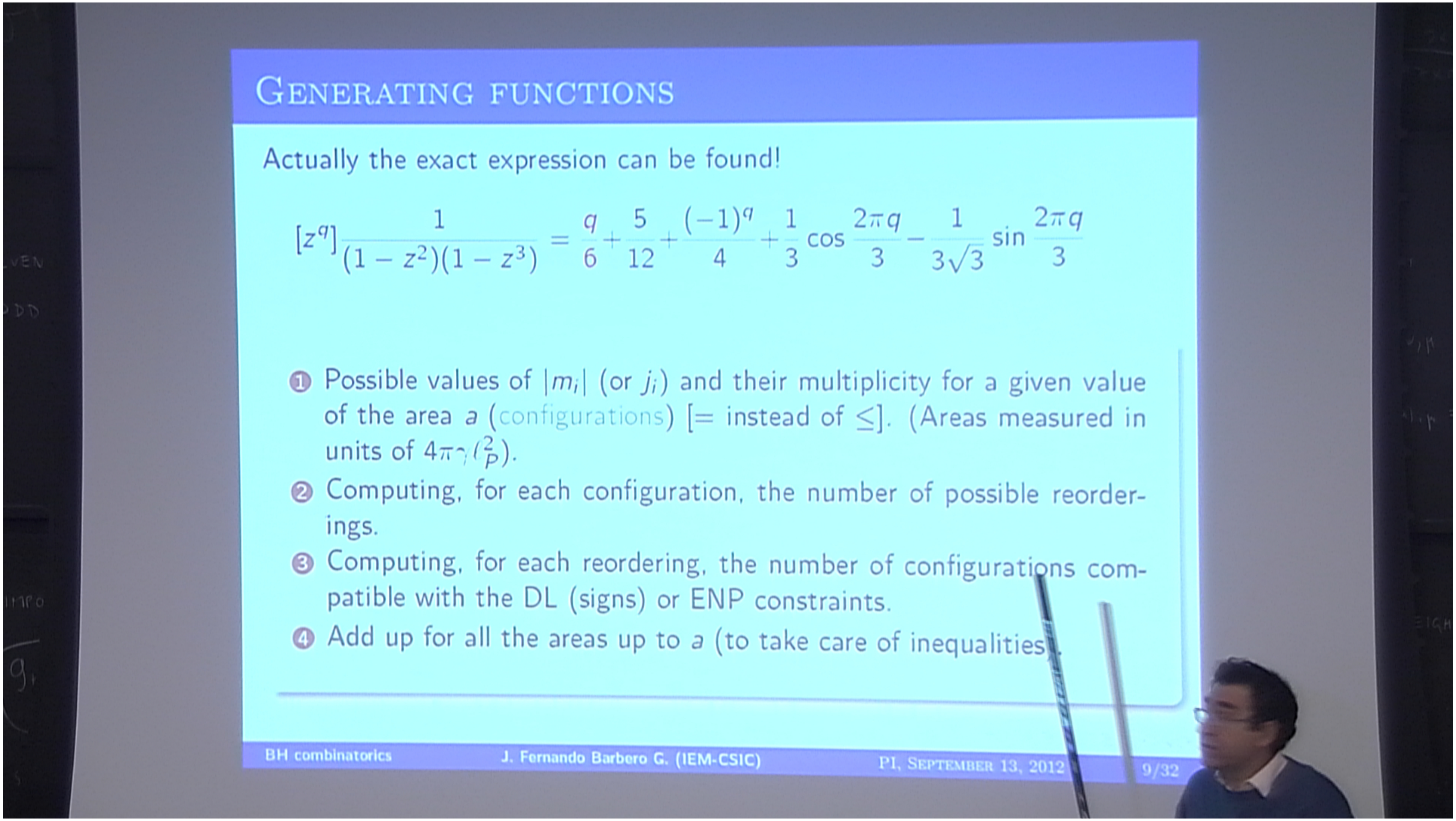
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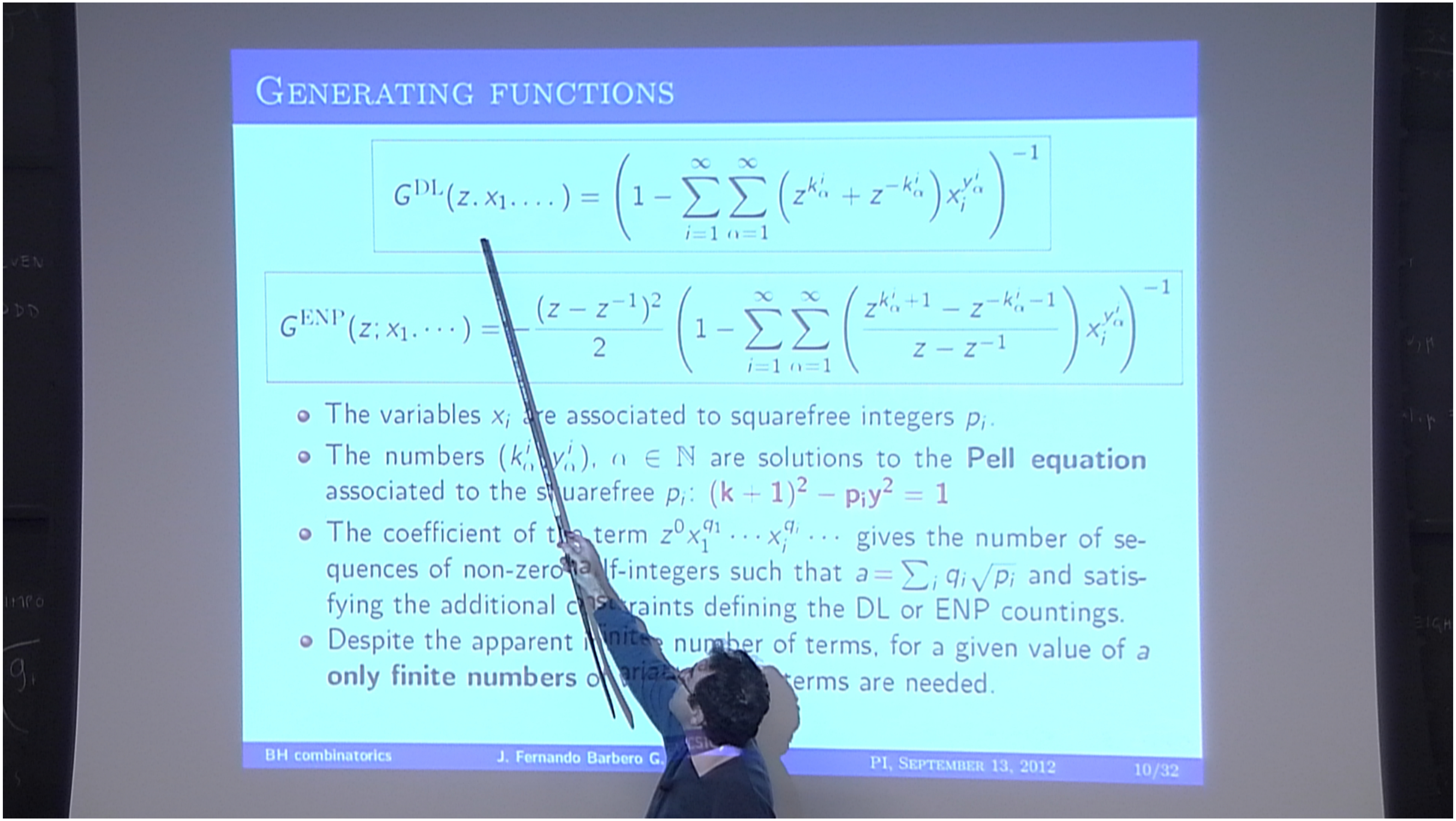
GENERATING FUNCTIONS
$G^{\mathrm{DL}}(z, x_1, \dots) = \left(1 - \sum_{i=1}^{\infty}\sum_{\alpha=1}^{\infty}\left(z^{k_\alpha^i} + z^{-k_\alpha^i}\right)x_i^{y_\alpha^i}\right)^{-1}$
$G^{\mathrm{ENP}}(z; x_1, \cdots) = -\frac{(z - z^{-1})^2}{2}\left(1 - \sum_{i=1}^{\infty}\sum_{\alpha=1}^{\infty}\left(\frac{z^{k_\alpha^i+1} - z^{-k_\alpha^i-1}}{z - z^{-1}}\right)x_i^{y_\alpha^i}\right)^{-1}$
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The coefficient of the term $z^0 x_1^{q_1} \cdots x_i^{q_i} \cdots$ gives the number of sequences of non-zero half-integers such that $a = \sum_i q_i\sqrt{p_i}$ and satisfying the additional constraints defining the DL or ENP countings.
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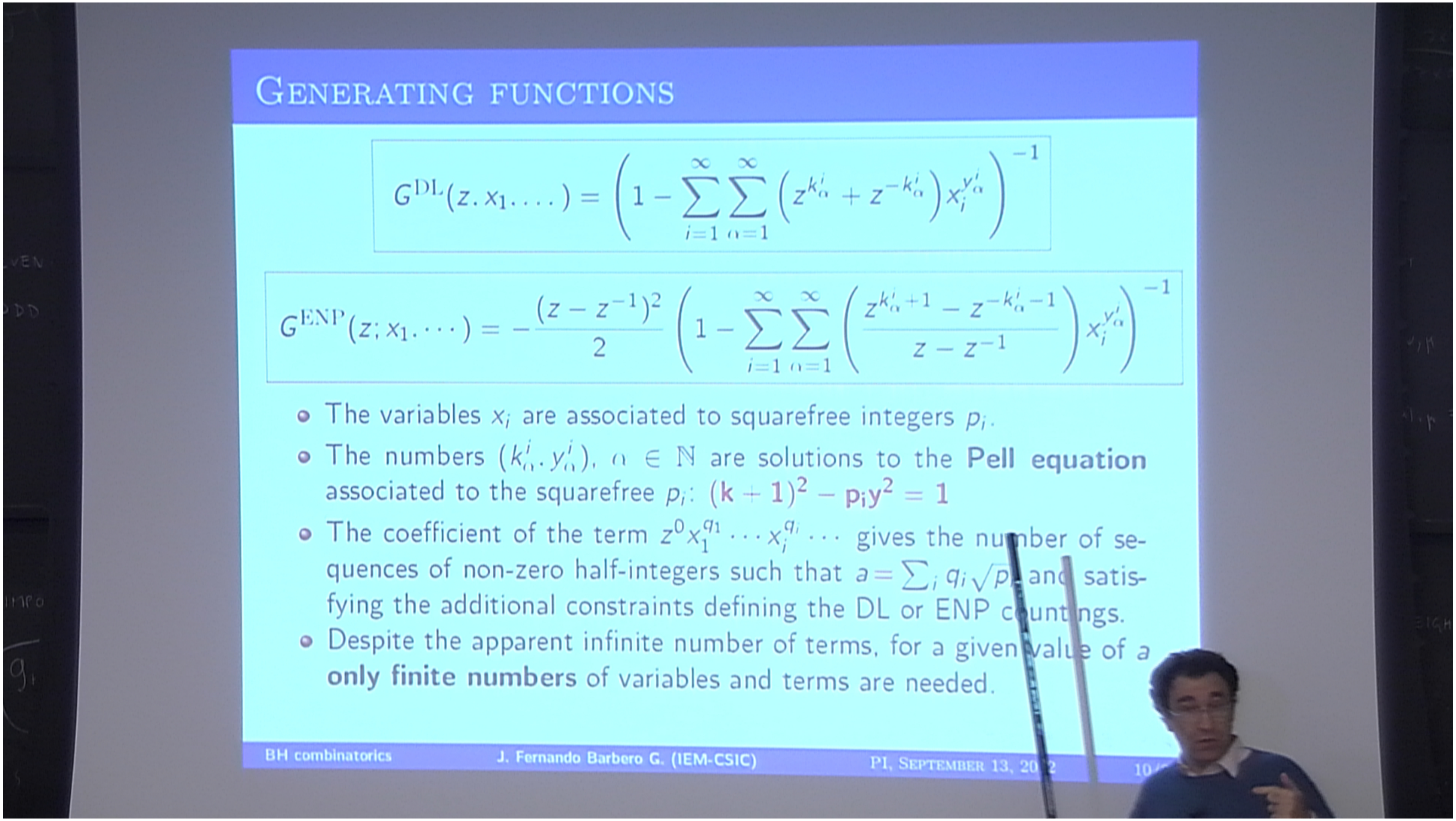
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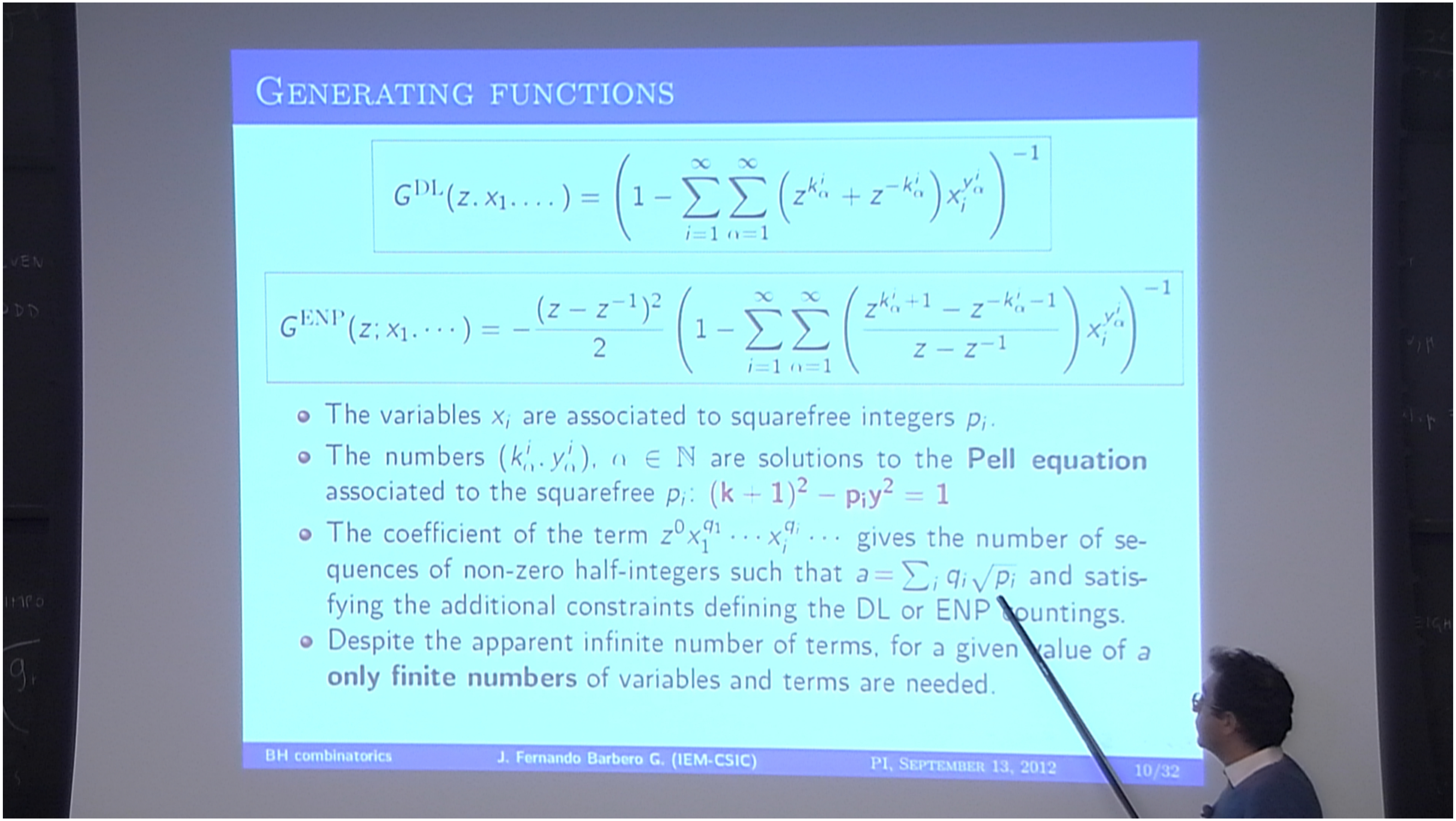

Generating functions
$$G^{\mathrm{DL}}(z, x_1, \dots) = \left(1 - \sum_{i=1}^{\infty}\sum_{\alpha=1}^{\infty}\left(z^{k_\alpha^i} + z^{-k_\alpha^i}\right)x_i^{y_\alpha^i}\right)^{-1}$$
$$G^{\mathrm{ENP}}(z; x_1, \cdots) = -\frac{(z - z^{-1})^2}{2}\left(1 - \sum_{i=1}^{\infty}\sum_{\alpha=1}^{\infty}\left(\frac{z^{k_\alpha^i+1} - z^{-k_\alpha^i-1}}{z - z^{-1}}\right)x_i^{y_\alpha^i}\right)^{-1}$$
The variables x_i are associated to squarefree integers p_i.
The numbers (k_α^i, y_α^i), $\alpha \in \mathbb{N}$ are solutions to the **Pell equation** associated to the squarefree p_i: $(\mathbf{k}+\mathbf{1})^2 - \mathbf{p_i y^2} = \mathbf{1}$
The coefficient of the term $z^0 x_1^{q_1} \cdots x_i^{q_i} \cdots$ gives the number of sequences of non-zero half-integers such that $a = \sum_i q_i \sqrt{p_i}$ and satisfying the additional constraints defining the DL or ENP countings.
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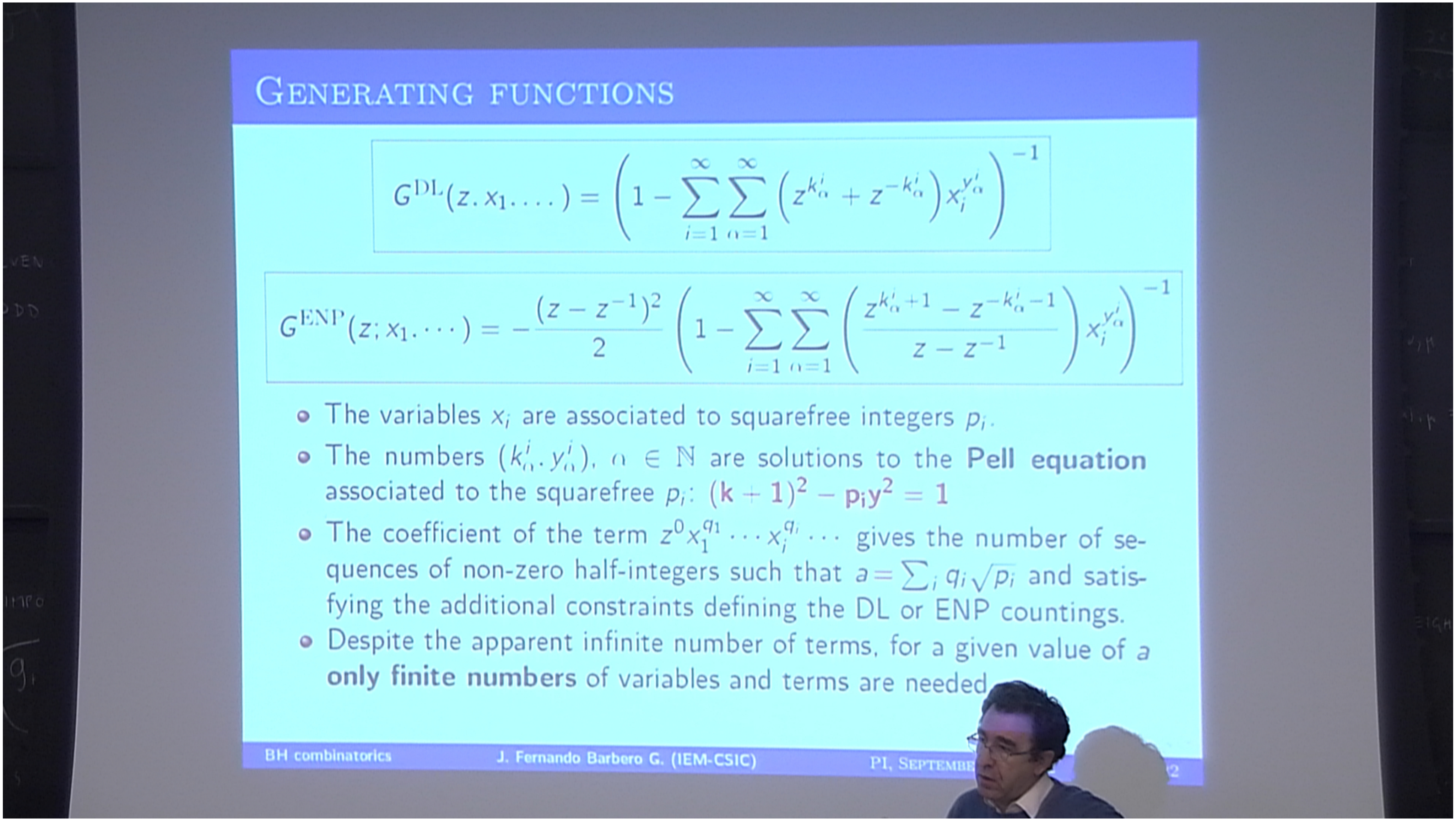
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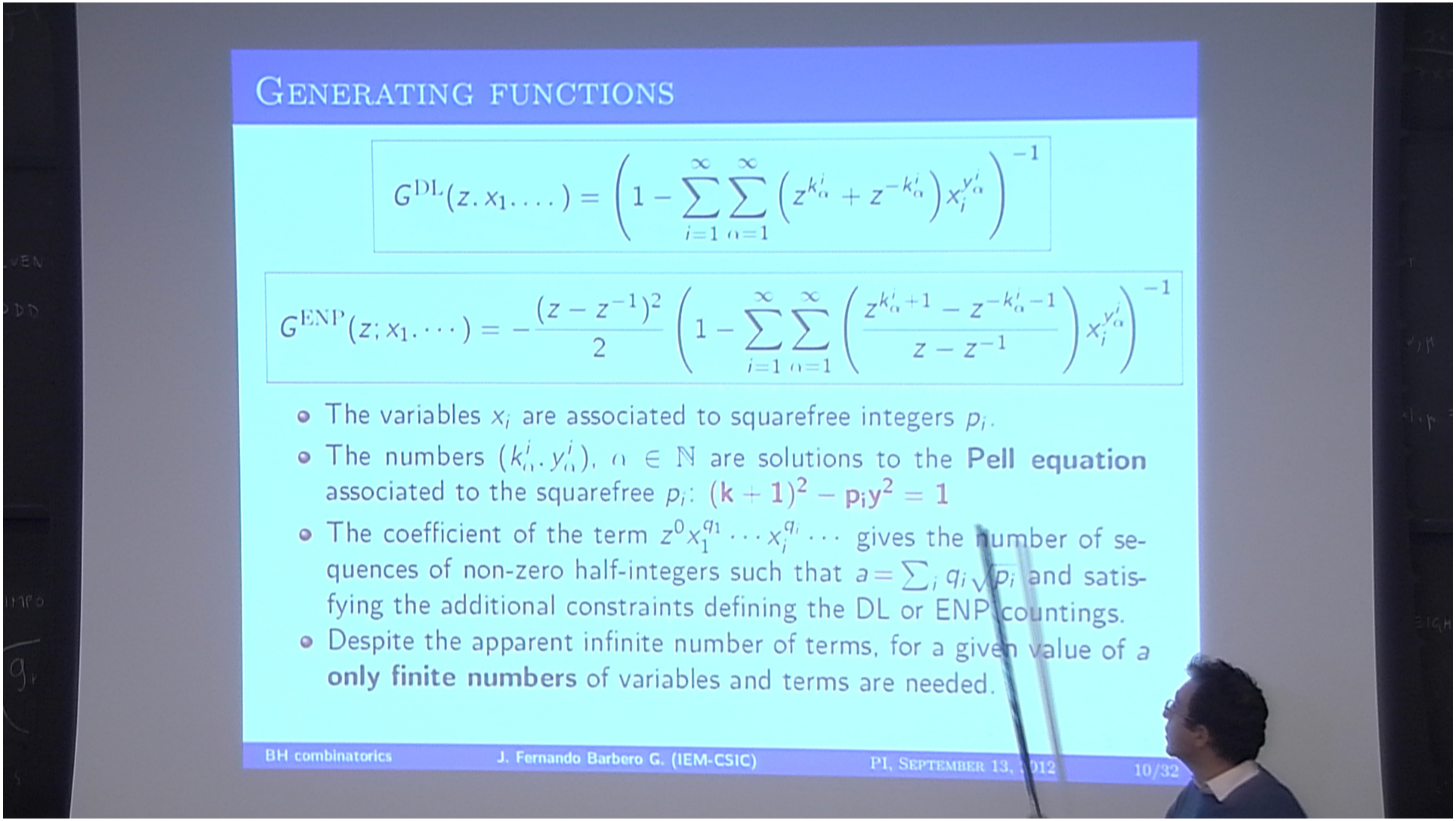
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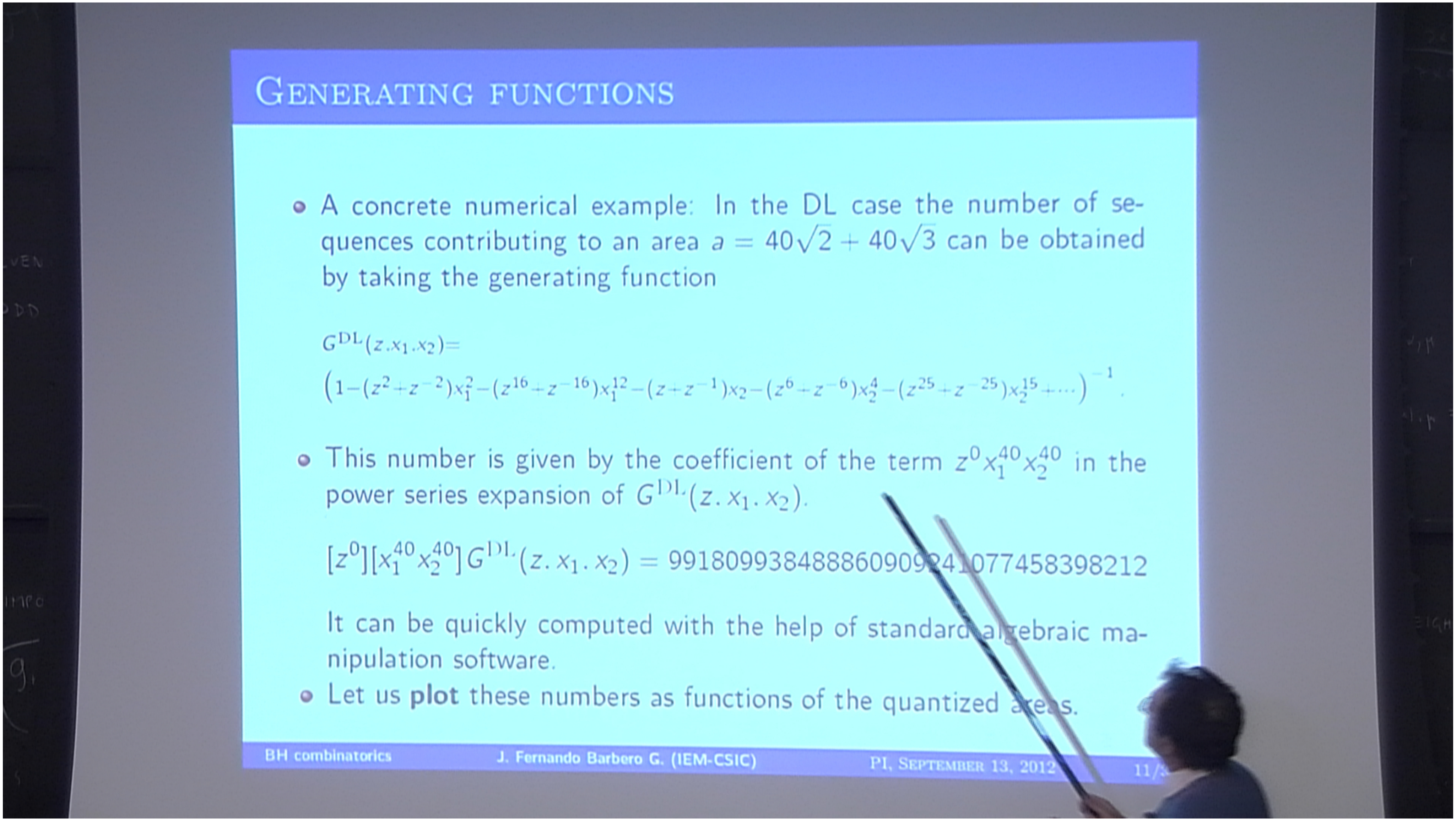
GENERATING FUNCTIONS
A concrete numerical example: In the DL case the number of sequences contributing to an area $a = 40\sqrt{2} + 40\sqrt{3}$ can be obtained by taking the generating function
$G^{\mathrm{DL}}(z, x_1, x_2) =$
$\left(1-(z^2+z^{-2})x_1^2-(z^{16}+z^{-16})x_1^{12}-(z+z^{-1})x_2-(z^6+z^{-6})x_2^4-(z^{25}+z^{-25})x_2^{15}+\cdots\right)^{-1}.$
This number is given by the coefficient of the term $z^0 x_1^{40} x_2^{40}$ in the power series expansion of $G^{\mathrm{DL}}(z, x_1, x_2)$.
$[z^0][x_1^{40}x_2^{40}]G^{\mathrm{DL}}(z, x_1, x_2) = 991809938488860909241077458398212$
It can be quickly computed with the help of standard algebraic manipulation software.
Let us **plot** these numbers as functions of the quantized areas.
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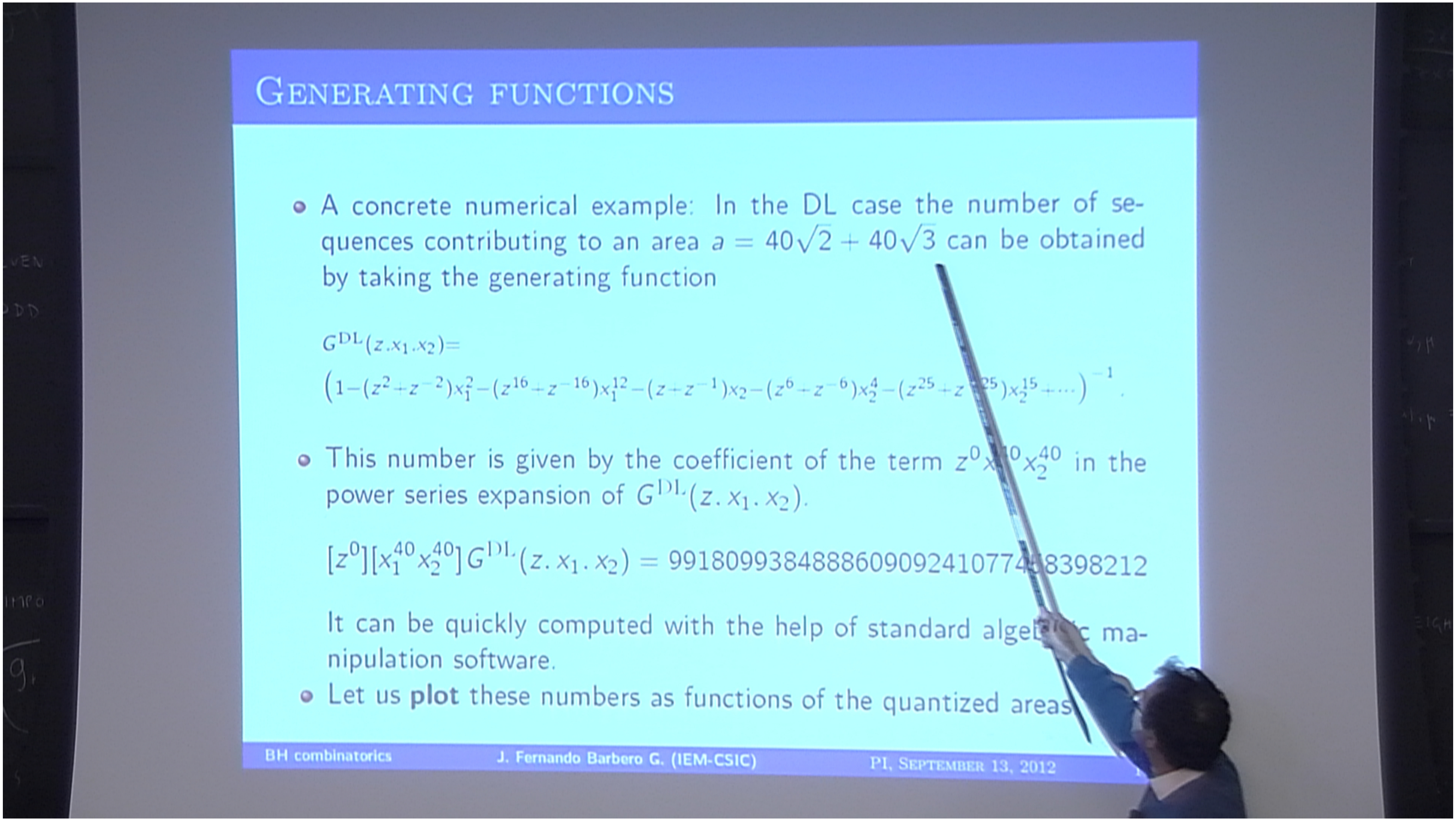
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8398212
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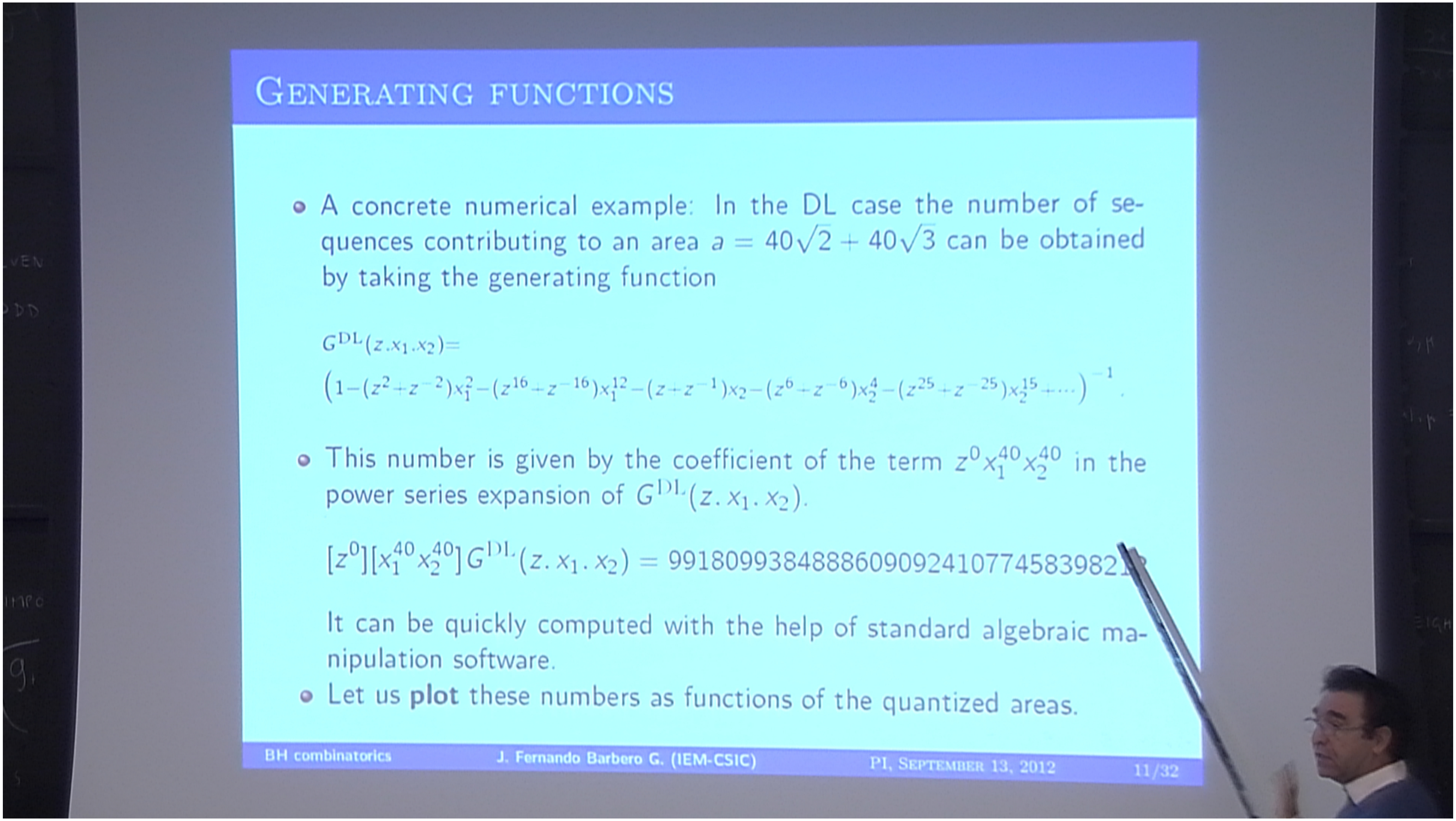
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ENTROPY
10.0
9.5
9.0
8.5
8.0
7.5
7.0
14
15
16
17
18
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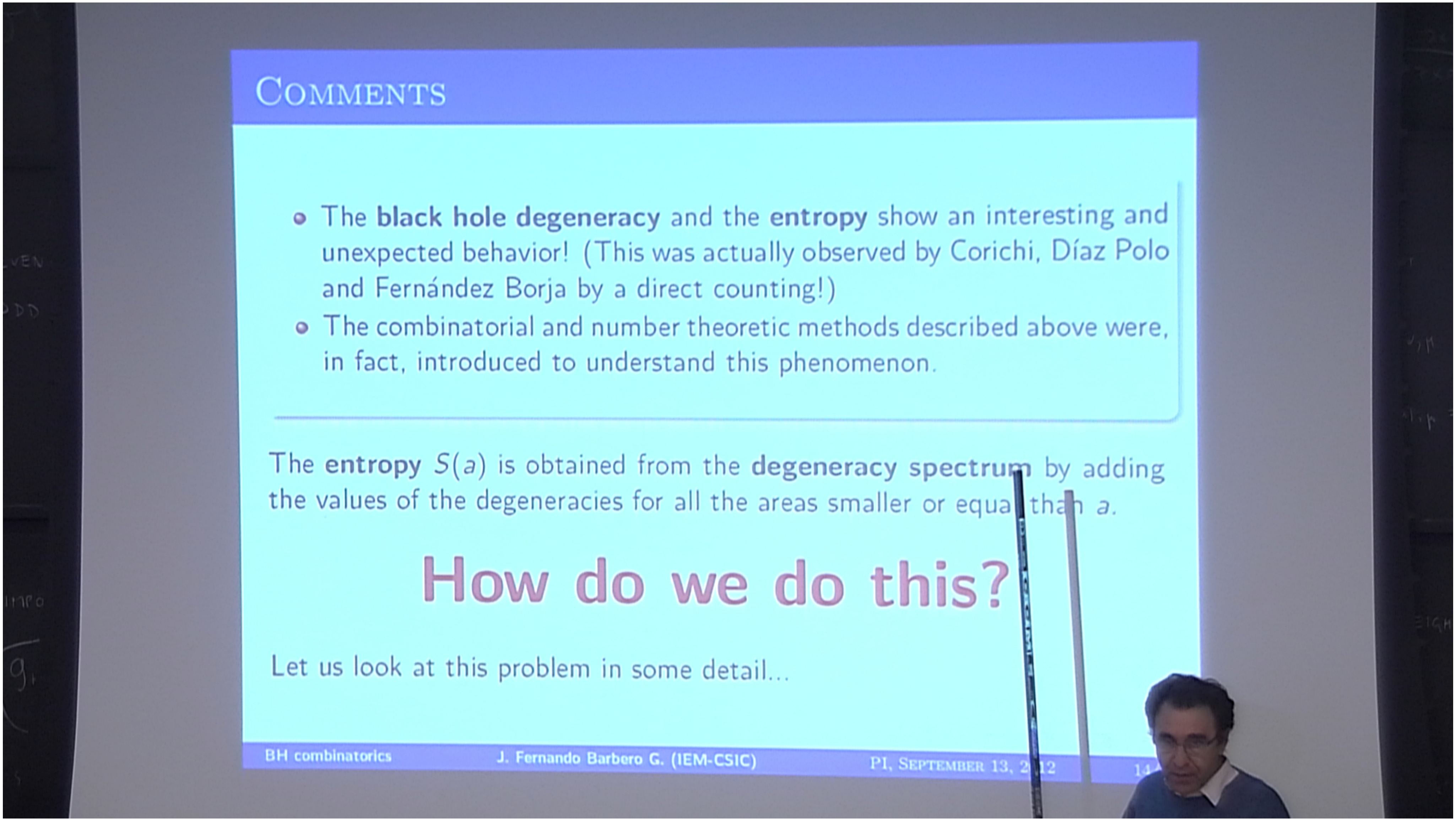
Comments
The black hole degeneracy and the entropy show an interesting and unexpected behavior! (This was actually observed by Corichi, Díaz Polo and Fernández Borja by a direct counting!)
The combinatorial and number theoretic methods described above were, in fact, introduced to understand this phenomenon.
The entropy S(a) is obtained from the degeneracy spectrum by adding the values of the degeneracies for all the areas smaller or equal than a.
How do we do this?
Let us look at this problem in some detail...
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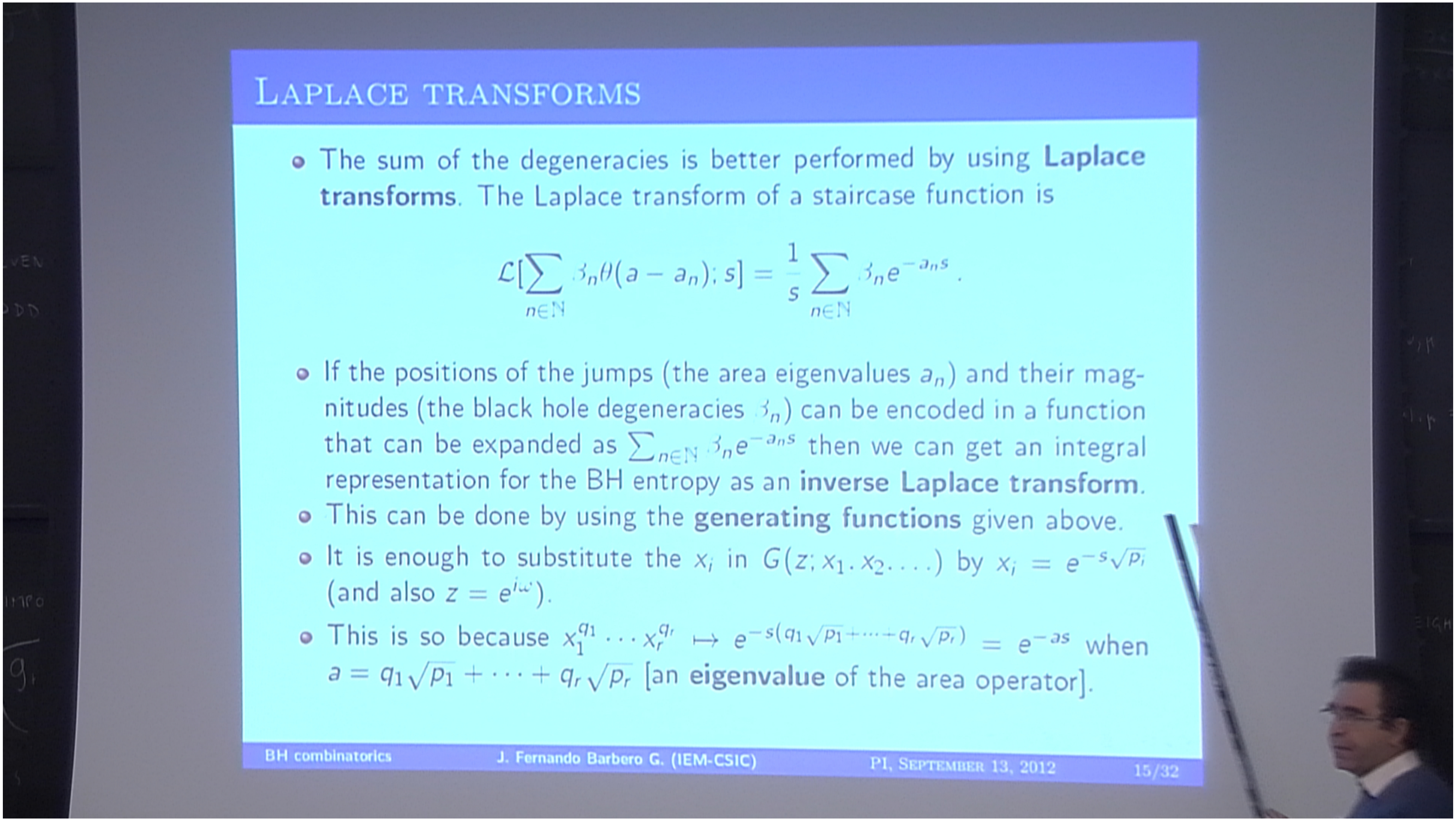
Laplace transforms
The sum of the degeneracies is better performed by using Laplace transforms. The Laplace transform of a staircase function is
$\mathcal{L}[\sum_{n\in\mathbb{N}} \beta_n \theta(a-a_n); s] = \frac{1}{s}\sum_{n\in\mathbb{N}} \beta_n e^{-a_n s}$.
If the positions of the jumps (the area eigenvalues a_n) and their magnitudes (the black hole degeneracies β_n) can be encoded in a function that can be expanded as $\sum_{n\in\mathbb{N}} \beta_n e^{-a_n s}$ then we can get an integral representation for the BH entropy as an inverse Laplace transform.
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This is so because $x_1^{q_1} \cdots x_r^{q_r} \mapsto e^{-s(q_1\sqrt{p_1}+\cdots+q_r\sqrt{p_r})} = e^{-as}$ when $a = q_1\sqrt{p_1} + \cdots + q_r\sqrt{p_r}$ [an eigenvalue of the area operator].
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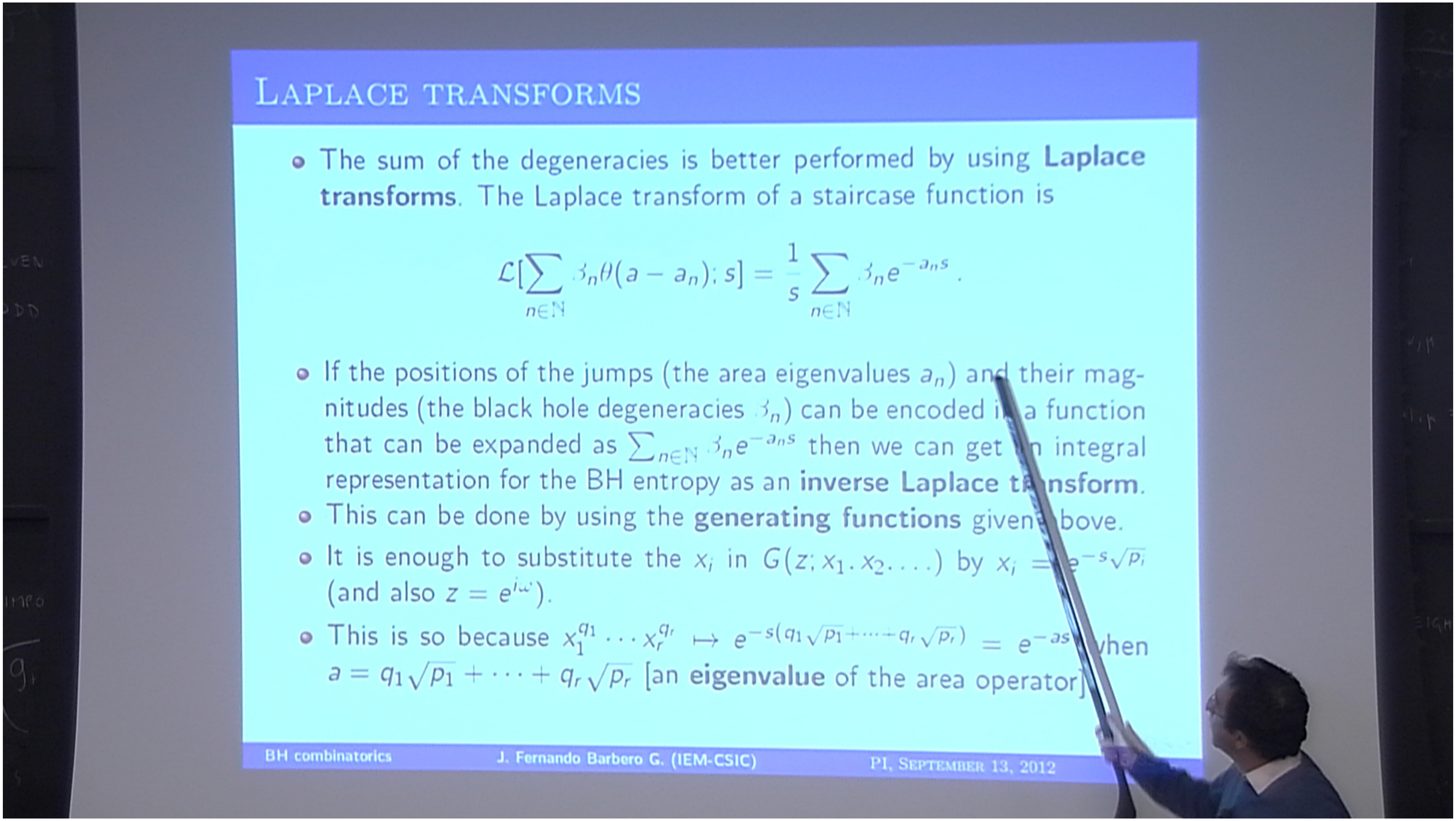
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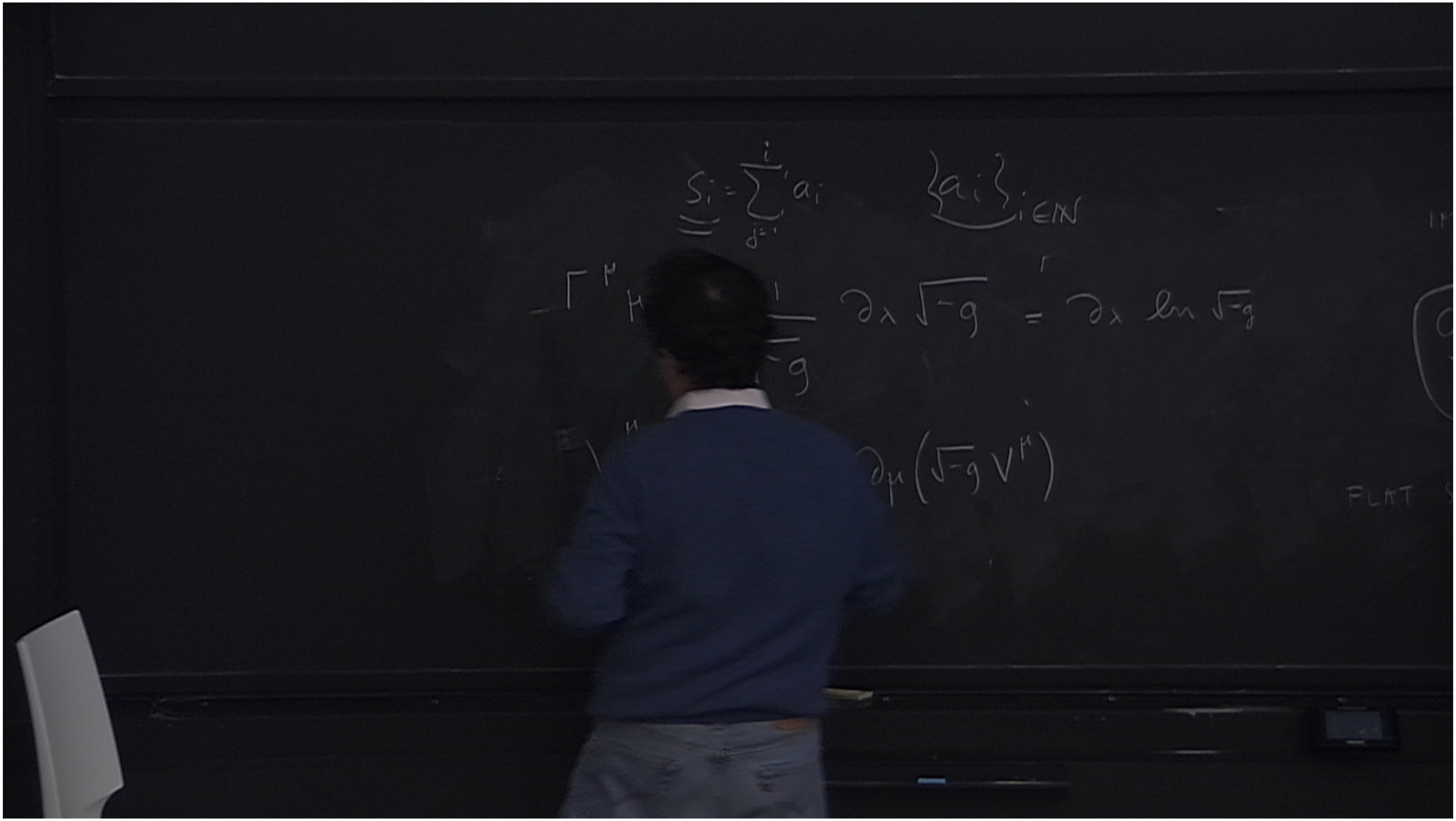
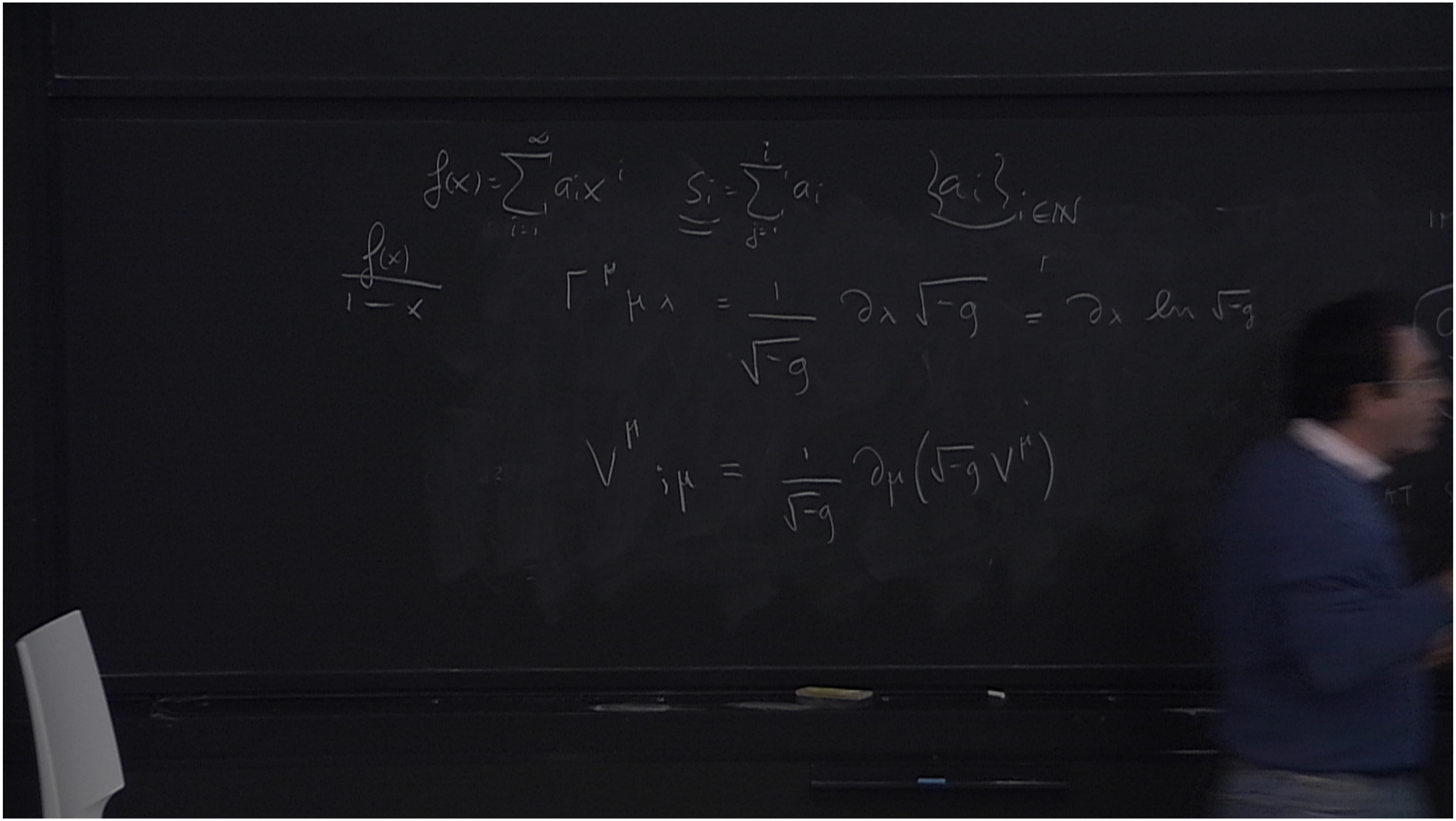
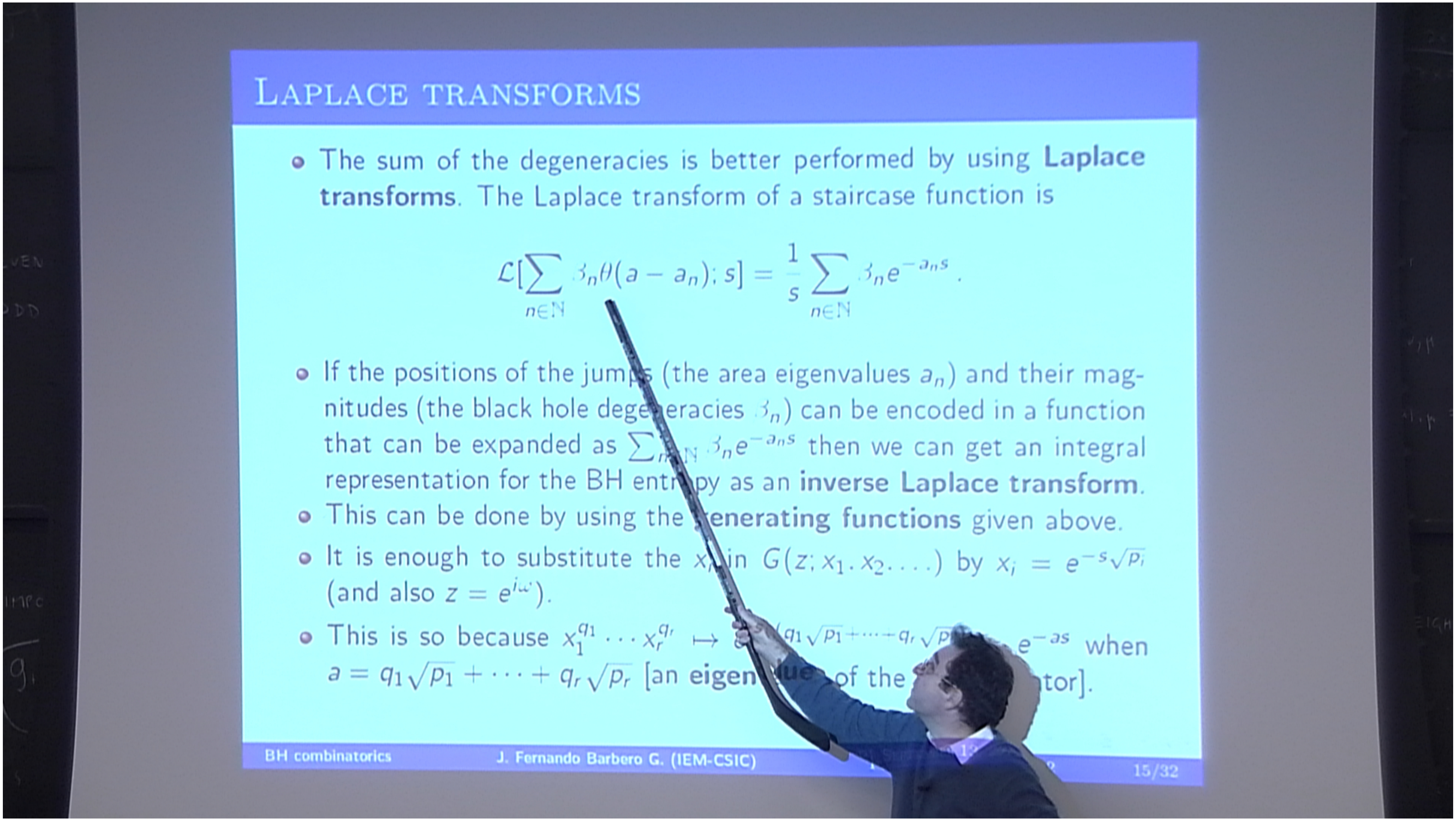

FLAT


$f(x) = \sum_{i=1}^{\infty} a_i x^i$
$S_i = \sum_{j=1}^{i} a_j$
$\{a_i\}_{i \in \mathbb{N}}$
$\frac{f(x)}{1-x}$
$\Gamma^{\mu}{}_{\mu\lambda} = \frac{1}{\sqrt{-g}} \partial_\lambda \sqrt{-g} = \partial_\lambda \ln \sqrt{-g}$
$V^{\mu}{}_{;\mu} = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\, V^{\mu} \right)$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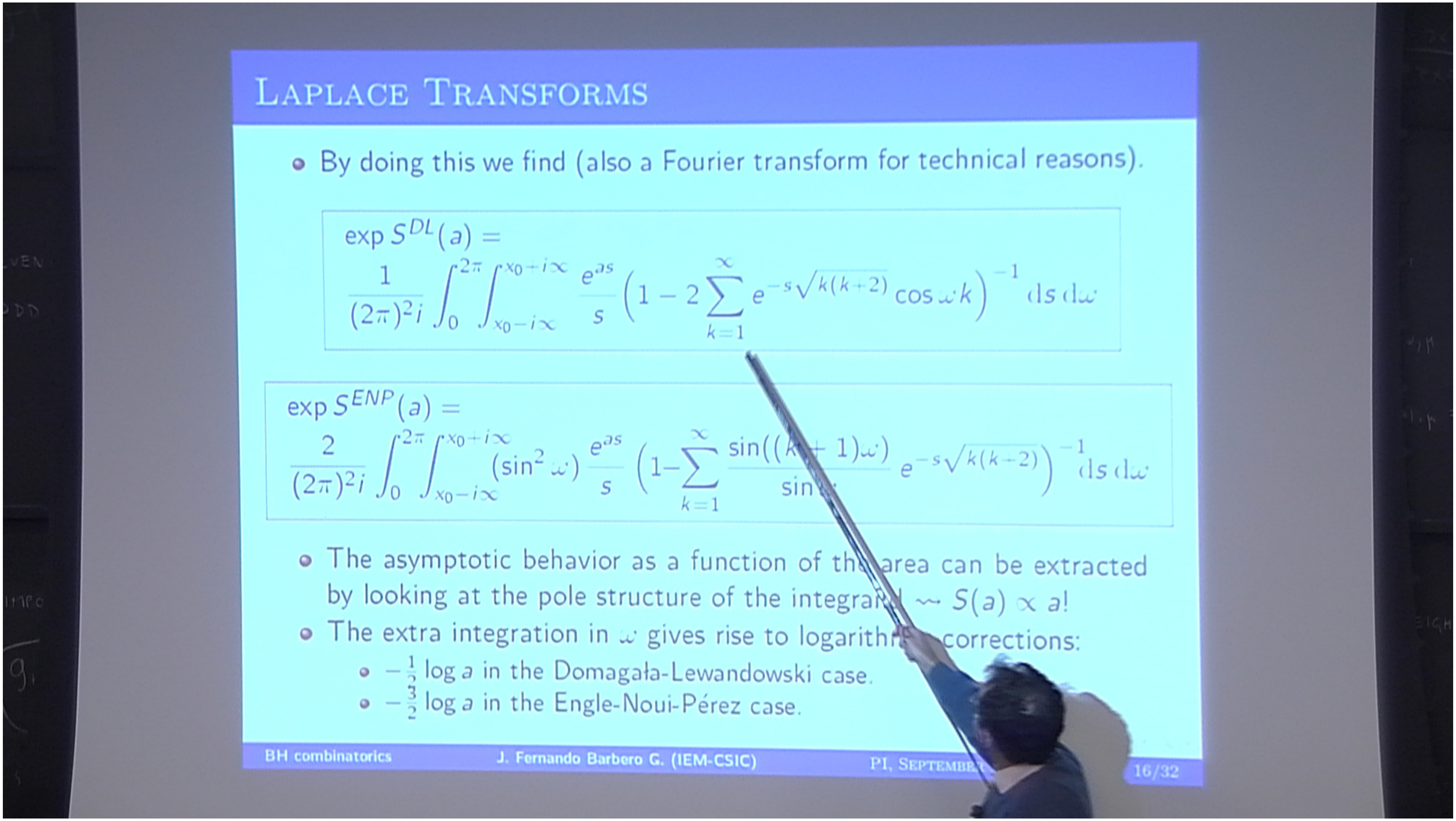
Laplace Transforms
By doing this we find (also a Fourier transform for technical reasons).
$\exp S^{DL}(a) = \frac{1}{(2\pi)^2 i}\int_0^{2\pi}\int_{x_0-i\infty}^{x_0+i\infty}\frac{e^{as}}{s}\left(1-2\sum_{k=1}^{\infty}e^{-s\sqrt{k(k+2)}}\cos\omega k\right)^{-1}\mathrm{d}s\,\mathrm{d}\omega$
$\exp S^{ENP}(a) = \frac{2}{(2\pi)^2 i}\int_0^{2\pi}\int_{x_0-i\infty}^{x_0+i\infty}(\sin^2\omega)\frac{e^{as}}{s}\left(1-\sum_{k=1}^{\infty}\frac{\sin((k+1)\omega)}{\sin\omega}e^{-s\sqrt{k(k+2)}}\right)^{-1}\mathrm{d}s\,\mathrm{d}\omega$
The asymptotic behavior as a function of the area can be extracted by looking at the pole structure of the integrand $\rightsquigarrow S(a) \propto a$!
The extra integration in ω gives rise to logarithmic corrections:
$-\frac{1}{2}\log a$ in the Domagała-Lewandowski case.
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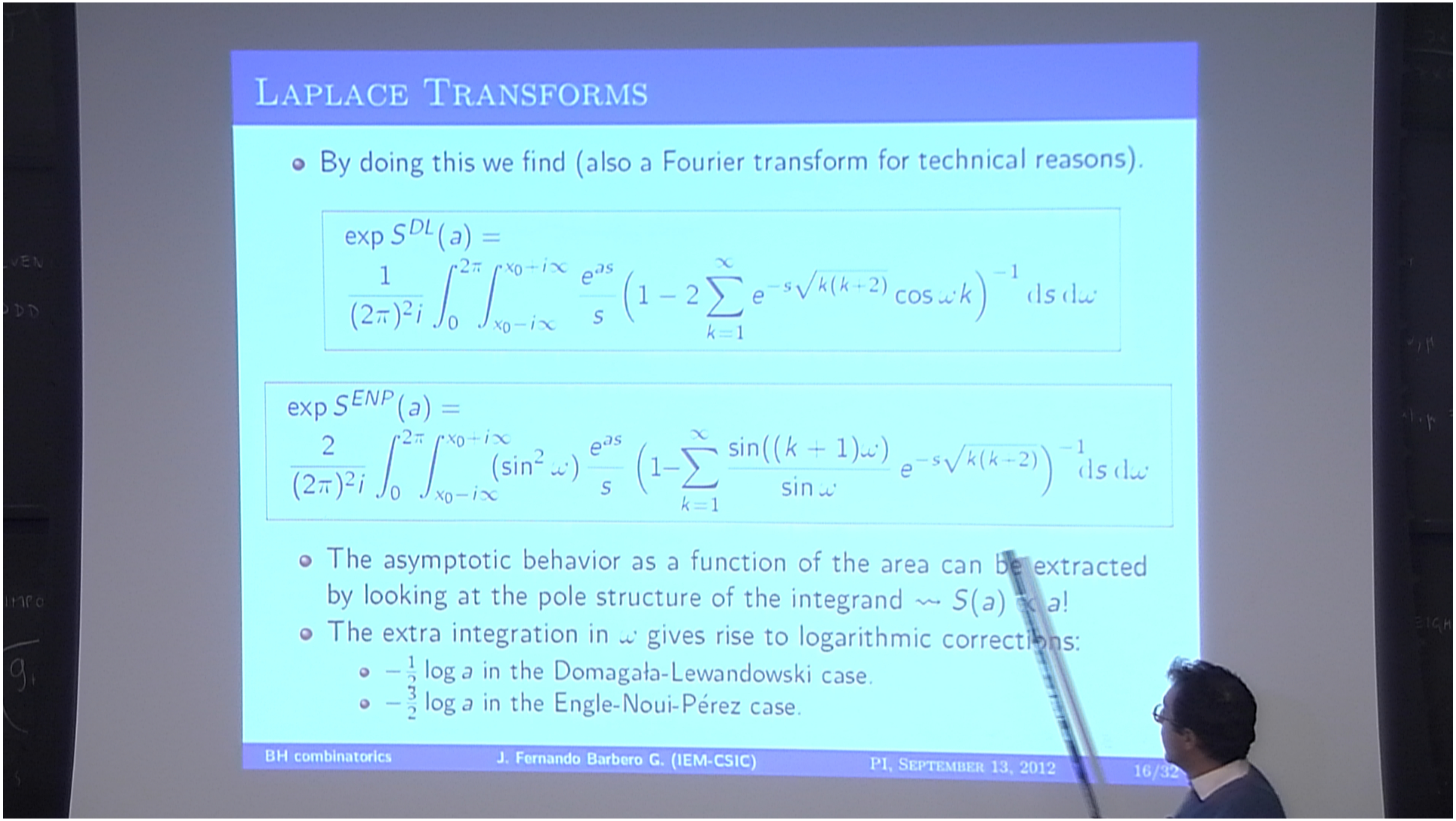
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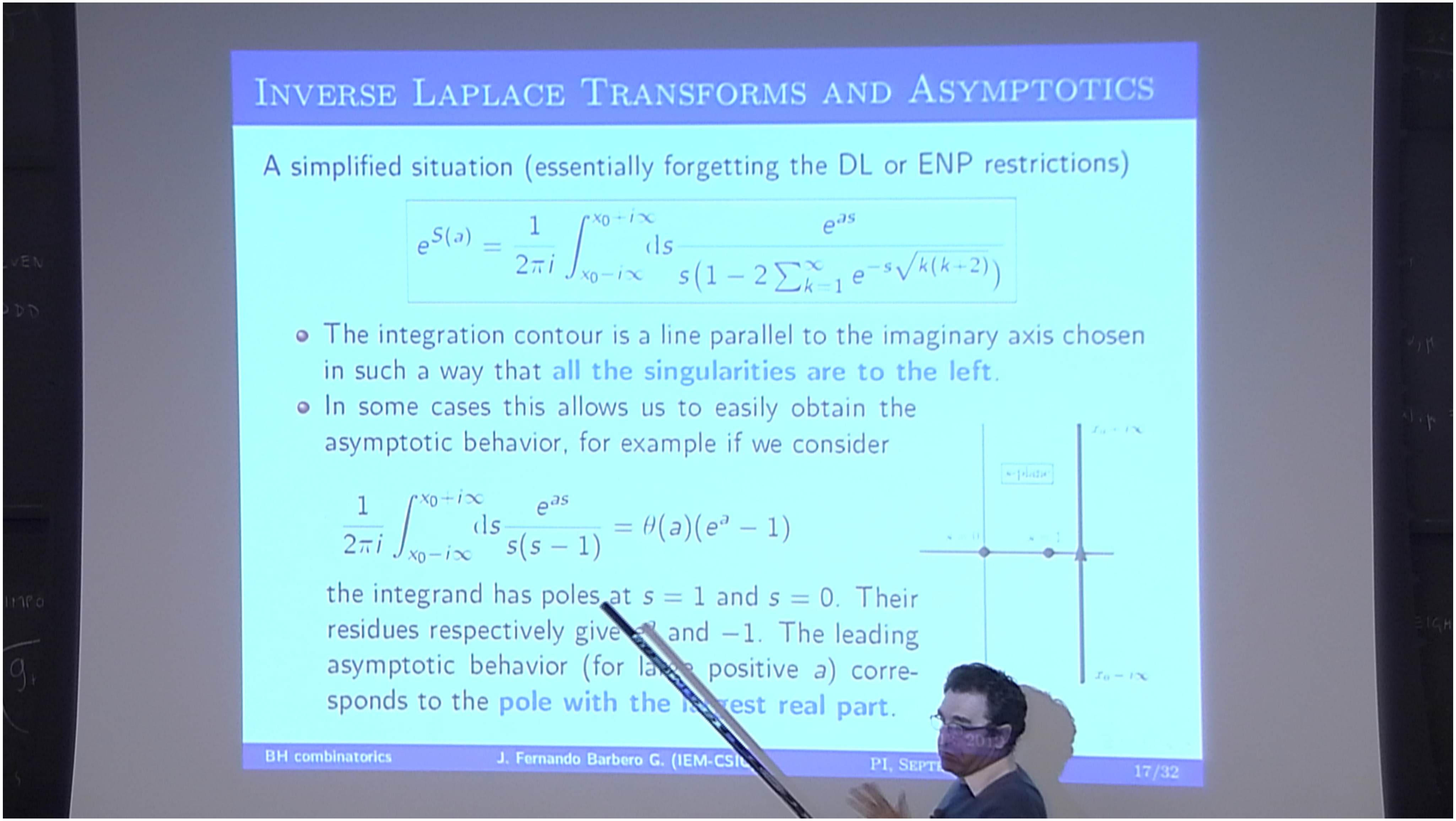
Inverse Laplace Transforms and Asymptotics
A simplified situation (essentially forgetting the DL or ENP restrictions)
$e^{S(a)} = \frac{1}{2\pi i}\int_{x_0-i\infty}^{x_0+i\infty} ds \frac{e^{as}}{s\left(1-2\sum_{k=1}^{\infty} e^{-s\sqrt{k(k+2)}}\right)}$
The integration contour is a line parallel to the imaginary axis chosen in such a way that all the singularities are to the left.
In some cases this allows us to easily obtain the asymptotic behavior, for example if we consider
$\frac{1}{2\pi i}\int_{x_0-i\infty}^{x_0+i\infty} ds \frac{e^{as}}{s(s-1)} = \theta(a)(e^a - 1)$
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Inverse Laplace Transforms and Asymptotics

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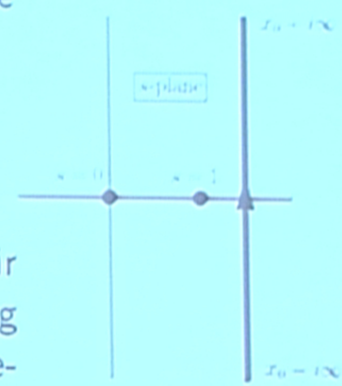
the integrand has poles at $s=1$ and $s=0$. Their residues respectively give e^{a} and -1. The leading asymptotic behavior (for large positive a) corresponds to the pole with the largest real part.



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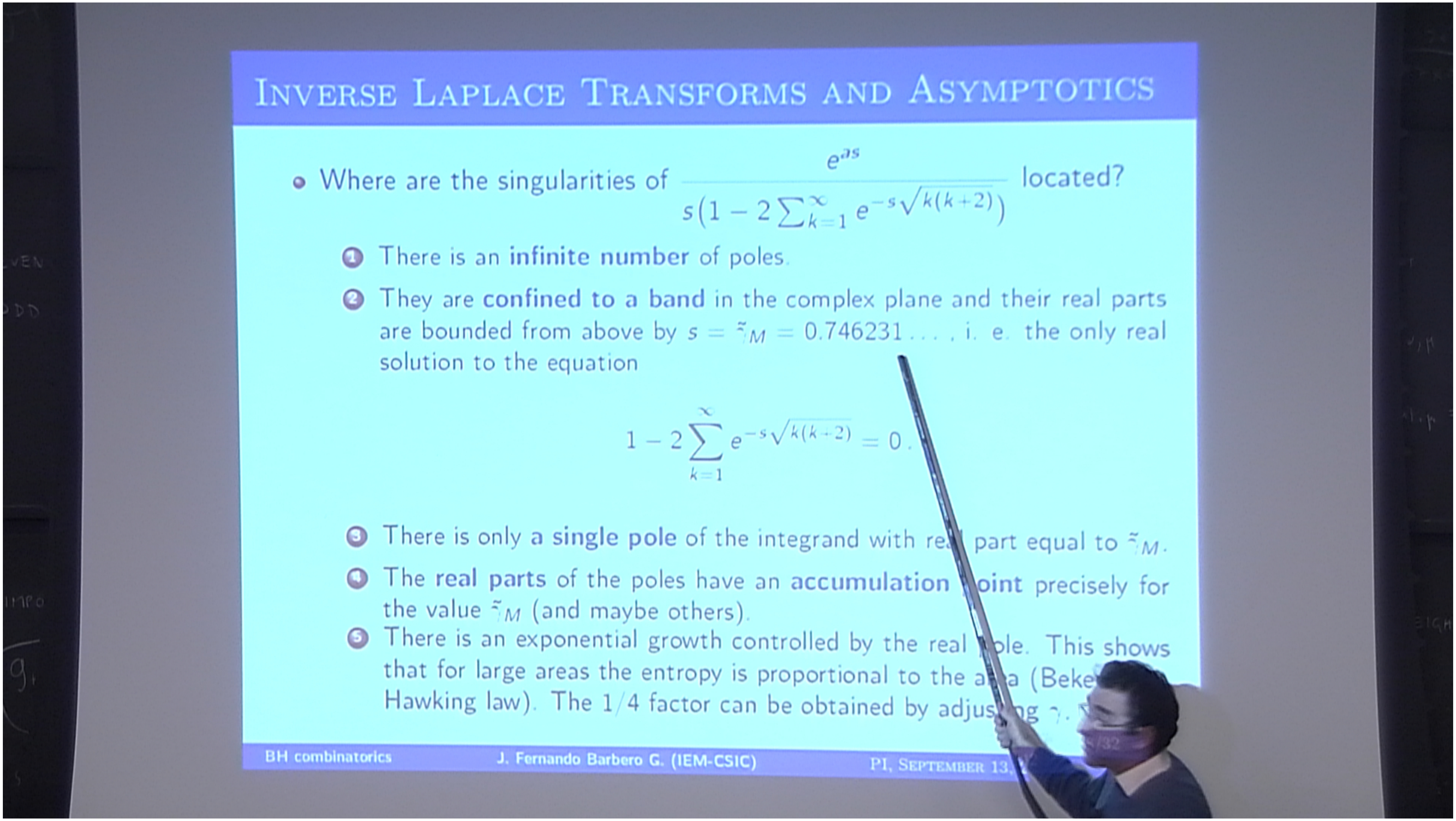

INVERSE LAPLACE TRANSFORMS AND ASYMPTOTICS
Where are the singularities of $\frac{e^{as}}{s\left(1-2\sum_{k=1}^{\infty} e^{-s\sqrt{k(k+2)}}\right)}$ located?
There is an infinite number of poles.
They are confined to a band in the complex plane and their real parts are bounded from above by $s = \tilde{\gamma}_M = 0.746231\ldots$, i. e. the only real solution to the equation
$$1-2\sum_{k=1}^{\infty} e^{-s\sqrt{k(k+2)}} = 0\,.$$
There is only a single pole of the integrand with re part equal to $\tilde{\gamma}_M$.
The real parts of the poles have an accumulation oint precisely for the value $\tilde{\gamma}_M$ (and maybe others).
There is an exponential growth controlled by the real ole. This shows that for large areas the entropy is proportional to the a a (Beke Hawking law). The 1/4 factor can be obtained by adjus ng γ.
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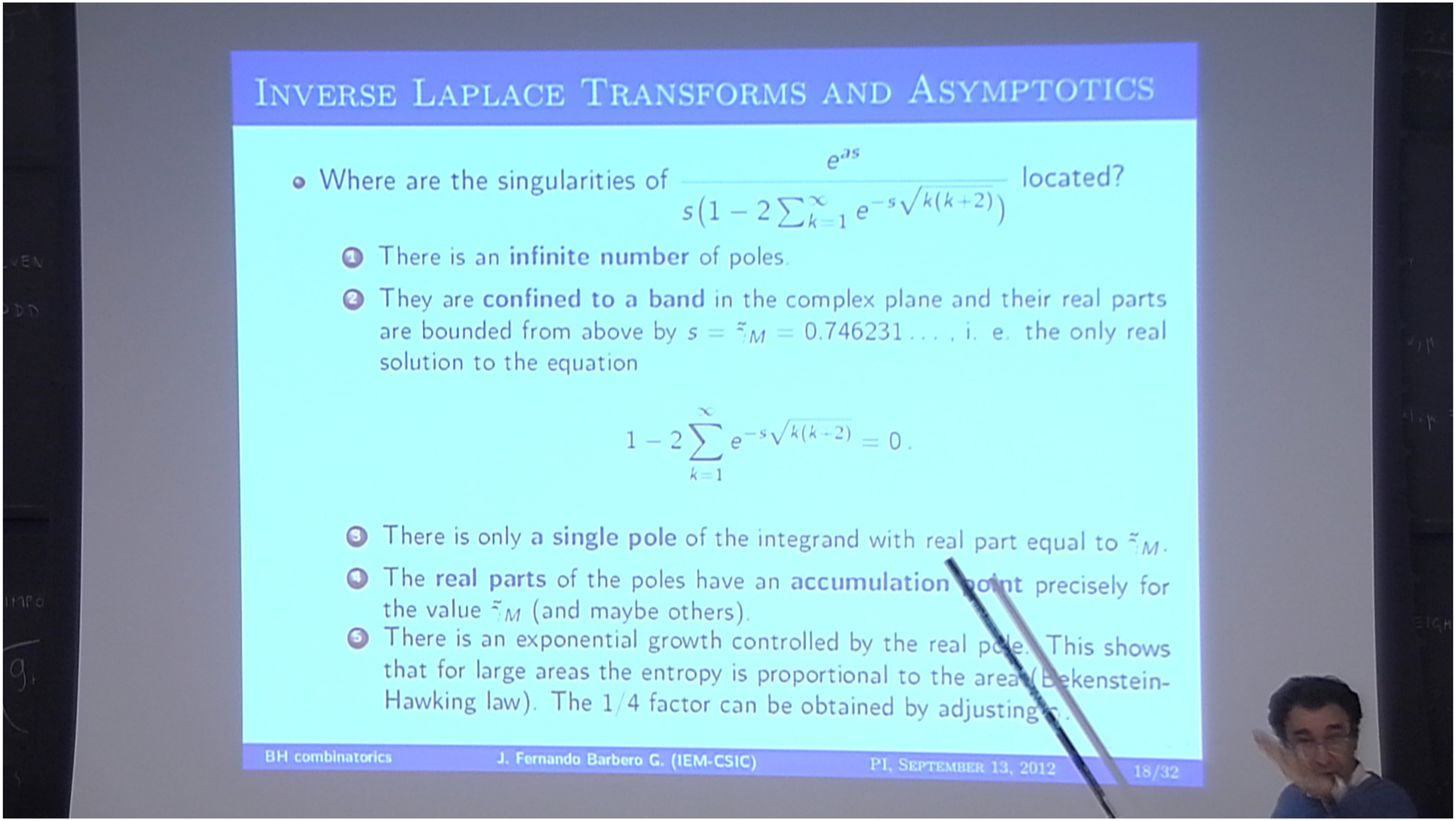
PI, SEPTEMBER 13,



Inverse Laplace Transforms and Asymptotics
Where are the singularities of $\frac{e^{as}}{s(1-2\sum_{k=1}^{\infty} e^{-s\sqrt{k(k+2)}})}$ located?
There is an **infinite number** of poles.
They are **confined to a band** in the complex plane and their real parts are bounded from above by $s = \tilde{\ }_M = 0.746231\ldots$, i. e. the only real solution to the equation
$$1-2\sum_{k=1}^{\infty} e^{-s\sqrt{k(k+2)}} = 0\,.$$
There is only **a single pole** of the integrand with real part equal to $\tilde{\ }_M$.
The **real parts** of the poles have an **accumulation point** precisely for the value $\tilde{\ }_M$ (and maybe others).
There is an exponential growth controlled by the real pole. This shows that for large areas the entropy is proportional to the area (Bekenstein-Hawking law). The 1/4 factor can be obtained by adjusting
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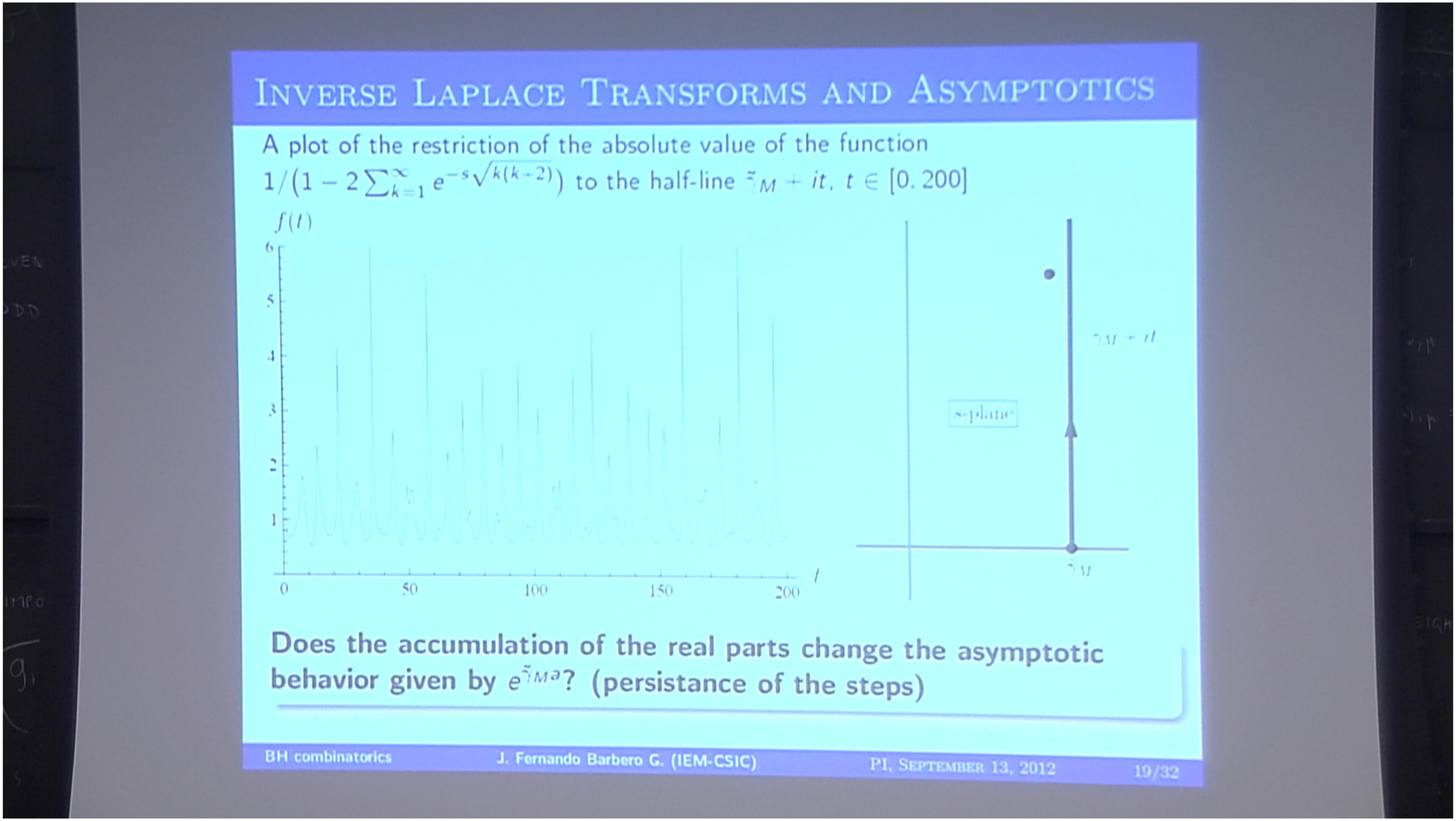
INVERSE LAPLACE TRANSFORMS AND ASYMPTOTICS
A plot of the restriction of the absolute value of the function
$1/(1-2\sum_{k=1}^{\infty} e^{-s\sqrt{k(k+2)}})$ to the half-line $\tilde{\gamma}_M + it$, $t \in [0, 200]$
$f(t)$
0
50
100
150
200
t
s-plane
$\tilde{\gamma}_M + it$
$\tilde{\gamma}_M$
Does the accumulation of the real parts change the asymptotic behavior given by $e^{\tilde{\gamma}_M a}$? (persistance of the steps)
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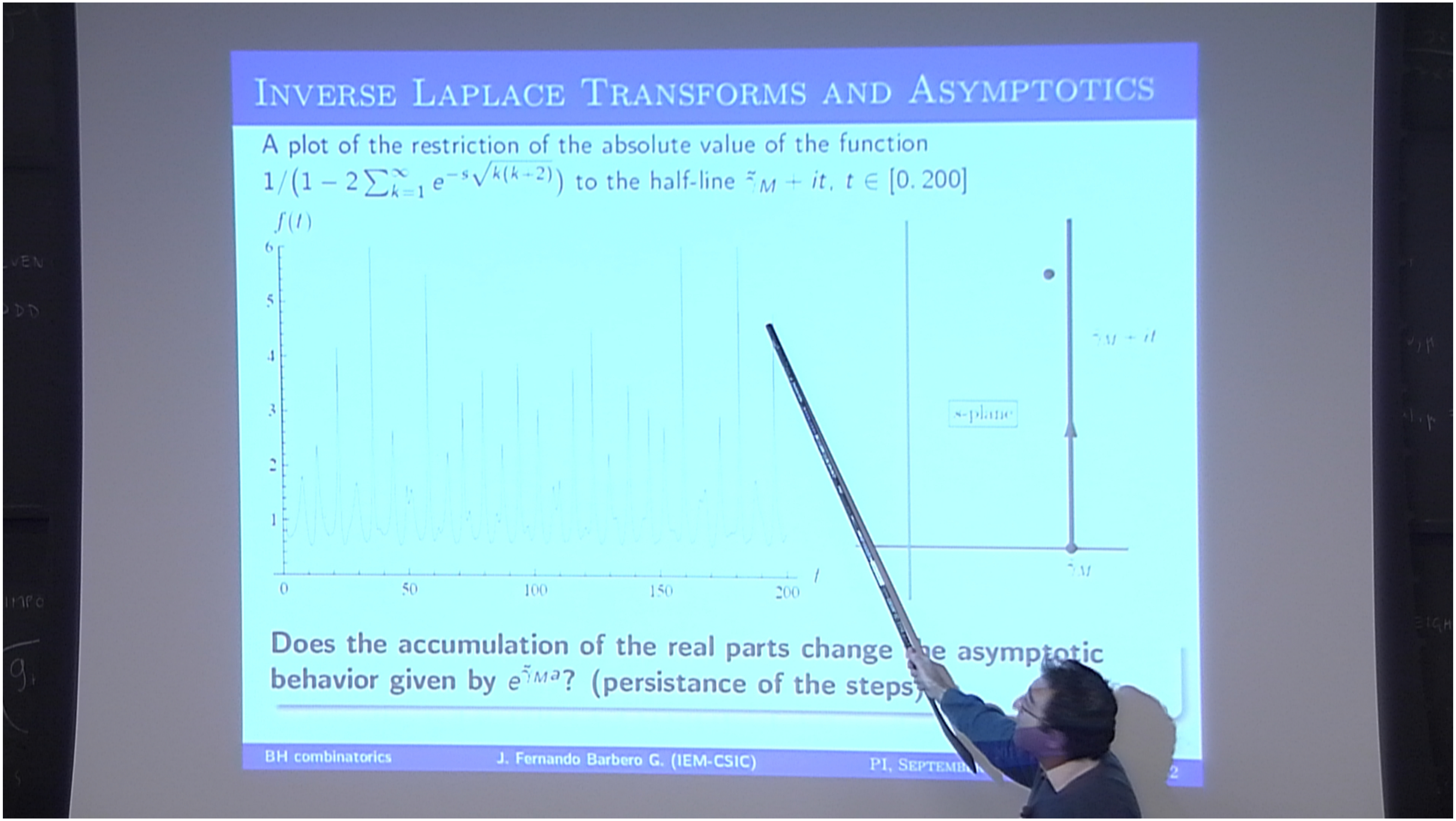
INVERSE LAPLACE TRANSFORMS AND ASYMPTOTICS
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$f(t)$
6
5
4
3
2
1
0
50
100
150
200
t
s-plane
$\tilde{\gamma}_M + it$
$\tilde{\gamma}_M$
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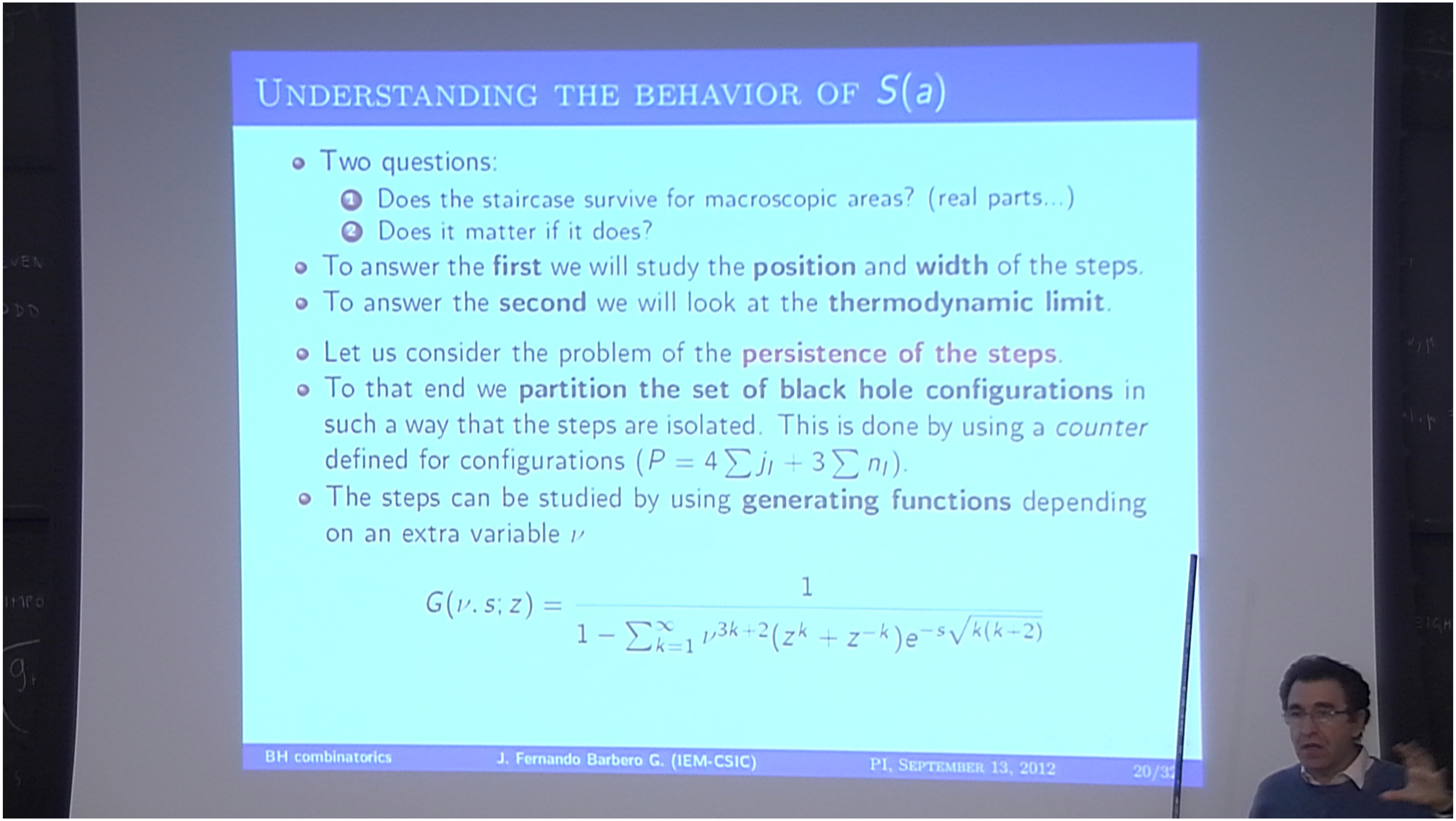
PI, SEPTEMB


UNDERSTANDING THE BEHAVIOR OF $S(a)$
Two questions:
1 Does the staircase survive for macroscopic areas? (real parts...)
2 Does it matter if it does?
To answer the first we will study the position and width of the steps.
To answer the second we will look at the thermodynamic limit.
Let us consider the problem of the persistence of the steps.
To that end we partition the set of black hole configurations in such a way that the steps are isolated. This is done by using a counter defined for configurations ($P = 4\sum j_I + 3\sum n_I$).
The steps can be studied by using generating functions depending on an extra variable ν
$G(\nu, s; z) = \frac{1}{1 - \sum_{k=1}^{\infty} \nu^{3k+2}(z^k + z^{-k})e^{-s\sqrt{k(k-2)}}}$
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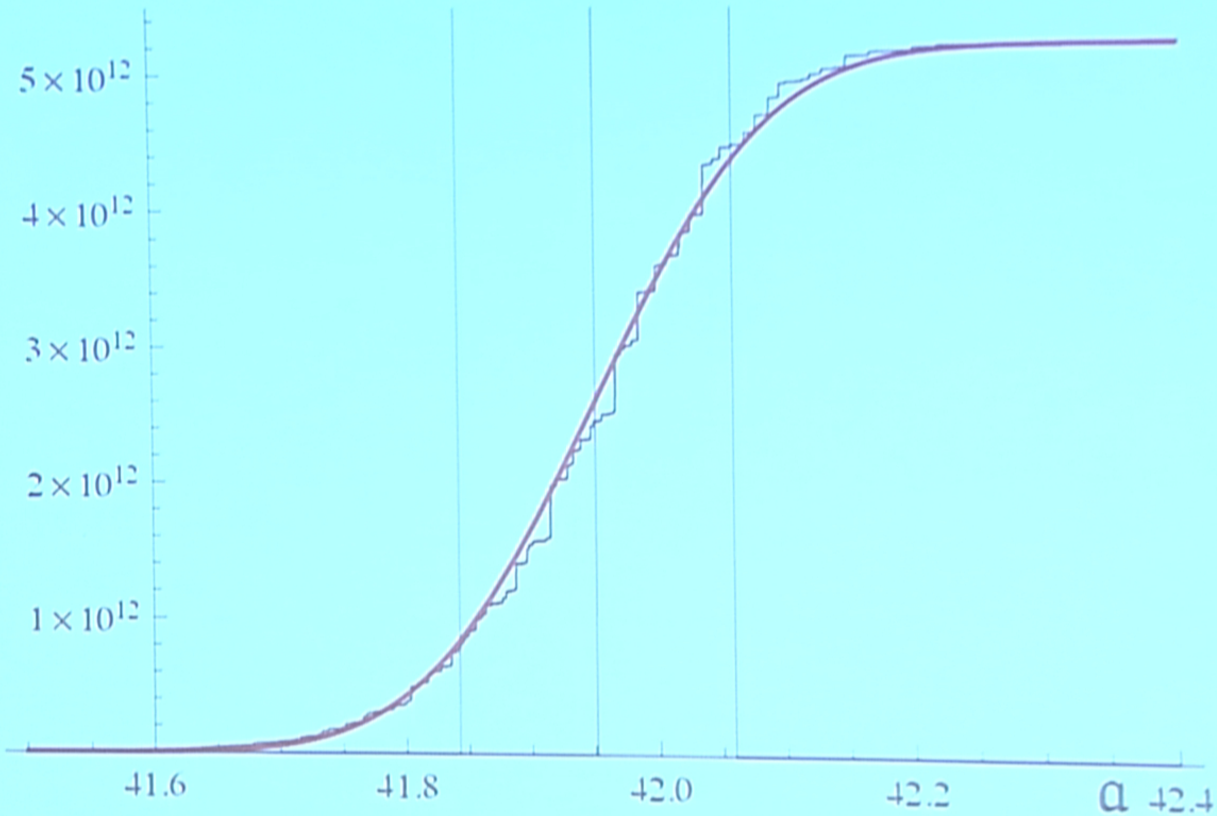
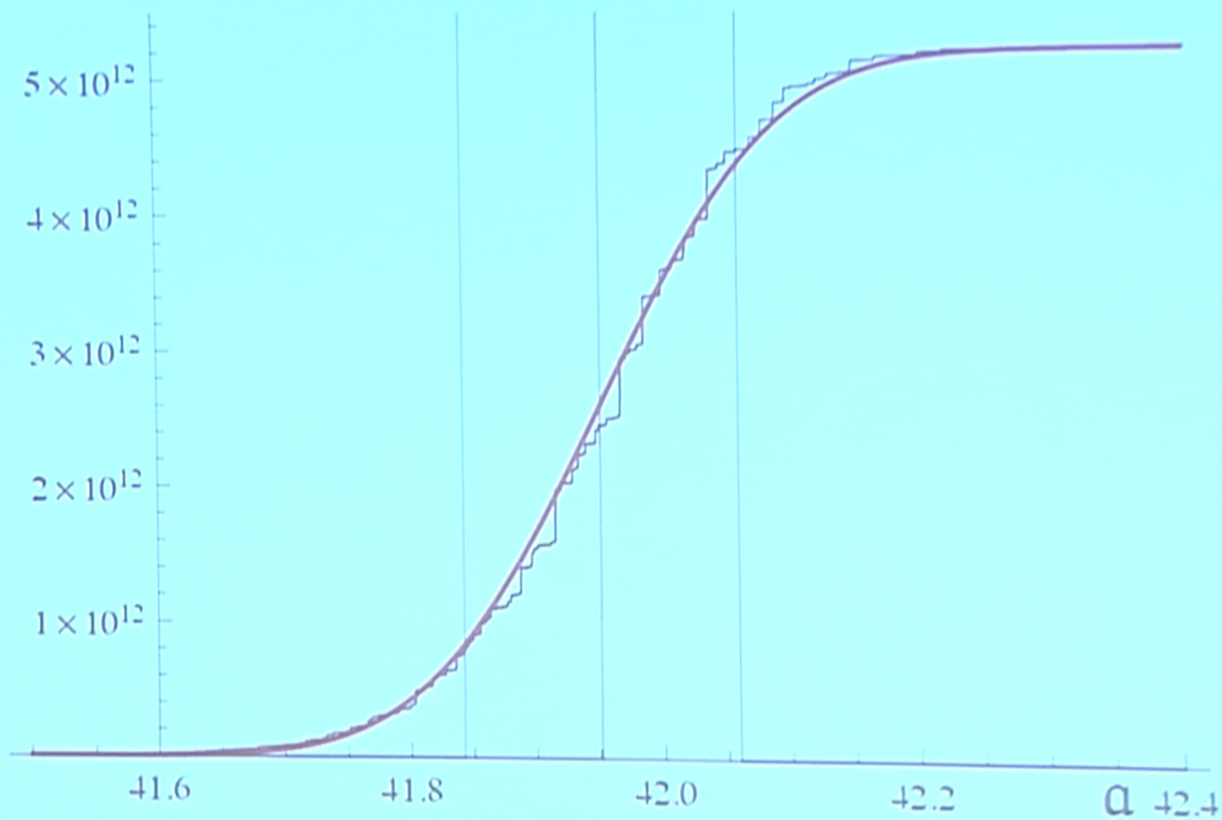
Understanding the behavior of $S(a)$

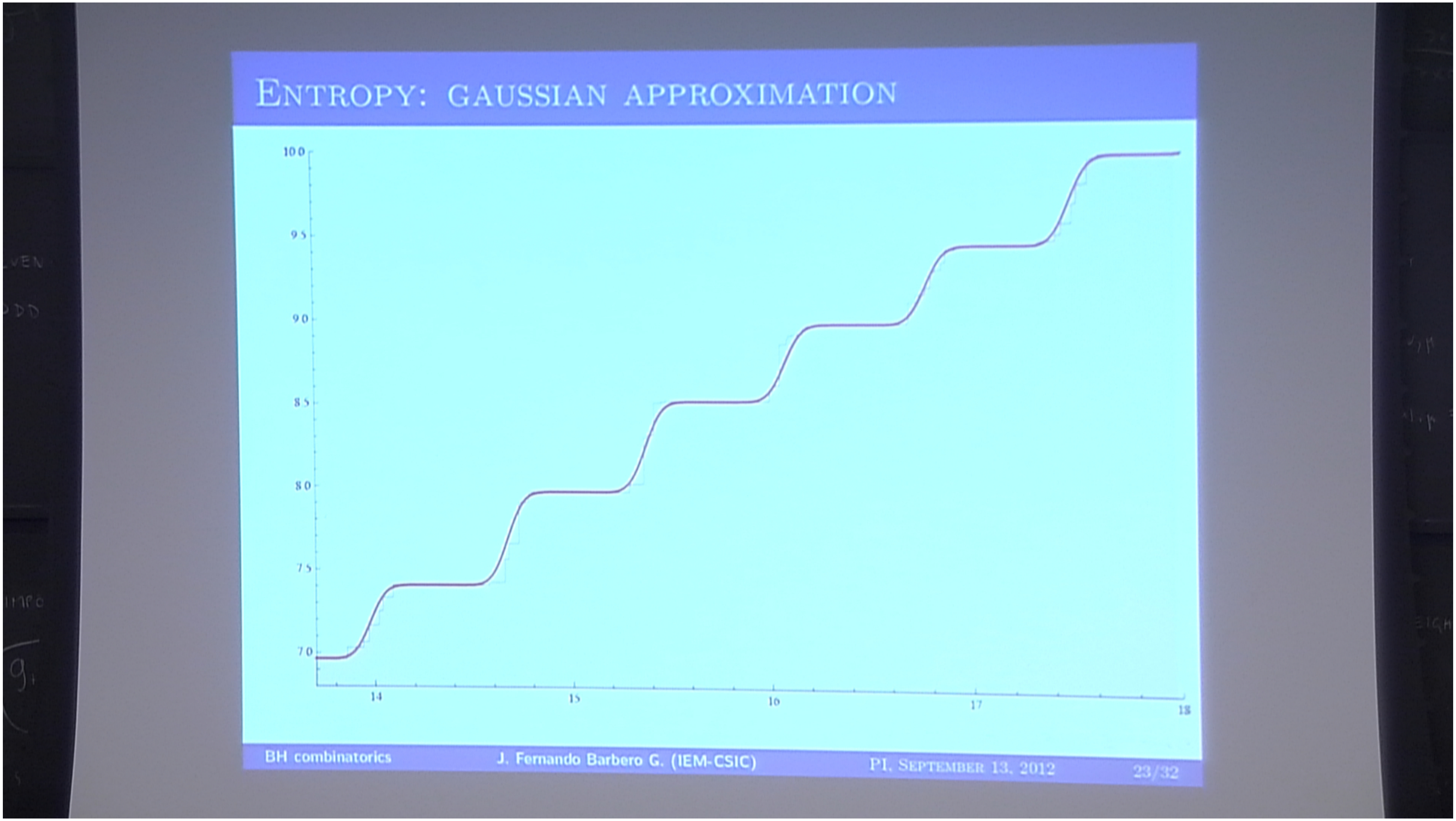
- The steps can be interpreted as (non-normalized) probability distributions.
- We can study their properties (the **mean** and the **variance**). These, in particular, can be used to **define** the position and width of the steps).
- We can introduce a "Gaussian" (error function) approximation for the steps. The entropy can be then approximated as a sum of these smoothed steps. This is very useful to study the behavior of the entropy for **large areas**.
- The standard and proper way to compute the mean and the variance is to use generating functions... but **this is precisely what we have**!!

Smoothing of the steps



Smoothing of the steps




ENTROPY: GAUSSIAN APPROXIMATION
100
95
90
85
80
75
70
14
15
16
17
18
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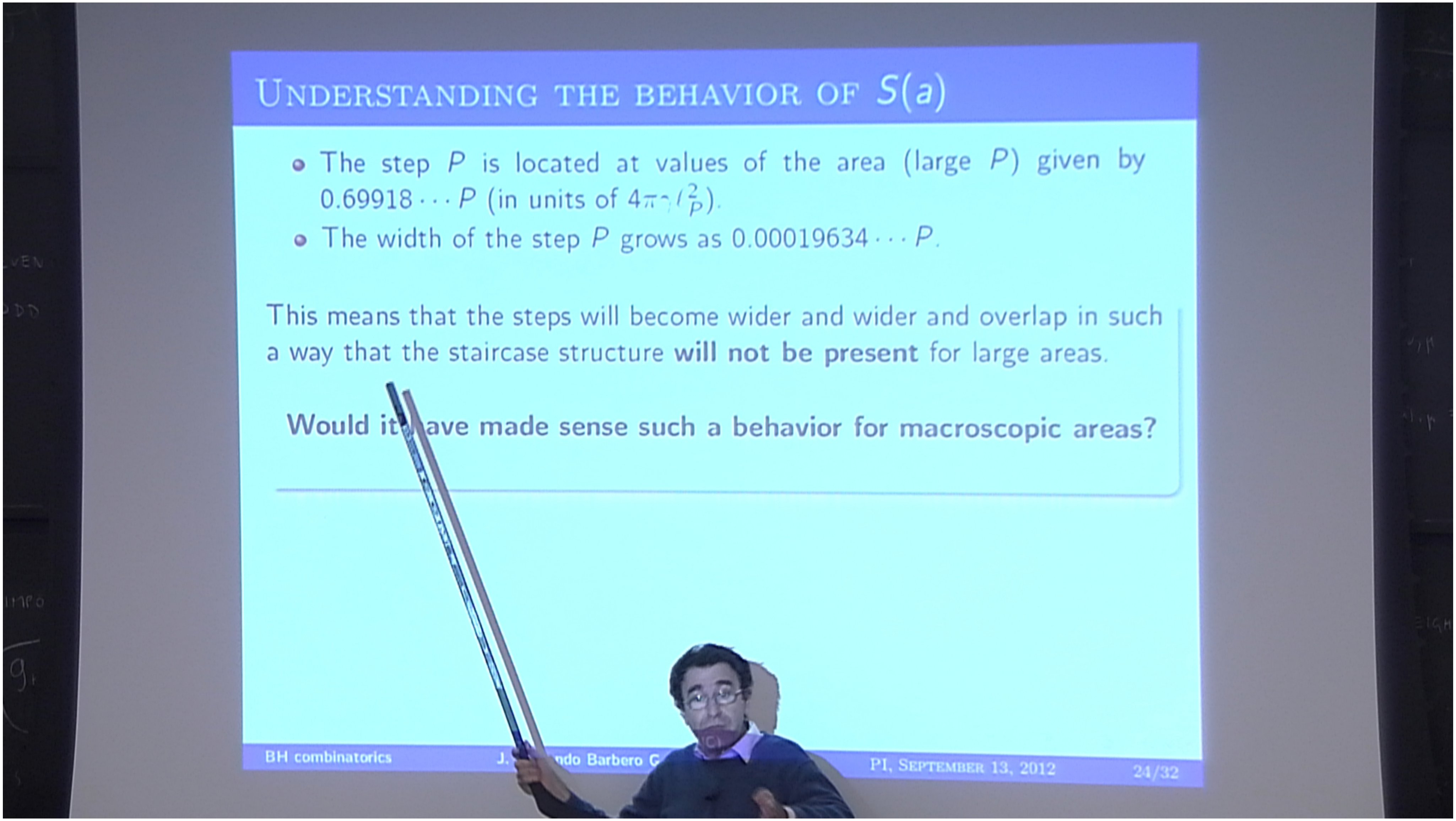
Understanding the behavior of S(a)
The step P is located at values of the area (large P) given by 0.69918 · · · P (in units of 4πγℓ²_P).
The width of the step P grows as 0.00019634 · · · P.
This means that the steps will become wider and wider and overlap in such a way that the staircase structure will not be present for large areas.
Would it have made sense such a behavior for macroscopic areas?
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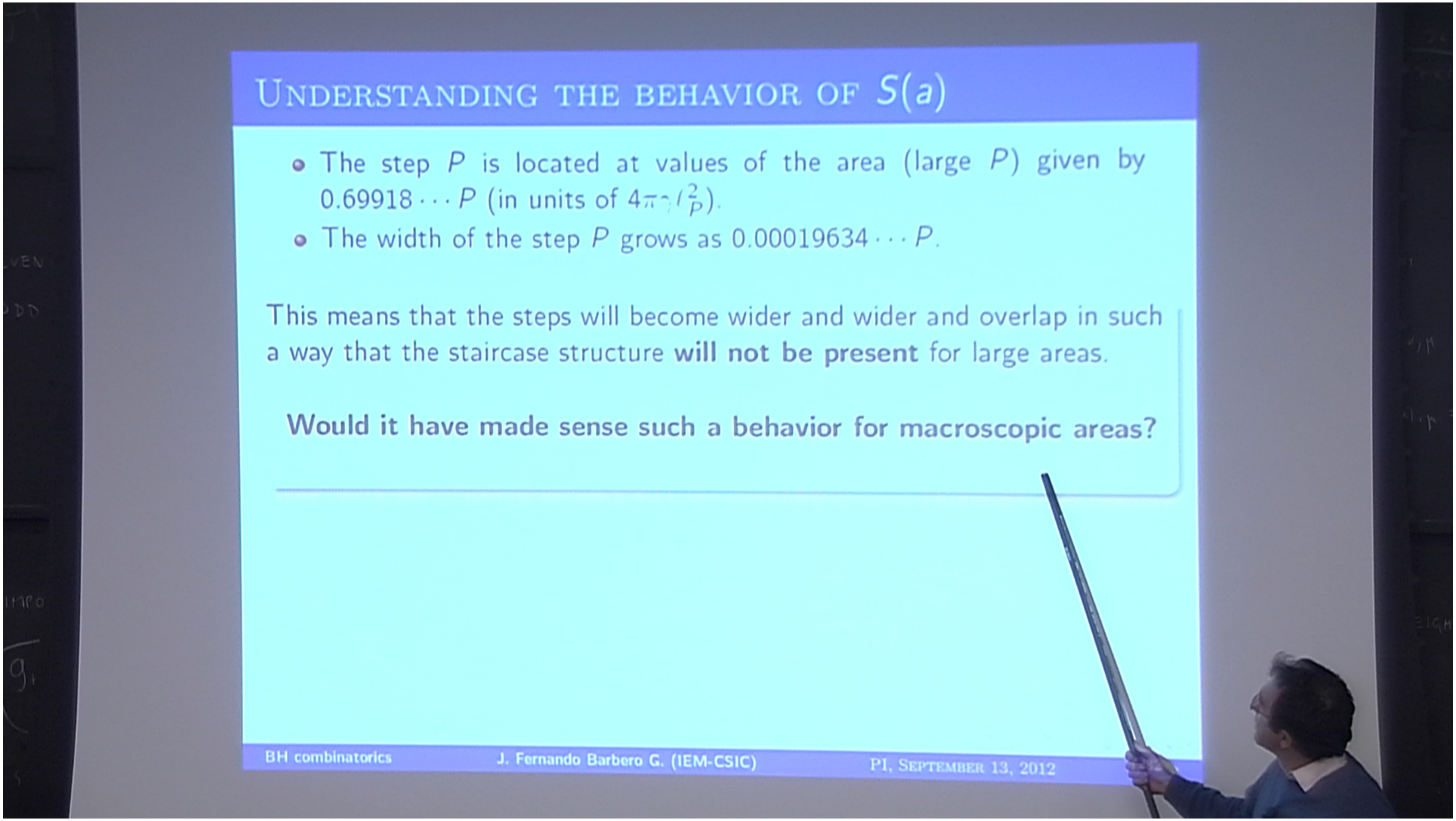
Understanding the behavior of $S(a)$
The step P is located at values of the area (large P) given by $0.69918 \cdots P$ (in units of $4\pi\gamma\ell_P^2$).
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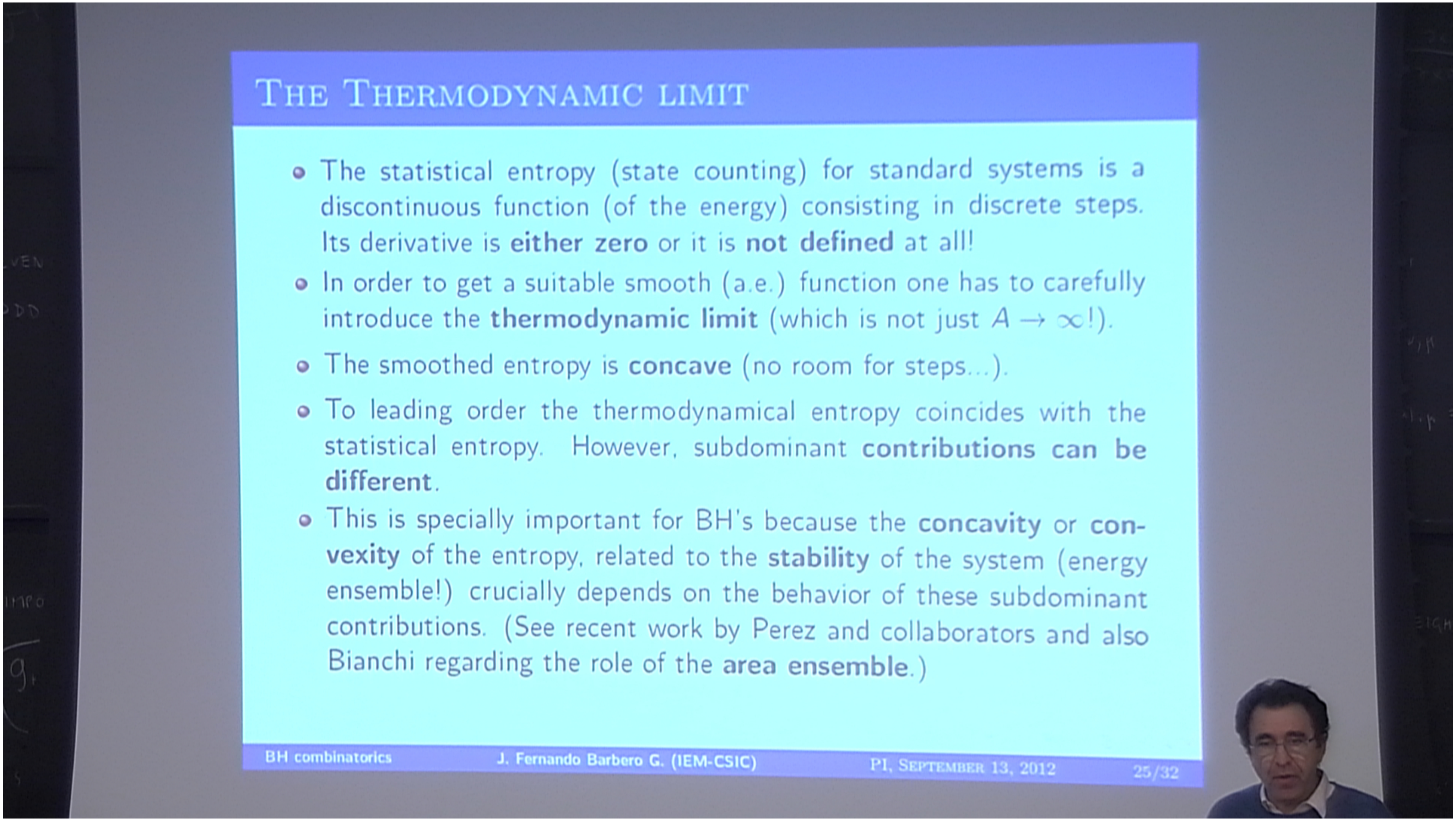
THE THERMODYNAMIC LIMIT
The statistical entropy (state counting) for standard systems is a discontinuous function (of the energy) consisting in discrete steps. Its derivative is either zero or it is not defined at all!
In order to get a suitable smooth (a.e.) function one has to carefully introduce the thermodynamic limit (which is not just $A \to \infty$!).
The smoothed entropy is concave (no room for steps...).
To leading order the thermodynamical entropy coincides with the statistical entropy. However, subdominant contributions can be different.
This is specially important for BH's because the concavity or convexity of the entropy, related to the stability of the system (energy ensemble!) crucially depends on the behavior of these subdominant contributions. (See recent work by Perez and collaborators and also Bianchi regarding the role of the area ensemble.)
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The Thermodynamic limit

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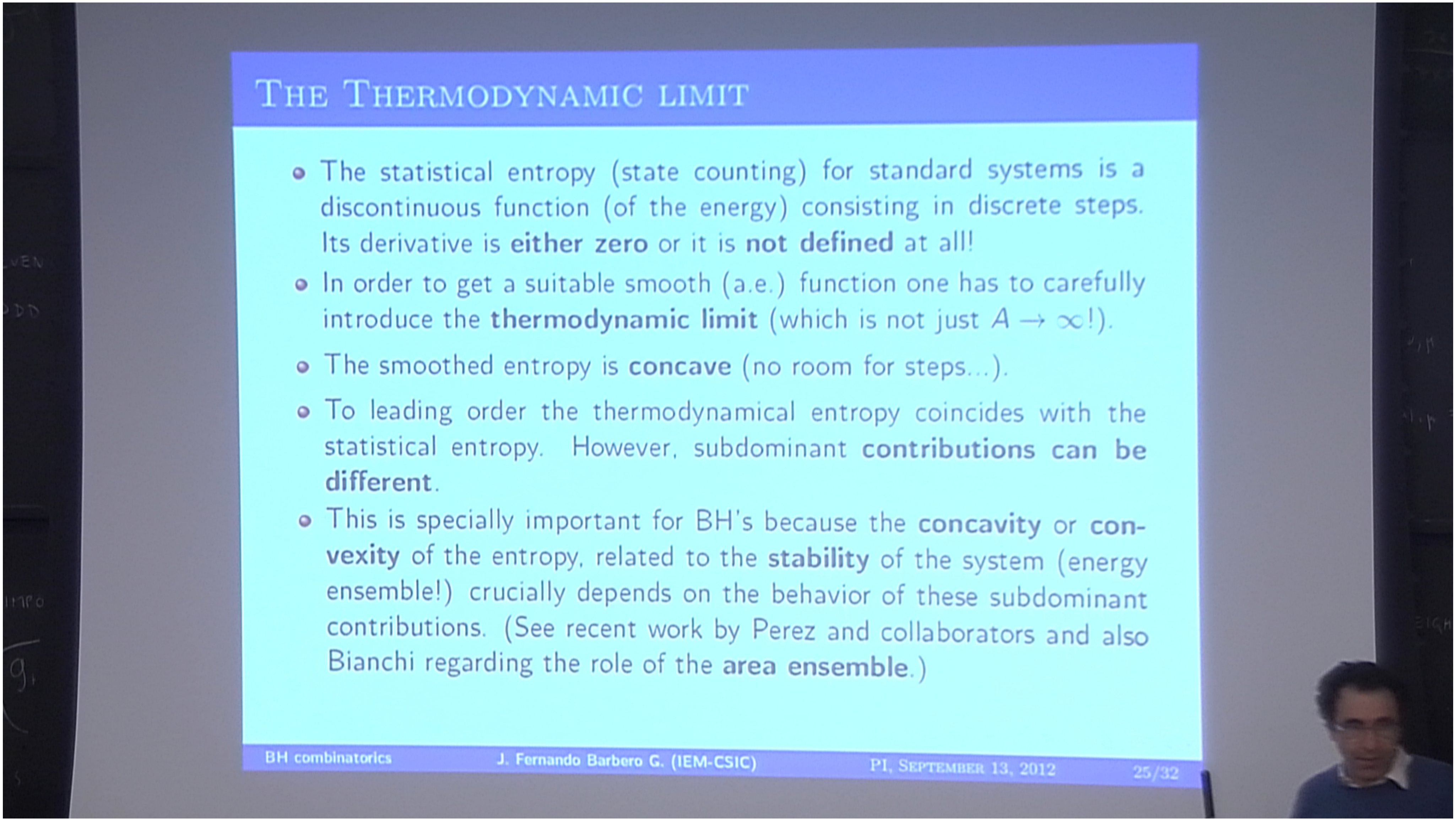
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The Thermodynamic limit

- We will explore this by using the **area canonical ensemble** used in the LQG model for BH's.
- The partition function in the **canonical area ensemble** of LQG is essentially given by the BH generating functions.
- In the thermodynamic limit, the partition function per-particle Z for the canonical area ensemble of non-interacting objects (i.e. Einstein crystal) can be computed as the **Laplace transform** of the corresponding number of microstates Ω for a single object:

$$\Omega(A) = \sum_{n=1}^{\infty} D_n \theta(A - A_n) \Leftrightarrow$$

$$Z(\alpha) = \alpha \int_0^{\infty} e^{-\alpha A} \Omega(A)\, dA = \sum_{n=1}^{\infty} D_n e^{-\alpha A_n}\,.$$

- The D_n encode the black hole degeneracies associated with the area eigenvalues A_n.

The Thermodynamic limit

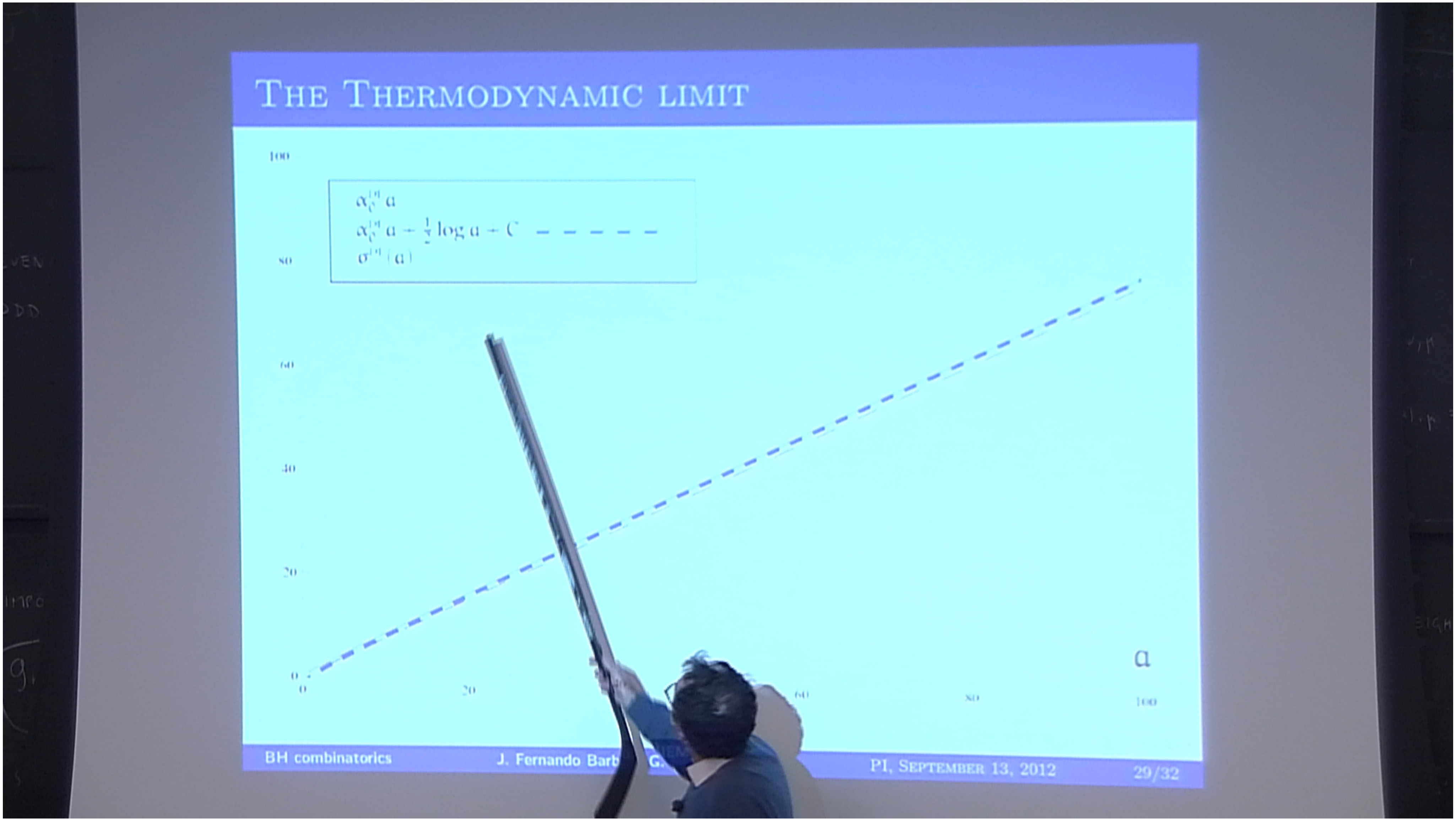
- These are the numbers that we obtain from the solution of the combinatorial problems that I have described at the beginning of the talk!
- The parameter α is conjugate to the area (hence is not the inverse of a temperature, which is conjugate to the energy).
- In the thermodynamic limit the average area is given by

$$a(\alpha) = -\frac{\mathrm{d}}{\mathrm{d}\alpha} \log Z(\alpha)$$

  and the (smoothed) entropy $\tilde{\sigma}$ as a function of α can be computed as

$$\tilde{\sigma}(\alpha) := \alpha a(\alpha) + \log Z(\alpha)\,.$$

- In practice, in order to express (and plot) the entropy as a function of the area, $a \mapsto \sigma(a)$, it is convenient to consider the parametrized curve $\alpha \mapsto (a(\alpha), \tilde{\sigma}(\alpha))$.


THE THERMODYNAMIC LIMIT
100
80
60
40
20
0
0
20
60
80
100
a
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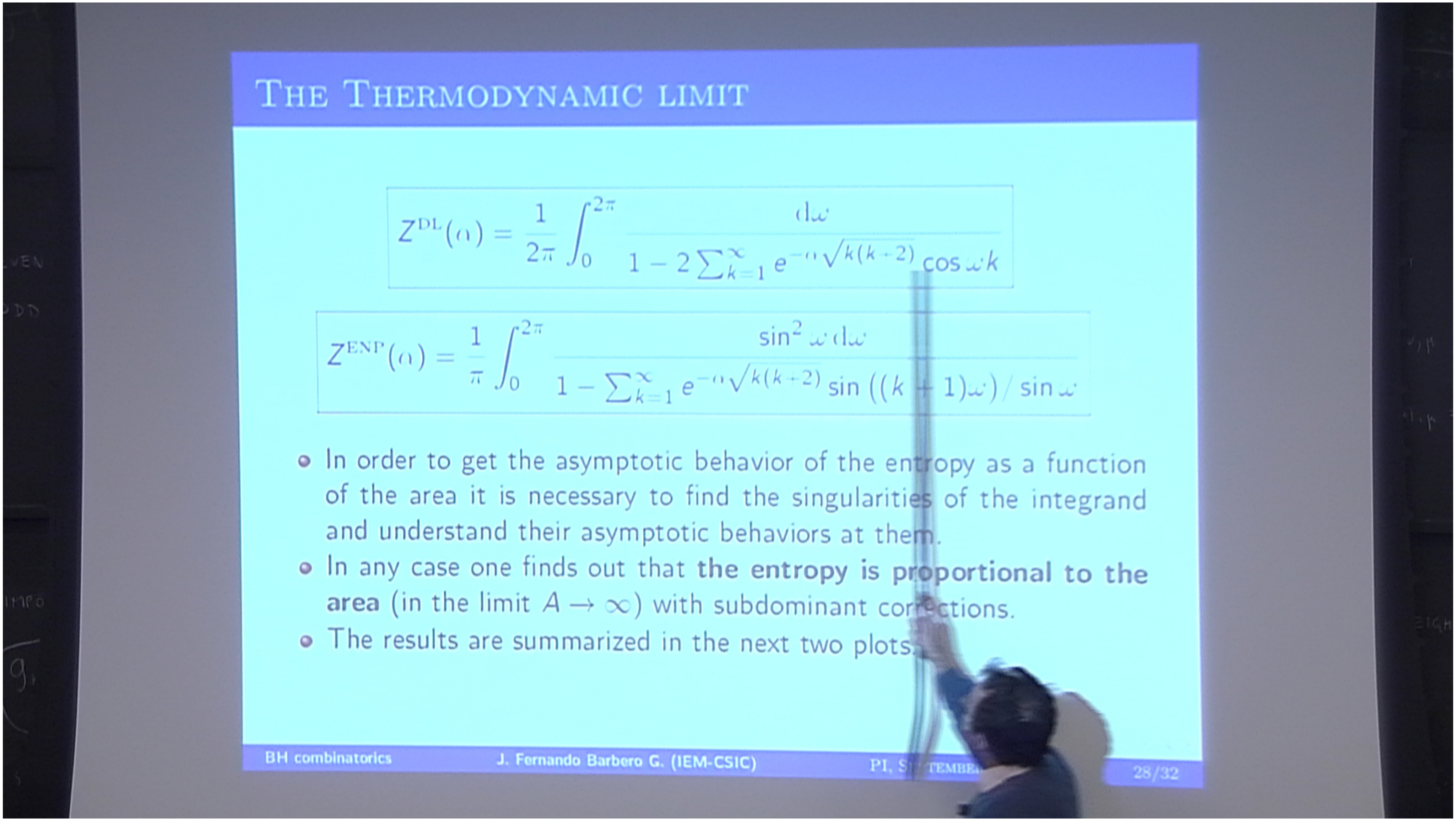
THE THERMODYNAMIC LIMIT
$Z^{\mathrm{DL}}(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} \frac{d\omega}{1 - 2\sum_{k=1}^{\infty} e^{-\alpha\sqrt{k(k+2)}}\cos\omega k}$
$Z^{\mathrm{ENP}}(\alpha) = \frac{1}{\pi}\int_0^{2\pi} \frac{\sin^2\omega\, d\omega}{1 - \sum_{k=1}^{\infty} e^{-\alpha\sqrt{k(k+2)}}\sin((k+1)\omega)/\sin\omega}$
In order to get the asymptotic behavior of the entropy as a function of the area it is necessary to find the singularities of the integrand and understand their asymptotic behaviors at them.
In any case one finds out that the entropy is proportional to the area (in the limit $A \to \infty$) with subdominant corrections.
The results are summarized in the next two plots.
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The Thermodynamic limit

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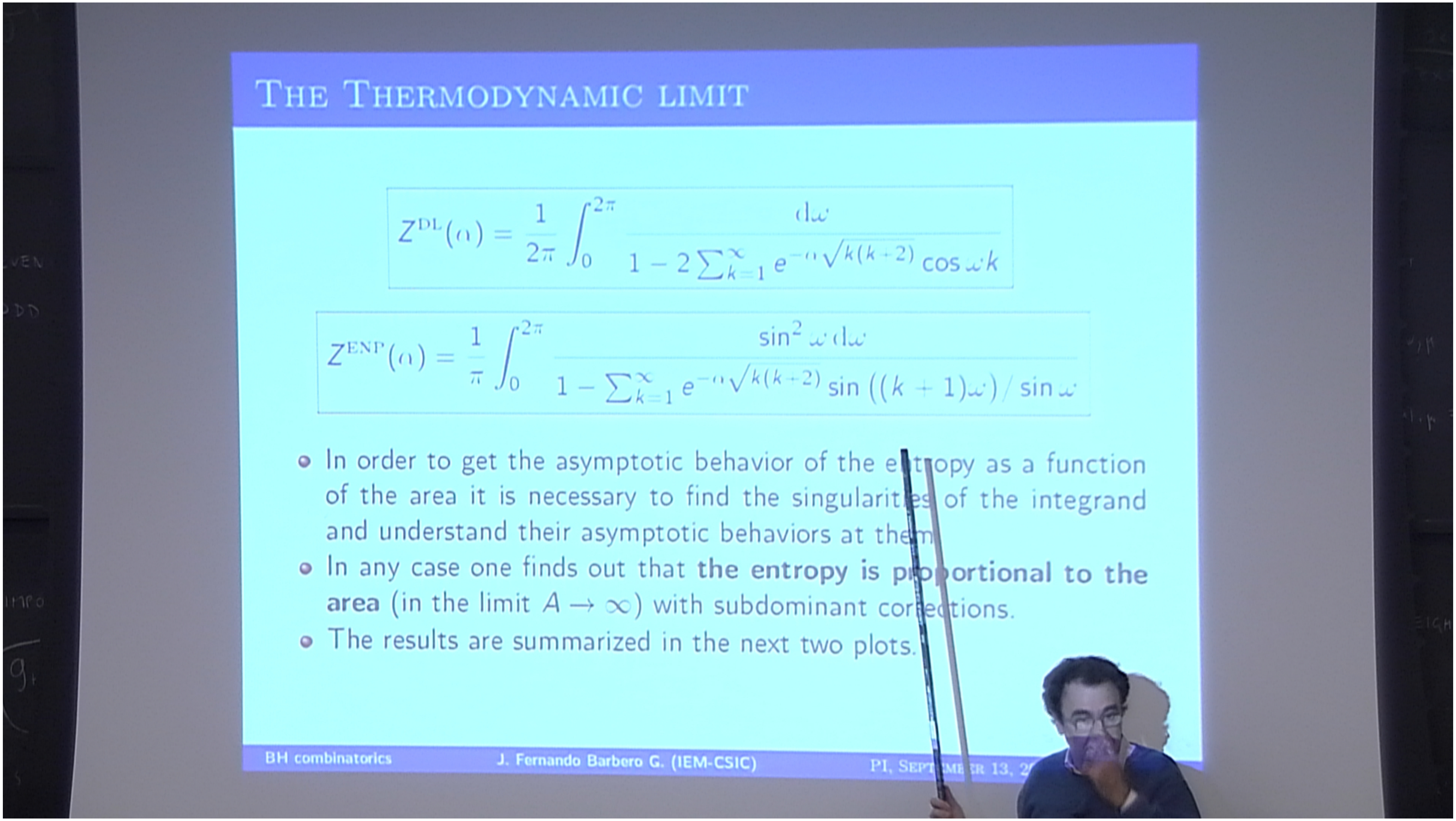
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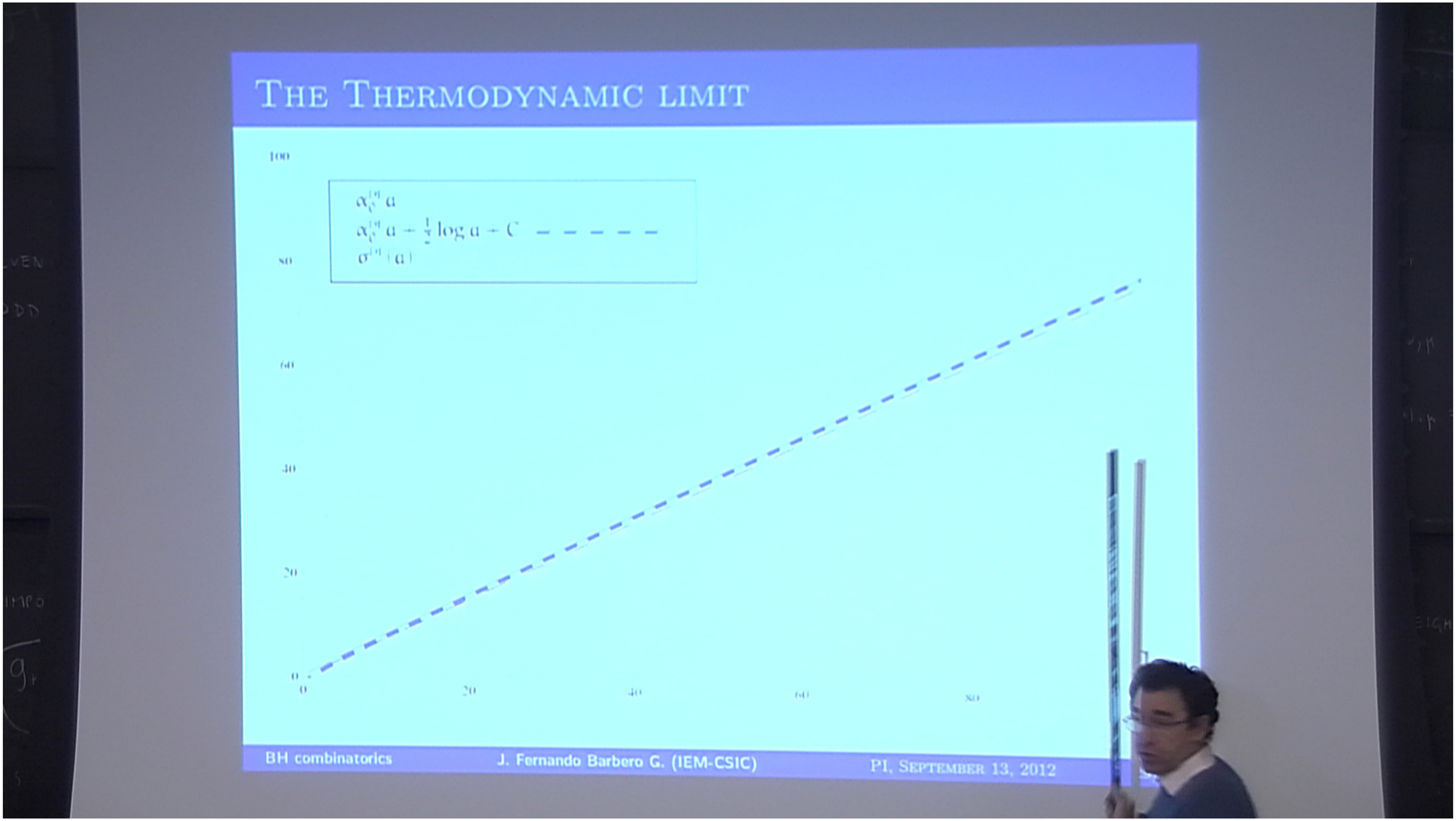
THE THERMODYNAMIC LIMIT
100
80
60
40
20
0
0
20
40
60
80
BH combinatorics
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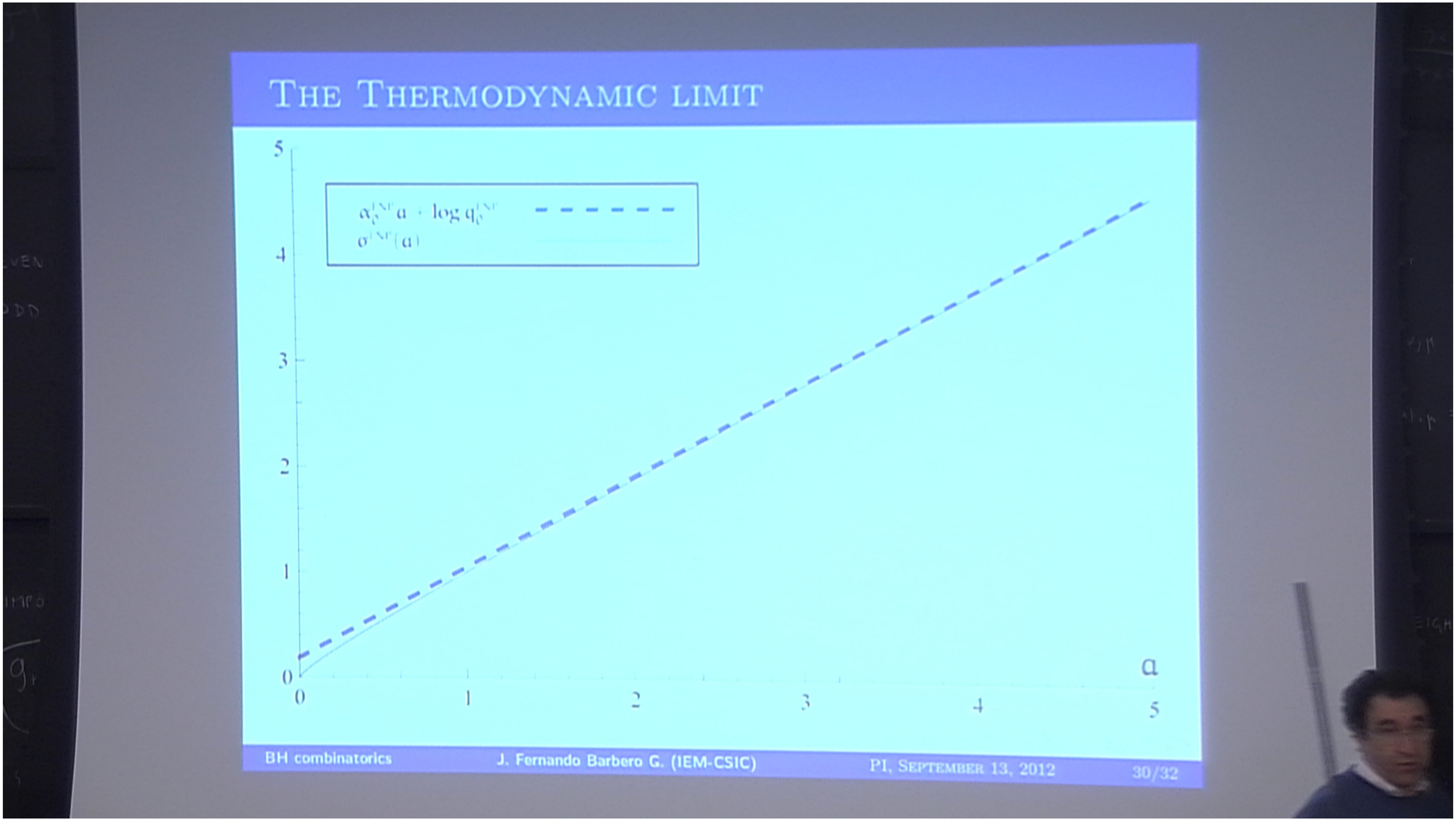
THE THERMODYNAMIC LIMIT
0
1
2
3
4
5
a
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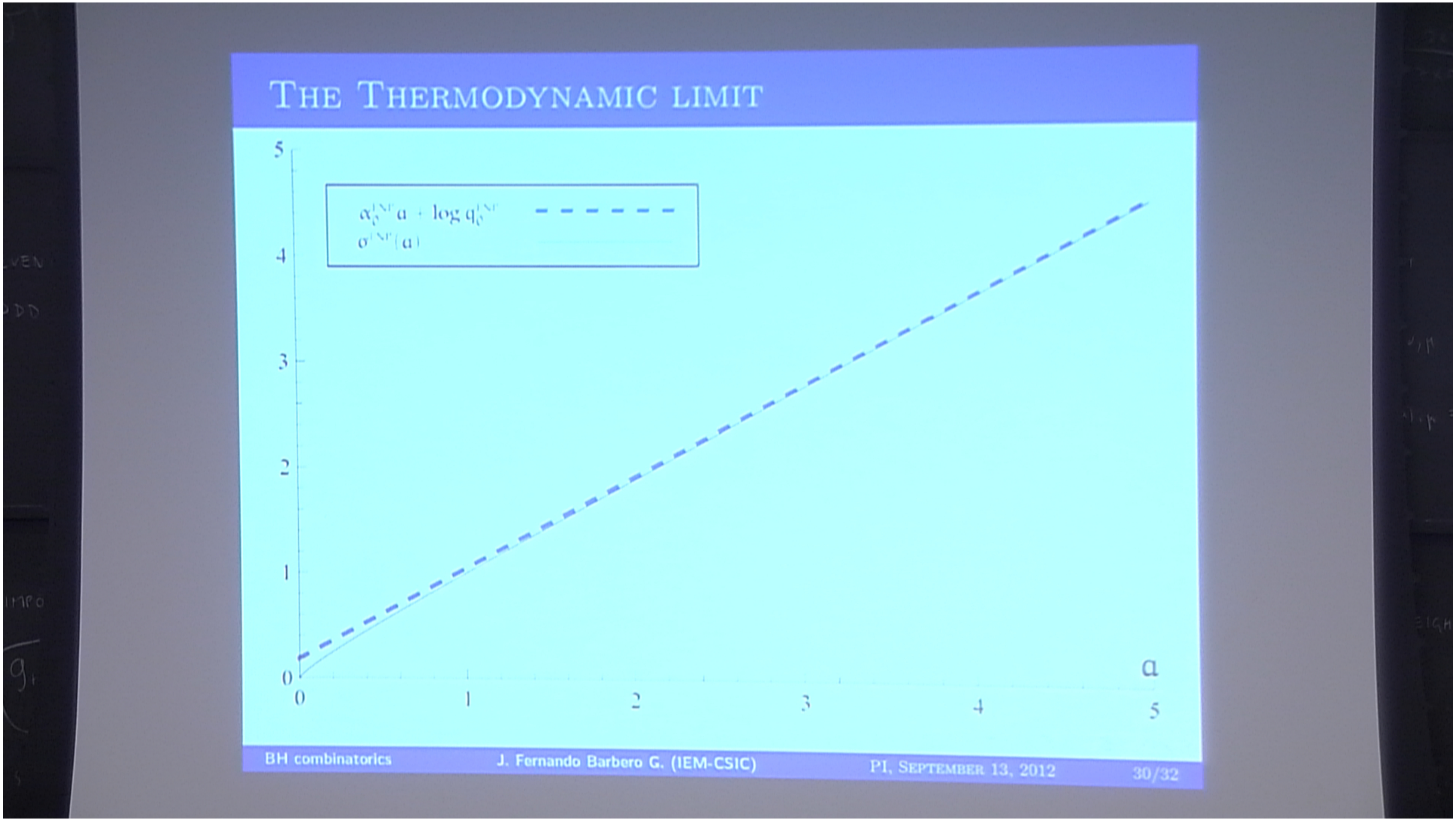
THE THERMODYNAMIC LIMIT
$\alpha_c^{LNP} a + \log q_c^{LNP}$
$\sigma^{LNP}(a)$
a
0
1
2
3
4
5
BH combinatorics
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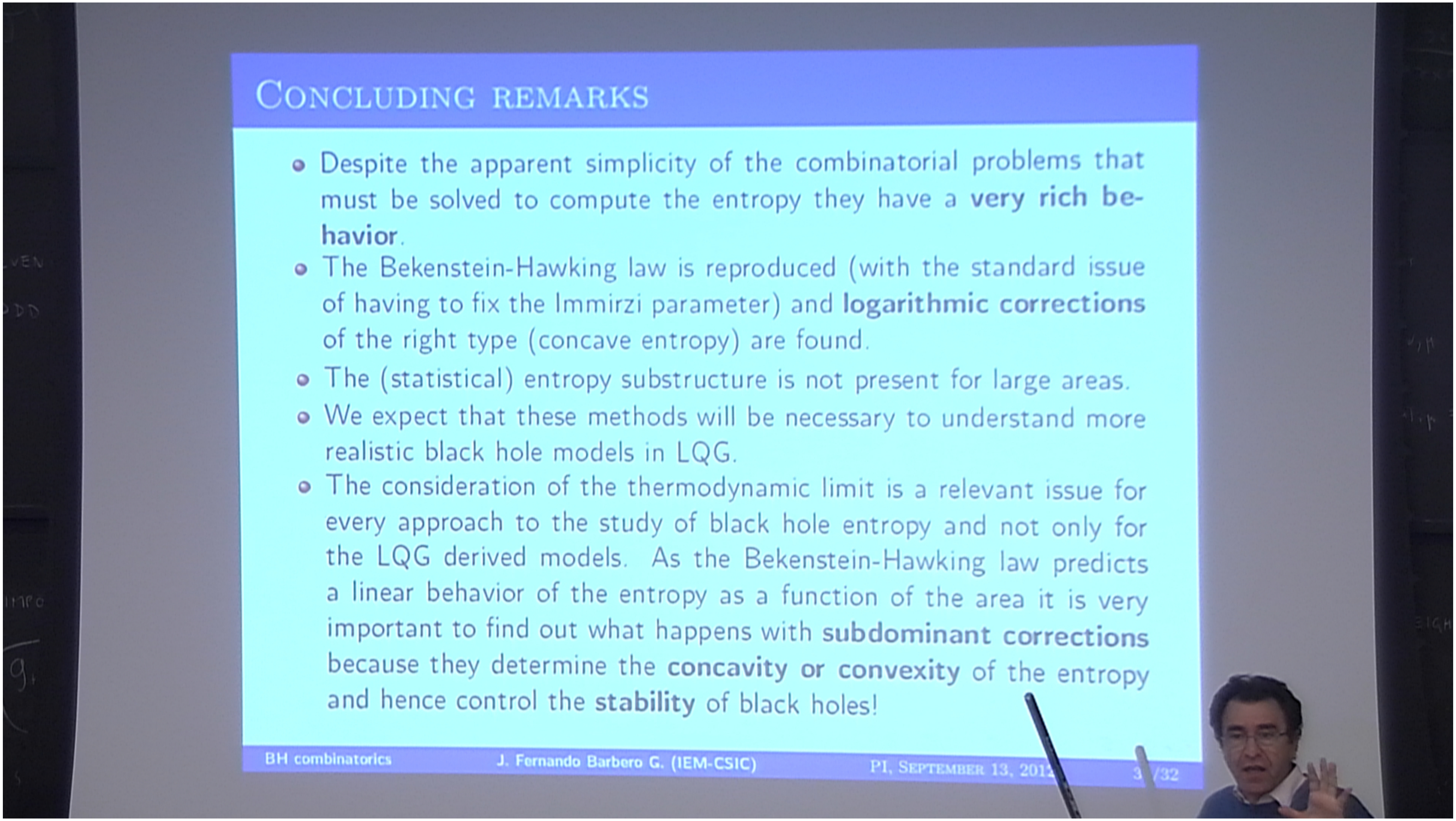

CONCLUDING REMARKS
Despite the apparent simplicity of the combinatorial problems that must be solved to compute the entropy they have a very rich behavior.
The Bekenstein-Hawking law is reproduced (with the standard issue of having to fix the Immirzi parameter) and logarithmic corrections of the right type (concave entropy) are found.
The (statistical) entropy substructure is not present for large areas.
We expect that these methods will be necessary to understand more realistic black hole models in LQG.
The consideration of the thermodynamic limit is a relevant issue for every approach to the study of black hole entropy and not only for the LQG derived models. As the Bekenstein-Hawking law predicts a linear behavior of the entropy as a function of the area it is very important to find out what happens with subdominant corrections because they determine the concavity or convexity of the entropy and hence control the stability of black holes!
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