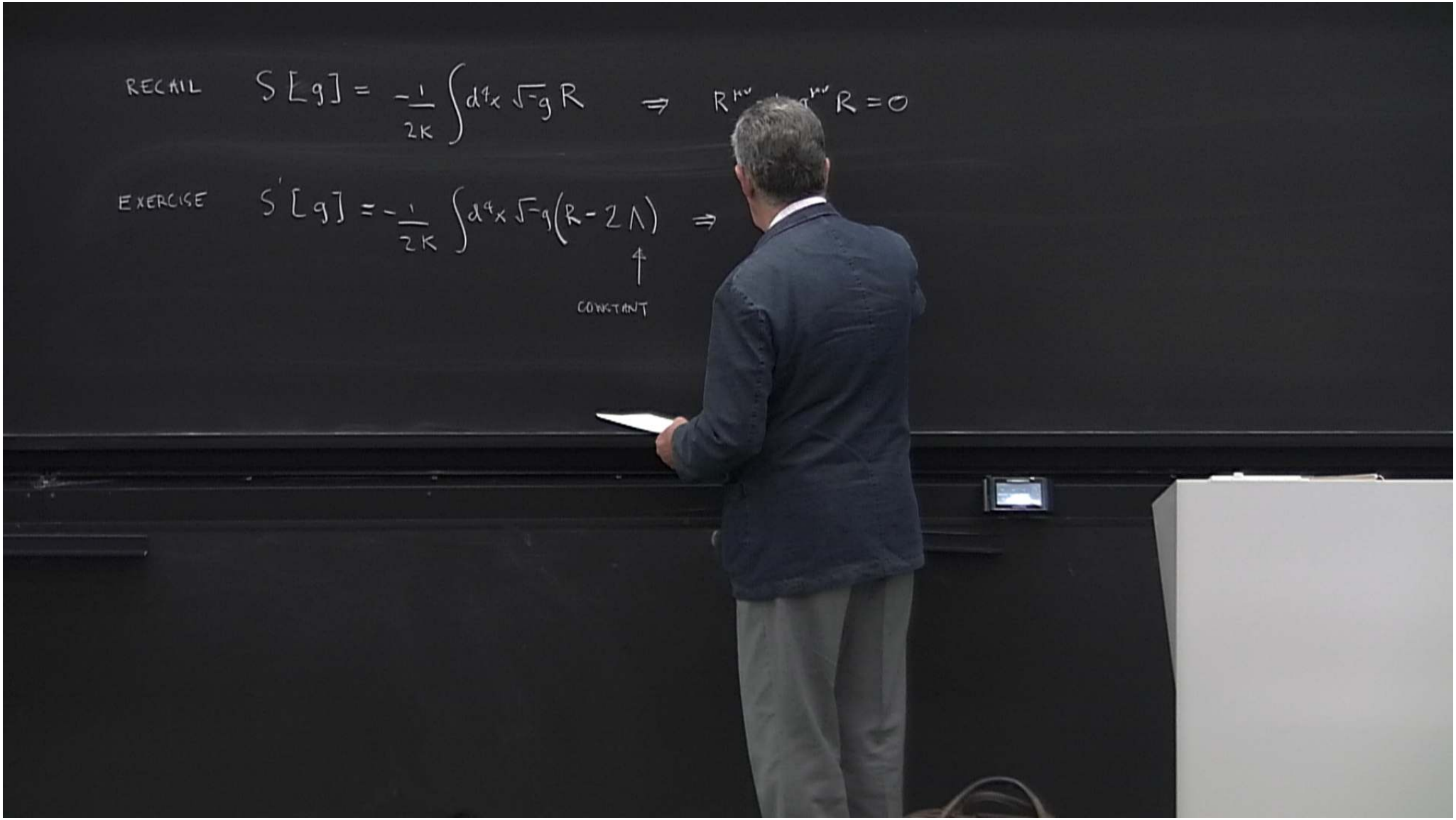
Title: Relativity - Lecture 12

Date: Sep 25, 2012 10:30 AM

URL: http://pirsa.org/12090034

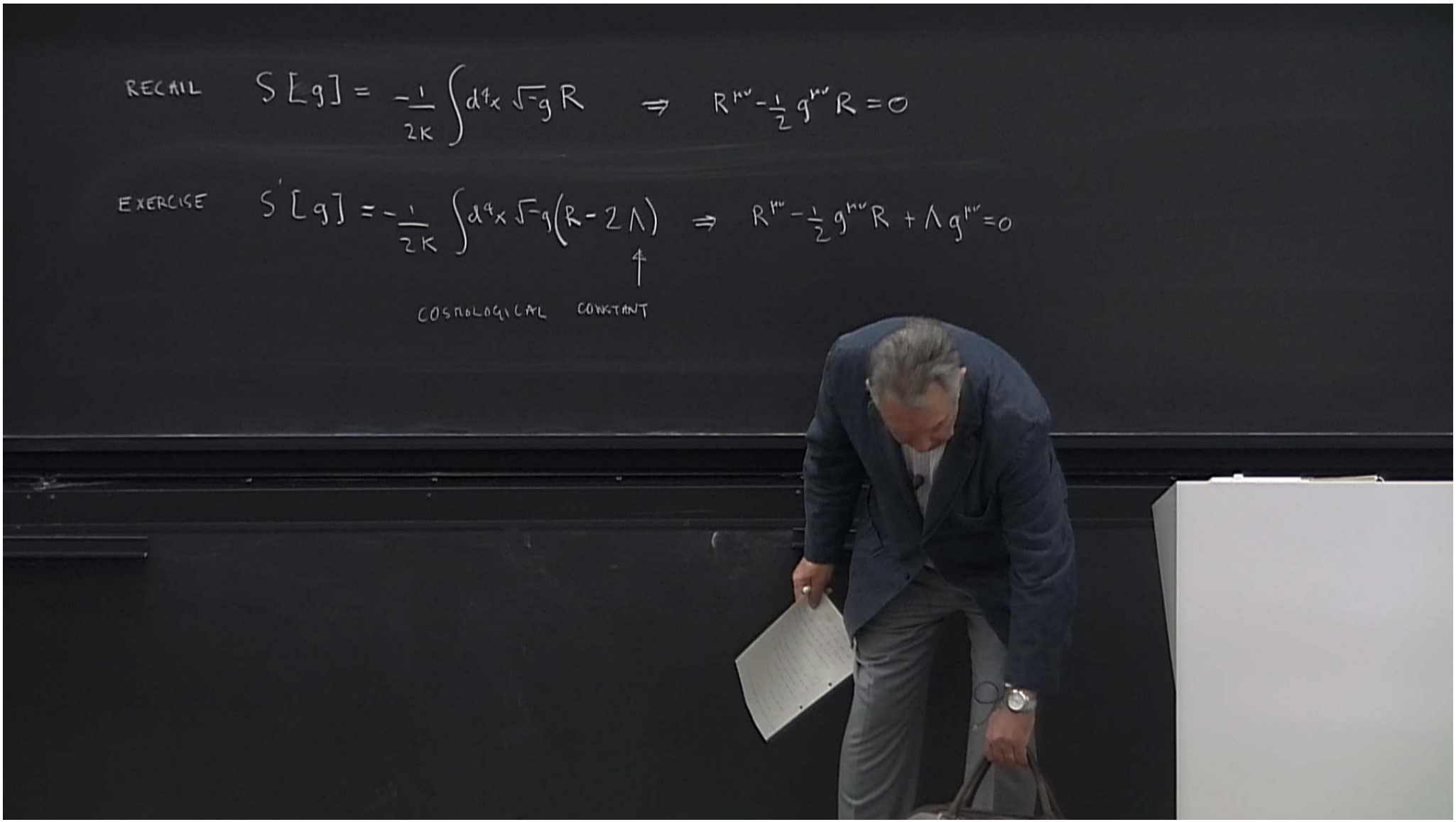
Abstract:


RECALL $S[g] = -\frac{1}{2\kappa}\int d^4x \sqrt{-g} R \Rightarrow R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 0$
EXERCISE $S'[g] = -\frac{1}{2\kappa}\int d^4x \sqrt{-g}(R - 2\Lambda) \Rightarrow$
CONSTANT

RECALL $S[g] = -\frac{1}{2K}\int d^4x \sqrt{-g} R \Rightarrow R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 0$

EXERCISE $S'[g] = -\frac{1}{2K}\int d^4x \sqrt{-g}(R - 2\Lambda) \Rightarrow R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R + \Lambda g^{\mu\nu} = 0$

COSMOLOGICAL CONSTANT



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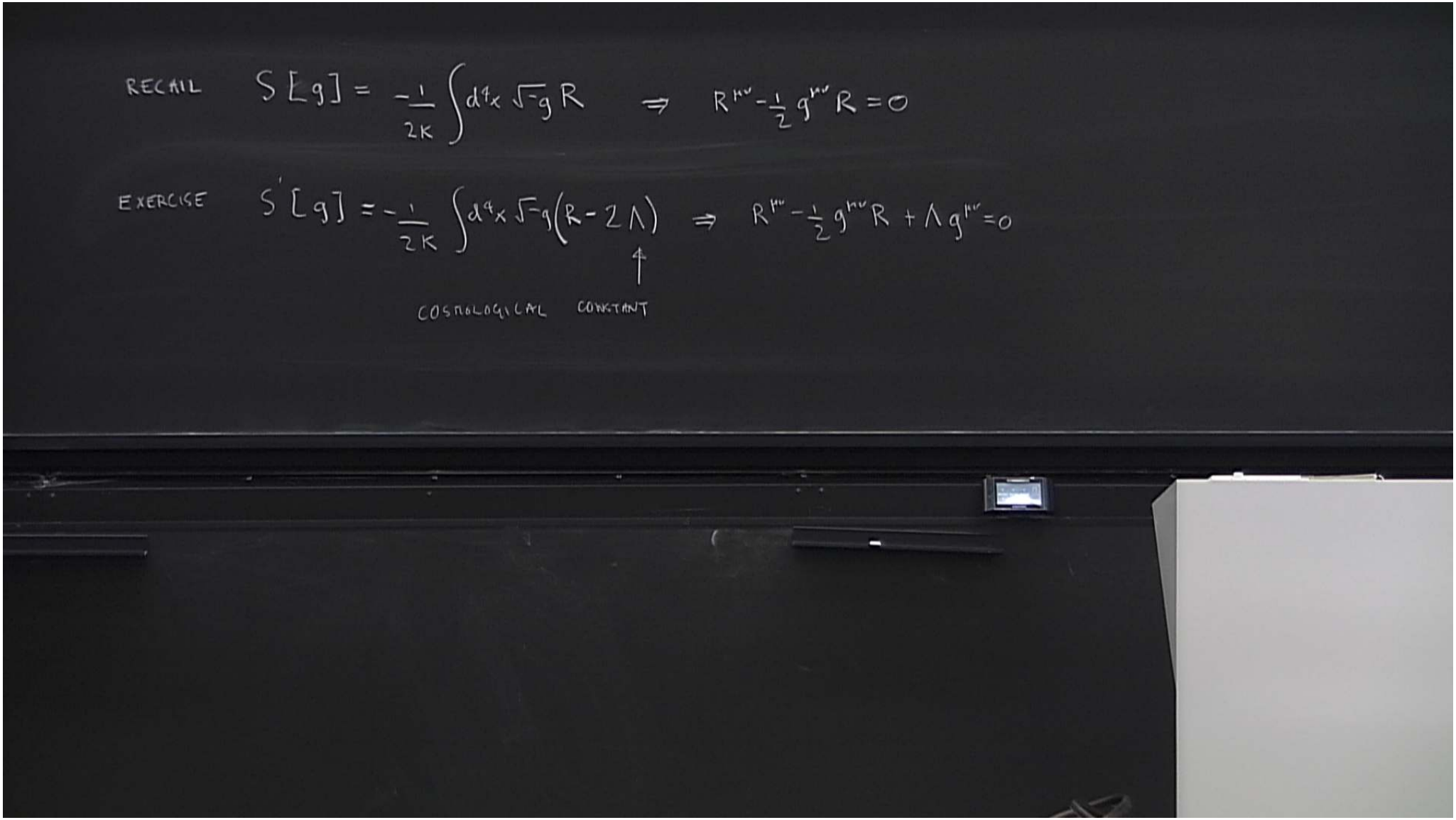
EXERCISE $S'[g] = -\frac{1}{2K}\int d^4x \sqrt{-g}(R - 2\Lambda) \Rightarrow R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R + \Lambda g^{\mu\nu} = 0$

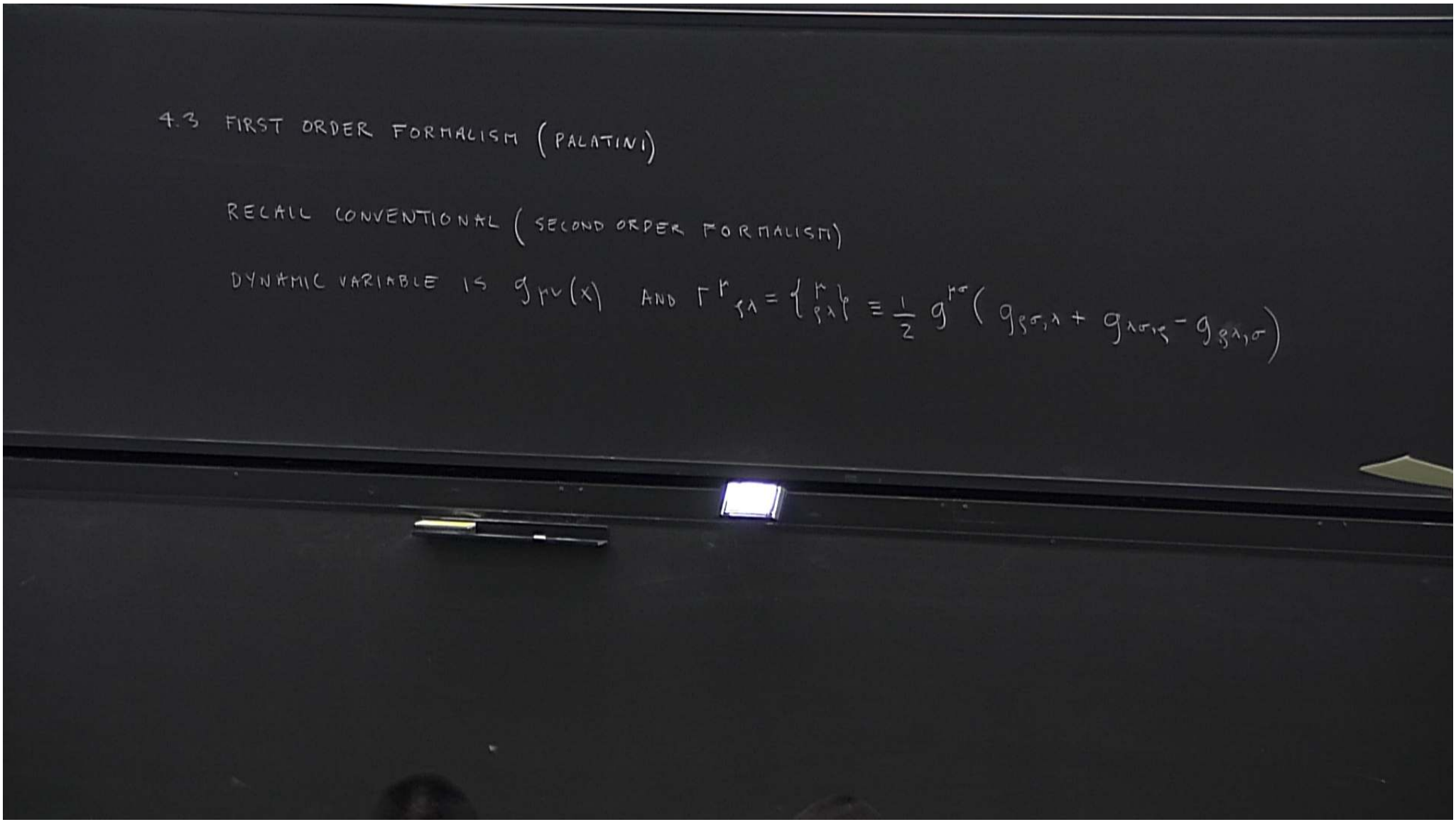
COSMOLOGICAL CONSTANT

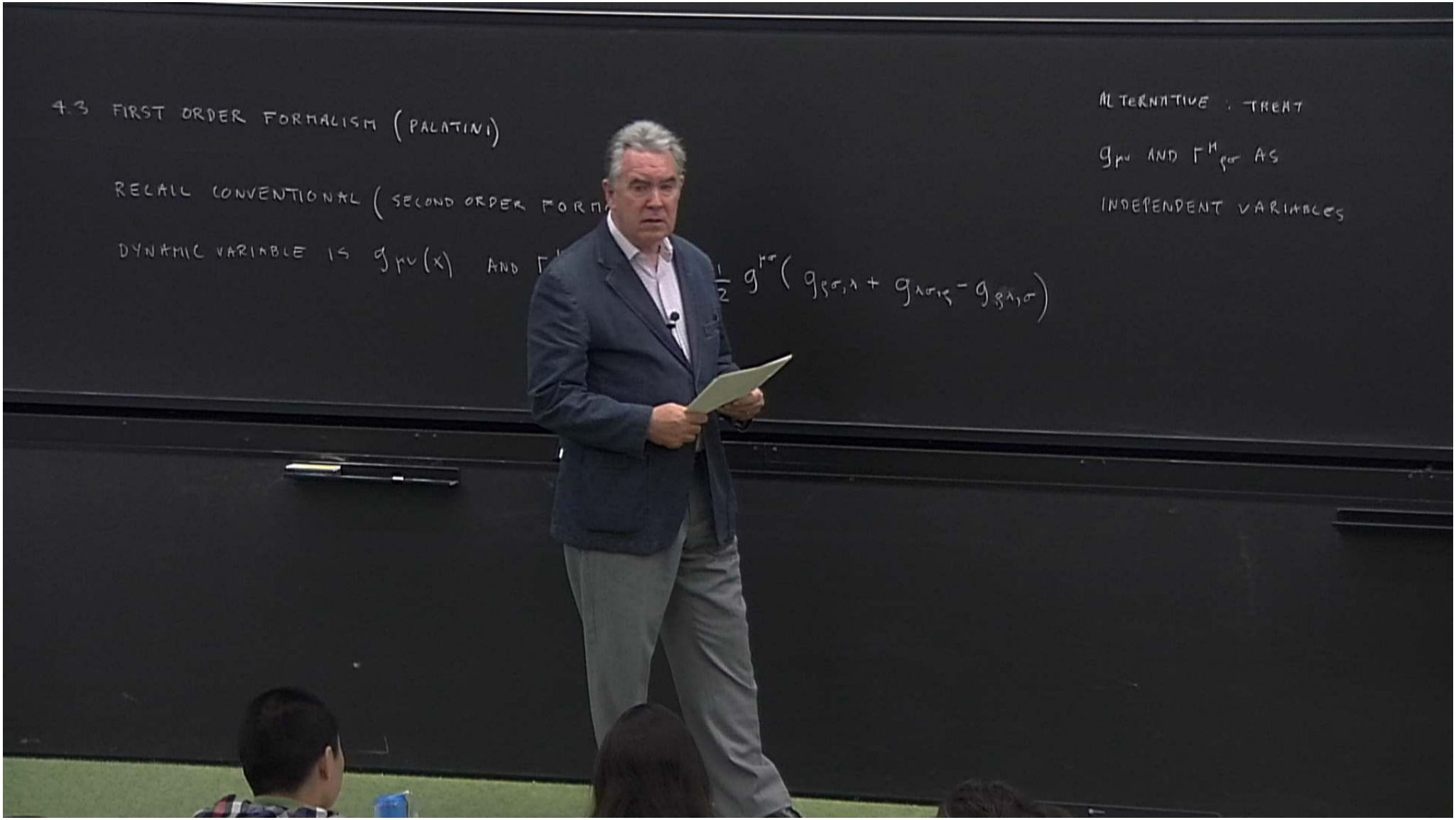


4.3 FIRST ORDER FORMALISM (PALATINI)

RECALL CONVENTIONAL (SECOND ORDER FORMALISM)

DYNAMIC VARIABLE IS $g_{\mu\nu}(x)$ AND $\Gamma^{\rho}_{\sigma\lambda} = \left\{ {\rho \atop \sigma\lambda} \right\} \equiv \frac{1}{2} g^{\rho\sigma} \left(g_{\sigma\sigma,\lambda} + g_{\lambda\sigma,\sigma} - g_{\sigma\lambda,\sigma} \right)$

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RECALL CONVENTIONAL (SECOND ORDER FORM
DYNAMIC VARIABLE IS $g_{\mu\nu}(x)$ AND
$\frac{1}{2} g^{\mu\sigma}(g_{\sigma\sigma,\lambda} + g_{\lambda\sigma,\sigma} - g_{\sigma\lambda,\sigma})$
ALTERNATIVE : TREAT
$g_{\mu\nu}$ AND $\Gamma^{\lambda}_{\sigma\sigma}$ AS
INDEPENDENT VARIABLES

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ALTERNATIVE: TREAT

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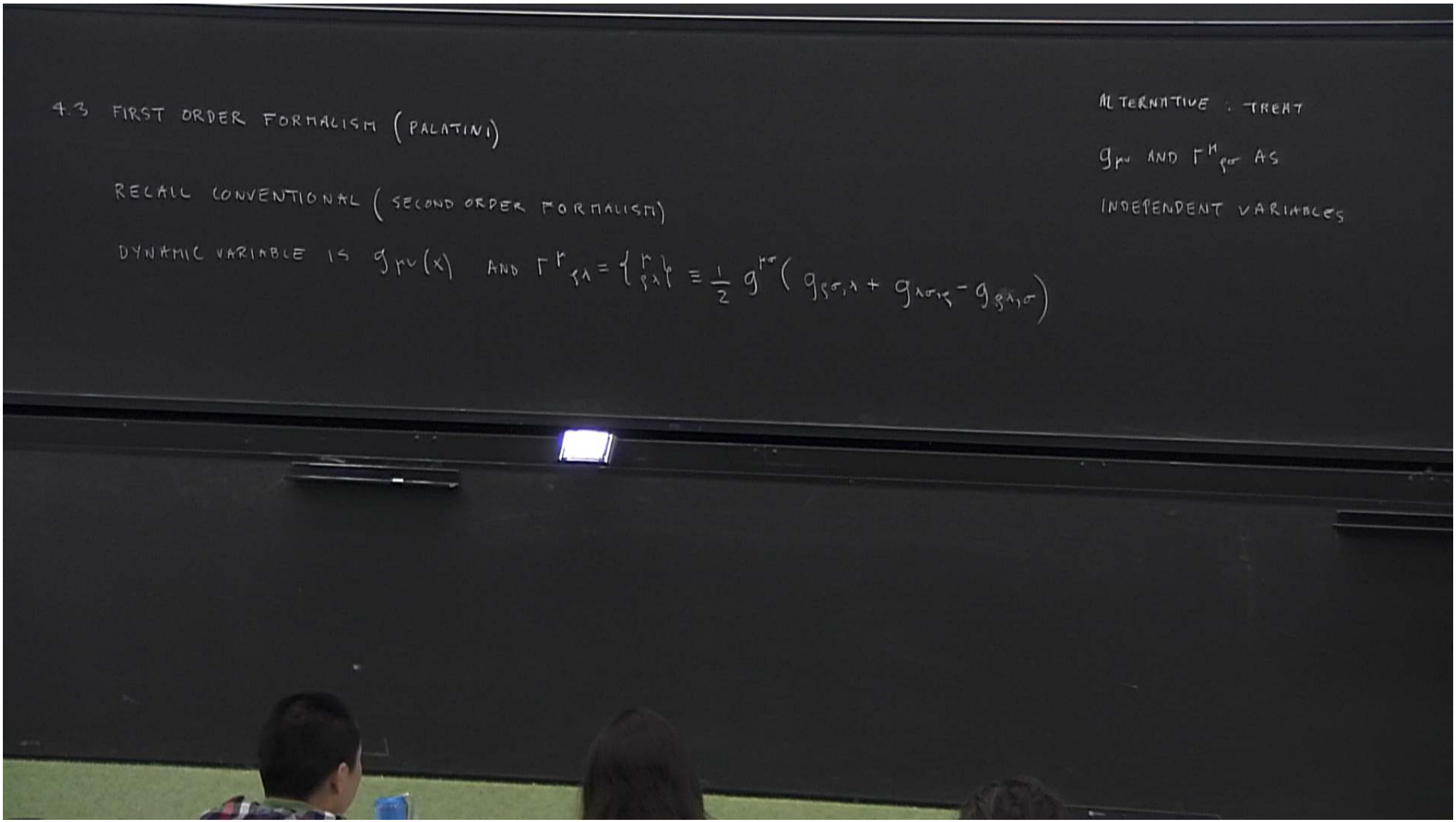
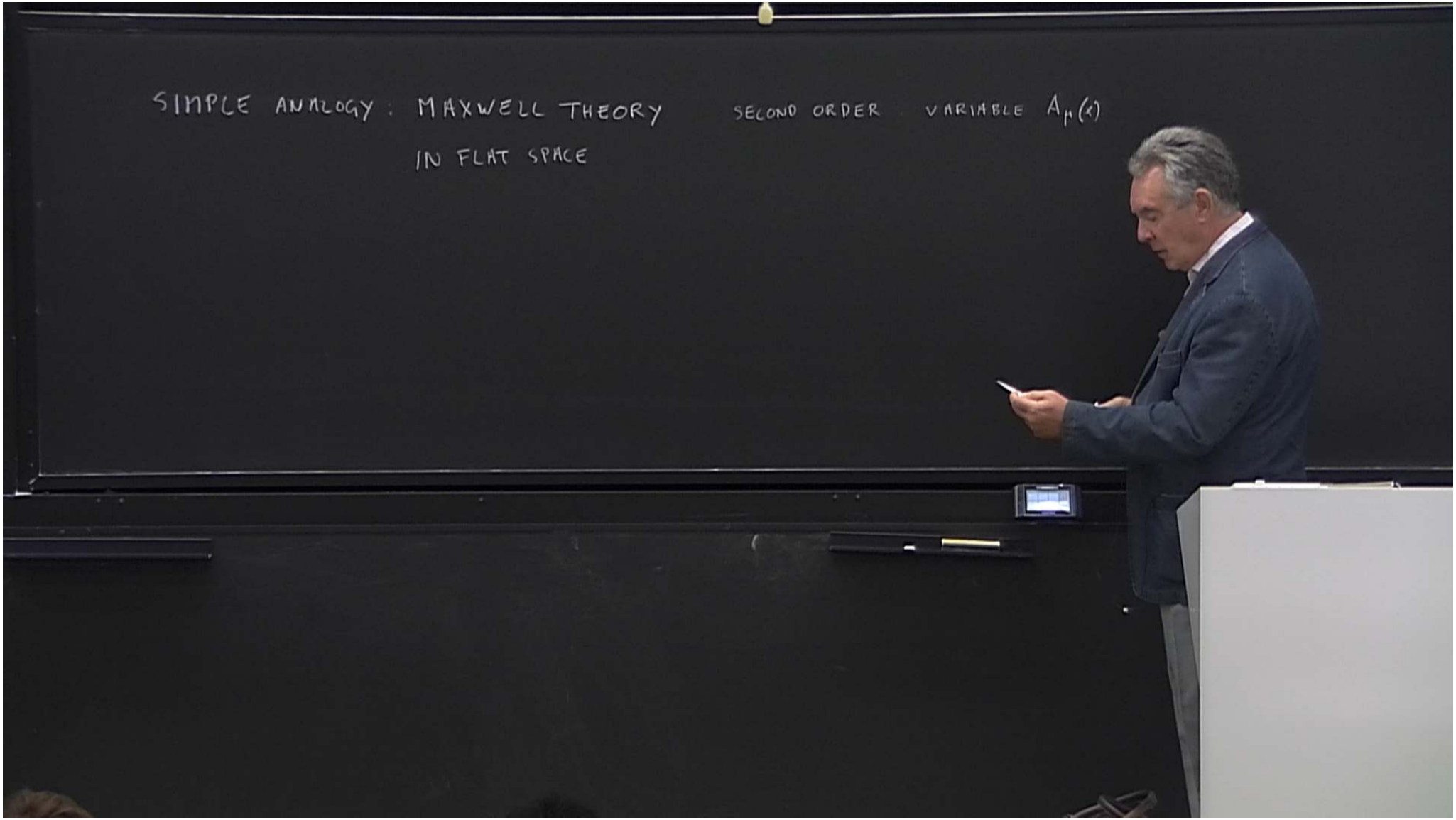
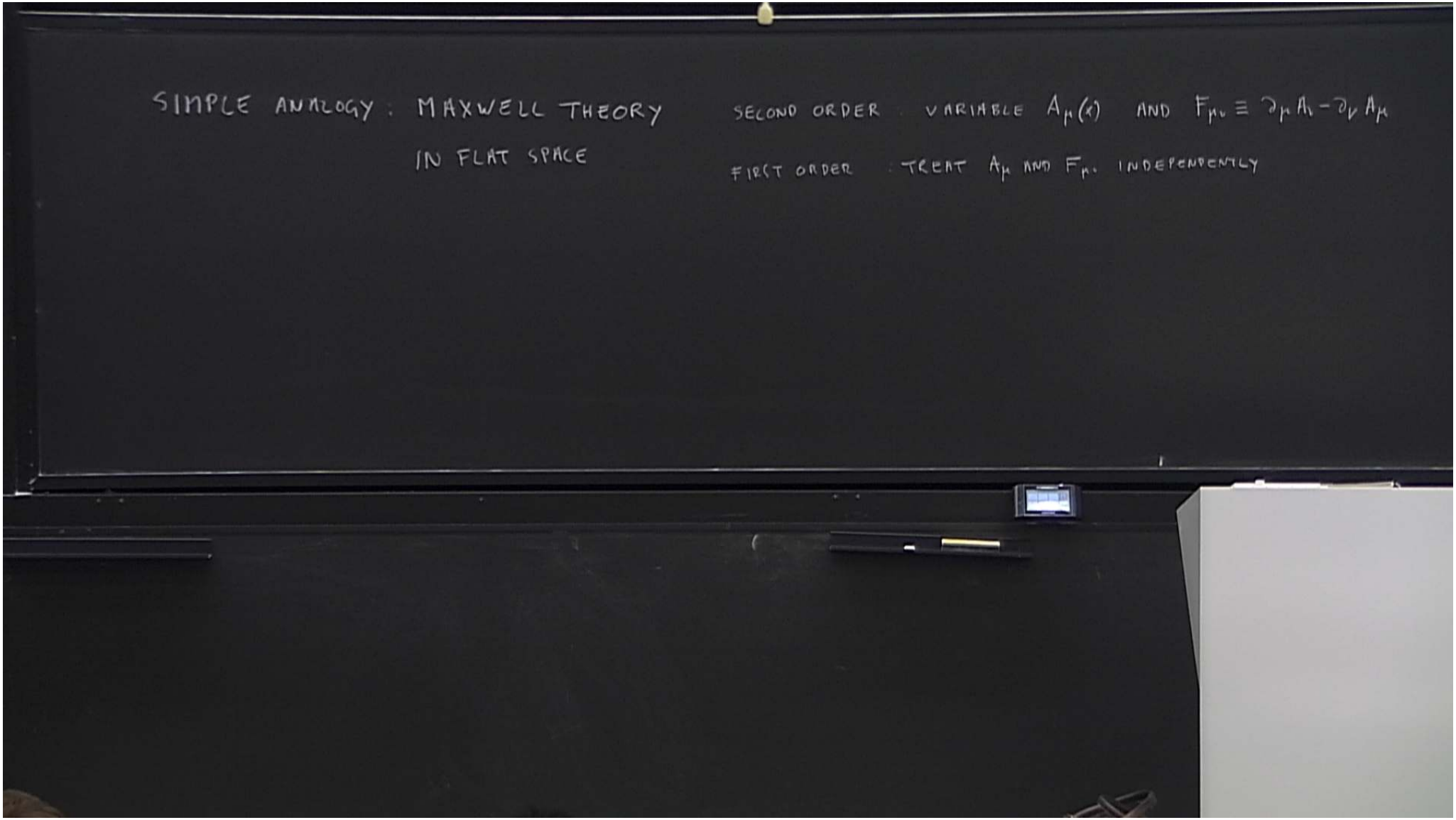
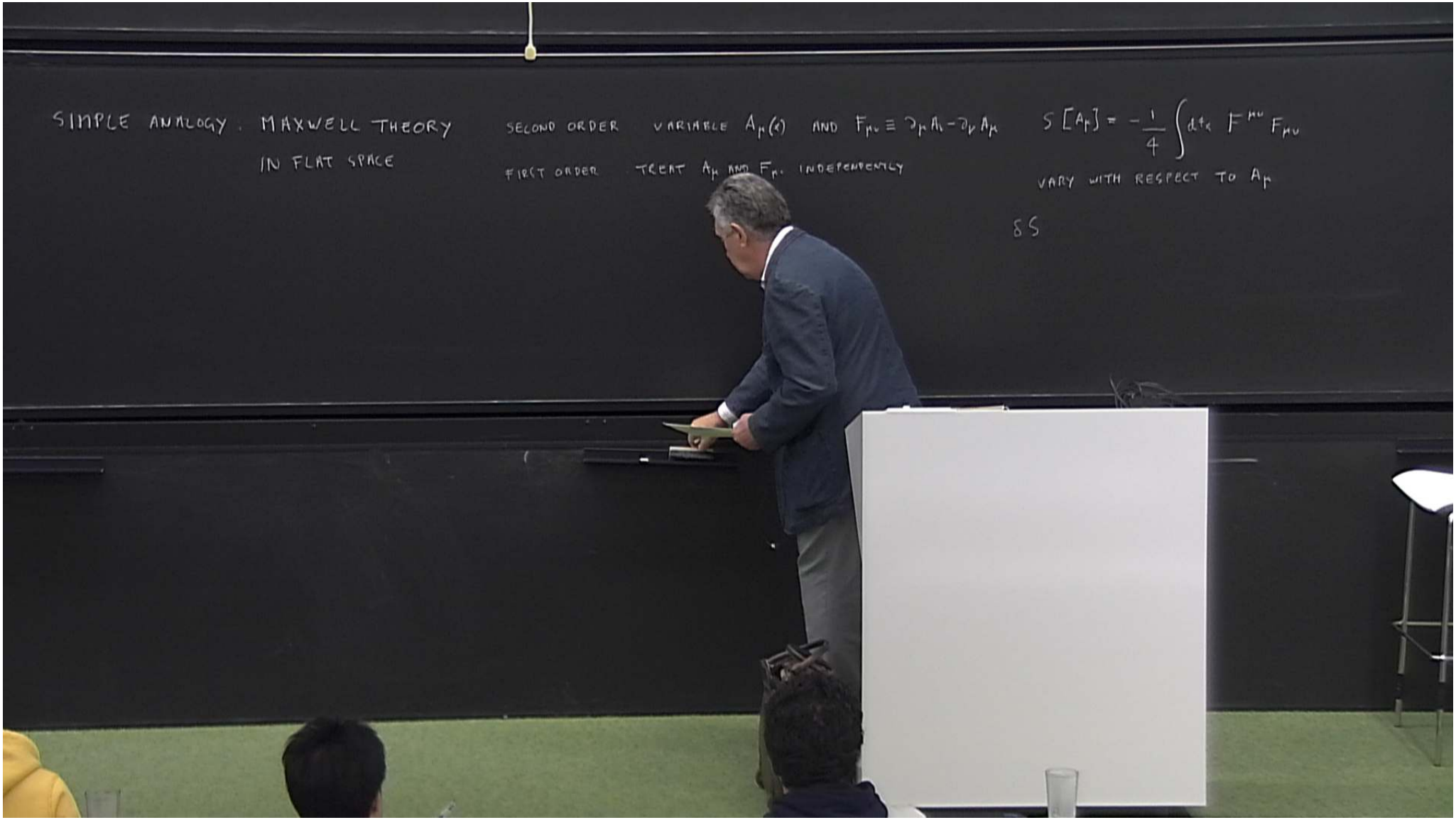
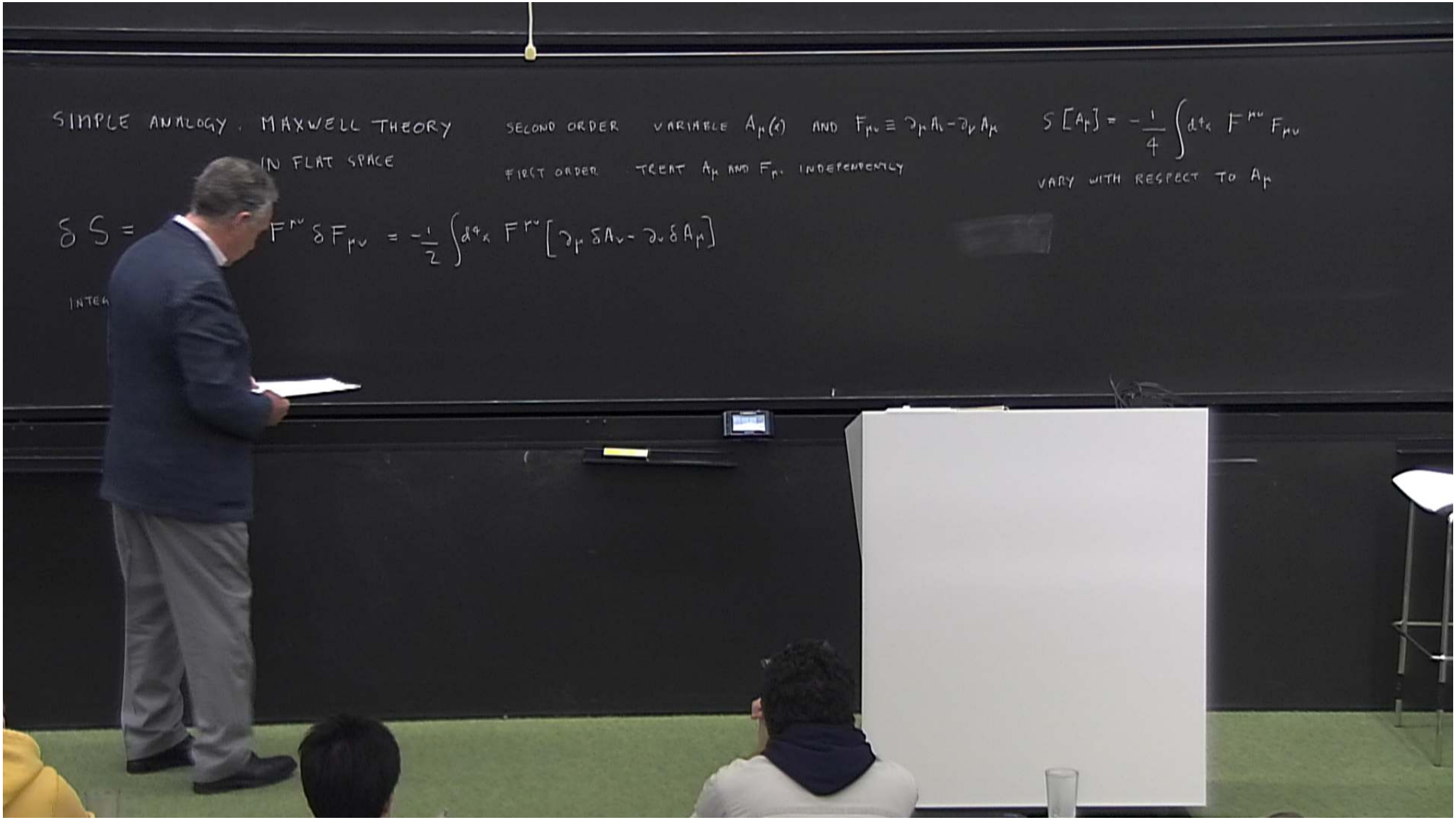



SIMPLE ANALOGY : MAXWELL THEORY SECOND ORDER VARIABLE $A_\mu(x)$
IN FLAT SPACE


SIMPLE ANALOGY : MAXWELL THEORY IN FLAT SPACE
SECOND ORDER : VARIABLE $A_\mu(x)$ AND $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$
FIRST ORDER : TREAT A_μ AND $F_{\mu\nu}$ INDEPENDENTLY


SIMPLE ANALOGY . MAXWELL THEORY
IN FLAT SPACE
SECOND ORDER VARIABLE $A_\mu(x)$ AND $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$
$S[A_\mu] = -\frac{1}{4}\int d^4x\, F^{\mu\nu} F_{\mu\nu}$
FIRST ORDER TREAT A_μ AND $F_{\mu\nu}$ INDEPENDENTLY
VARY WITH RESPECT TO A_μ
δS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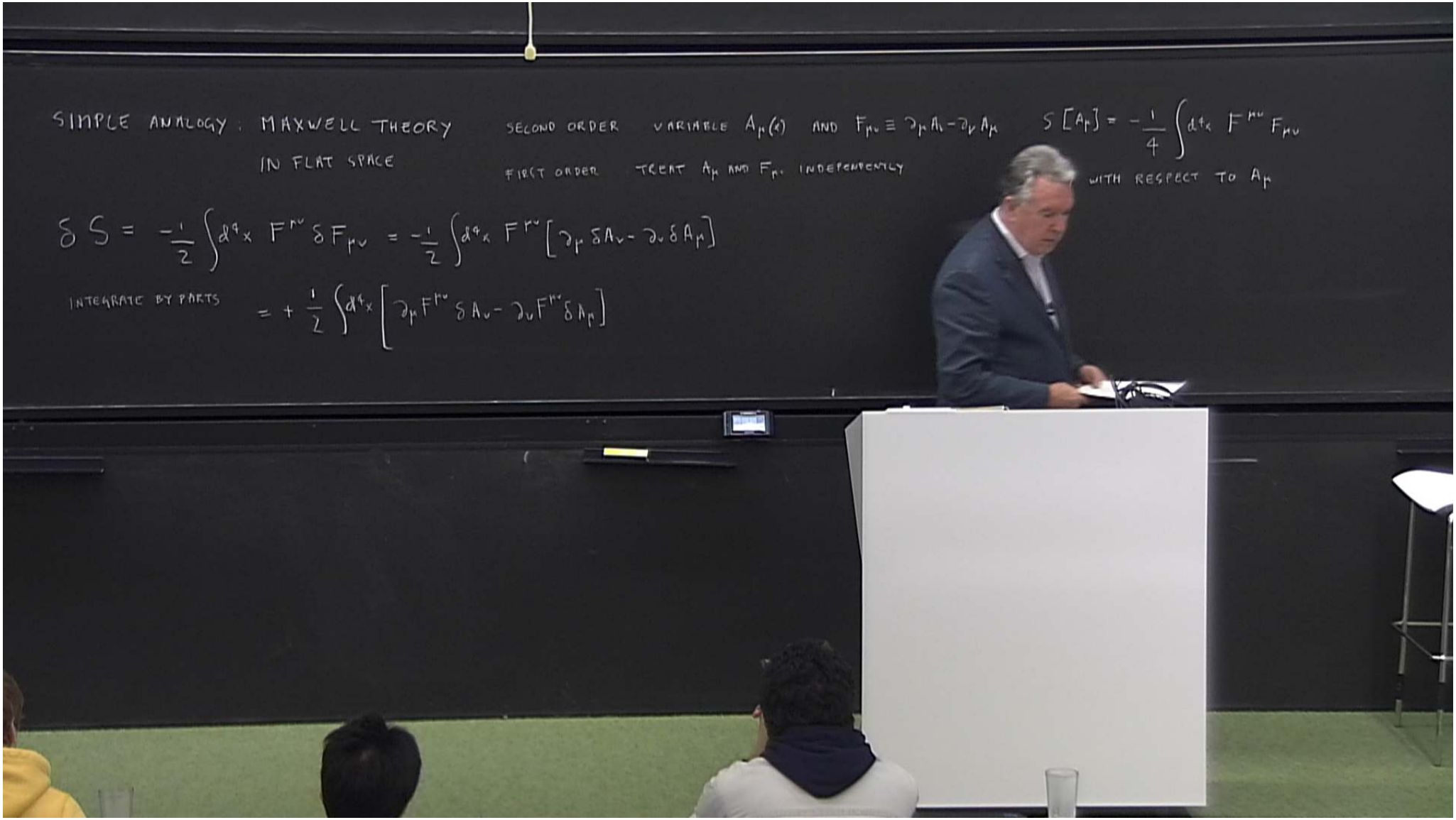
FIRST ORDER TREAT A_μ AND $F_{\mu\nu}$ INDEPENDENTLY
VARY WITH RESPECT TO A_μ
$\delta S =$
$F^{\mu\nu}\delta F_{\mu\nu} = -\frac{1}{2}\int d^4x\, F^{\mu\nu}\left[\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu\right]$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FIRST ORDER TREAT A_μ AND $F_{\mu\nu}$ INDEPENDENTLY
WITH RESPECT TO A_μ
$\delta S = -\frac{1}{2}\int d^4x\, F^{\mu\nu}\delta F_{\mu\nu} = -\frac{1}{2}\int d^4x\, F^{\mu\nu}[\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu]$
INTEGRATE BY PARTS
$= +\frac{1}{2}\int d^4x\,[\partial_\mu F^{\mu\nu}\delta A_\nu - \partial_\nu F^{\mu\nu}\delta A_\mu]$

SIMPLE ANALOGY: MAXWELL THEORY IN FLAT SPACE

SECOND ORDER VARIABLE $A_\mu(x)$ AND $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ $\quad S[A_\mu] = -\frac{1}{4}\int d^4x\, F^{\mu\nu} F_{\mu\nu}$

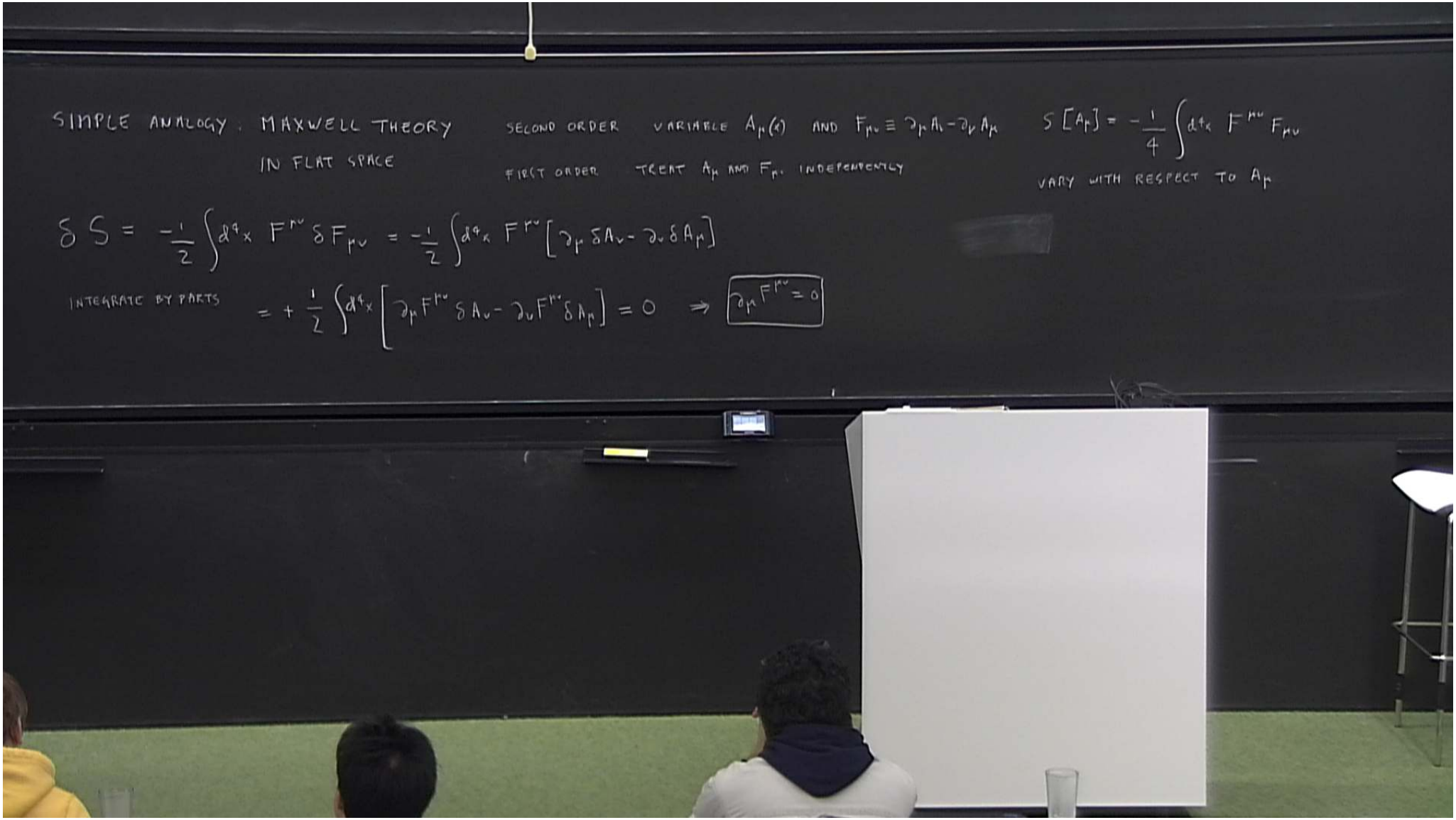
FIRST ORDER TREAT A_μ AND $F_{\mu\nu}$ INDEPENDENTLY

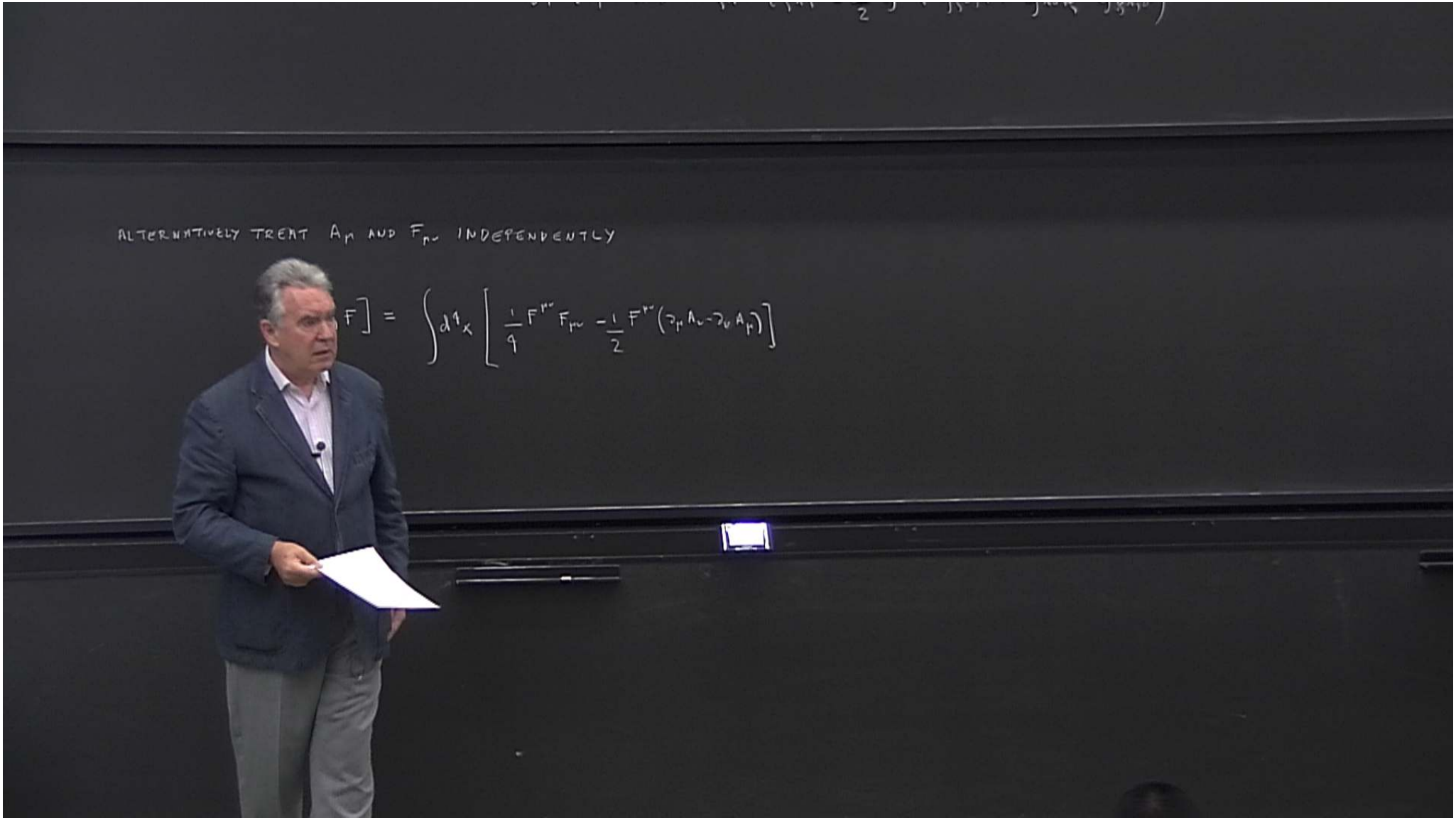
VARY WITH RESPECT TO A_μ

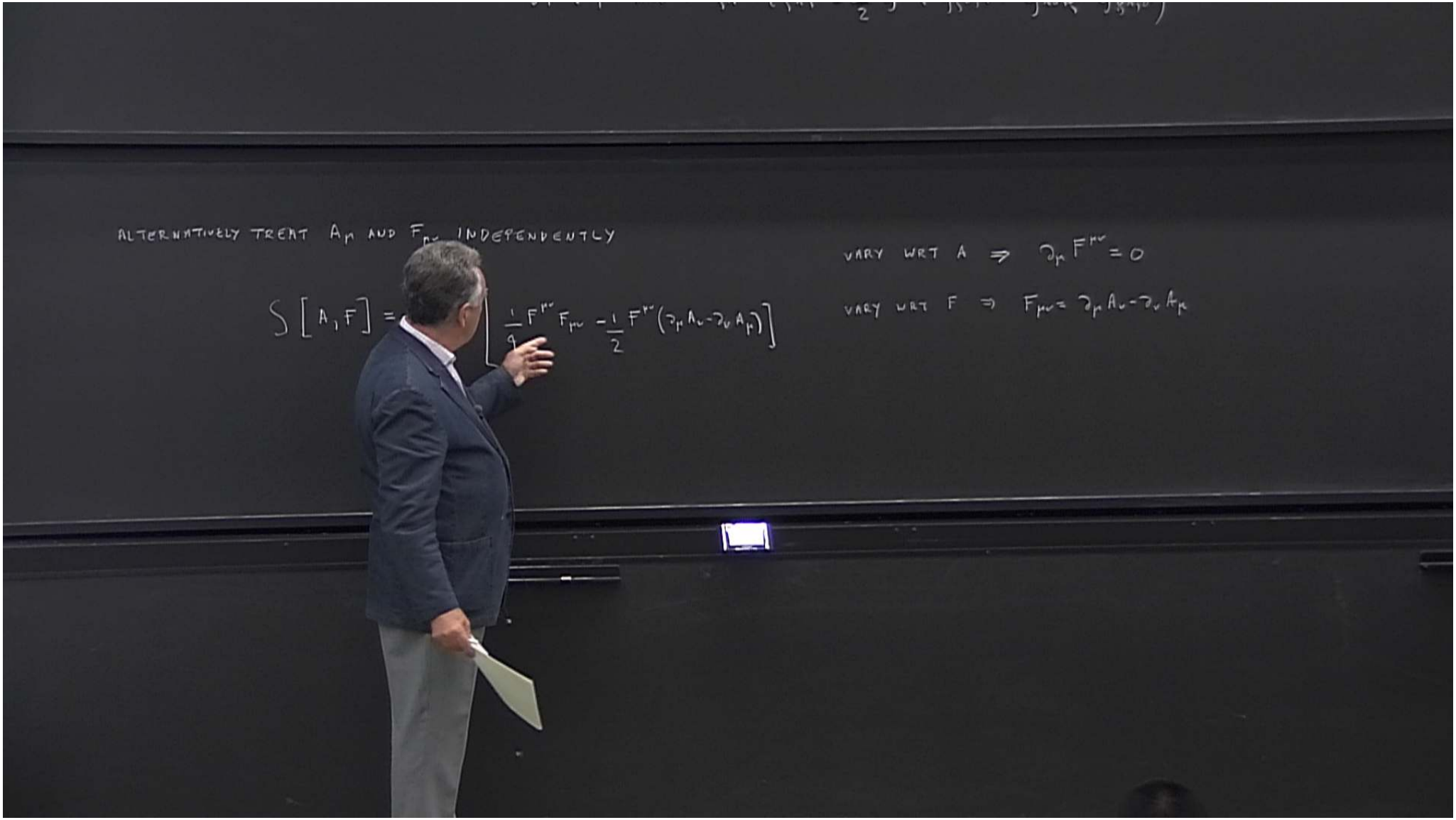
$$\delta S = -\frac{1}{2}\int d^4x\, F^{\mu\nu}\delta F_{\mu\nu} = -\frac{1}{2}\int d^4x\, F^{\mu\nu}\left[\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu\right]$$

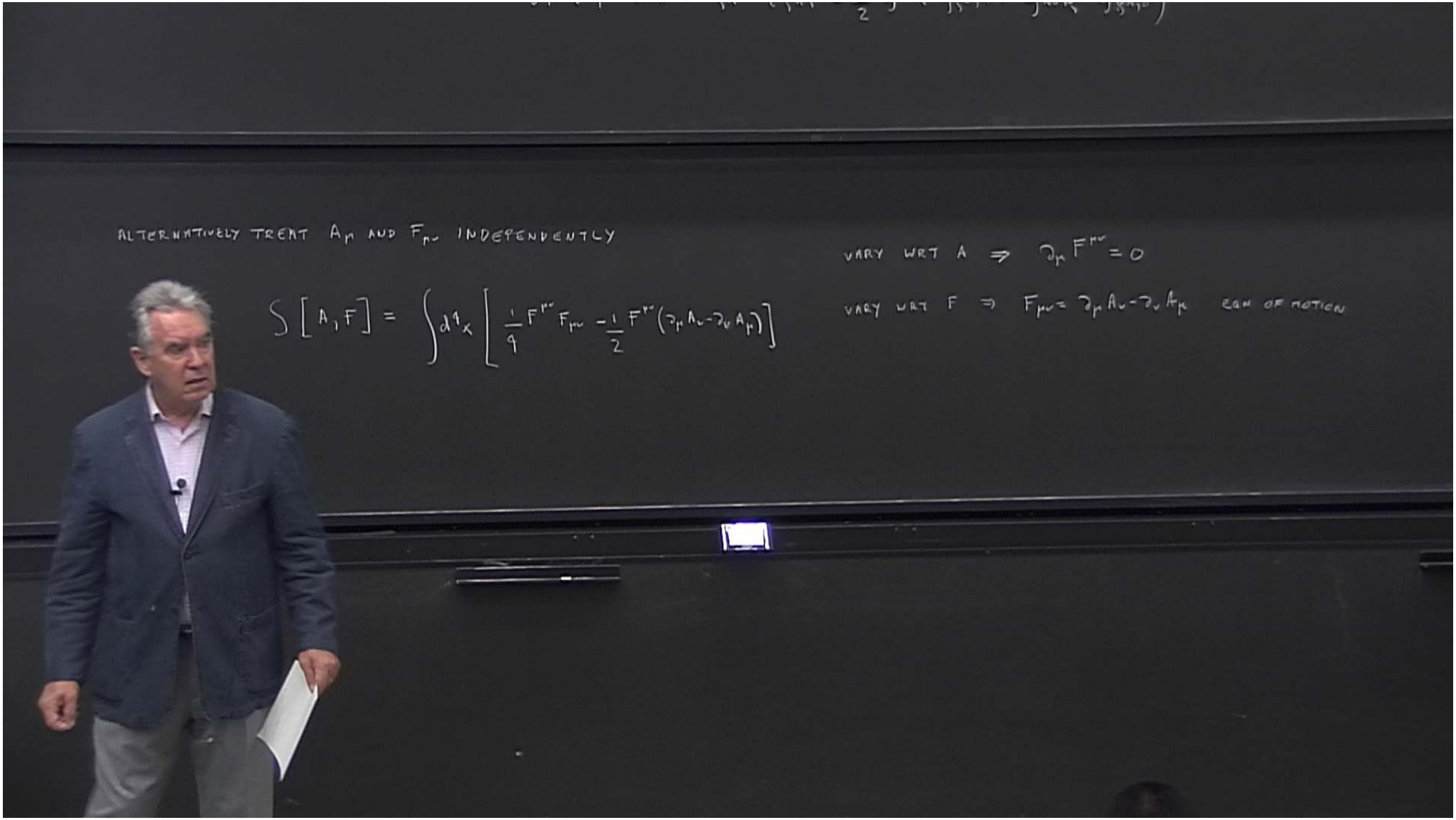
INTEGRATE BY PARTS

$$= +\frac{1}{2}\int d^4x \left[\partial_\mu F^{\mu\nu}\delta A_\nu - \partial_\nu F^{\mu\nu}\delta A_\mu\right] = 0 \quad \Rightarrow \quad \boxed{\partial_\mu F^{\mu\nu} = 0}$$




ALTERNATIVELY TREAT A_μ AND $F_{\mu\nu}$ INDEPENDENTLY
$F] = \int d^4x \left[\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} F^{\mu\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu) \right]$


ALTERNATIVELY TREAT A_μ AND $F_{\mu\nu}$ INDEPENDENTLY
$S[A, F] = \int \left[\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} F^{\mu\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu) \right]$
VARY WRT A $\Rightarrow$ $\partial_\mu F^{\mu\nu} = 0$
VARY WRT F $\Rightarrow$ $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$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VARY WRT A $\Rightarrow$ $\partial_\mu F^{\mu\nu} = 0$
VARY WRT F $\Rightarrow$ $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ EQN OF MOTION

ALTERNATIVELY TREAT A_μ AND $F_{\mu\nu}$ INDEPENDENTLY

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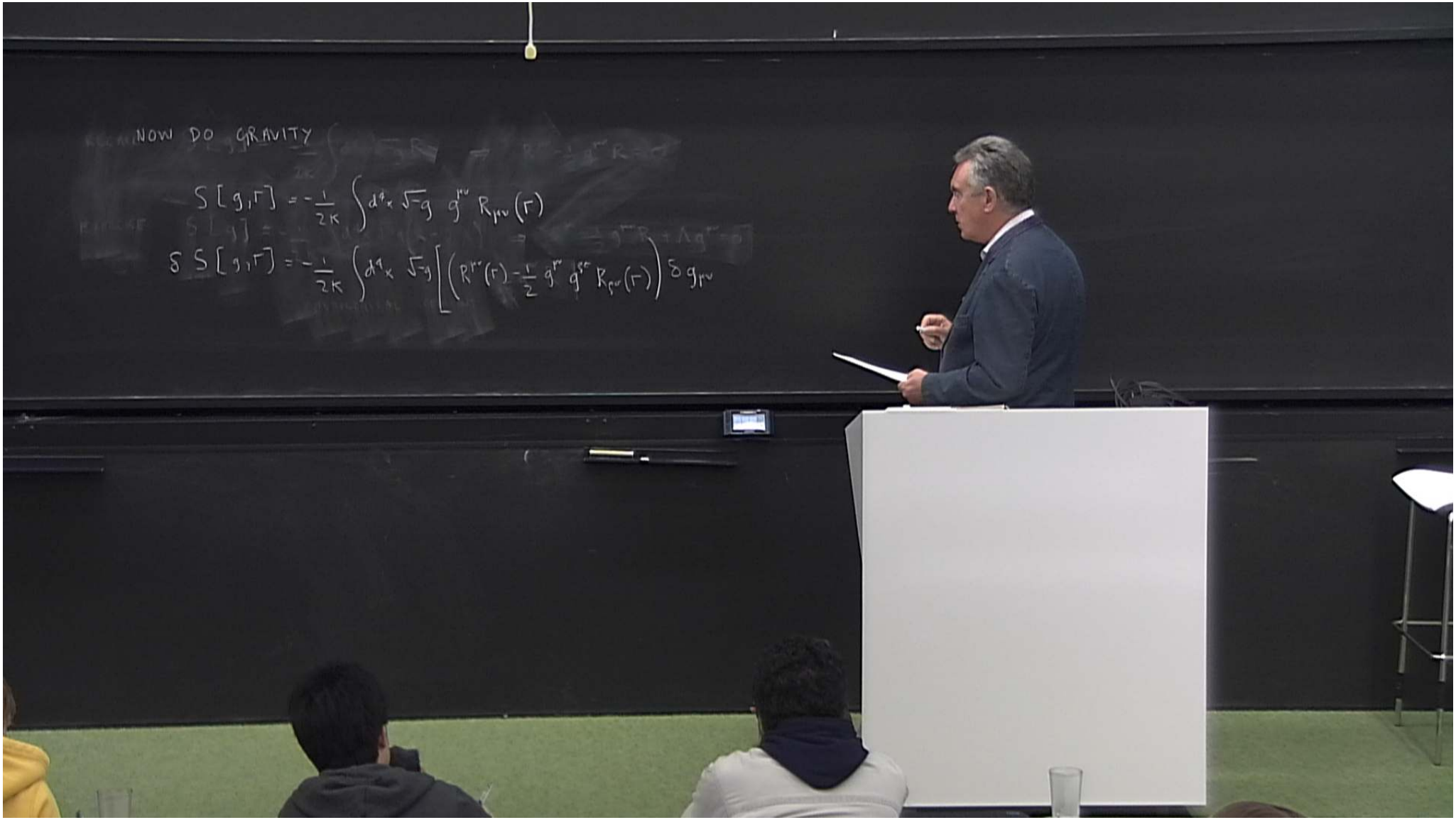
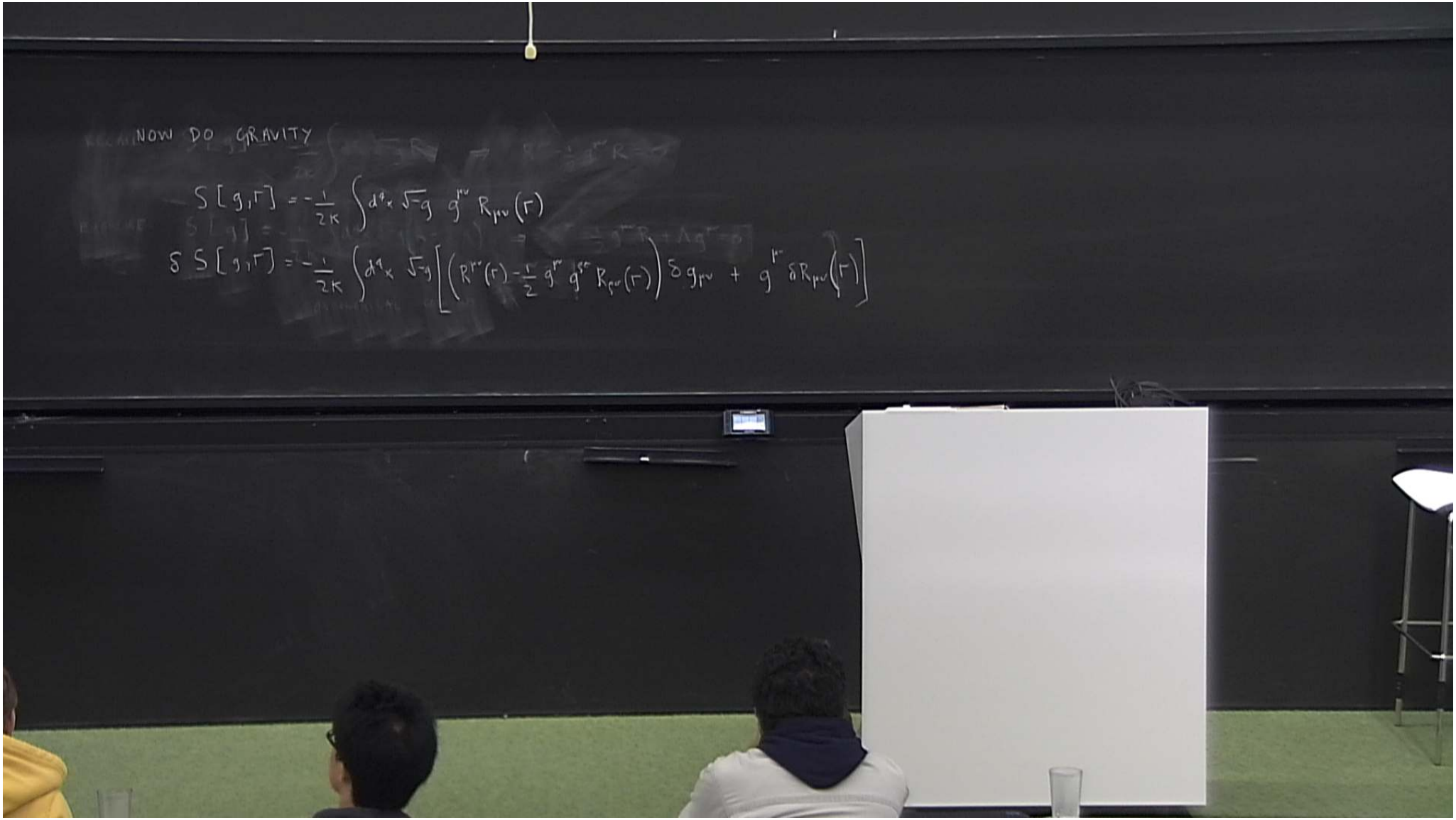
VARY WRT $A \Rightarrow \partial_\mu F^{\mu\nu} = 0$

VARY WRT $F \Rightarrow F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ EQN OF MOTION


NOW DO GRAVITY
$S[g,\Gamma] = -\frac{1}{2\kappa}\int d^4x\,\sqrt{-g}\;g^{\mu\nu}R_{\mu\nu}(\Gamma)$
$\delta S[g,\Gamma] = -\frac{1}{2\kappa}\int d^4x\,\sqrt{-g}\left[\left(R^{\mu\nu}(\Gamma) - \frac{1}{2}g^{\mu\nu}g^{\rho\sigma}R_{\rho\sigma}(\Gamma)\right)\delta g_{\mu\nu}\right.$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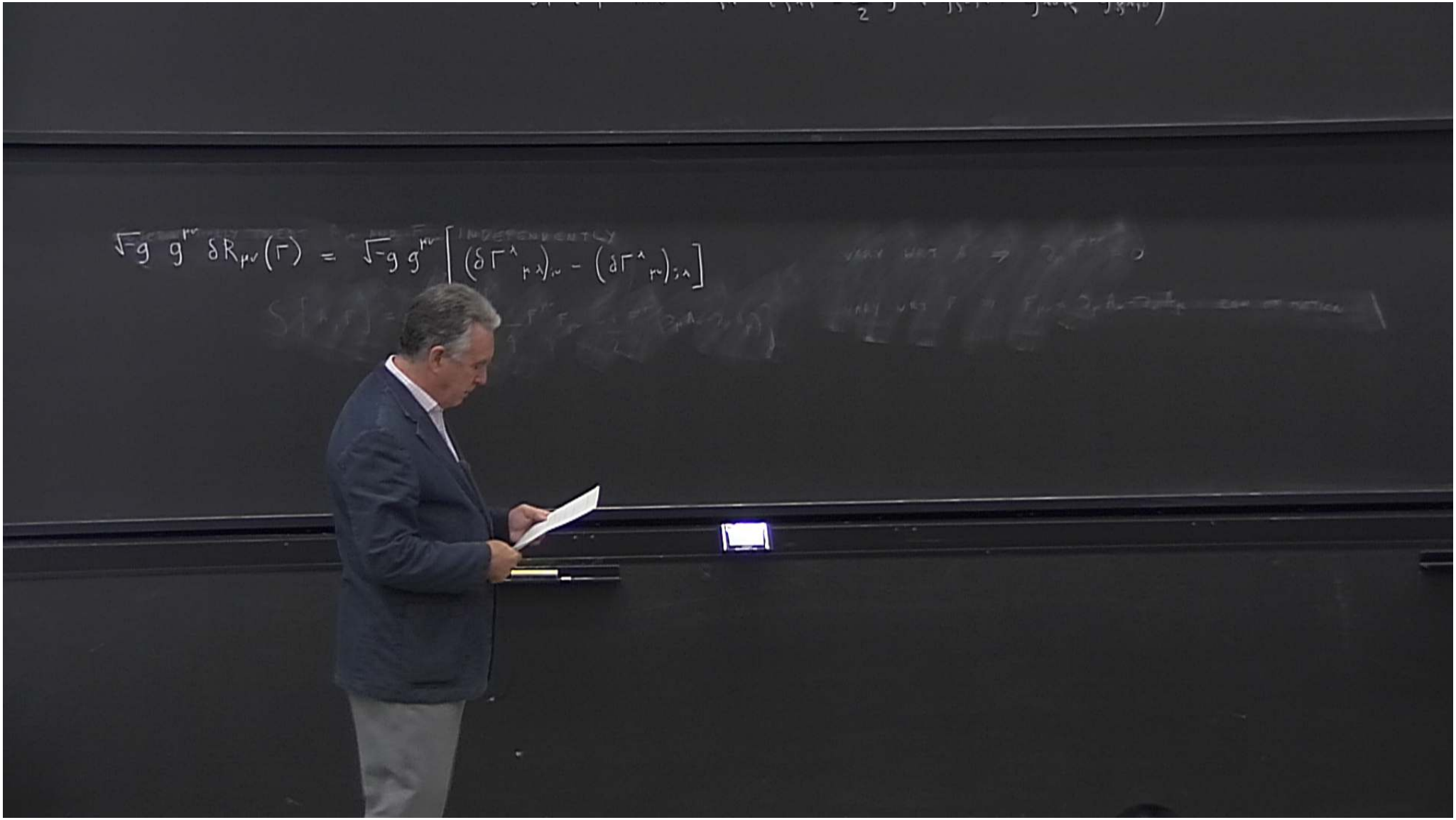
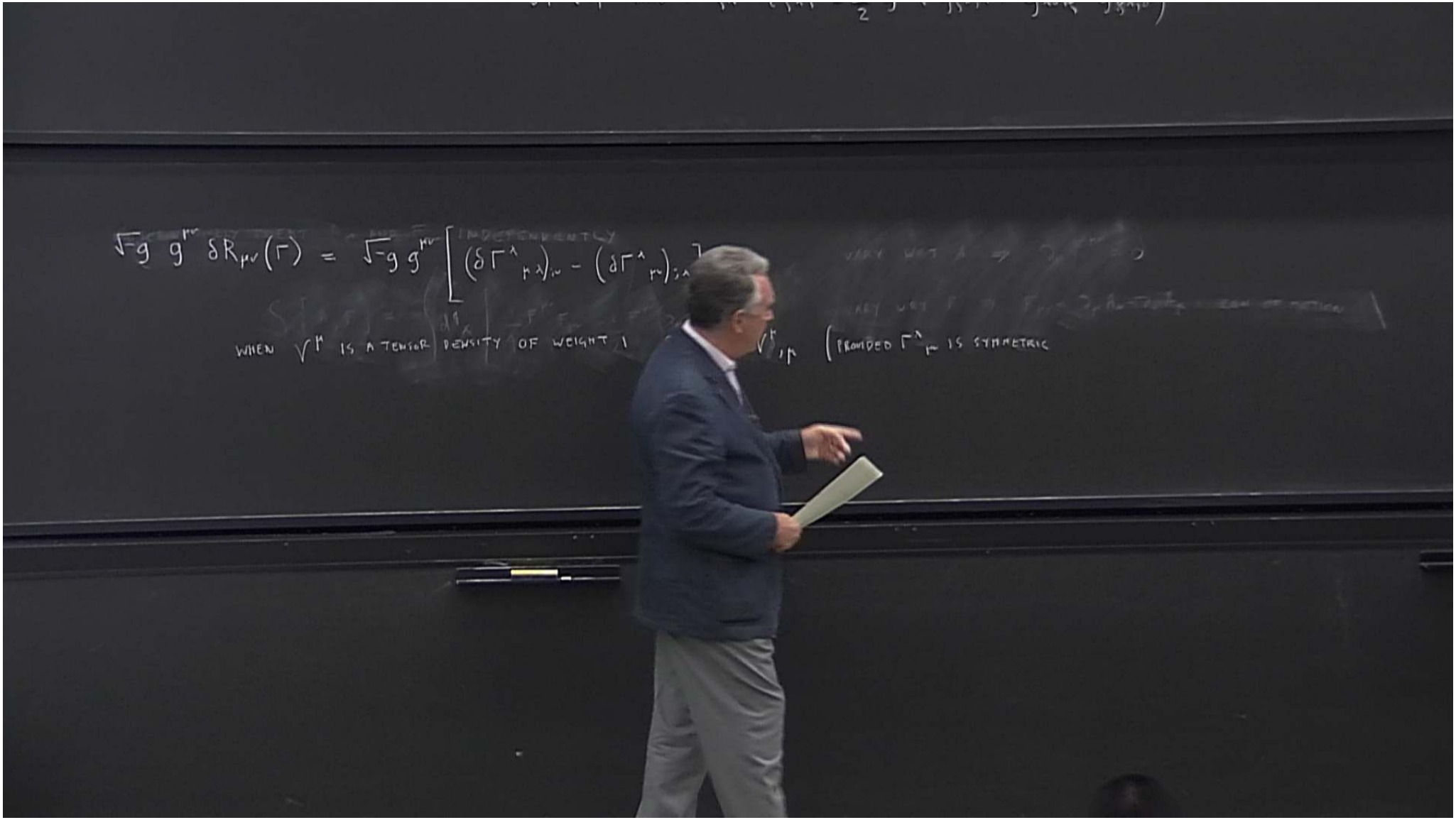
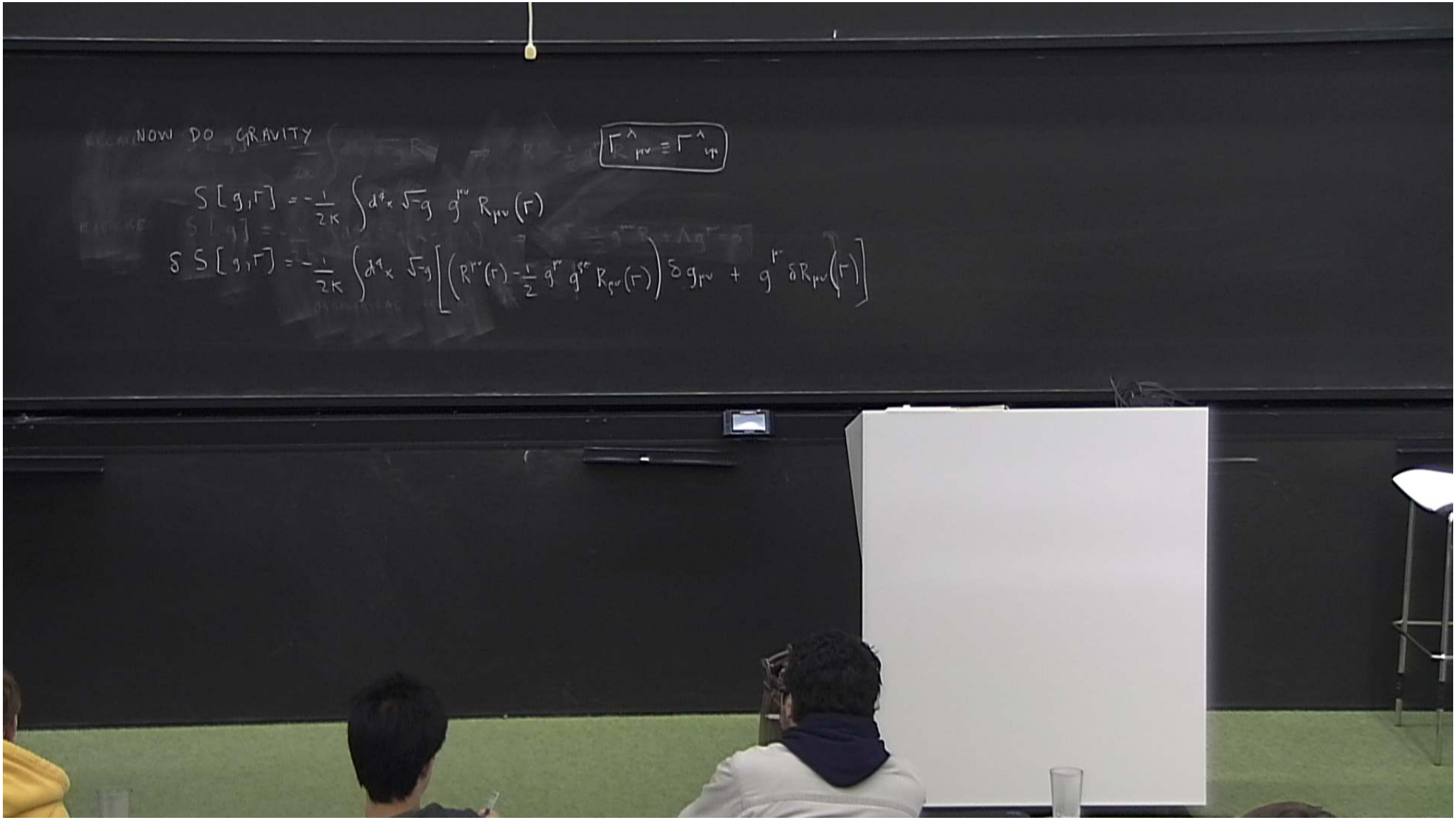
$\delta S[g,\Gamma] = -\frac{1}{2\kappa}\int d^4x\,\sqrt{-g}\left[\left(R^{\mu\nu}(\Gamma)-\frac{1}{2}g^{\mu\nu}g^{\rho\sigma}R_{\rho\sigma}(\Gamma)\right)\delta g_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}(\Gamma)\right]$

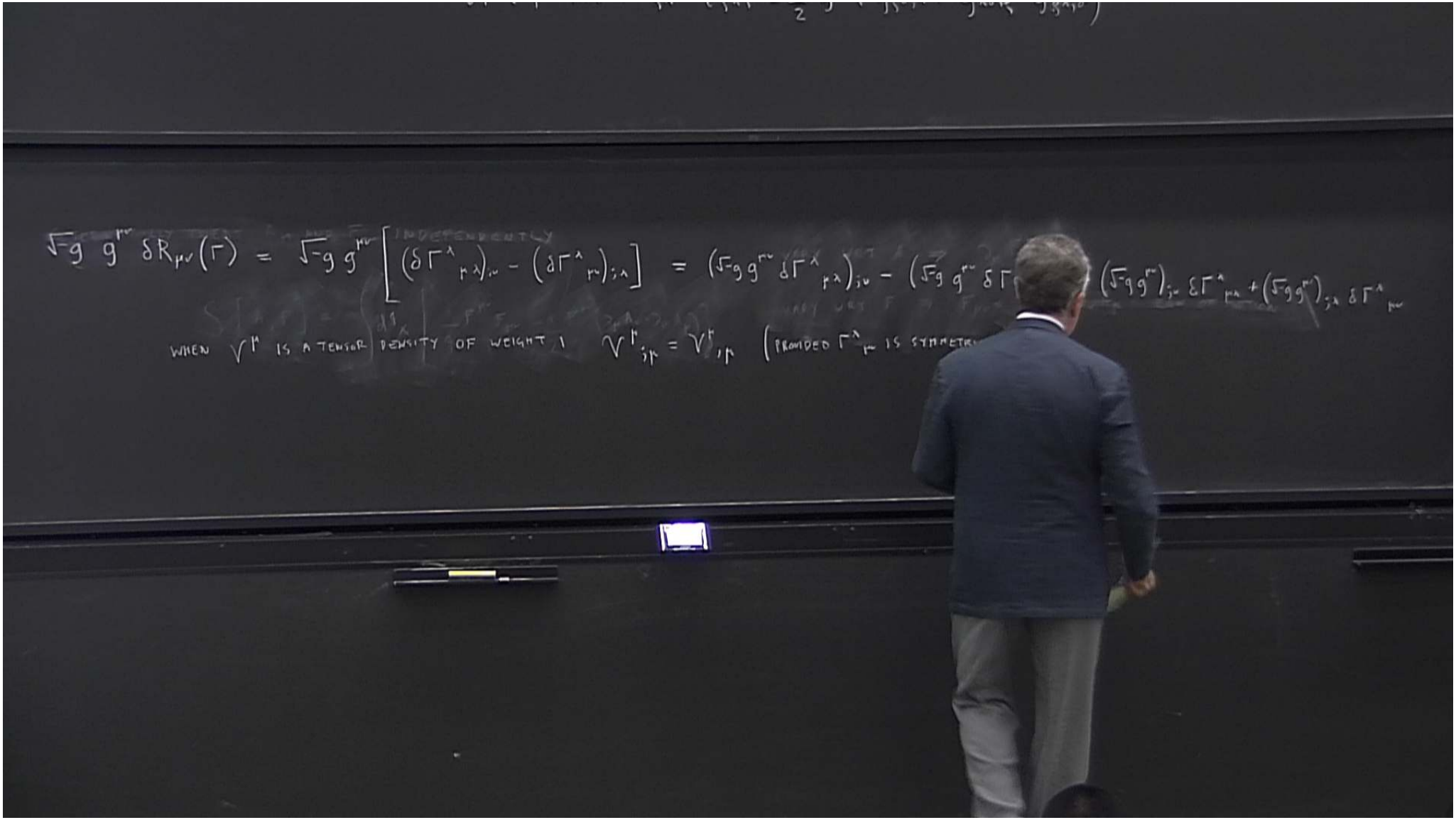

$\sqrt{-g}\, g^{\mu\nu}\, \delta R_{\mu\nu}(\Gamma) = \sqrt{-g}\, g^{\mu\nu} \left[(\delta \Gamma^{\lambda}_{\mu\lambda})_{;\nu} - (\delta \Gamma^{\lambda}_{\mu\nu})_{;\lambda} \right]$


$\sqrt{-g}\, g^{\mu\nu}\, \delta R_{\mu\nu}(\Gamma) = \sqrt{-g}\, g^{\mu\nu}\left[(\delta\Gamma^{\lambda}_{\mu\lambda})_{;\nu} - (\delta\Gamma^{\lambda}_{\mu\nu})_{;\lambda} \right]$
WHEN $\sqrt{g}^{\mu}$ IS A TENSOR DENSITY OF WEIGHT 1
$\sqrt{g}^{\mu}{}_{,\mu}$
(PROVIDED $\Gamma^{\lambda}_{\mu\nu}$ IS SYMMETRIC

$$\sqrt{-g}\, g^{\mu\nu}\, \delta R_{\mu\nu}(\Gamma) = \sqrt{-g}\, g^{\mu\nu}\left[\left(\delta \Gamma^{\lambda}_{\mu\lambda}\right)_{;\nu} - \left(\delta \Gamma^{\lambda}_{\mu\nu}\right)_{;\lambda} \right]$$

WHEN V^{μ} IS A TENSOR DENSITY OF WEIGHT 1 $\quad V^{\mu}_{\ ;\mu} = V^{\mu}_{\ ,\mu}$ (PROVIDED $\Gamma^{\lambda}_{\mu\nu}$ IS SYMMETRIC

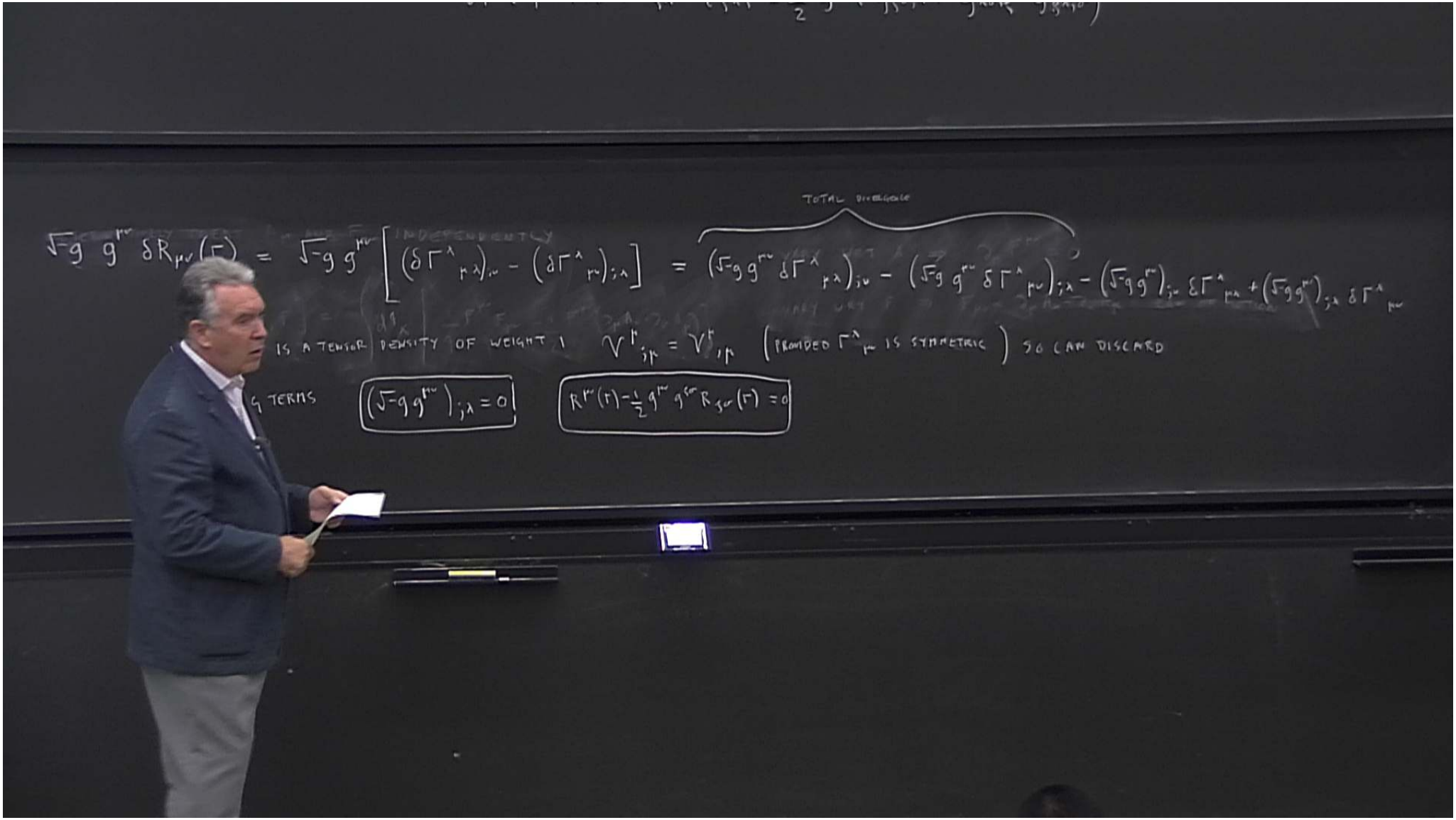

NOW DO GRAVITY
$\hat{\Gamma}^{\lambda}_{\mu\nu} \equiv \hat{\Gamma}^{\lambda}_{\nu\mu}$
$S[g,\Gamma] = -\frac{1}{2\kappa}\int d^4x\,\sqrt{-g}\; g^{\mu\nu} R_{\mu\nu}(\Gamma)$
$\delta S[g,\Gamma] = -\frac{1}{2\kappa}\int d^4x\,\sqrt{-g}\left[\left(R^{\mu\nu}(\Gamma) - \frac{1}{2} g^{\mu\nu} g^{\rho\sigma} R_{\rho\sigma}(\Gamma)\right)\delta g_{\mu\nu} + g^{\mu\nu}\,\delta R_{\mu\nu}(\Gamma)\right]$

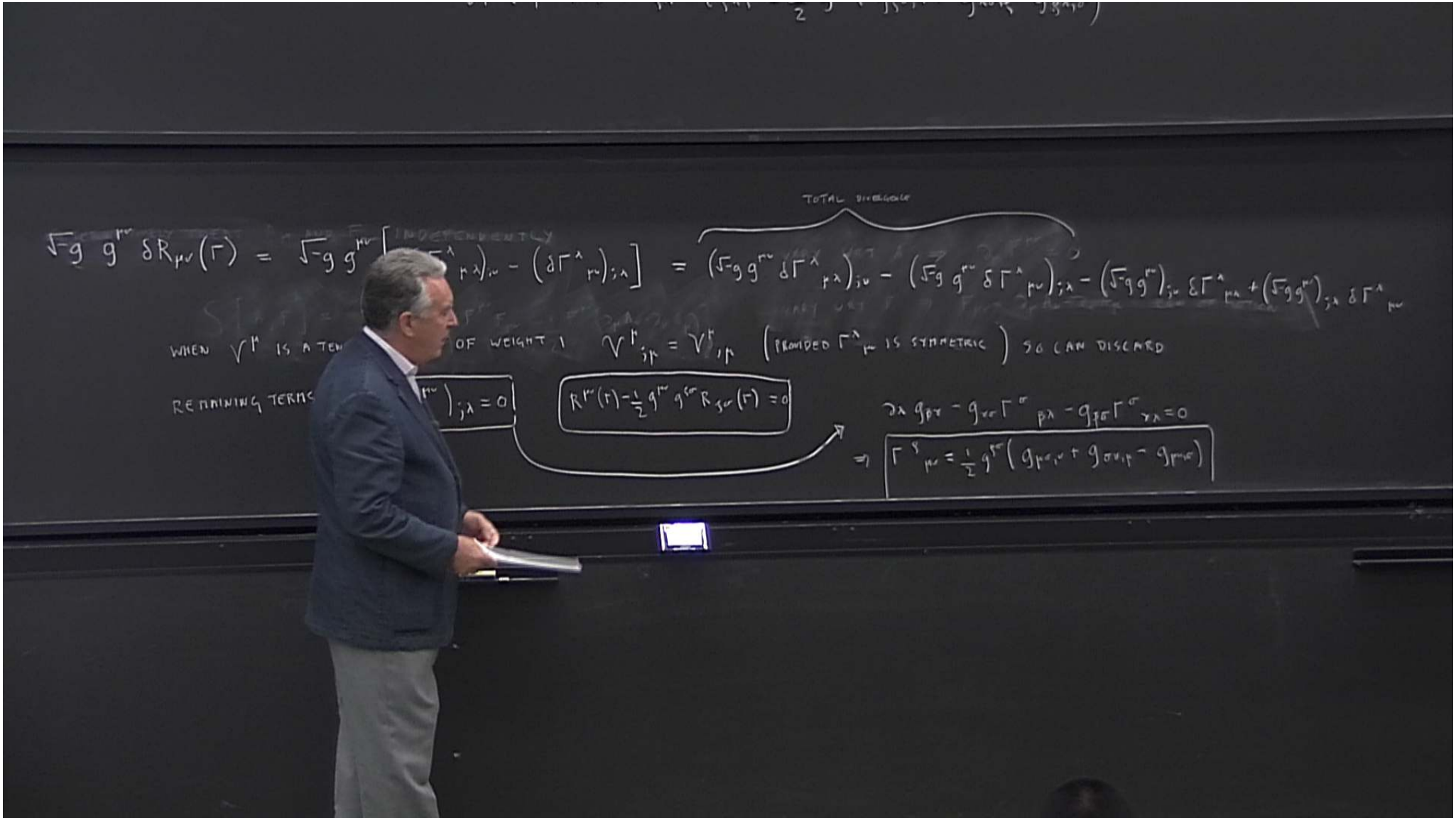

$\sqrt{-g}\, g^{\mu\nu}\, \delta R_{\mu\nu}(\Gamma) = \sqrt{-g}\, g^{\mu\nu}\left[\left(\delta \Gamma^{\lambda}_{\mu\lambda}\right)_{;\nu} - \left(\delta \Gamma^{\lambda}_{\mu\nu}\right)_{;\lambda} \right] = \left(\sqrt{-g}\, g^{\mu\nu}\, \delta \Gamma^{\lambda}_{\mu\lambda}\right)_{;\nu} - \left(\sqrt{-g}\, g^{\mu\nu}\, \delta \Gamma$
INDEPENDENTLY
$\left(\sqrt{-g}\, g^{\mu\nu}\right)_{;\nu} \delta \Gamma^{\lambda}_{\mu\lambda} + \left(\sqrt{-g}\, g^{\mu\nu}\right)_{;\lambda} \delta \Gamma^{\lambda}_{\mu\nu}$
WHEN V^{μ} IS A TENSOR DENSITY OF WEIGHT 1 $V^{\mu}_{;\mu} = V^{\mu}_{,\mu}$ (PROVIDED $\Gamma^{\lambda}_{\mu\nu}$ IS SYMMETR

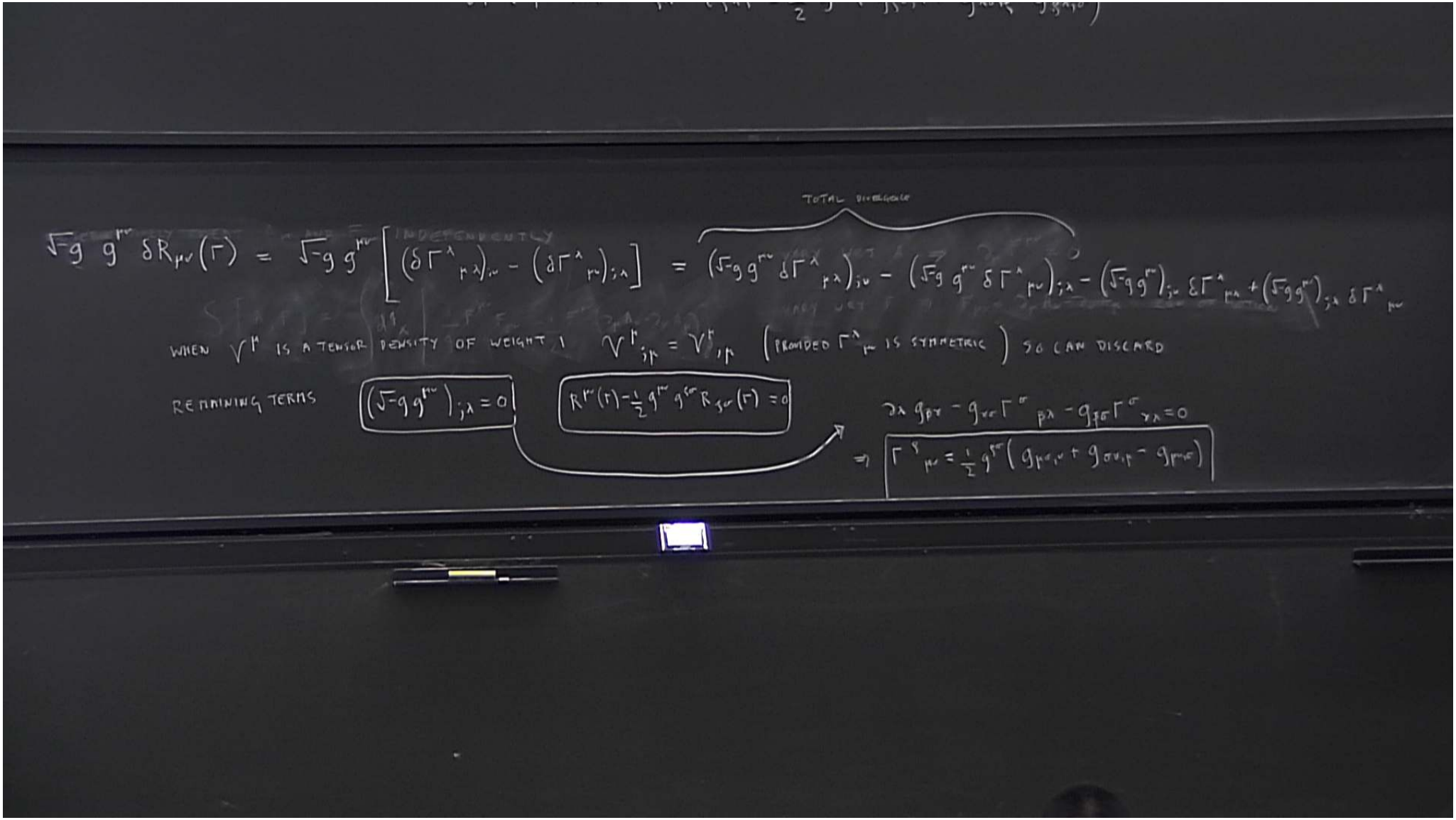
$$\sqrt{-g}\, g^{\mu\nu}\, \delta R_{\mu\nu}(\Gamma) = \sqrt{-g}\, g^{\mu\nu}\left[\left(\delta\Gamma^{\lambda}_{\mu\lambda}\right)_{;\nu} - \left(\delta\Gamma^{\lambda}_{\mu\nu}\right)_{;\lambda}\right] = \left(\sqrt{-g}\, g^{\mu\nu}\, \delta\Gamma^{\lambda}_{\mu\lambda}\right)_{;\nu} - \left(\sqrt{-g}\, g^{\mu\nu}\, \delta\Gamma^{\lambda}_{\mu\nu}\right)_{;\lambda} - \left(\sqrt{-g}\, g^{\mu\nu}\right)_{;\nu} \delta\Gamma^{\lambda}_{\mu\lambda} + \left(\sqrt{-g}\, g^{\mu\nu}\right)_{;\lambda} \delta\Gamma^{\lambda}_{\mu\nu}$$

INDEPENDENTLY

WHEN V^{μ} IS A TENSOR DENSITY OF WEIGHT 1 $V^{\mu}_{\ ;\mu} = V^{\mu}_{\ ,\mu}$ (PROVIDED $\Gamma^{\lambda}_{\mu\nu}$ IS SYMMETRIC


TOTAL DIVERGENCE
IS A TENSOR DENSITY OF WEIGHT 1
PROVIDED $\Gamma^{\lambda}_{\mu\nu}$ IS SYMMETRIC
SO CAN DISCARD
4 TERMS
$\left(\sqrt{-g}\, g^{\mu\nu}\right)_{;\lambda} = 0$
$R^{\mu\nu}(\Gamma) - \frac{1}{2} g^{\mu\nu} g^{\sigma\rho} R_{\sigma\rho}(\Gamma) = 0$


TOTAL DIVERGENCE
WHEN V^μ IS A TEN
OF WEIGHT 1
$V^\mu_{;\mu} = V^\mu_{,\mu}$
(PROVIDED $\Gamma^\lambda_{\mu\nu}$ IS SYMMETRIC)
SO CAN DISCARD
RETAINING TERMS
$R^{\mu\nu}(\Gamma) - \frac{1}{2} g^{\mu\nu} g^{\sigma\tau} R_{\sigma\tau}(\Gamma) = 0$
$\partial_\lambda g_{\rho\nu} - g_{\sigma\nu}\Gamma^\sigma_{\rho\lambda} - g_{\rho\sigma}\Gamma^\sigma_{\nu\lambda} = 0$
$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}\left(g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho}\right)$

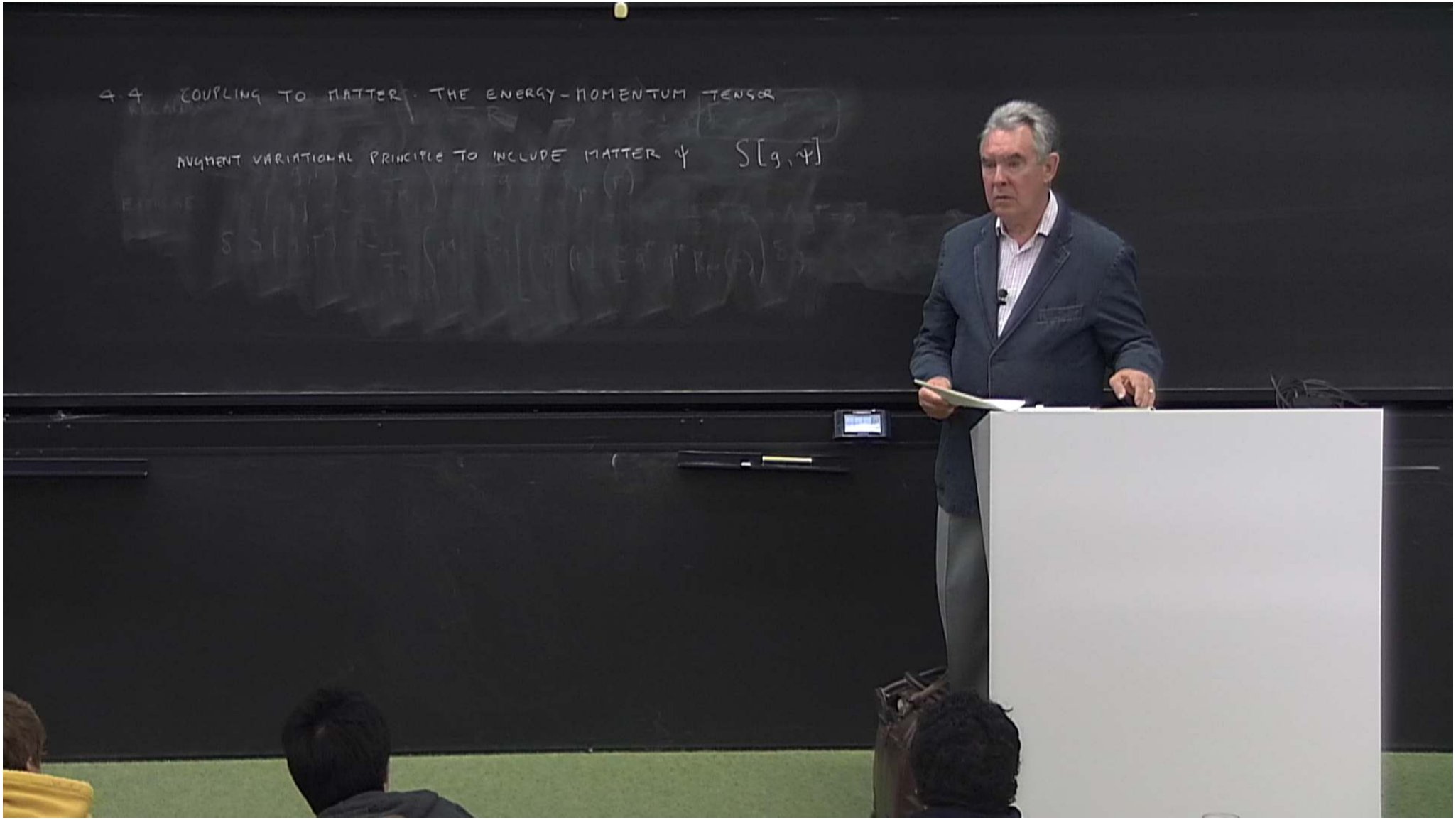


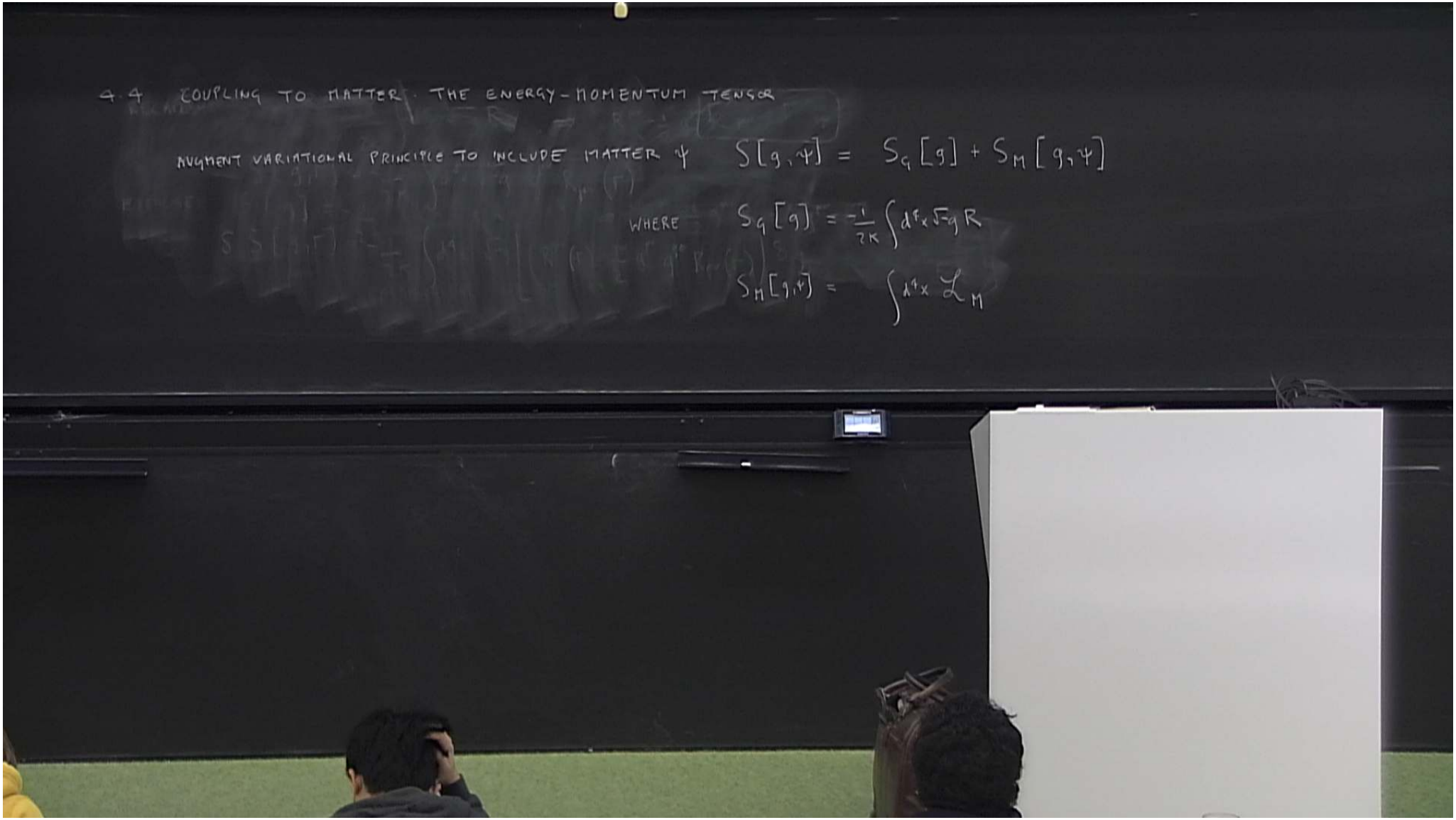
$$\sqrt{-g}\, g^{\mu\nu}\, \delta R_{\mu\nu}(\Gamma) = \sqrt{-g}\, g^{\mu\nu}\left[\left(\delta\Gamma^\lambda{}_{\mu\lambda}\right)_{;\nu} - \left(\delta\Gamma^\lambda{}_{\mu\nu}\right)_{;\lambda}\right] = \left(\sqrt{-g}\, g^{\mu\nu}\, \delta\Gamma^\lambda{}_{\mu\lambda}\right)_{;\nu} - \left(\sqrt{-g}\, g^{\mu\nu}\, \delta\Gamma^\lambda{}_{\mu\nu}\right)_{;\lambda} - \left(\sqrt{-g}\, g^{\mu\nu}\right)_{;\nu} \delta\Gamma^\lambda{}_{\mu\lambda} + \left(\sqrt{-g}\, g^{\mu\nu}\right)_{;\lambda} \delta\Gamma^\lambda{}_{\mu\nu}$$

$$\left(\sqrt{-g}\, g^{\mu\nu}\right)_{;\lambda} = 0 \qquad R^{\mu\nu}(\Gamma) - \tfrac{1}{2} g^{\mu\nu} g^{\sigma\tau} R_{\sigma\tau}(\Gamma) = 0$$

$$\partial_\lambda g_{\rho\nu} - g_{\nu\sigma}\Gamma^\sigma{}_{\rho\lambda} - g_{\rho\sigma}\Gamma^\sigma{}_{\nu\lambda} = 0$$

$$\Rightarrow \boxed{\Gamma^\lambda{}_{\mu\nu} = \tfrac{1}{2} g^{\lambda\sigma}\left(g_{\rho\sigma,\nu} + g_{\sigma\nu,\rho} - g_{\rho\nu,\sigma}\right)}$$


4.4 COUPLING TO MATTER · THE ENERGY-MOMENTUM TENSOR
AUGMENT VARIATIONAL PRINCIPLE TO INCLUDE MATTER ψ S[g, ψ]


4.4 COUPLING TO MATTER · THE ENERGY-MOMENTUM TENSOR
AUGMENT VARIATIONAL PRINCIPLE TO INCLUDE MATTER ψ
$S[g,\psi] = S_G[g] + S_M[g,\psi]$
WHERE
$S_G[g] = \frac{-1}{2\kappa}\int d^4x \sqrt{-g} R$
$S_M[g,\psi] = \int d^4x \mathcal{L}_M$

REMAINING TERMS

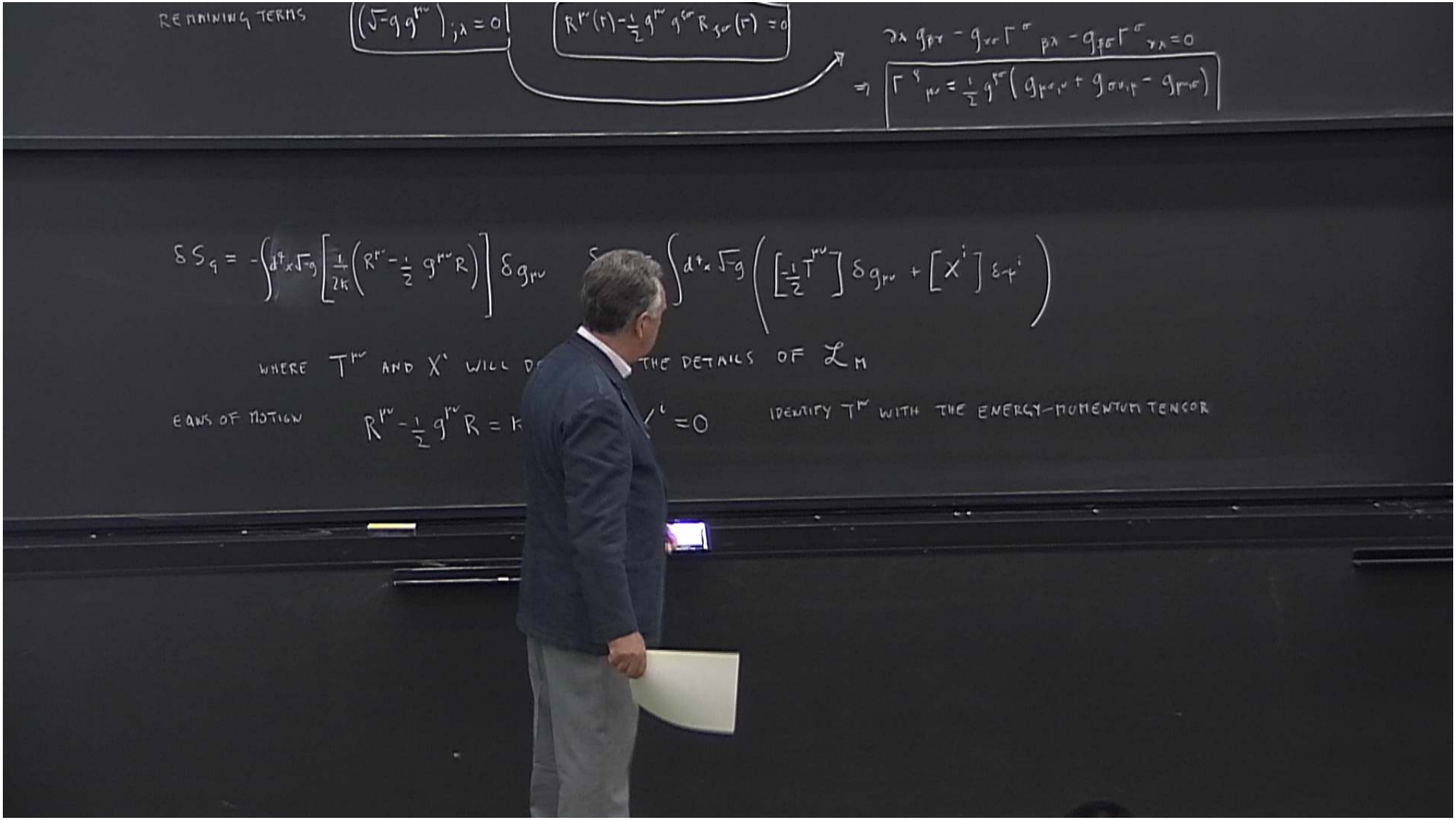
$$\left(\sqrt{-g}\, g^{\mu\nu}\right)_{;\lambda} = 0$$

$$R^{\mu\nu}(\Gamma) - \frac{1}{2} g^{\mu\nu} g^{\sigma\tau} R_{\sigma\tau}(\Gamma) = 0$$

$$\partial_\lambda g_{\beta\nu} - g_{\sigma\nu}\Gamma^{\sigma}_{\beta\lambda} - g_{\beta\sigma}\Gamma^{\sigma}_{\nu\lambda} = 0$$

$$\Rightarrow \boxed{\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}\left(g_{\mu\sigma,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma}\right)}$$

$$\delta S_g = -\int d^4x \sqrt{-g}\left[\frac{1}{2\kappa}\left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R\right)\right] \delta g_{\mu\nu} \qquad \delta S_M = \int d^4x \sqrt{-g}\left([\quad] \delta g_{\mu\nu} + [\quad] \delta_\mu \right.$$


REMAINING TERMS
$\left(\sqrt{-g}\, g^{\mu\nu}\right)_{;\lambda} = 0$
$R^{\mu\nu}(\Gamma) - \frac{1}{2} g^{\mu\nu} g^{\sigma\tau} R_{\sigma\tau}(\Gamma) = 0$
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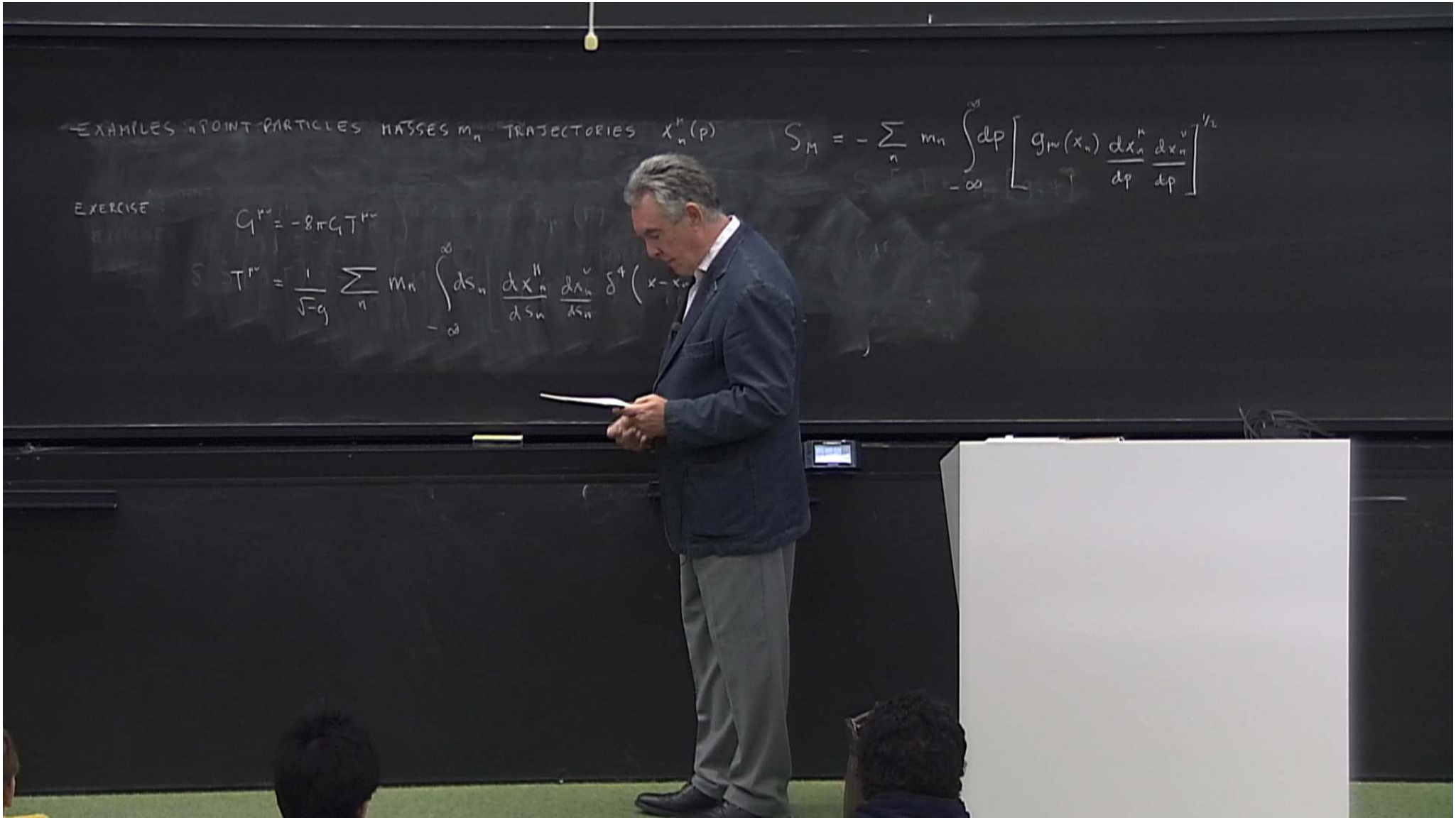
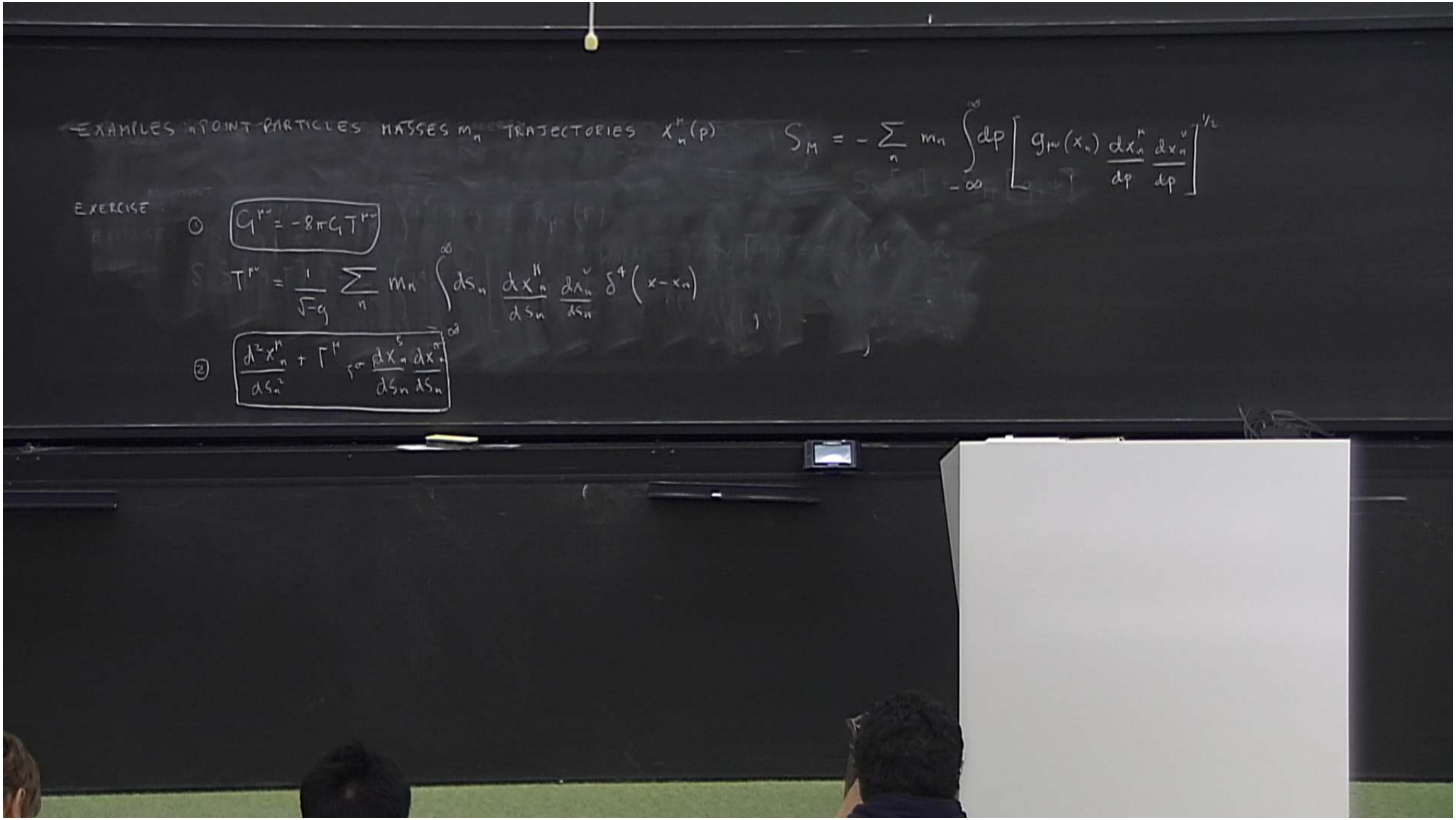
$\Rightarrow \Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}\left(g_{\beta\sigma,\nu} + g_{\sigma\nu,\beta} - g_{\beta\nu,\sigma}\right)$
$\delta S_g = -\int d^4x \sqrt{-g}\left[\frac{1}{2\kappa}\left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R\right)\right] \delta g_{\mu\nu}$
$\int d^4x \sqrt{-g}\left(\left[-\frac{1}{2} T^{\mu\nu}\right] \delta g_{\mu\nu} + \left[X^i\right] \varepsilon_i\right)$
WHERE $T^{\mu\nu}$ AND X^i WILL D
THE DETAILS OF $\mathcal{L}_M$
EQNS OF MOTION $R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = \kappa$
$/^i = 0$
IDENTIFY $T^{\mu\nu}$ WITH THE ENERGY-MOMENTUM TENSOR

EXAMPLES n POINT PARTICLES MASSES m_n TRAJECTORIES $x^\mu_n(p)$


EXAMPLES n POINT PARTICLES MASSES m_n TRAJECTORIES $x_n^\mu(p)$
$S_M = -\sum_n m_n \int_{-\infty}^{\infty} dp \left[g_{\mu\nu}(x_n) \frac{dx_n^\mu}{dp} \frac{dx_n^\nu}{dp} \right]^{1/2}$
EXERCISE
$G^{\mu\nu} = -8\pi G_1 T^{\mu\nu}$
$T^{\mu\nu} = \frac{1}{\sqrt{-g}} \sum_n m_n \int_{-\infty}^{\infty} ds_n \frac{dx_n^\mu}{ds_n} \frac{dx_n^\nu}{ds_n} \delta^4(x - x_n$


EXAMPLES: POINT PARTICLES MASSES m_n TRAJECTORIES $x_n^\mu(p)$
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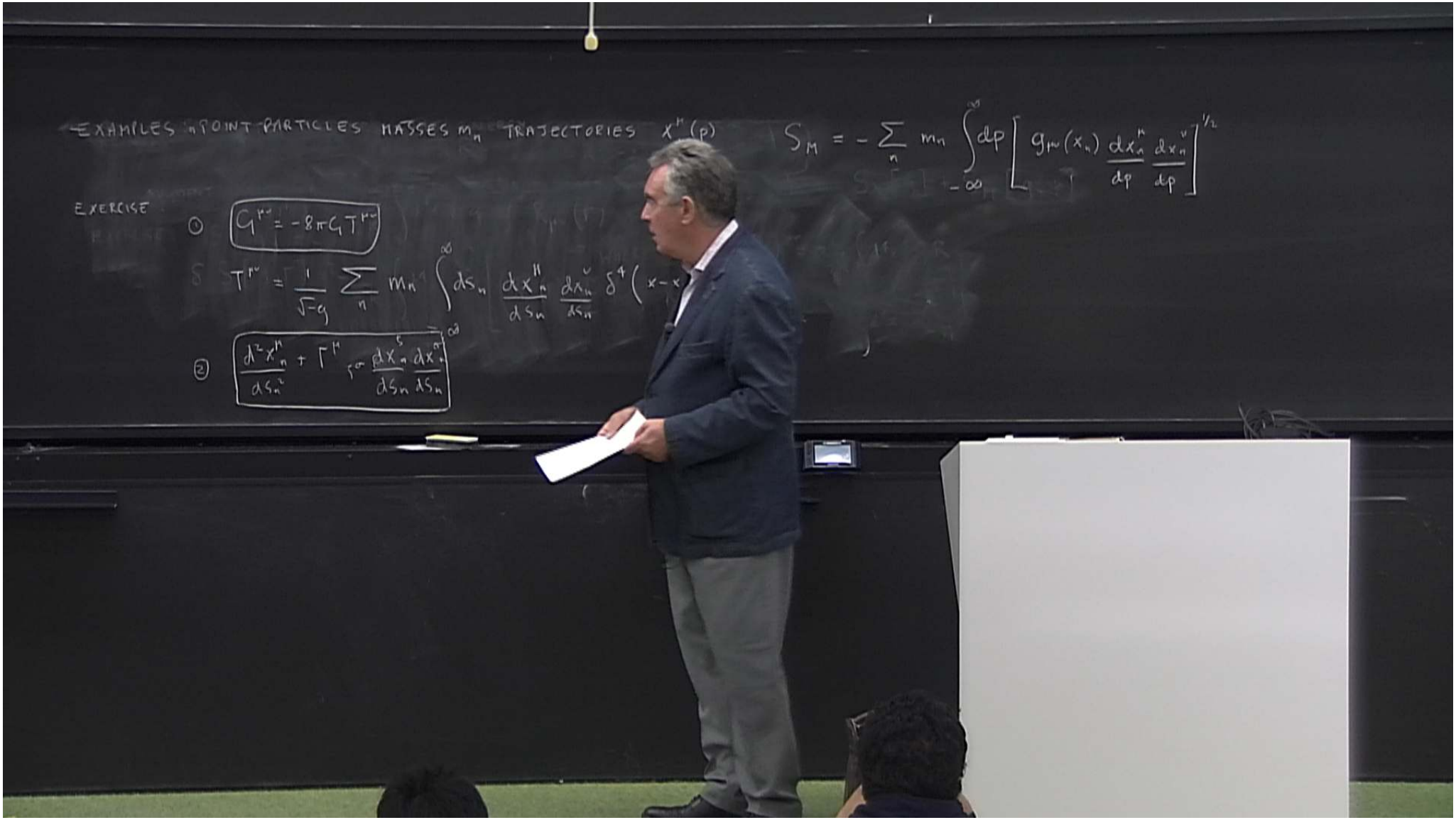
EXERCISE
① $G^{\mu\nu} = -8\pi G_N T^{\mu\nu}$
$T^{\mu\nu} = \frac{1}{\sqrt{-g}} \sum_n m_n \int ds_n \frac{dx_n^\mu}{ds_n} \frac{dx_n^\nu}{ds_n} \delta^4(x - x_n)$
② $\frac{d^2 x_n^\mu}{ds_n^2} + \Gamma^\mu_{\sigma\rho} \frac{dx_n^\sigma}{ds_n} \frac{dx_n^\rho}{ds_n} = 0$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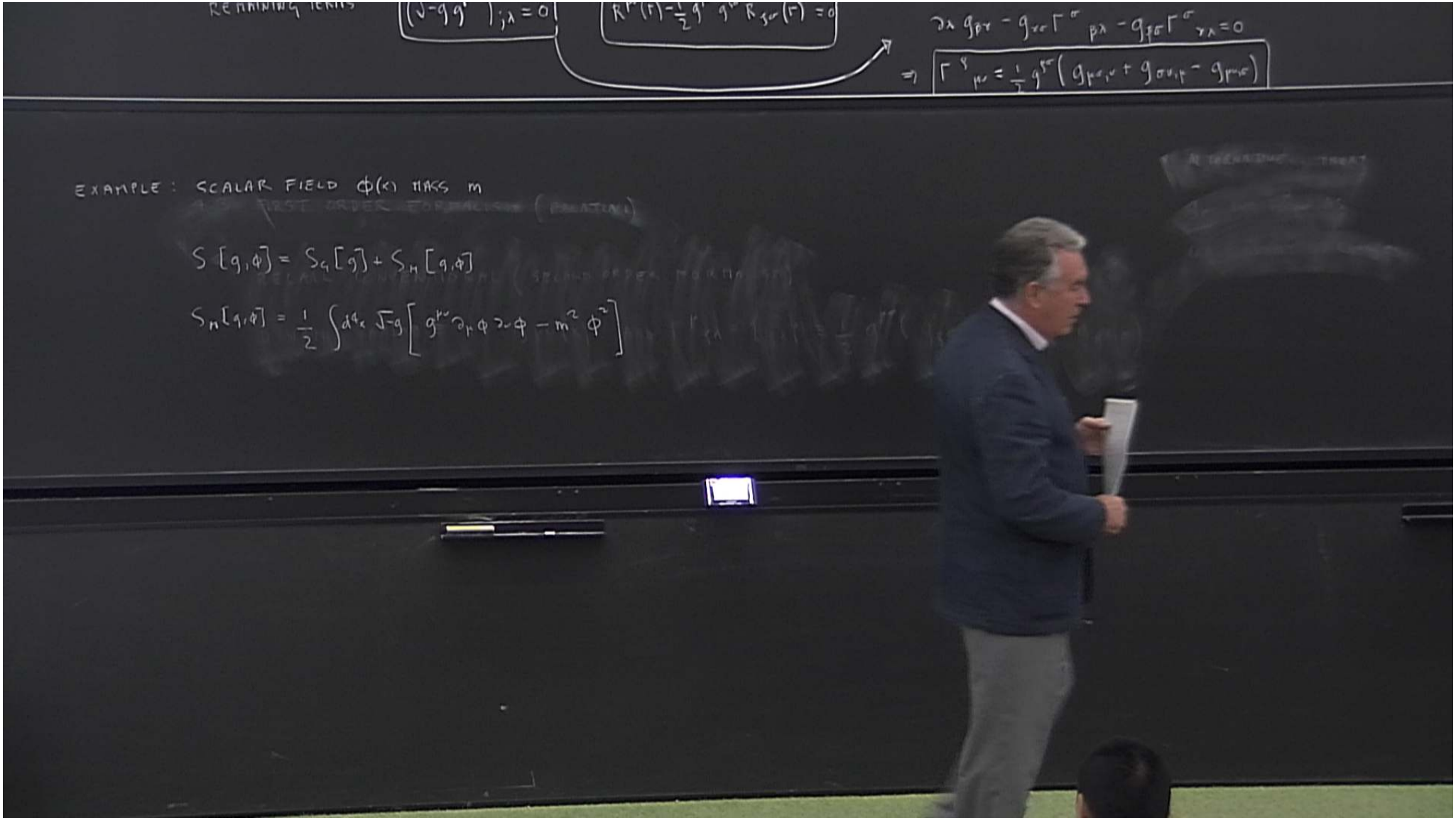
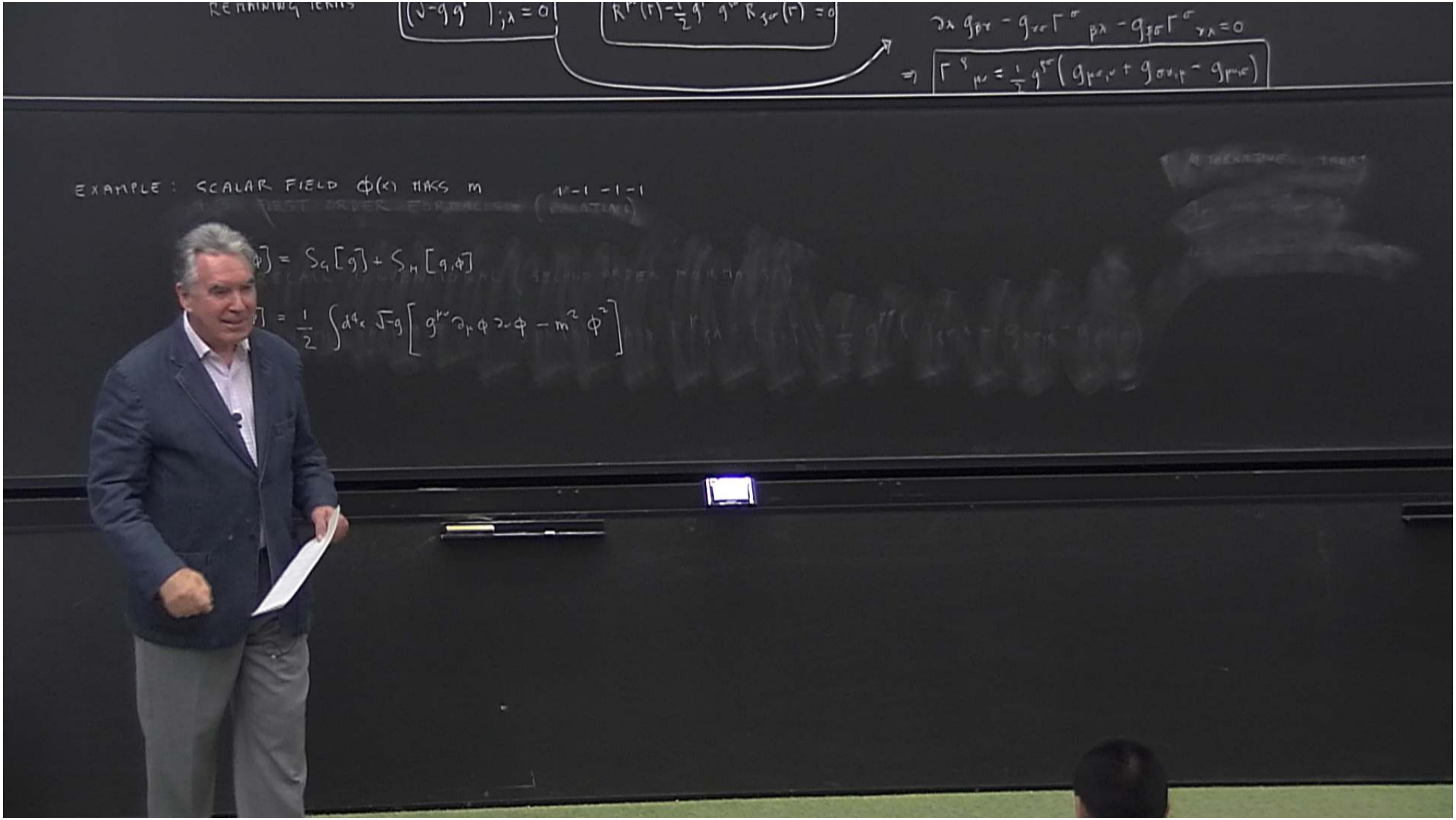

EXAMPLE : SCALAR FIELD $\phi(x)$ MASS m
$S[g,\phi] = S_G[g] + S_M[g,\phi]$
$S_M[g,\phi] = \frac{1}{2}\int d^4x \sqrt{-g}\left[g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - m^2\phi^2 \right]$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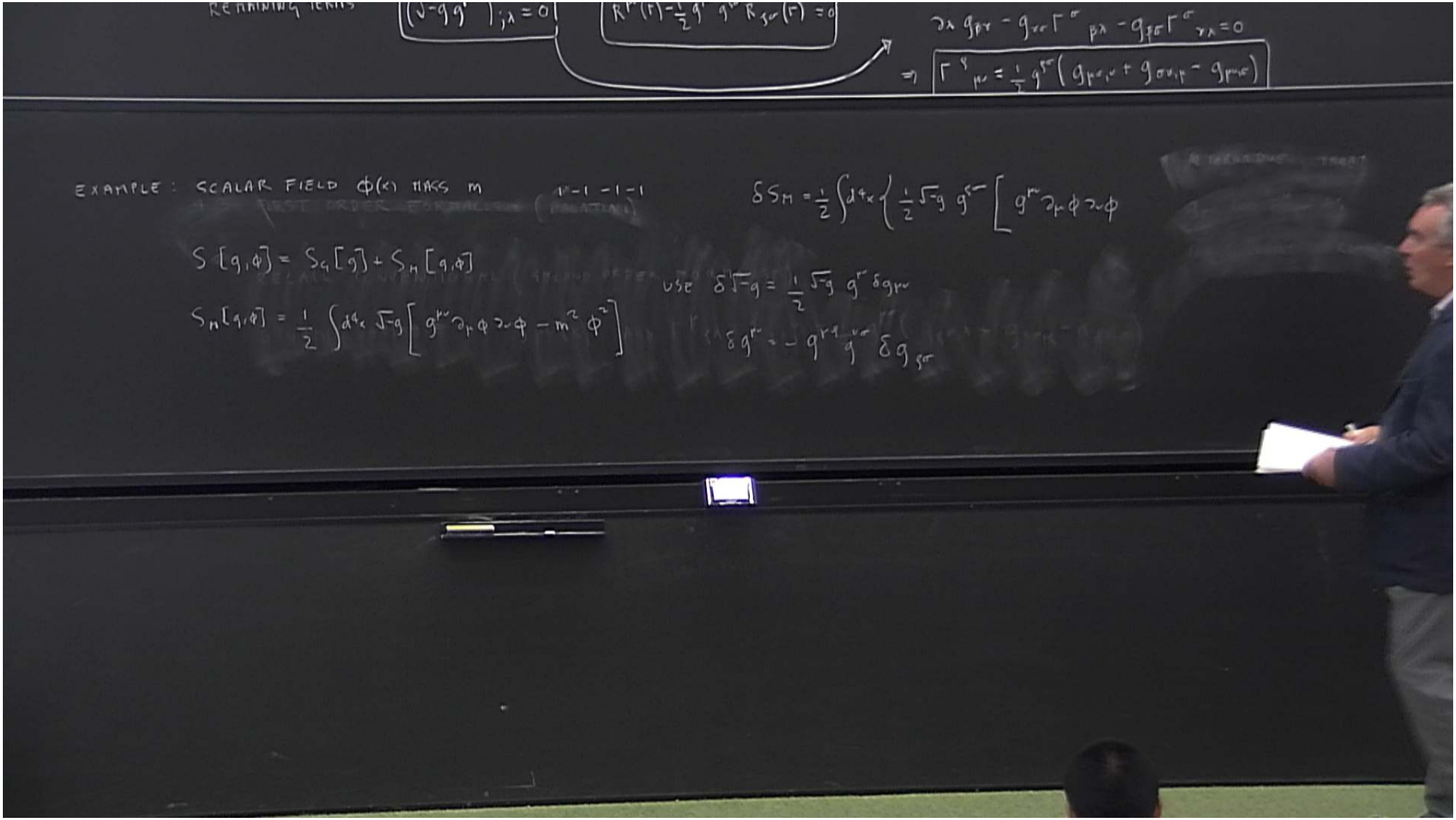
$\Rightarrow \Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}\left(g_{\beta\sigma,\nu} + g_{\sigma\nu,\beta} - g_{\beta\nu,\sigma} \right)$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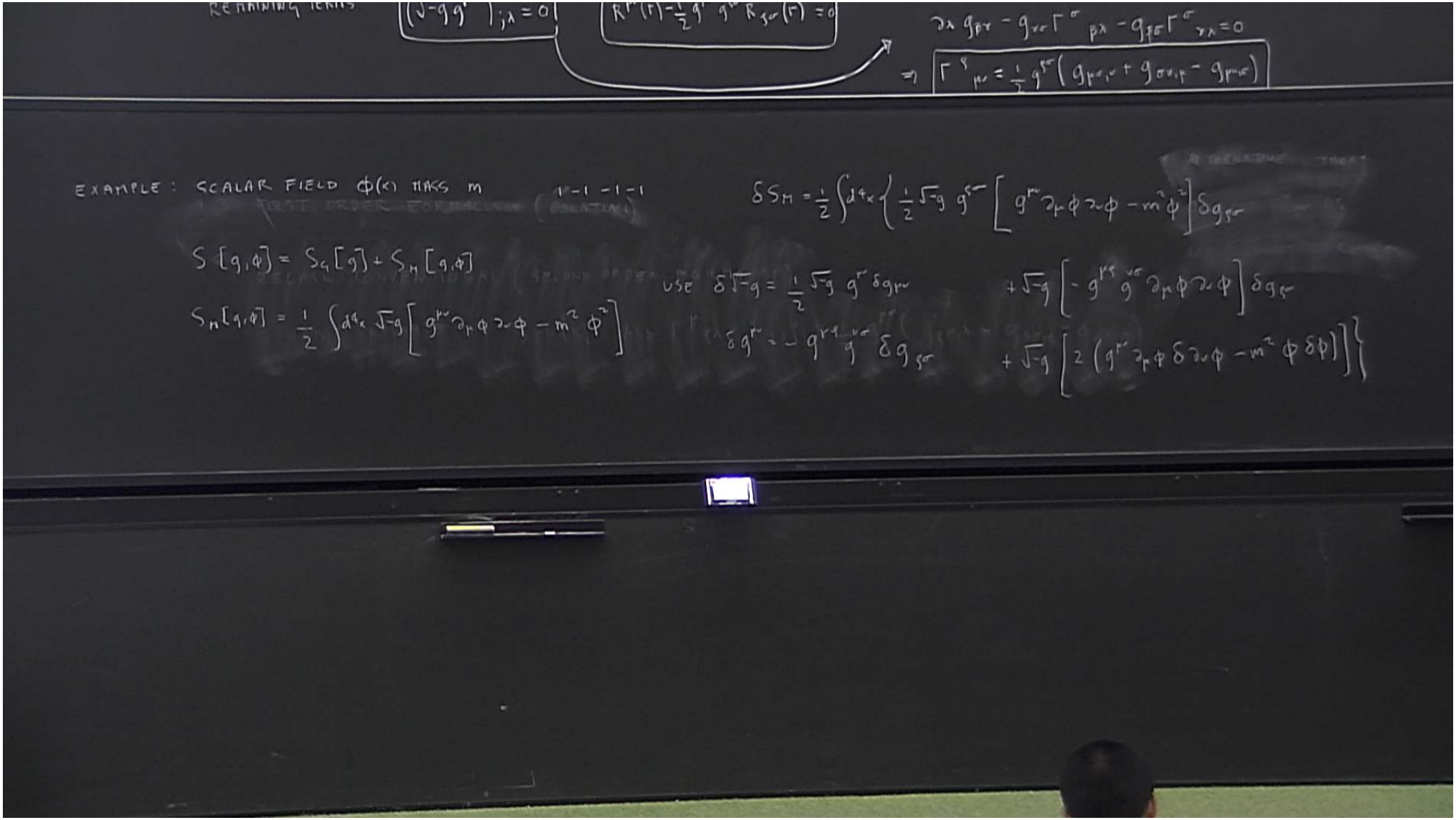
$\delta S_M = \frac{1}{2}\int d^4x \left(\frac{1}{2}\sqrt{-g}\, g^{\alpha\beta}\left[g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi\right.\right.$
use $\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}\, g^{\mu\nu}\delta g_{\mu\nu}$
$\delta g^{\mu\nu} = -g^{\mu\alpha}g^{\nu\beta}\delta g_{\alpha\beta}$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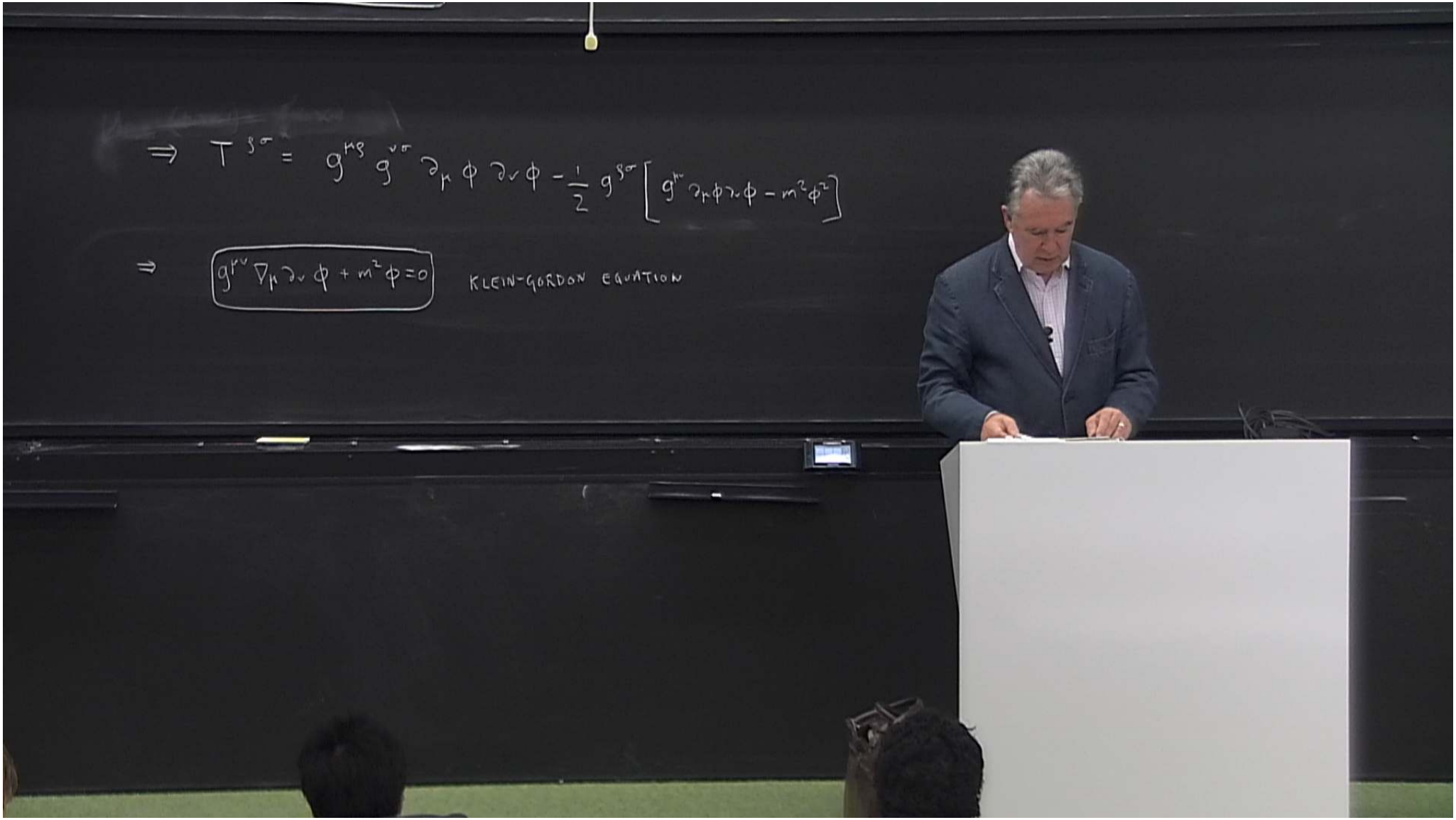
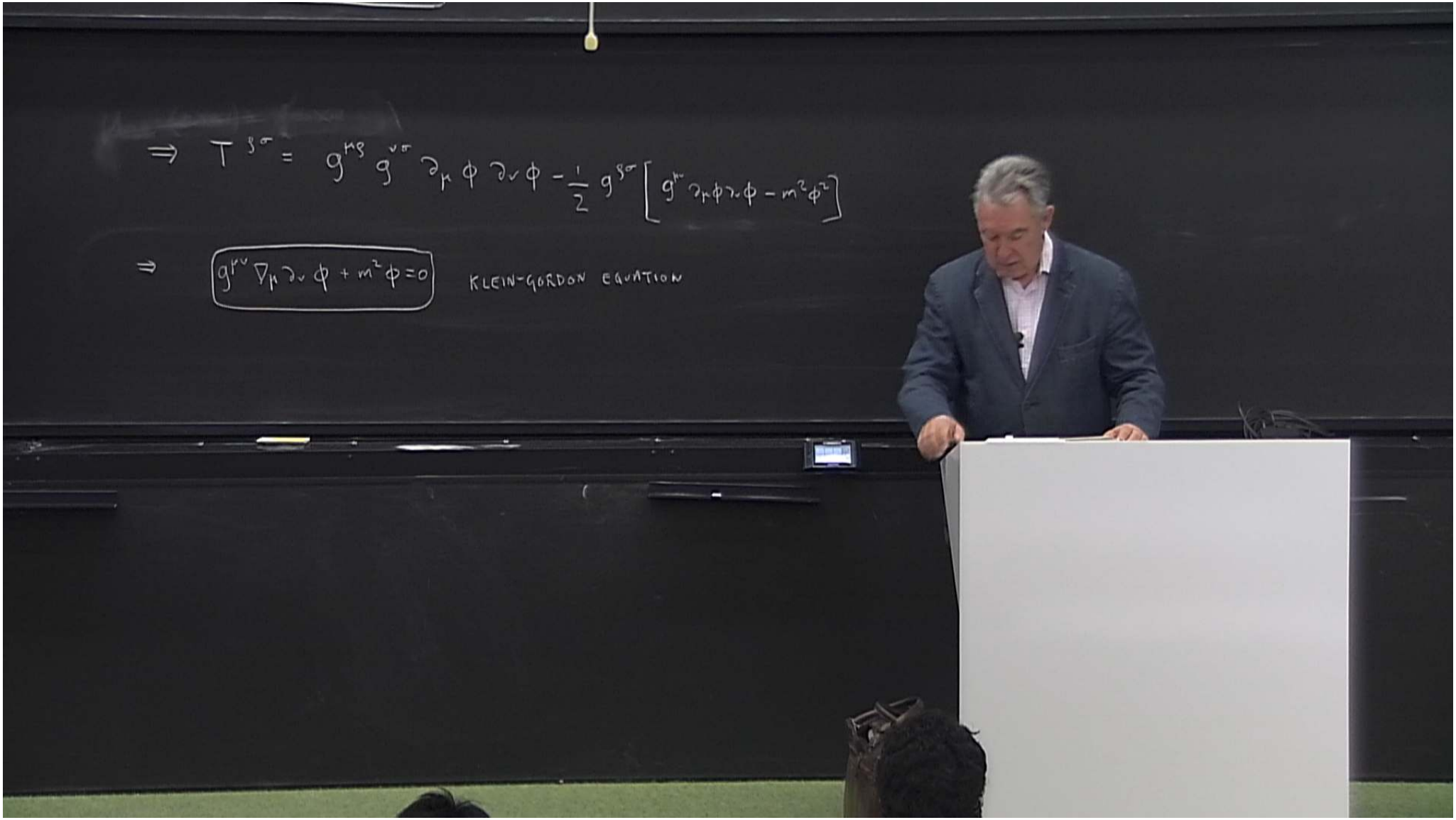
use $\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}\,g^{\mu\nu}\delta g_{\mu\nu}$
$\delta g^{\mu\nu} = -g^{\mu\rho}g^{\nu\sigma}\delta g_{\rho\sigma}$
$\delta S_M = \frac{1}{2}\int d^4x \left\{\frac{1}{2}\sqrt{-g}\,g^{\rho\sigma}\left[g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - m^2\phi^2\right]\delta g_{\rho\sigma}\right.$
$+\sqrt{-g}\left[-g^{\mu\rho}g^{\nu\sigma}\partial_\mu\phi\,\partial_\nu\phi\right]\delta g_{\rho\sigma}$
$\left.+\sqrt{-g}\left[2\left(g^{\mu\nu}\partial_\mu\phi\,\delta\partial_\nu\phi - m^2\phi\,\delta\phi\right)\right]\right\}$


$\Rightarrow T^{\mu\sigma} = g^{\mu\rho} g^{\nu\sigma} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g^{\mu\sigma} \left[g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - m^2 \phi^2 \right]$
$\Rightarrow \boxed{g^{\mu\nu} \nabla_\mu \partial_\nu \phi + m^2 \phi = 0}$
KLEIN-GORDON EQUATION


$\Rightarrow T^{\mu\sigma} = g^{\mu\beta} g^{\nu\sigma} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g^{\beta\sigma} \left[g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - m^2 \phi^2 \right]$
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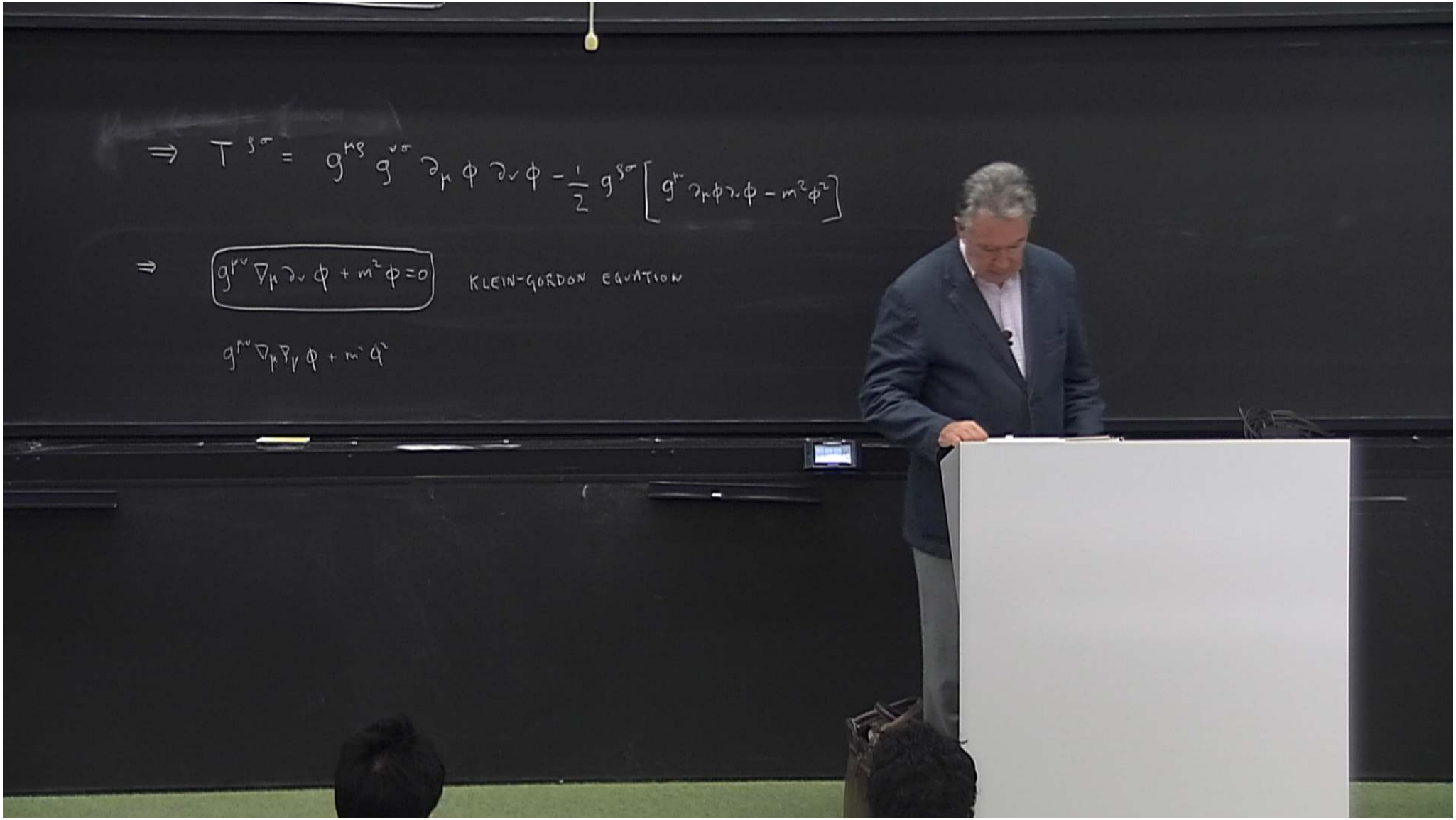
KLEIN-GORDON EQUATION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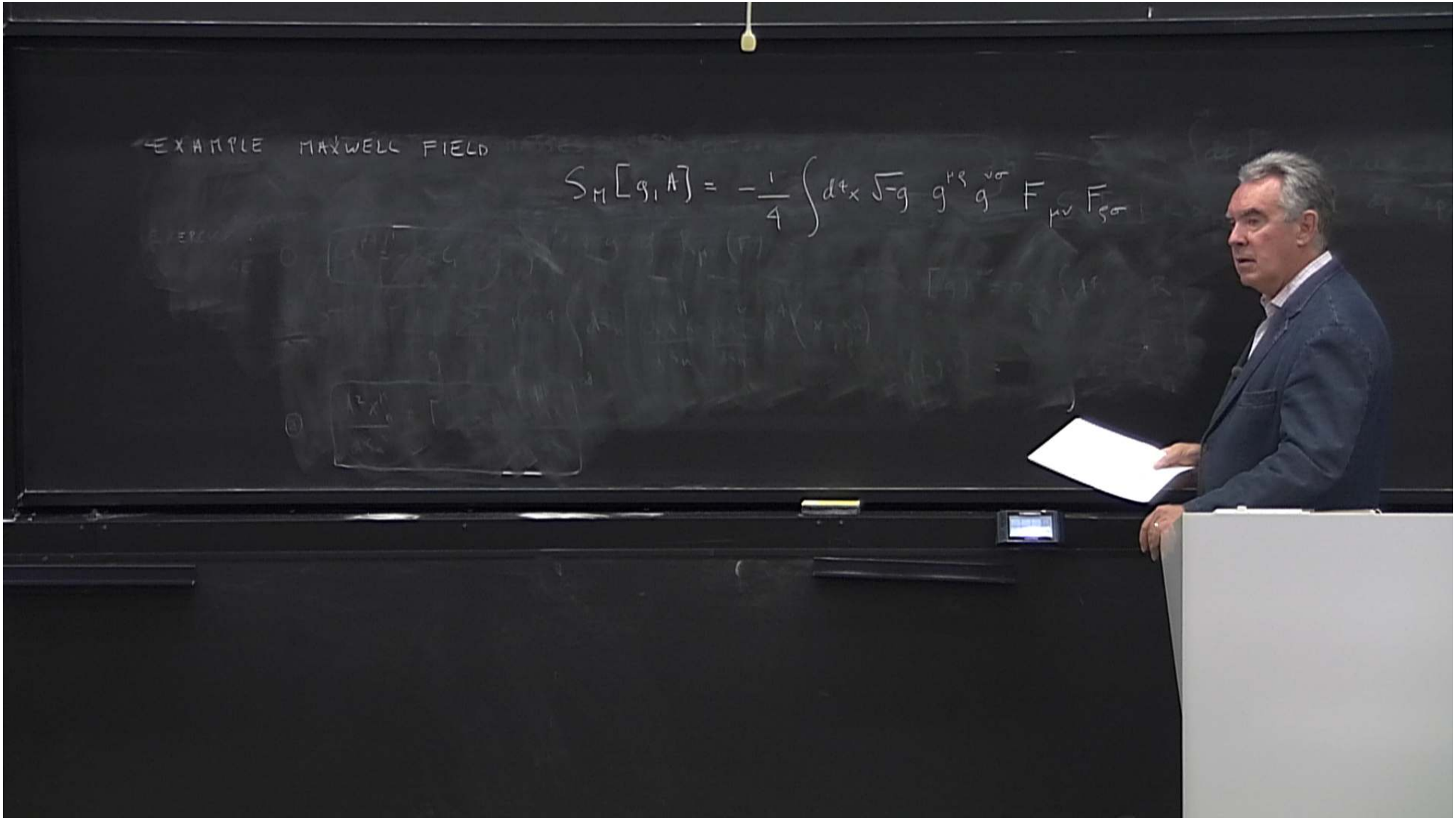
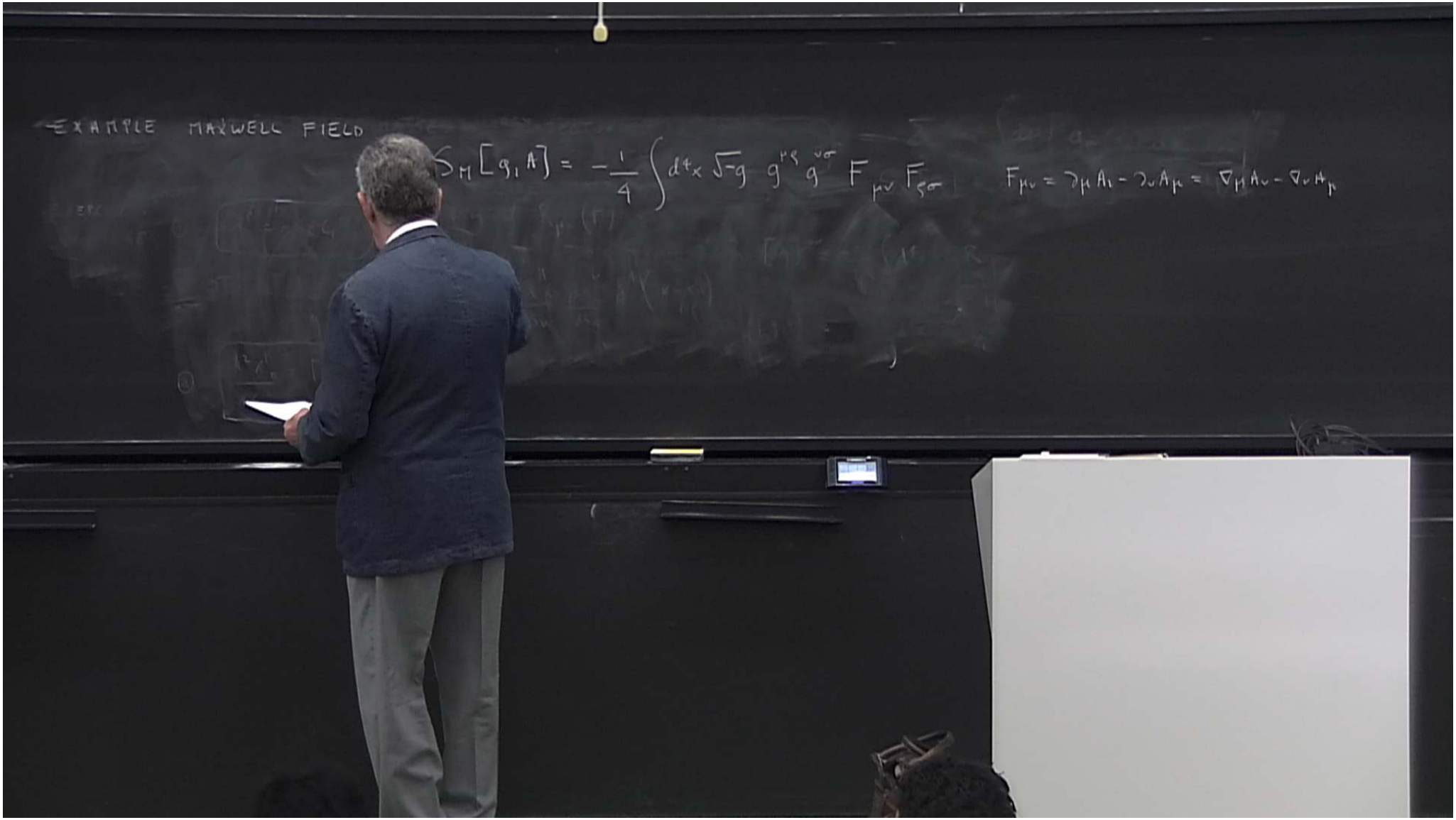
KLEIN-GORDON EQUATION
$g^{\mu\nu} \nabla_\mu \nabla_\nu \phi + m^2 \phi$


EXAMPLE MAXWELL FIELD
$S_M[g, A] = -\frac{1}{4}\int d^4x \sqrt{-g}\; g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma}$


EXAMPLE MAXWELL FIELD
$S_M[g, A] = -\frac{1}{4}\int d^4x \sqrt{-g}\; g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma}$
$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \nabla_\mu A_\nu - \nabla_\nu A_\mu$