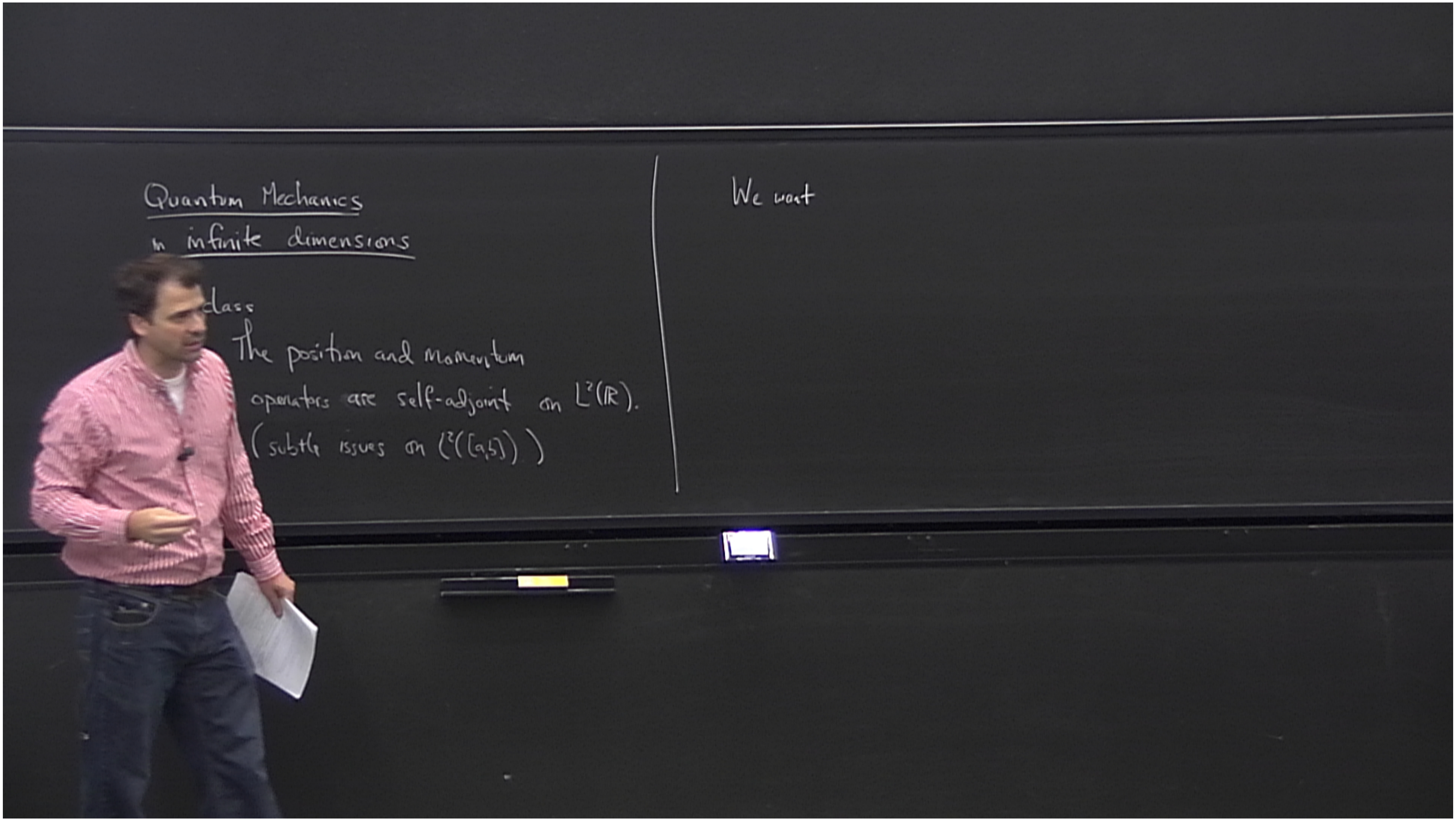
Title: Quantum Theory - Lecture 11

Date: Sep 24, 2012 09:00 AM

URL: http://pirsa.org/12090014

Abstract:


Quantum Mechanics
in infinite dimensions
class
The position and momentum
operators are self-adjoint on L²(ℝ).
(subtle issues on L²([a,b]))
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Quantum Mechanics in infinite dimensions

Last class

Lemma. The position and momentum operators are self-adjoint on $L^2(\mathbb{R})$.

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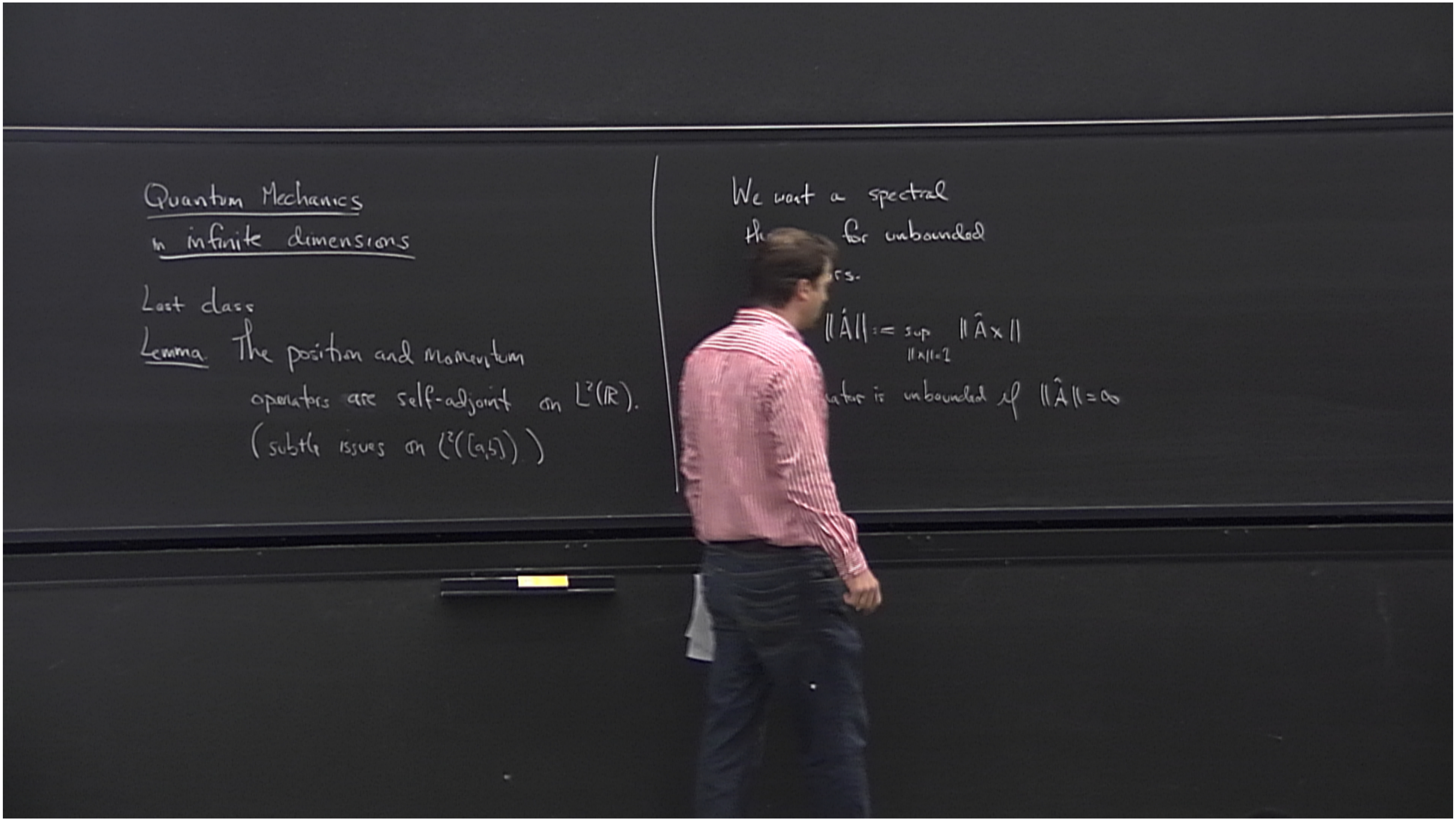
We want a spectral theorem for unbounded operators.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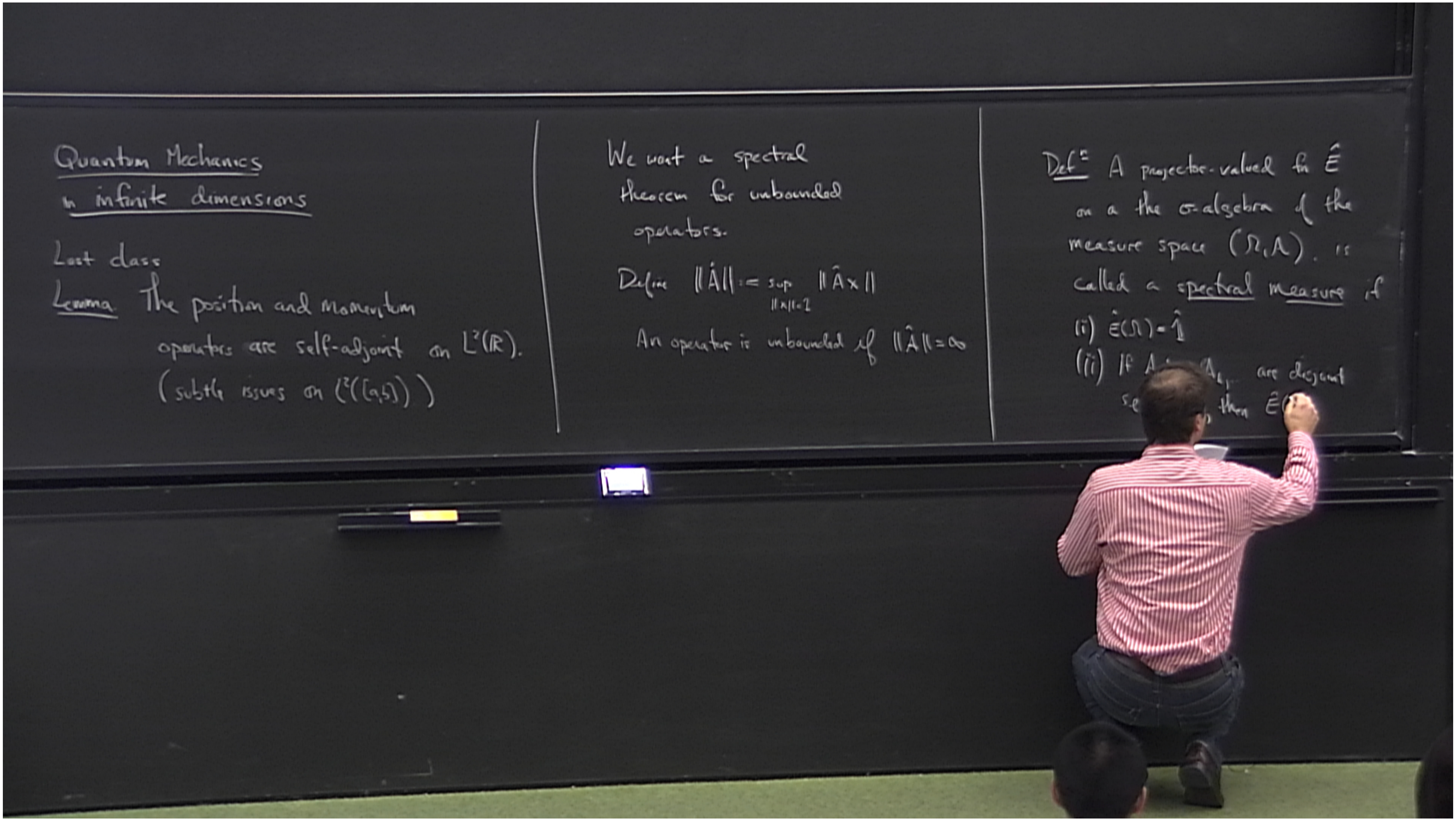
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Def$^{\underline{n}}$ A projector-valued fn $\hat{E}$ on a the σ-algebra of the measure space $(\Omega, \mathcal{A})$ is called a spectral measure if
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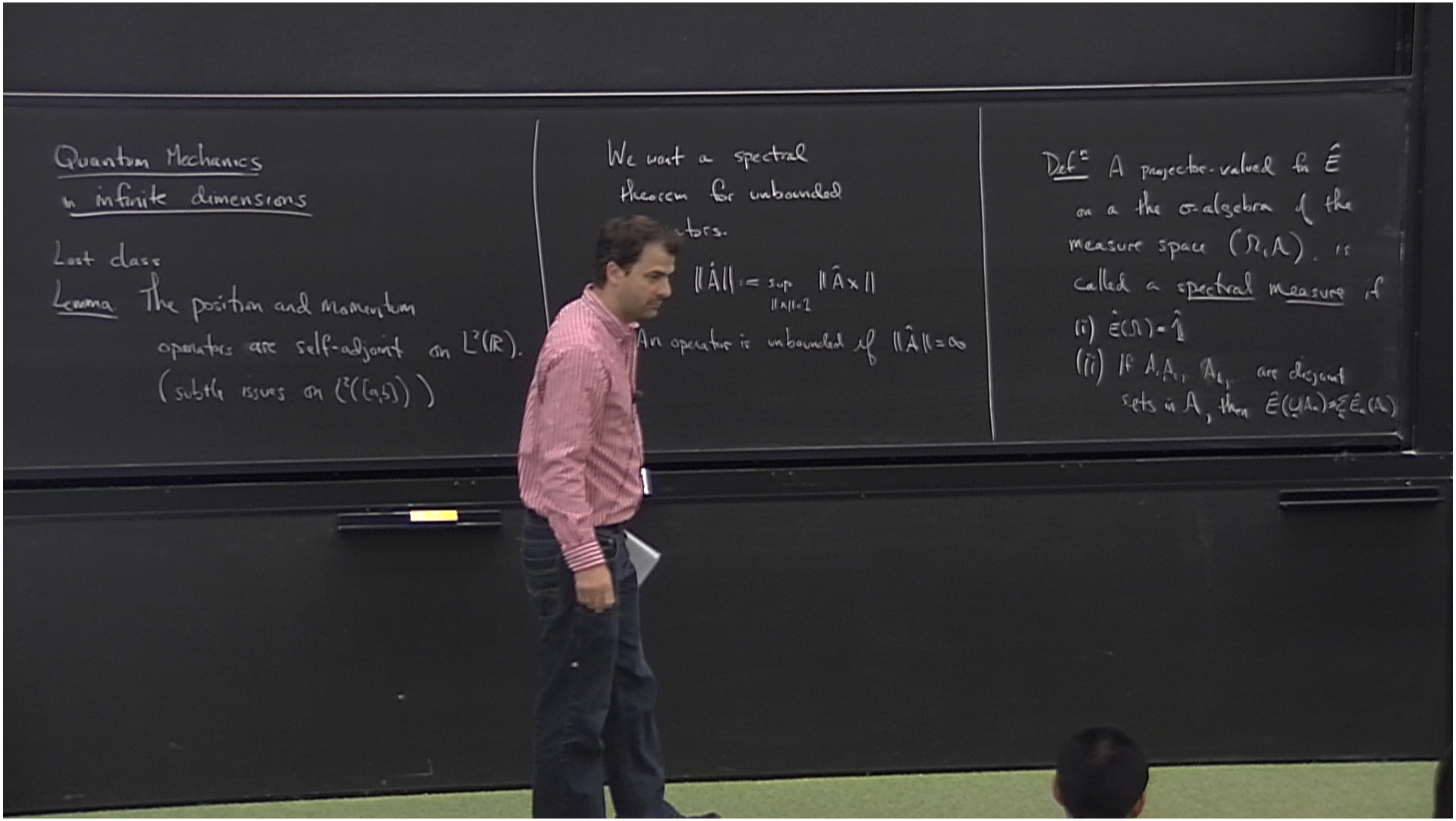
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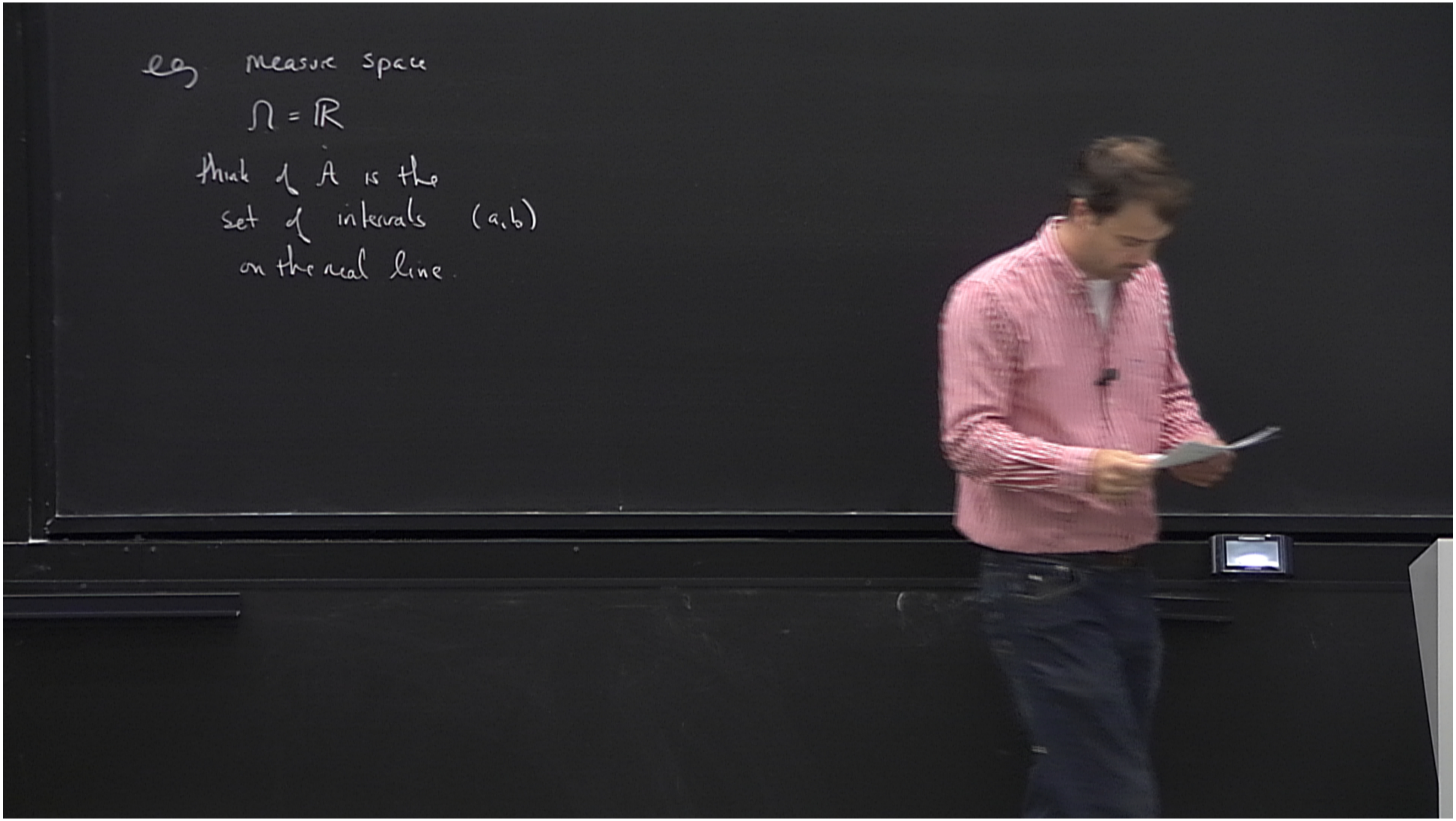
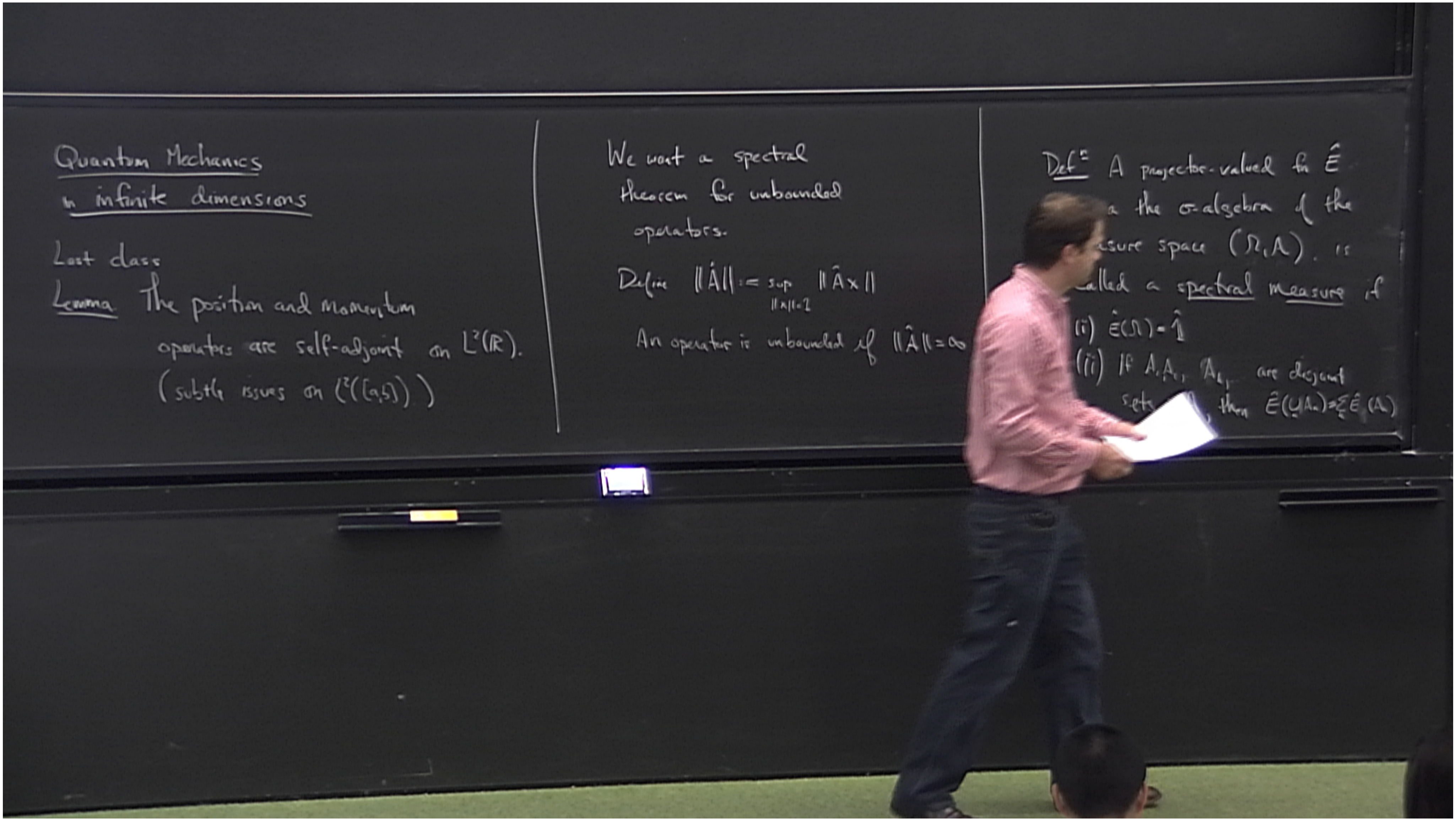

eg measure space
$\Omega = \mathbb{R}$
think of A is the
set of intervals (a,b)
on the real line.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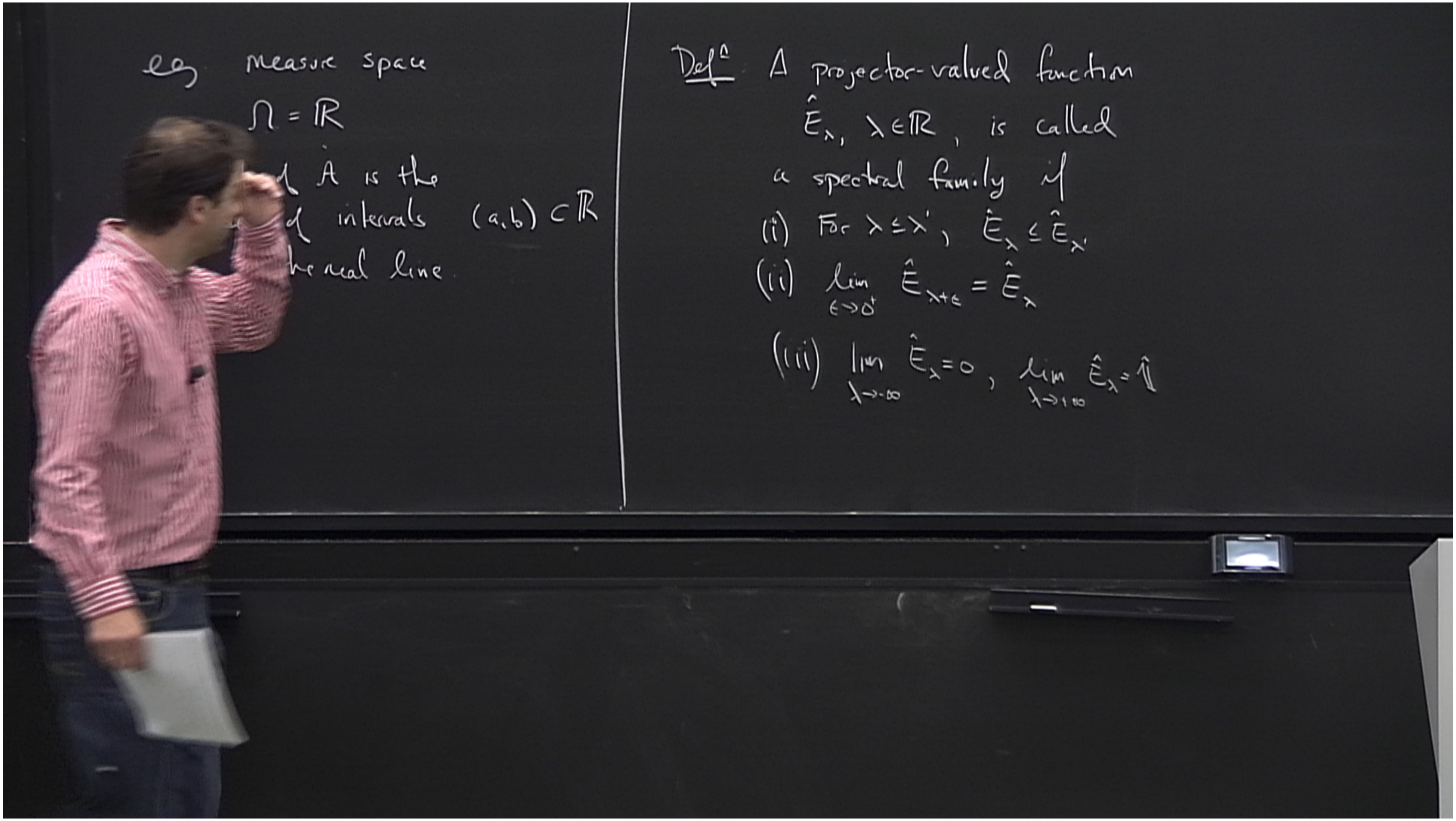
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(i) For $\lambda \leq \lambda'$, $\hat{E}_\lambda \leq \hat{E}_{\lambda'}$
(ii) $\lim_{\epsilon \to 0^+} \hat{E}_{\lambda+\epsilon} = \hat{E}_\lambda$
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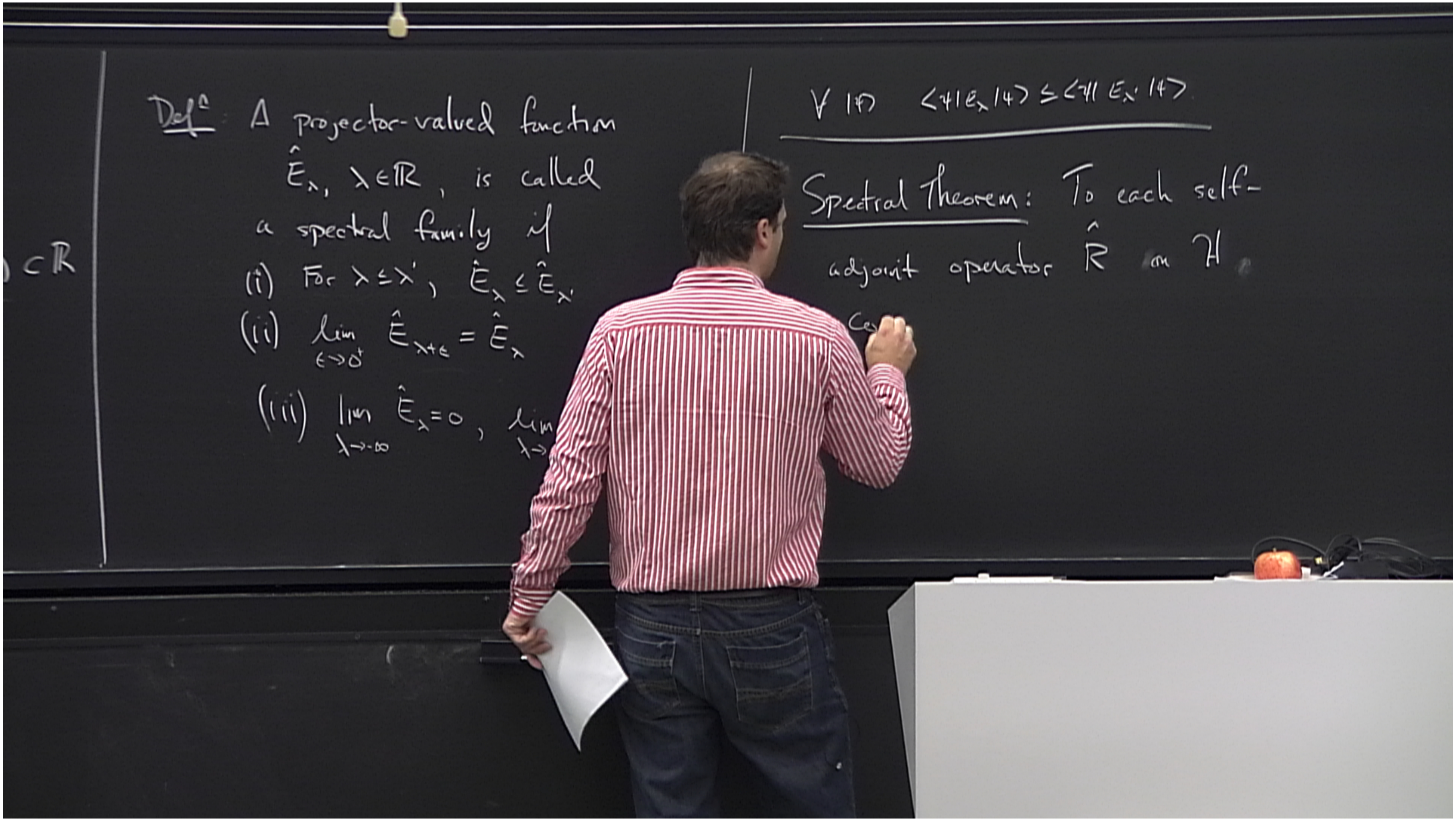
$\forall\, |\psi\rangle \quad \langle\psi|E_\lambda|\psi\rangle \le \langle\psi|E_{\lambda'}|\psi\rangle$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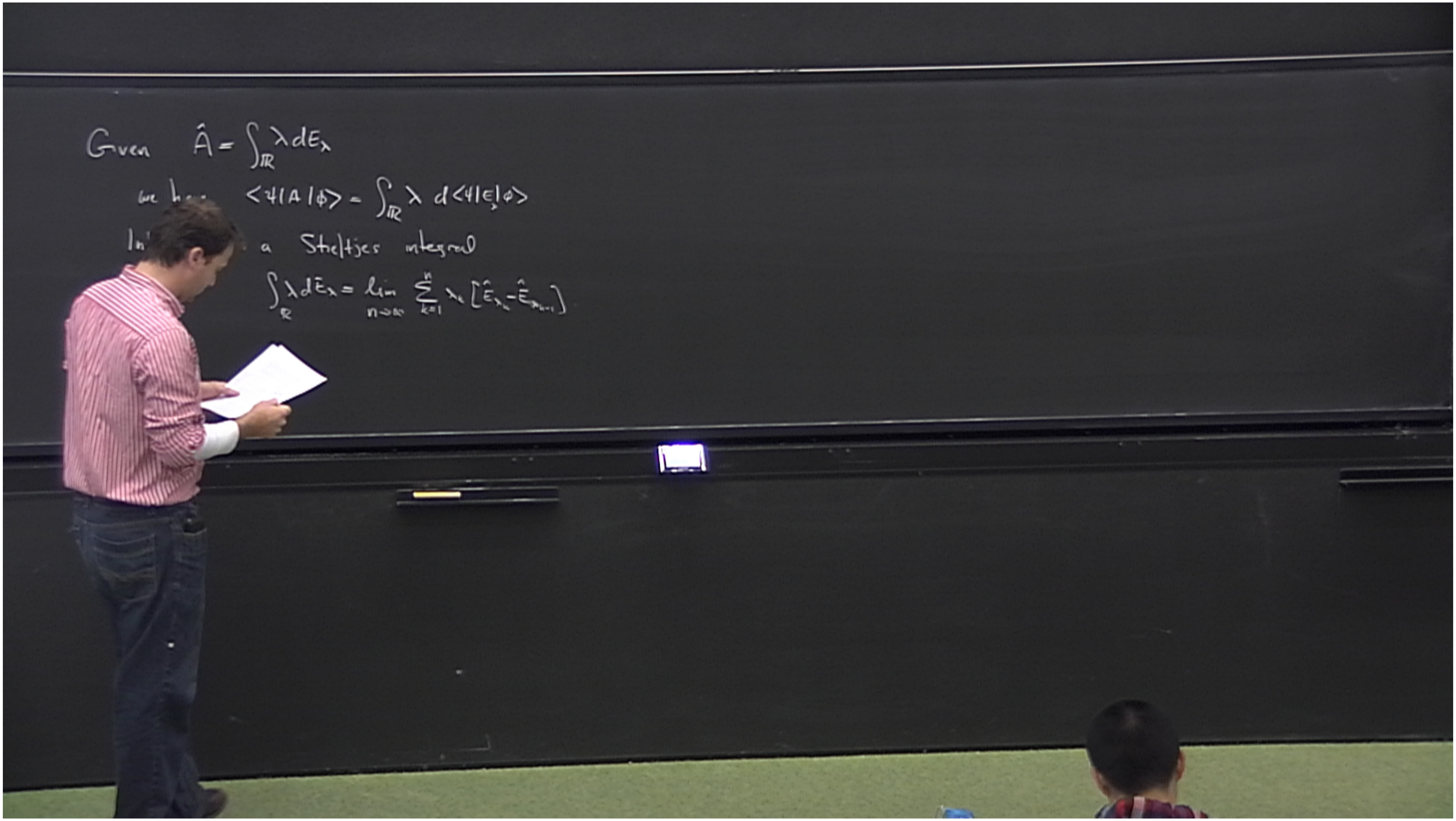
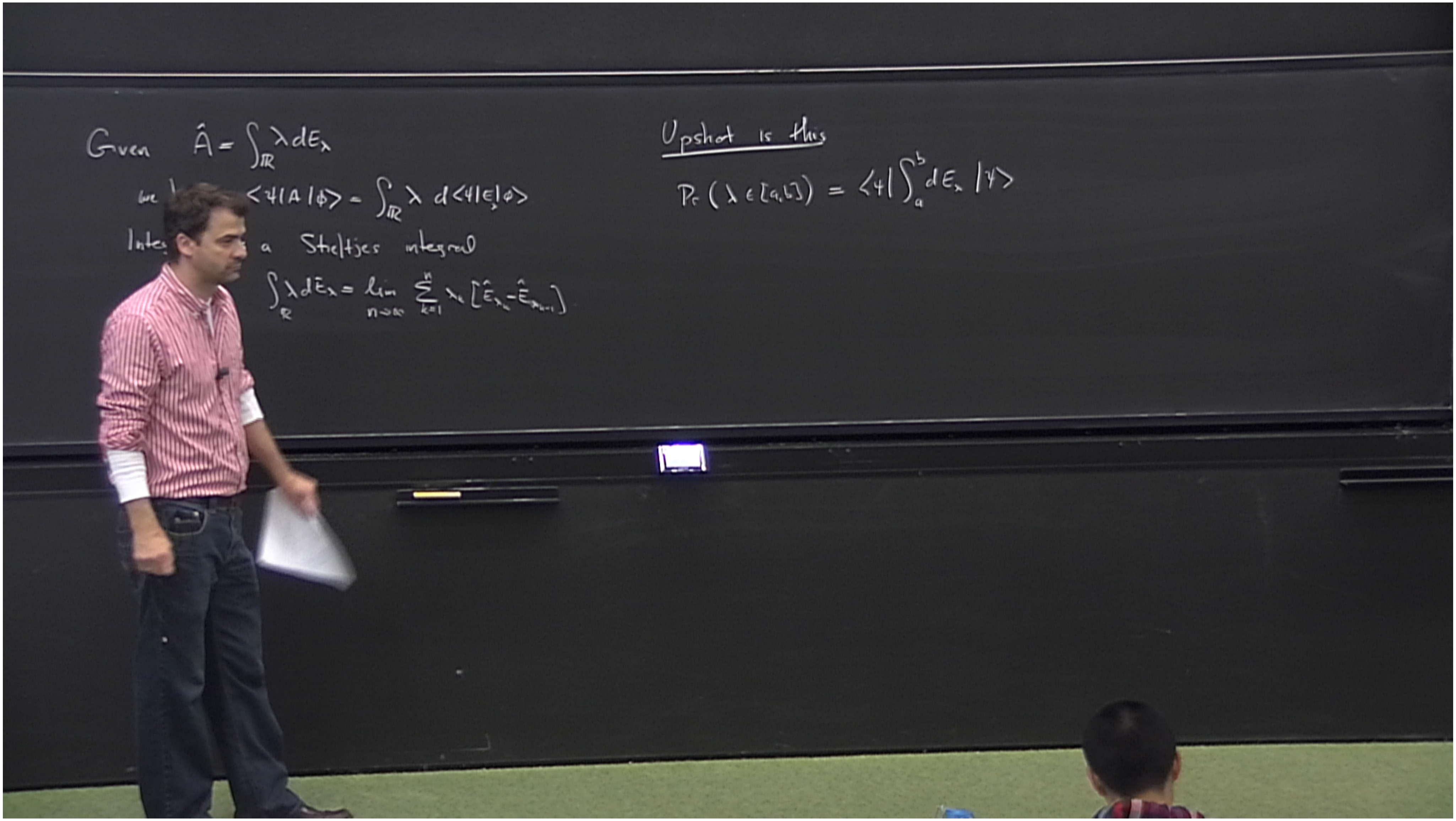
$$D(\hat{R}) = \left\{ \psi \in \mathcal{H} \,\middle|\, \int \lambda^2 \, d(\psi, E_\lambda \psi) < \infty \right\}$$

and for all $\psi \in D(\hat{R})$, $\phi \in \mathcal{H}$

$$(\phi, \hat{R}\psi) = \int_{\mathbb{R}} \lambda \, d(\phi, \hat{E}_\lambda \psi)$$


Given $\hat{A} = \int_{\mathbb{R}} \lambda \, dE_\lambda$
$\langle \psi | A | \phi \rangle = \int_{\mathbb{R}} \lambda \, d\langle \psi | E_\lambda | \phi \rangle$
a Stieltjes integral
$\int_{\mathbb{R}} \lambda \, d\hat{E}_\lambda = \lim_{n\to\infty} \sum_{k=1}^{n} \lambda_k [\hat{E}_{\lambda_k} - \hat{E}_{\lambda_{k-1}}]$



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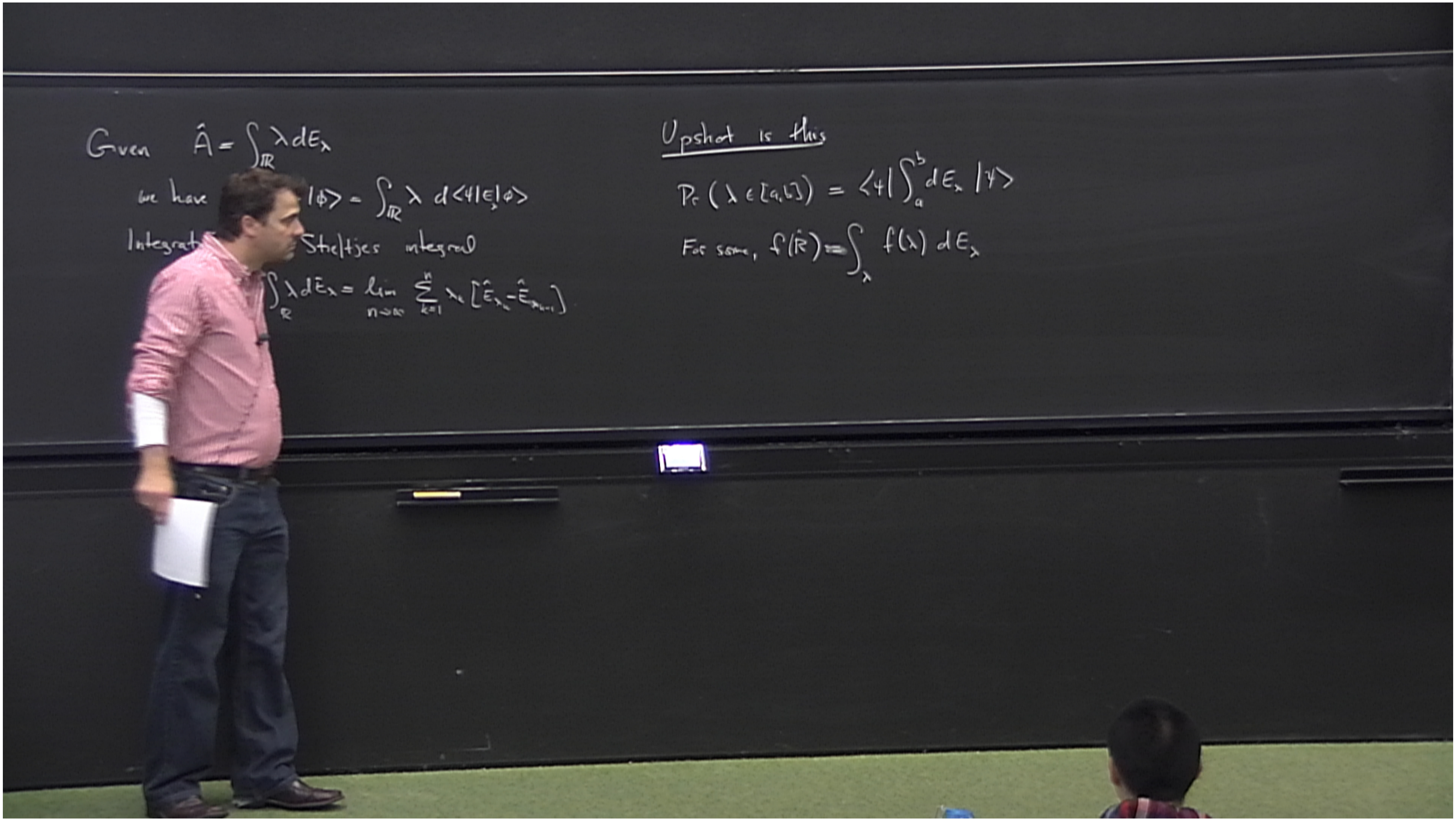
Upshot is this
$Pr(\lambda \in [a,b]) = \langle \psi | \int_a^b dE_\lambda | \psi \rangle$



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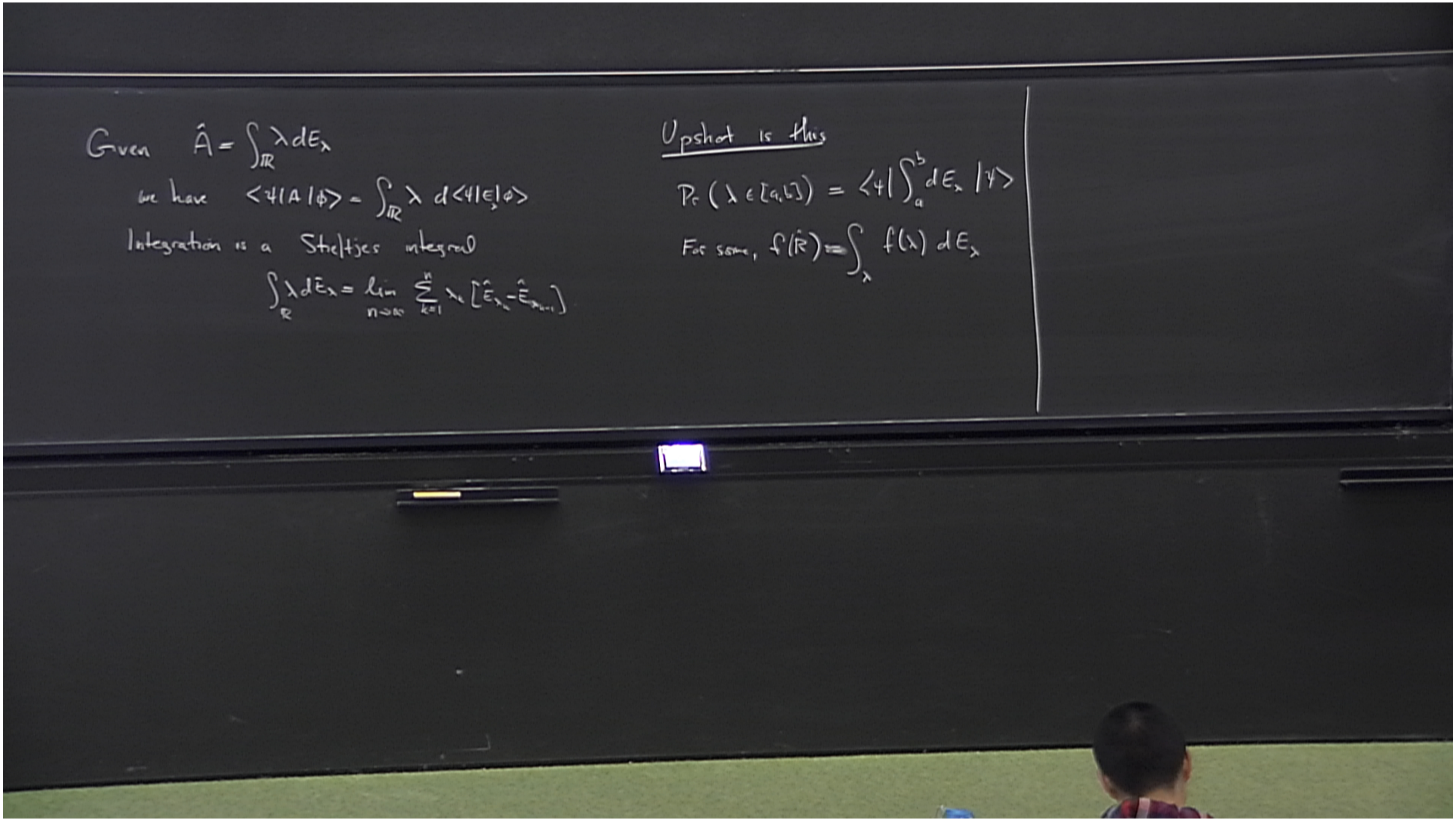
$\Pr(\lambda \in [a,b]) = \langle\psi| \int_a^b dE_\lambda |\psi\rangle$
For some, $f(\hat{R}) = \int_\lambda f(\lambda) \, dE_\lambda$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Integration is a Stieltjes integral
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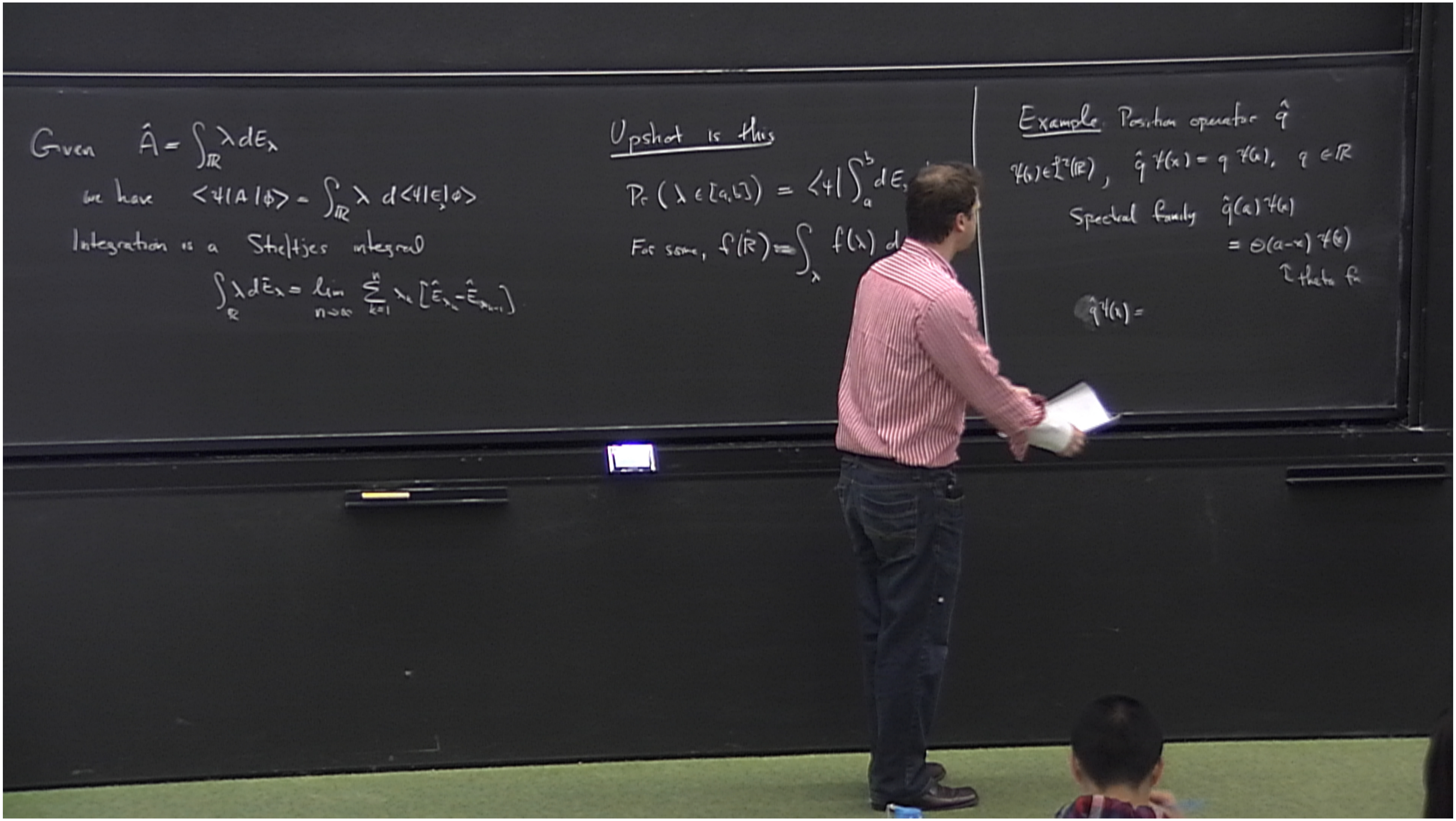
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Spectral family $\hat{q}(a)\psi(x)$
$= \Theta(a-x)\,\psi(x)$
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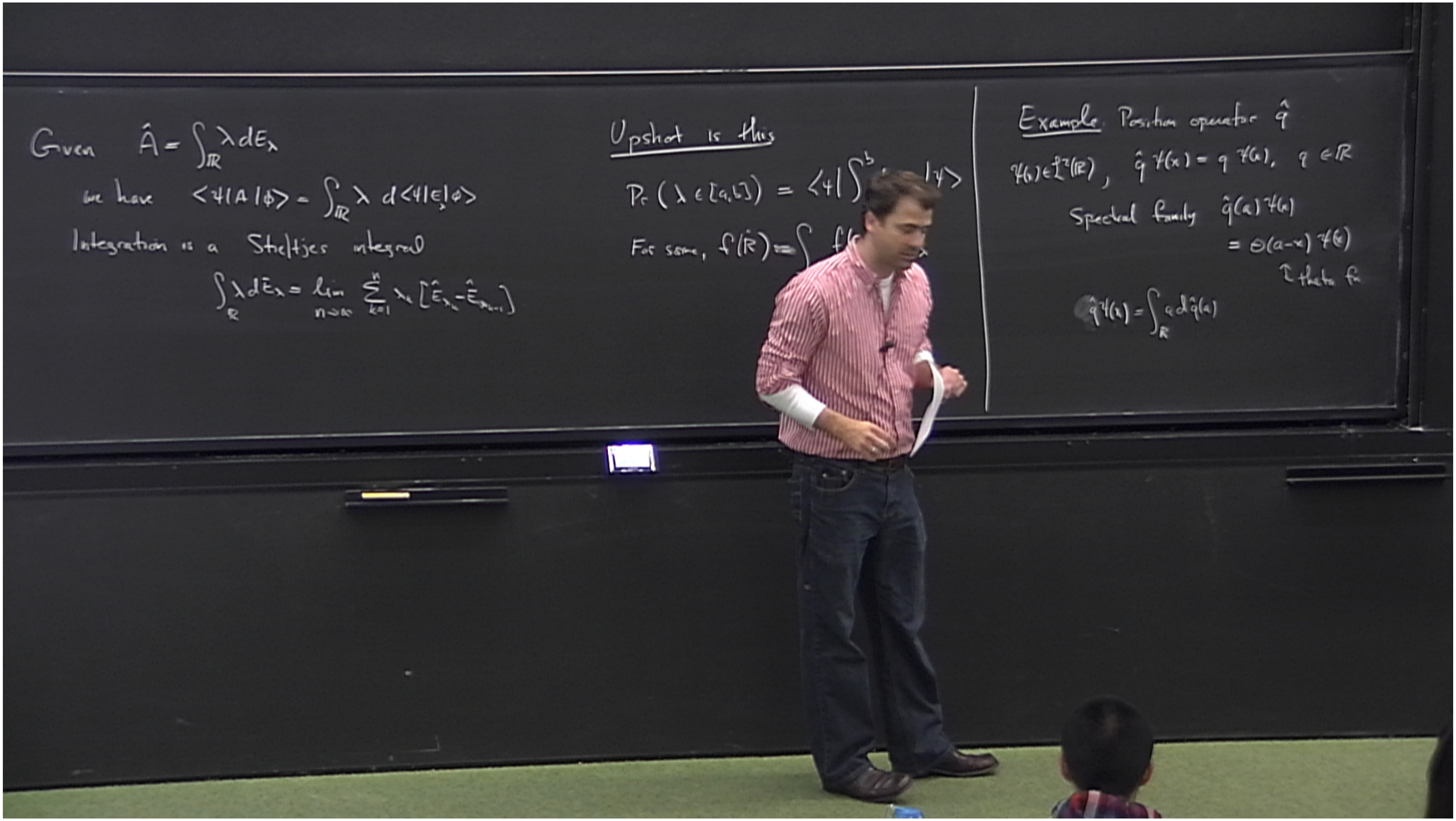
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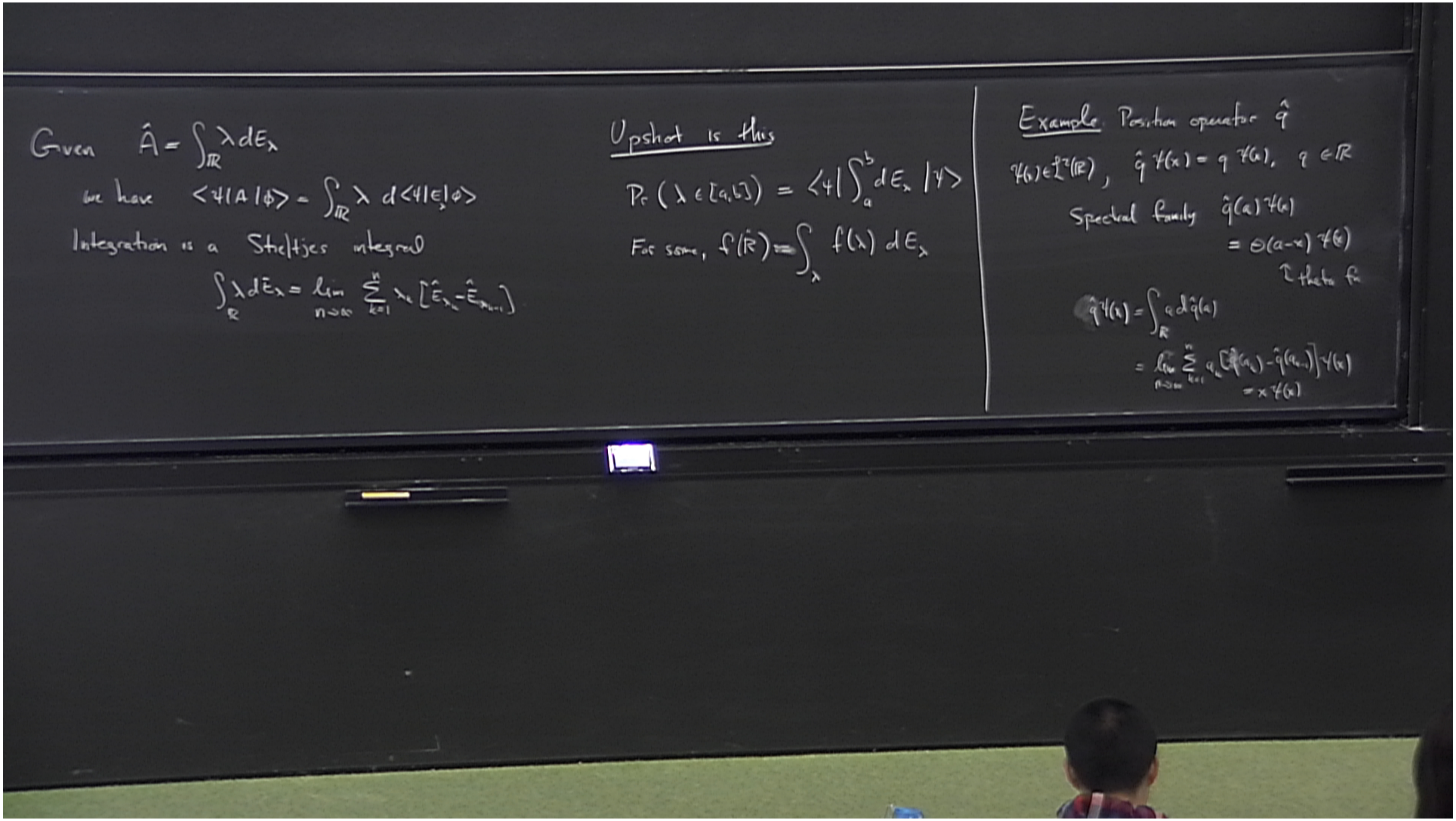
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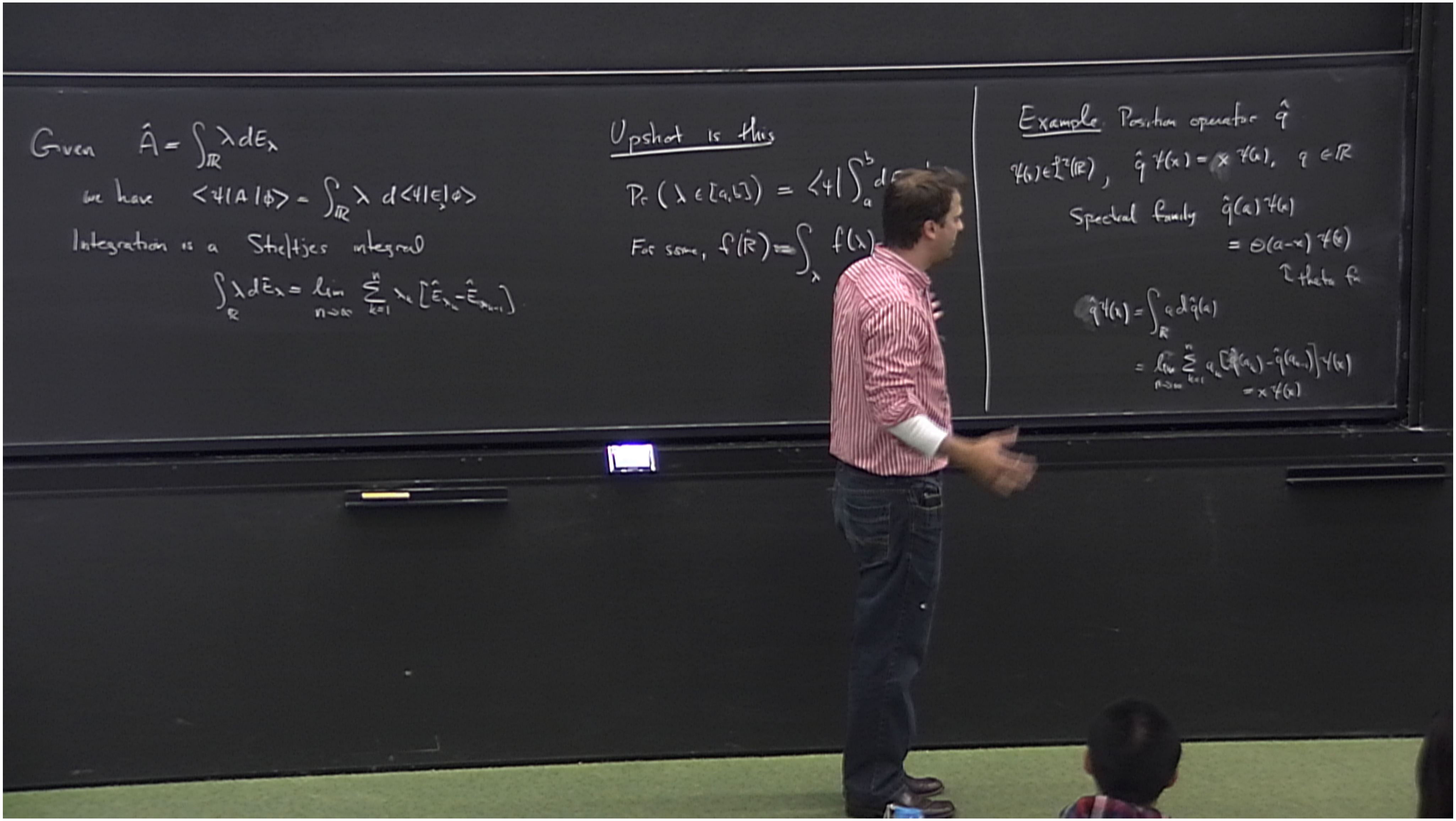
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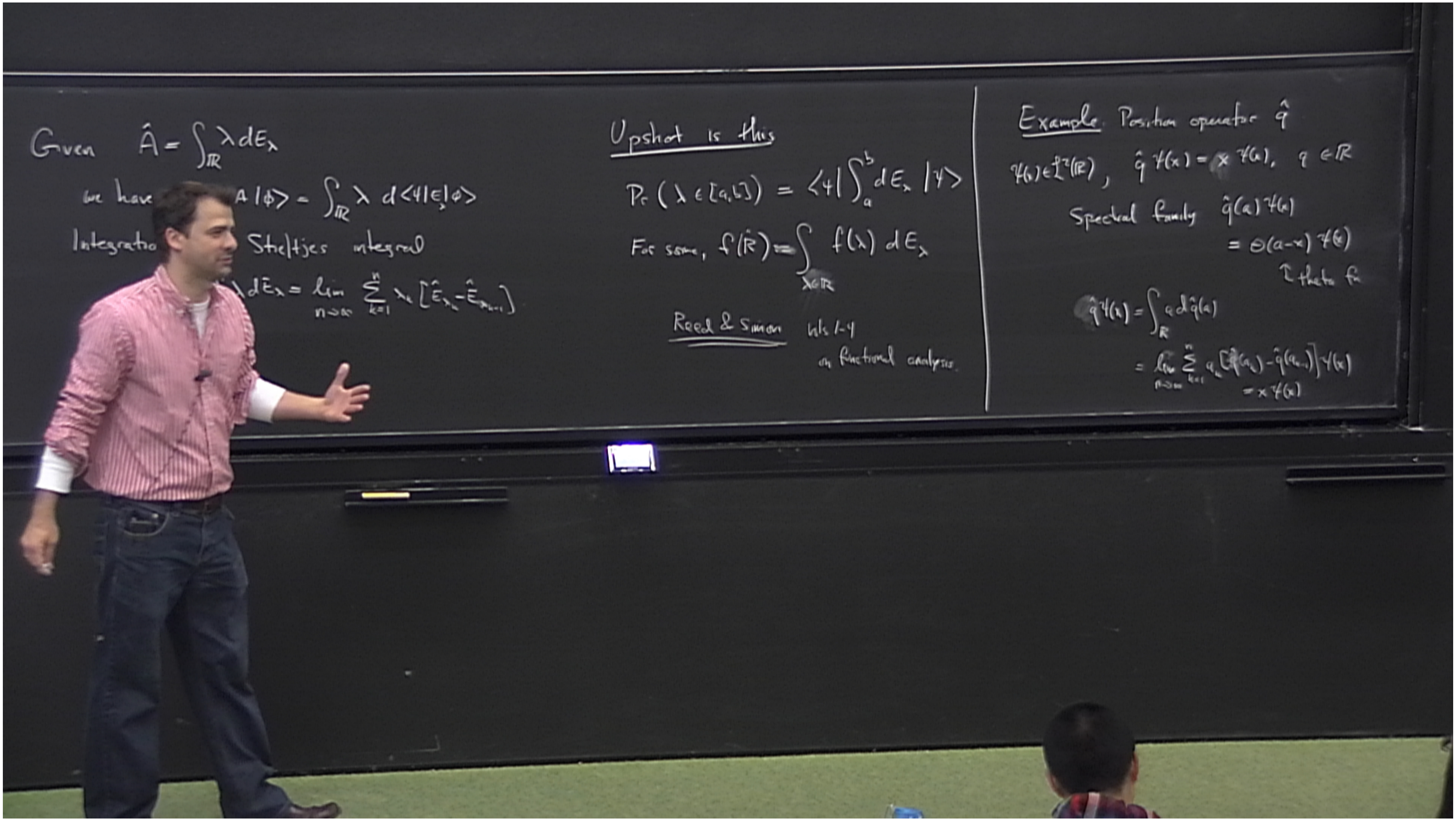
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Reed & Simon Vols 1-4
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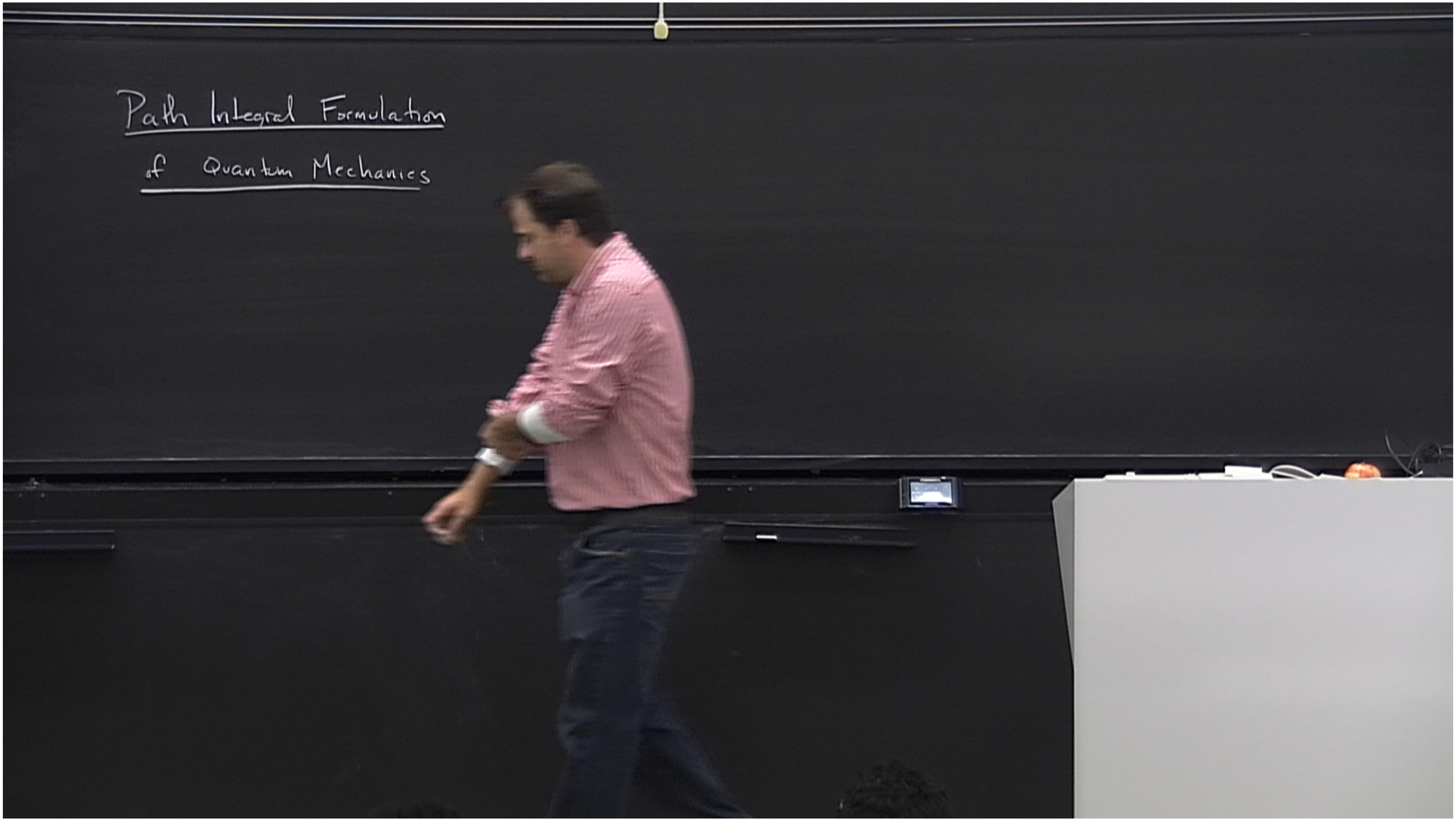
Path Integral Formulation
of Quantum Mechanics


Path Integral Formulation
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R. Feynman, Rev. Mod. Phys, 1948

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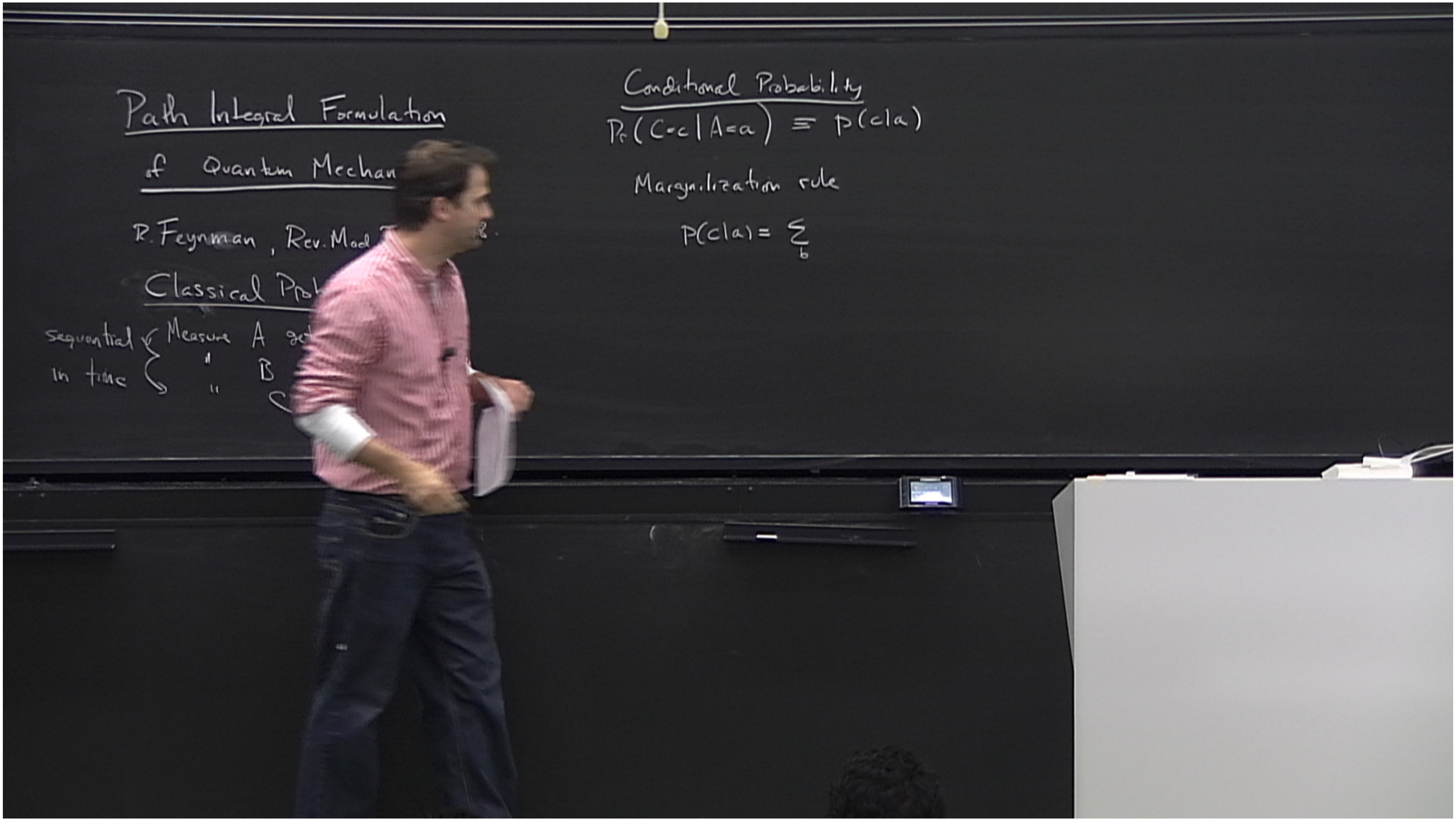
Classical Probability


Path Integral Formulation
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R. Feynman, Rev. Mod
Classical Prob
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Conditional Probability
Pr(C=c|A=a) ≡ p(c|a)
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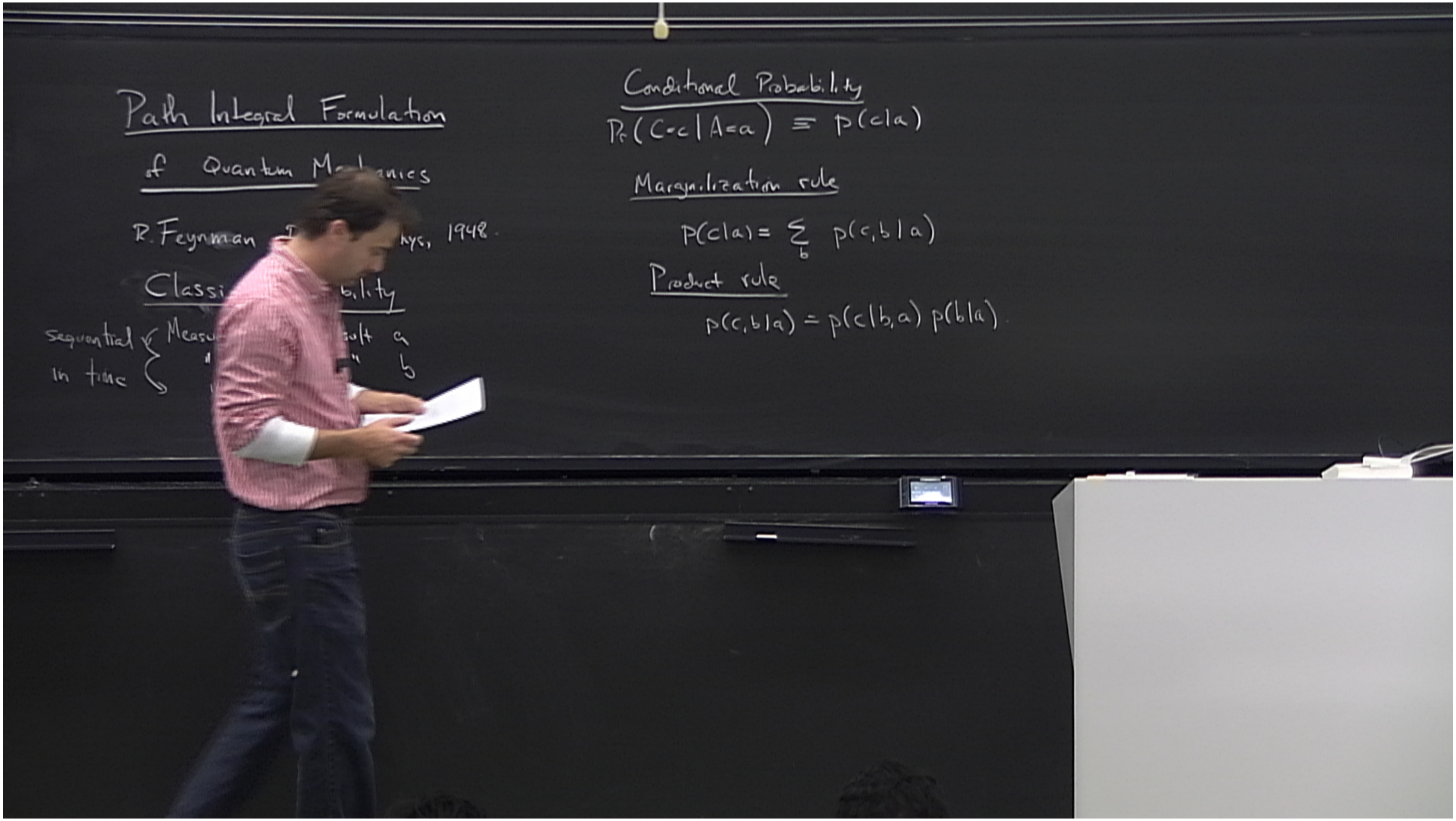
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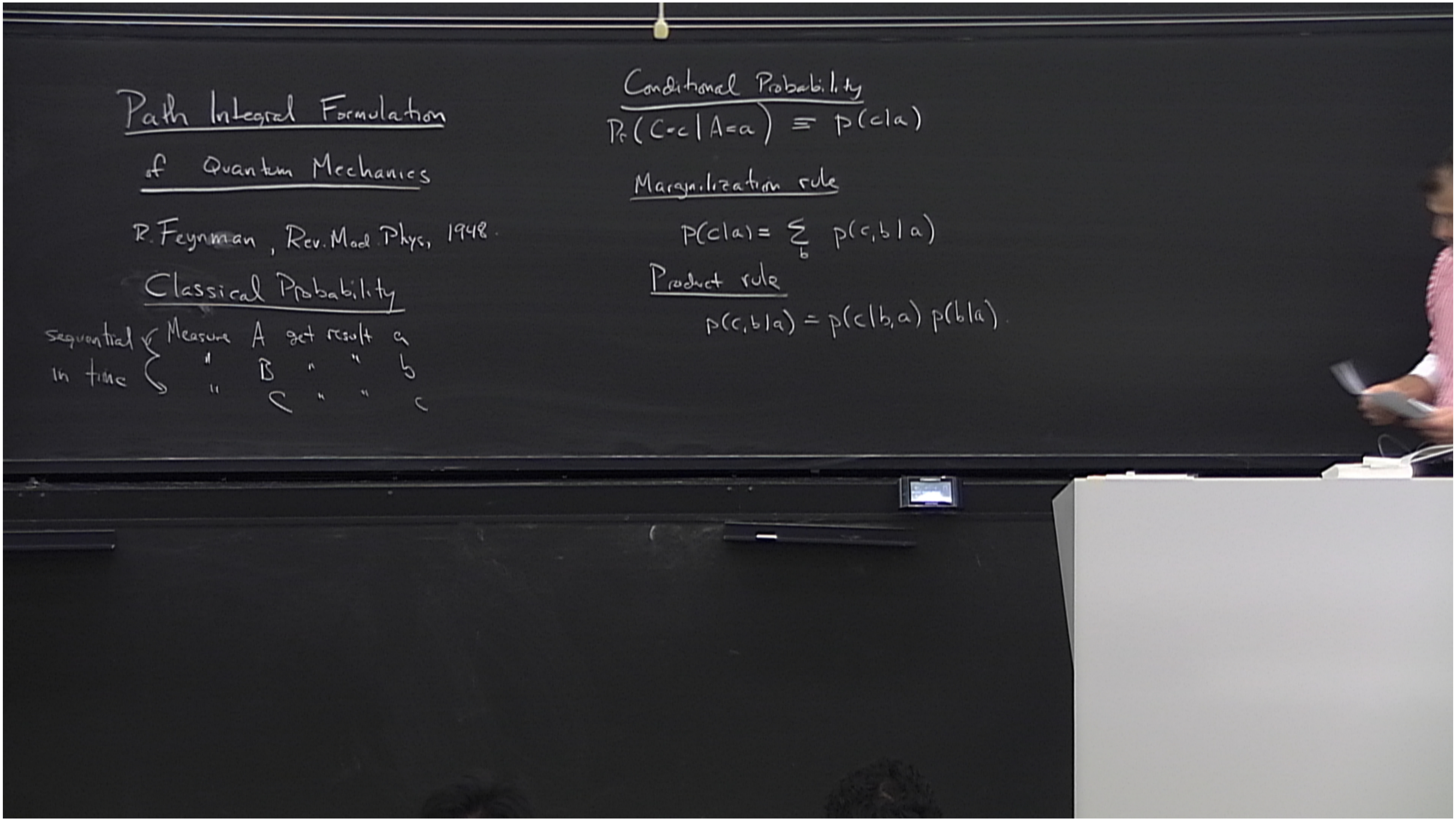
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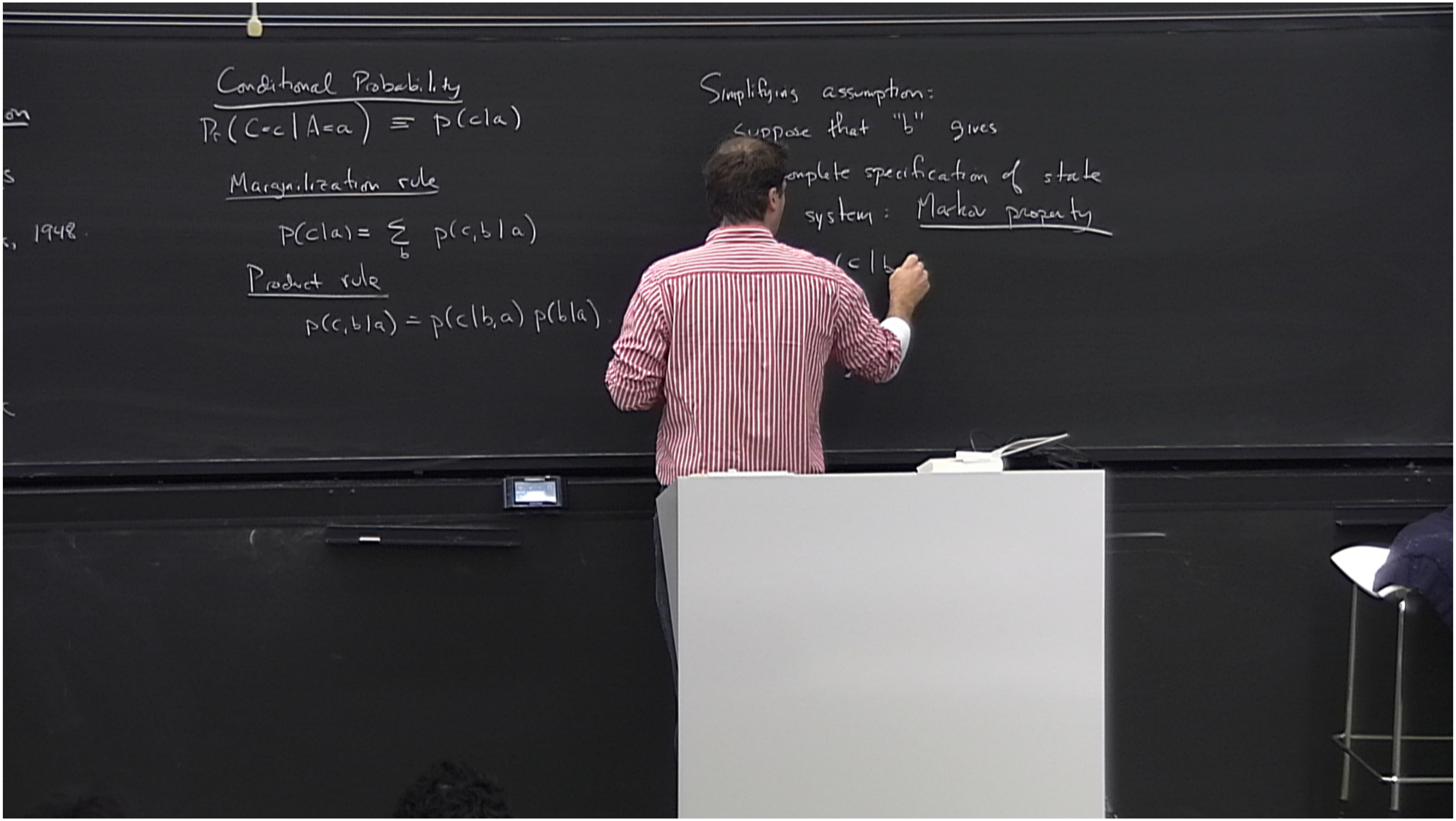
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1948
Simplifying assumption:
Suppose that "b" gives
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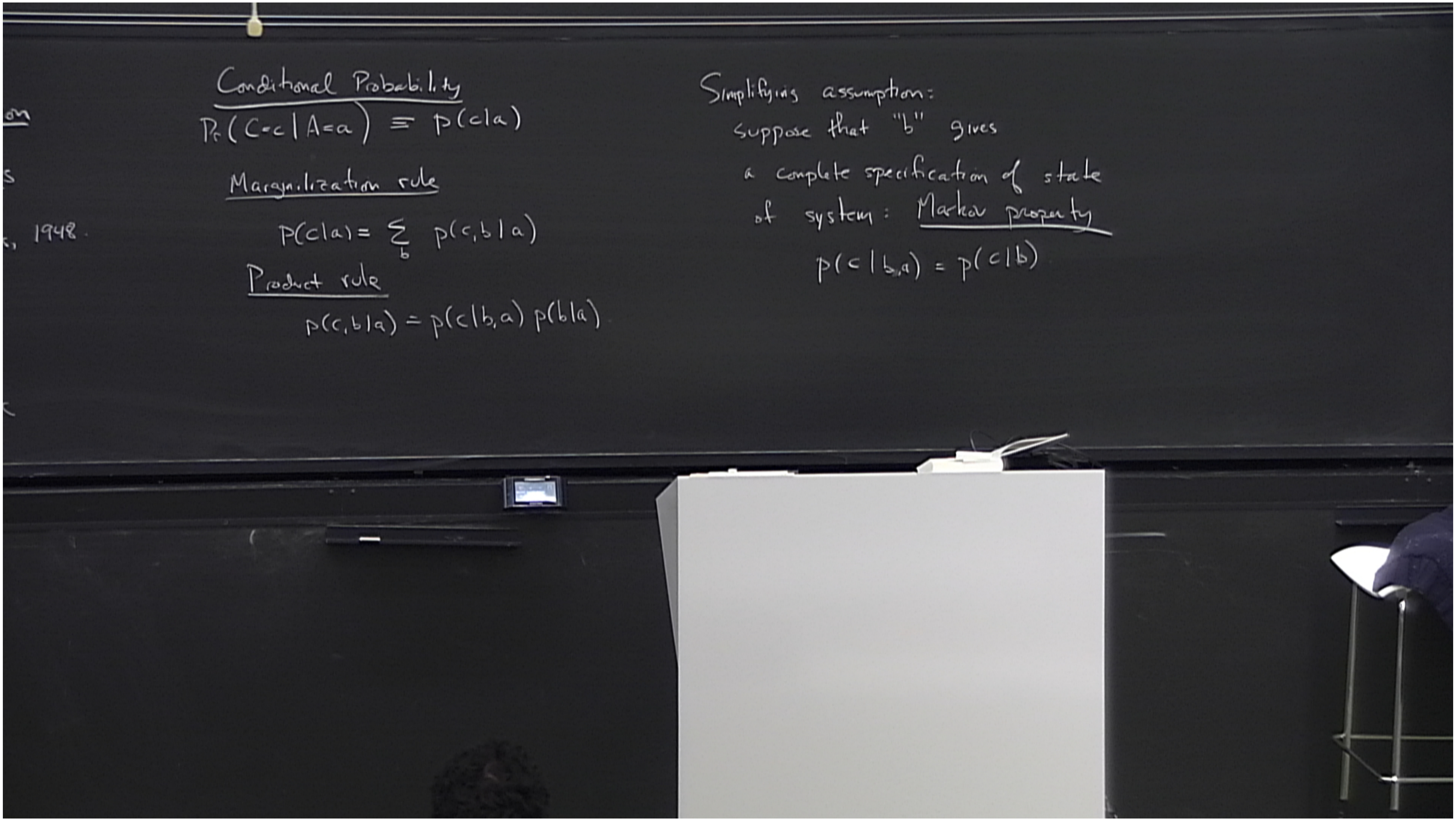
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Simplifying assumption:
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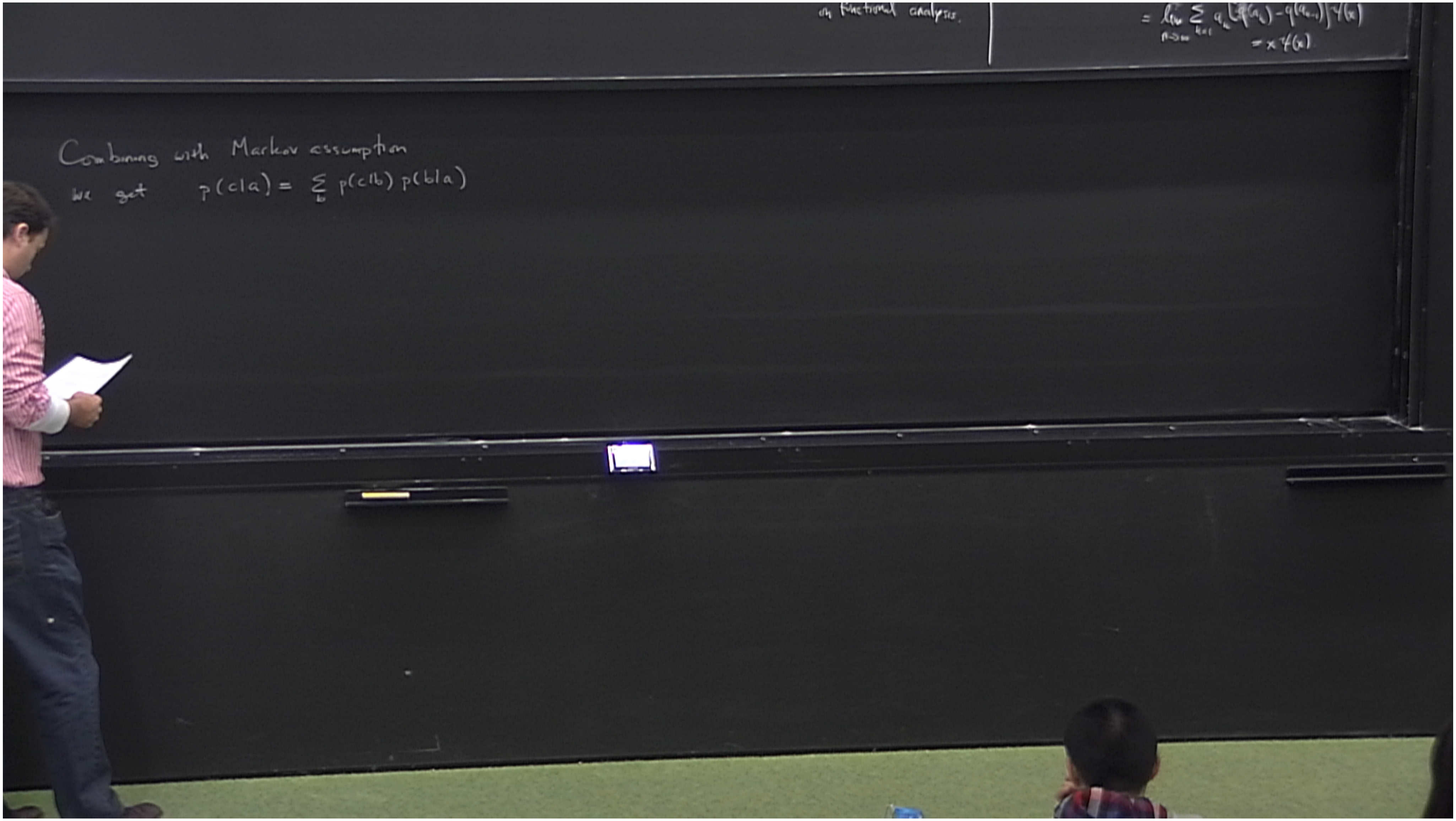
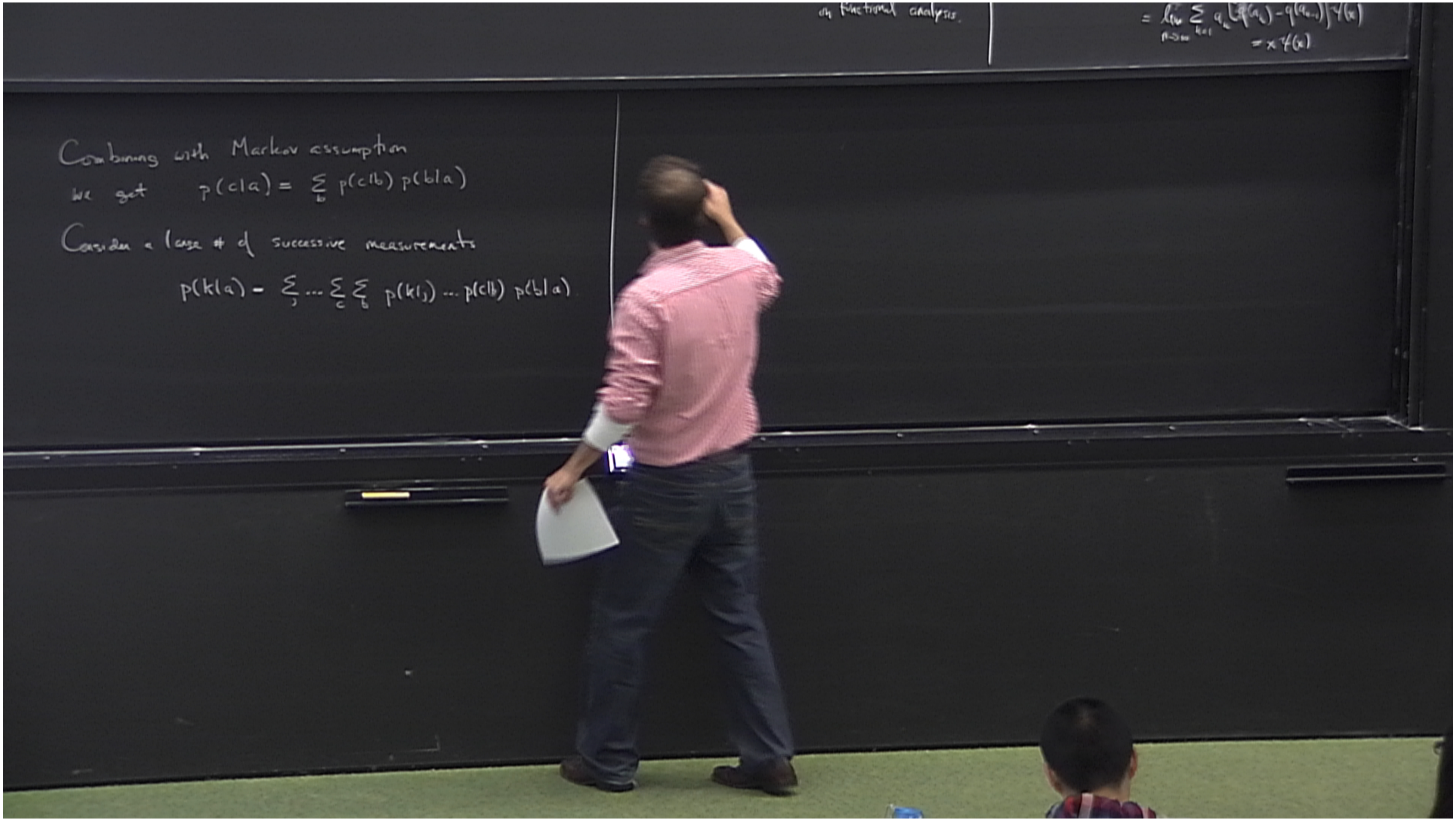
a complete specification of state
of system: Markov property
$p(c|b,a) = p(c|b)$


on functional analysis.
Combining with Markov assumption
we get p(c|a) = Σ_b p(c|b) p(b|a)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Consider a large # of successive measurements
p(k|a) = Σ_j ... Σ_c Σ_b p(k|j) ... p(c|b) p(b|a)

on functional analysis.

$$= \lim_{n\to\infty} \sum_{k=1}^{n} q_k \left[\theta(q_k) - \theta(q_{k-1})\right] \psi(x)$$
$$= x\,\psi(x)$$

Combining with Markov assumption we get

$$p(c|a) = \sum_b p(c|b)\, p(b|a)$$

Consider a large # of successive measurements

$$p(k|a) = \sum_j \cdots \sum_c \sum_b p(k|j) \cdots p(c|b)\, p(b|a)$$

Generalize this to continuous outcomes

$$p(x_N|x_0) = \int dx_{N-1} \cdots \int dx_2 \int dx_1\, p(x_N|x_{N-1}) \cdots p(x_2|x_1)\, p(x_1|x_0)$$

on functional analysis.

$$= \lim_{n\to\infty} \sum_{k=1}^{n} q_k \left[\theta(q_k) - \theta(q_{k-1})\right] \psi(x) = x\,\psi(x)$$

Combining with Markov assumption we get $p(c|a) = \sum_b p(c|b)\,p(b|a)$

Consider a large # of successive measurements

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this is called a "path integral" and is interpreted as a "sum over paths", expressed symbolically as

$$p(x|x_0) = \int \mathcal{D}[x(t)]\; p(x(t)|x_0)$$

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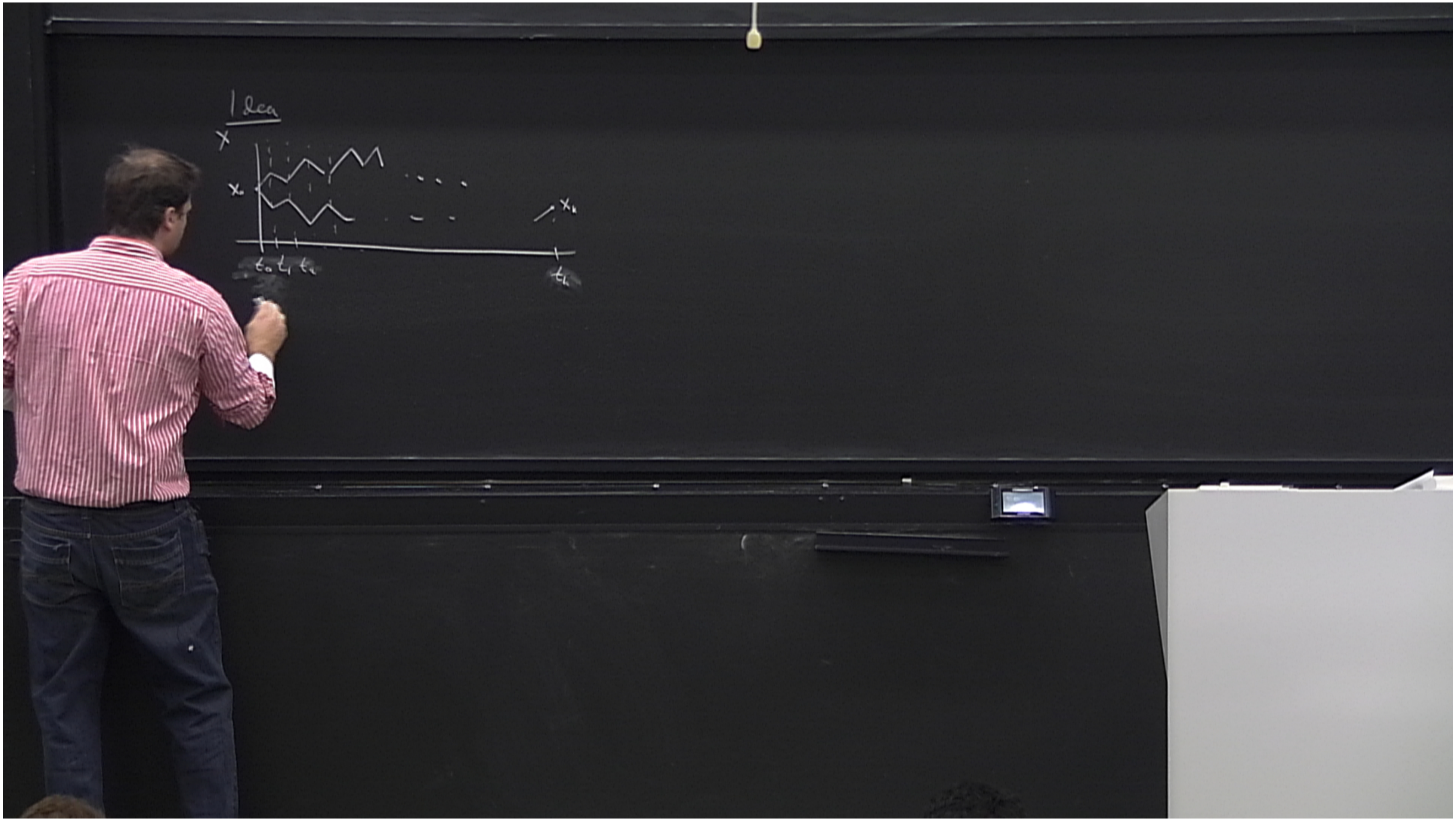
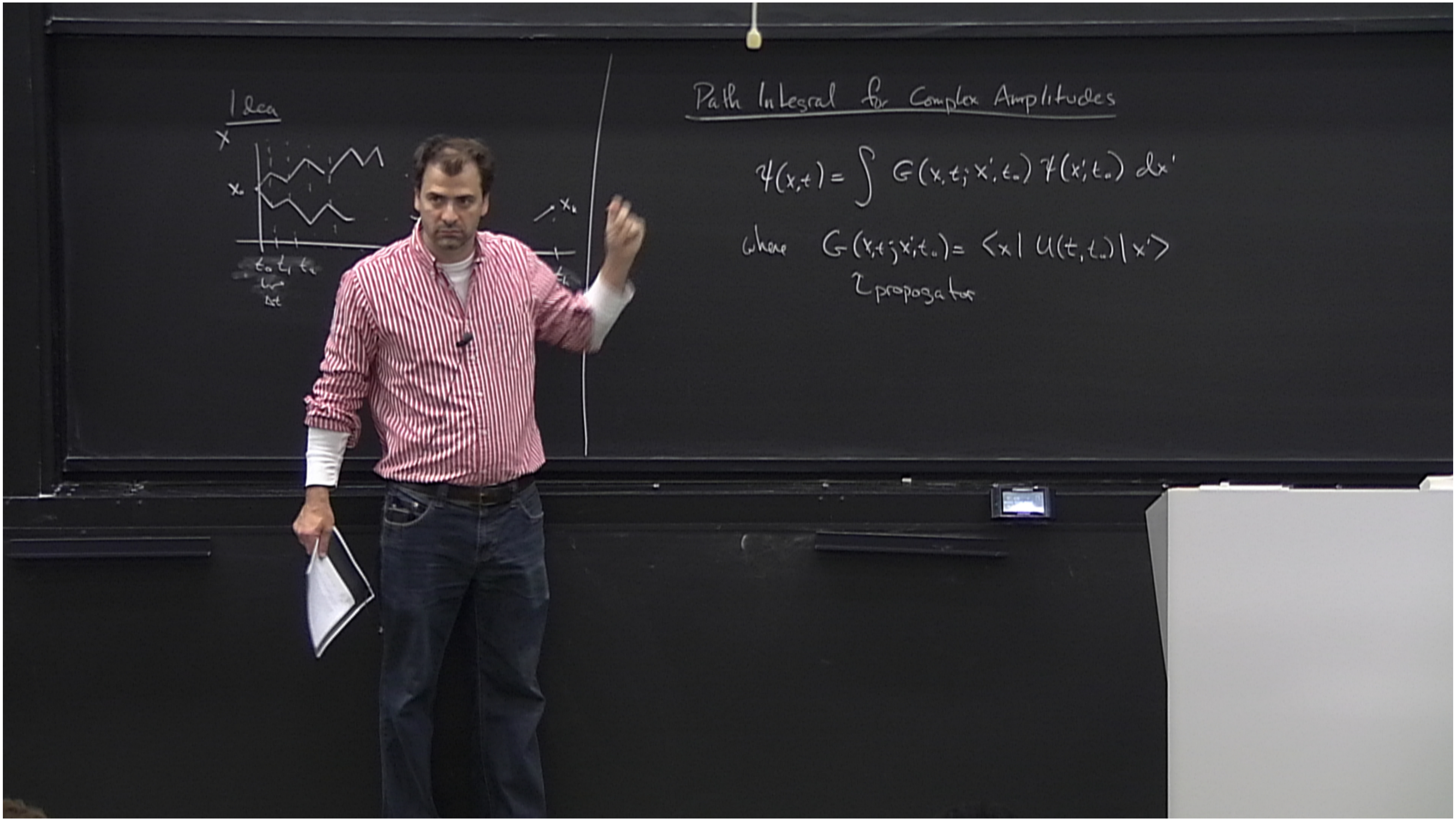
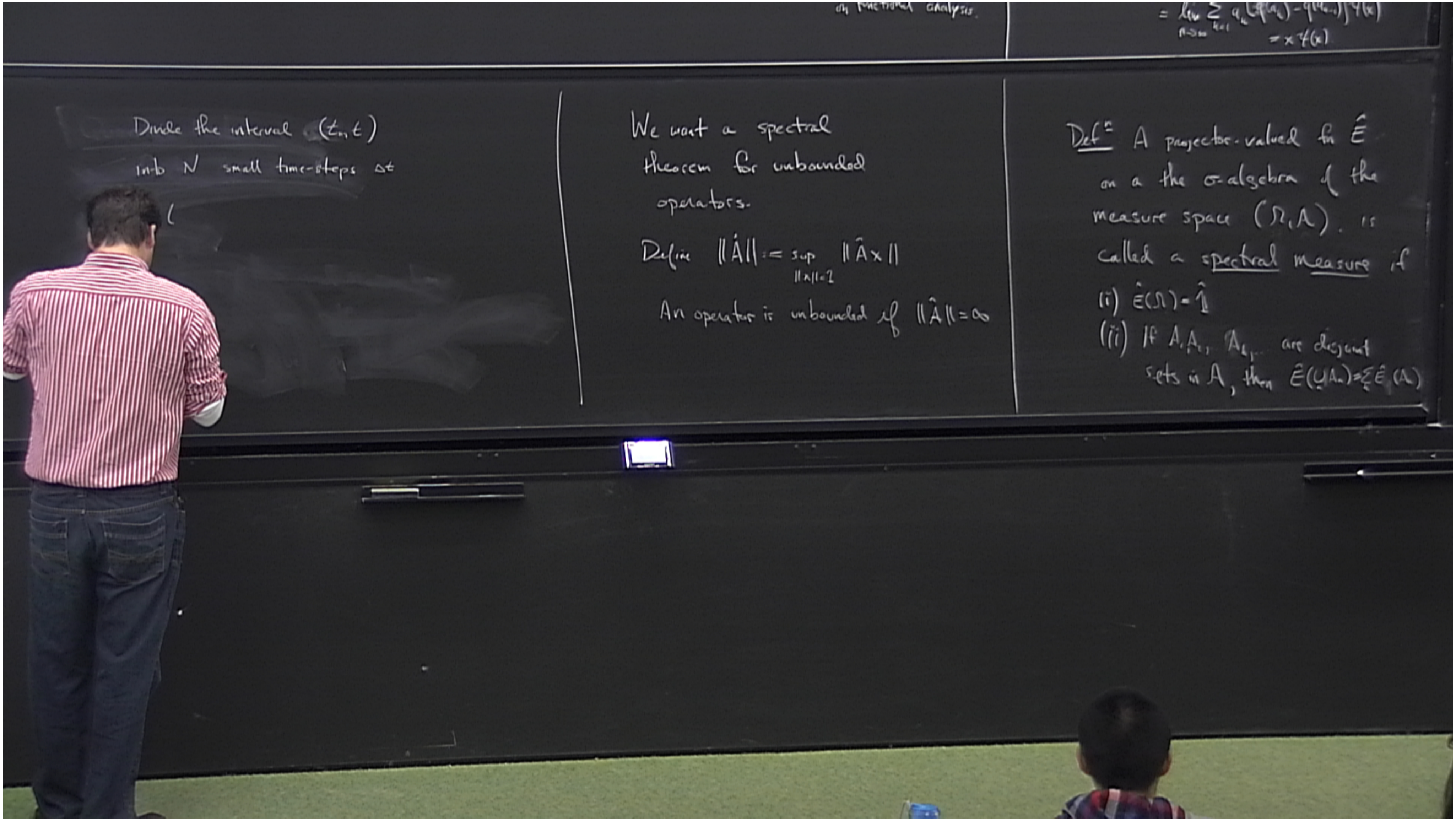
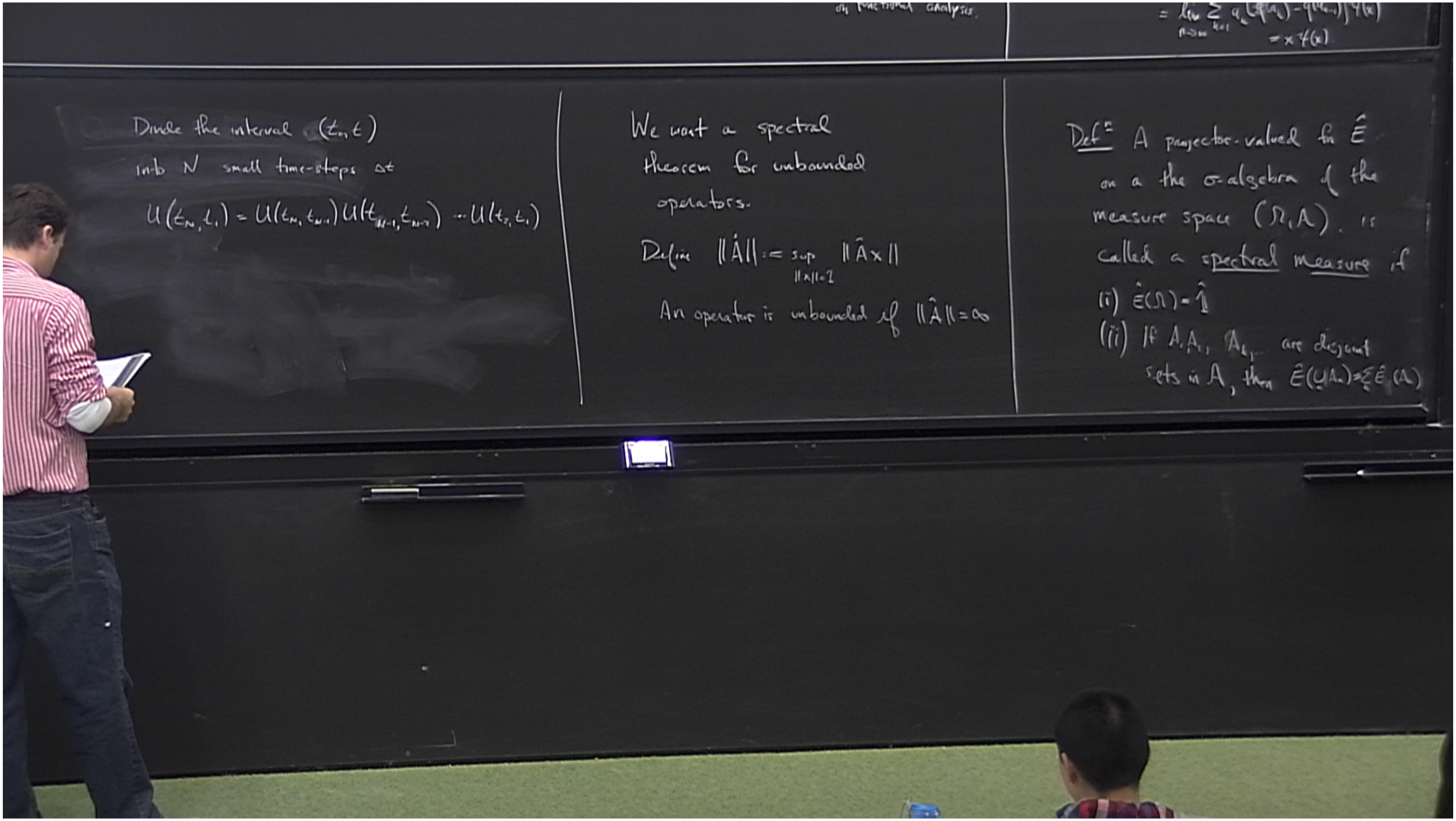
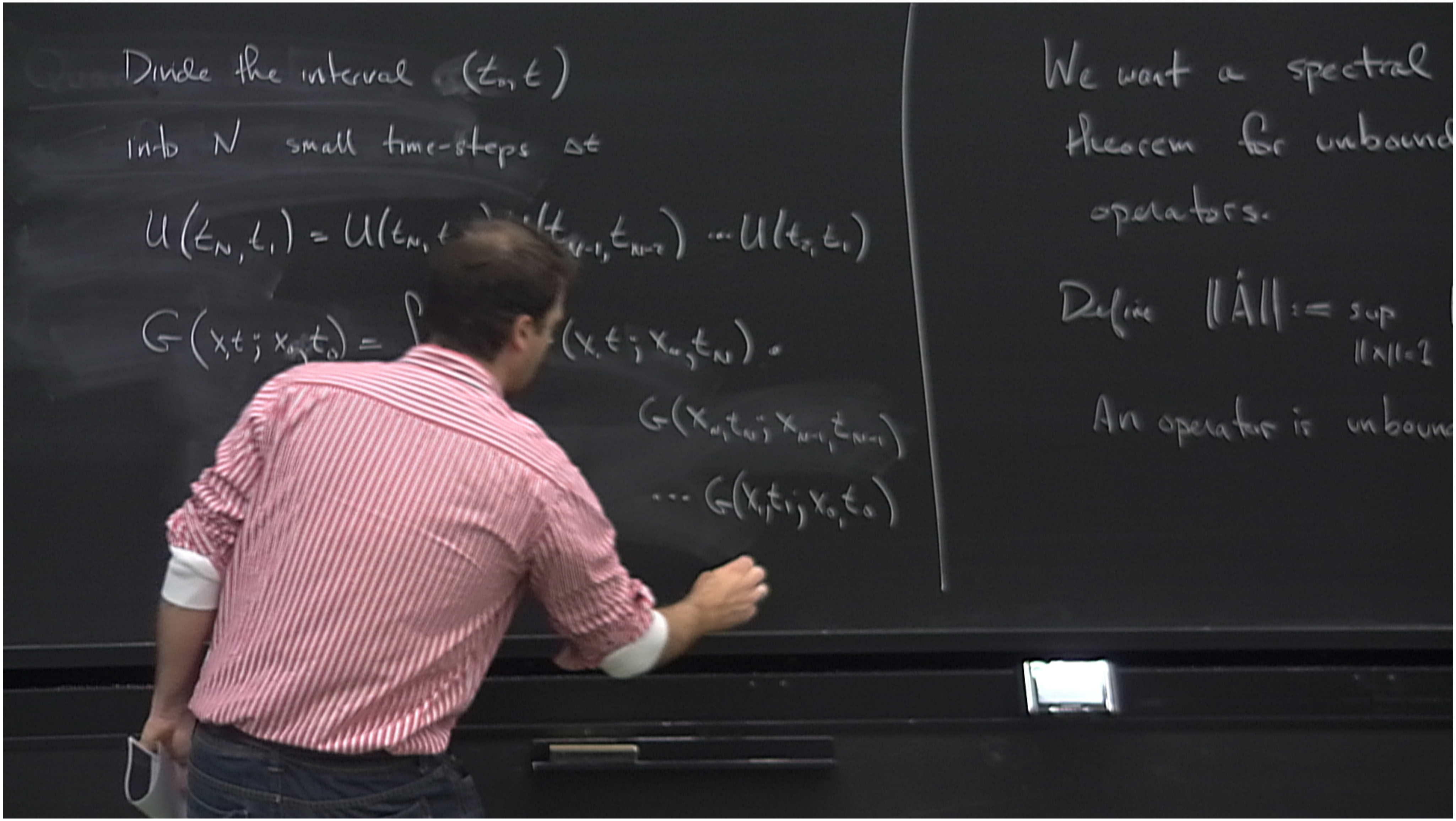

Idea
x
x_0
x_N
t_0 t_1 t_2
t_N


Idea
Path Integral for Complex Amplitudes
ψ(x,t) = ∫ G(x,t;x',t₀) ψ(x',t₀) dx'
where G(x,t;x',t₀) = <x| U(t,t₀) |x'>
propogator


Divide the interval (t0, t)
into N small time-steps Δt
We want a spectral
theorem for unbounded
operators.
Define ||Â|| := sup ||x||=1 ||Âx||
An operator is unbounded if ||Â|| = ∞
Def^n A projector-valued fn Ê
on a the σ-algebra of the
measure space (Ω, A) is
called a spectral measure if
(i) Ê(Ω) = 1̂
(ii) If A, A1, A2, ... are disjoint
sets in A, then Ê(∪An) = Σ Ê(An)


Divide the interval (t_0, t)
into N small time-steps Δt
$U(t_N, t_1) = U(t_N, t_{N-1})U(t_{N-1}, t_{N-2}) \cdots U(t_2, t_1)$
We want a spectral
theorem for unbounded
operators.
Define $\|\hat{A}\| := \sup_{\|x\|=1} \|\hat{A}x\|$
An operator is unbounded if $\|\hat{A}\| = \infty$
Def^n A projector-valued fn $\hat{E}$
on a the σ-algebra of the
measure space (Ω, A) is
called a spectral measure if
(i) $\hat{E}(\Omega) = \hat{1}$
(ii) If $A_1, A_2, \ldots$ are disjoint
sets in A, then $\hat{E}(\bigcup A_n) = \sum \hat{E}(A_n)$
$= \lim_{n\to\infty} \sum_{k=1} a_k (q(a_k) - q(a_{k-1}))\psi(x)$
$= x\psi(x)$


Divide the interval (t_0, t)
into N small time-steps Δt
G(x_{N-1}, t_{N-1}; x_{N-2}, t_{N-2})
... G(x_1, t_1; x_0, t_0)
We want a spectral
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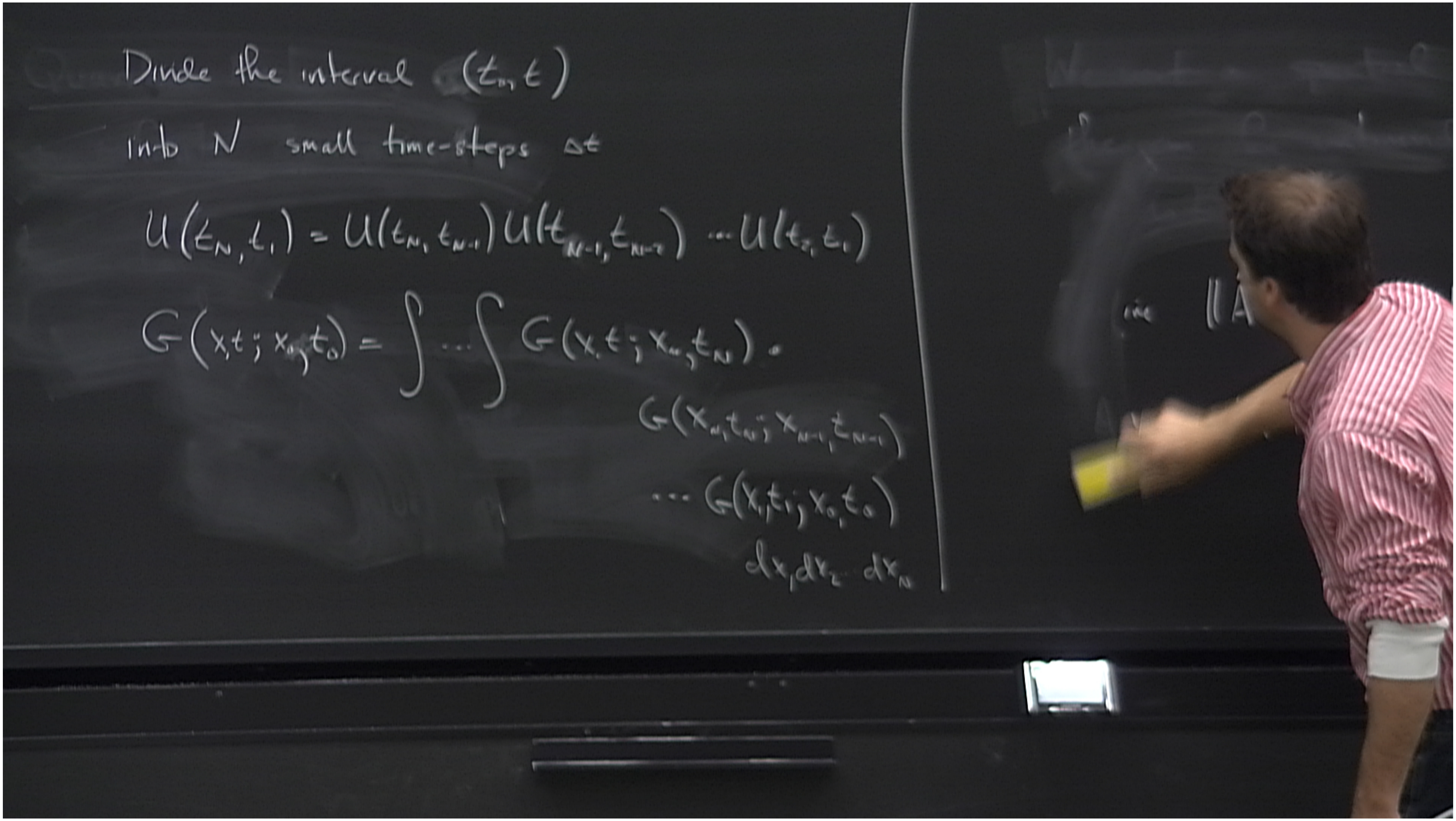
Divide the interval (t_0, t)
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$U(t_N, t_1) = U(t_N, t_{N-1})\,U(t_{N-1}, t_{N-2}) \cdots U(t_2, t_1)$
$G(x,t;x_0,t_0) = \int \cdots \int G(x,t;x_N,t_N) \cdot G(x_N,t_N;x_{N-1},t_{N-1}) \cdots G(x_1,t_1;x_0,t_0)\, dx_1 dx_2 \cdots dx_N$

Divide the interval (t_0, t)

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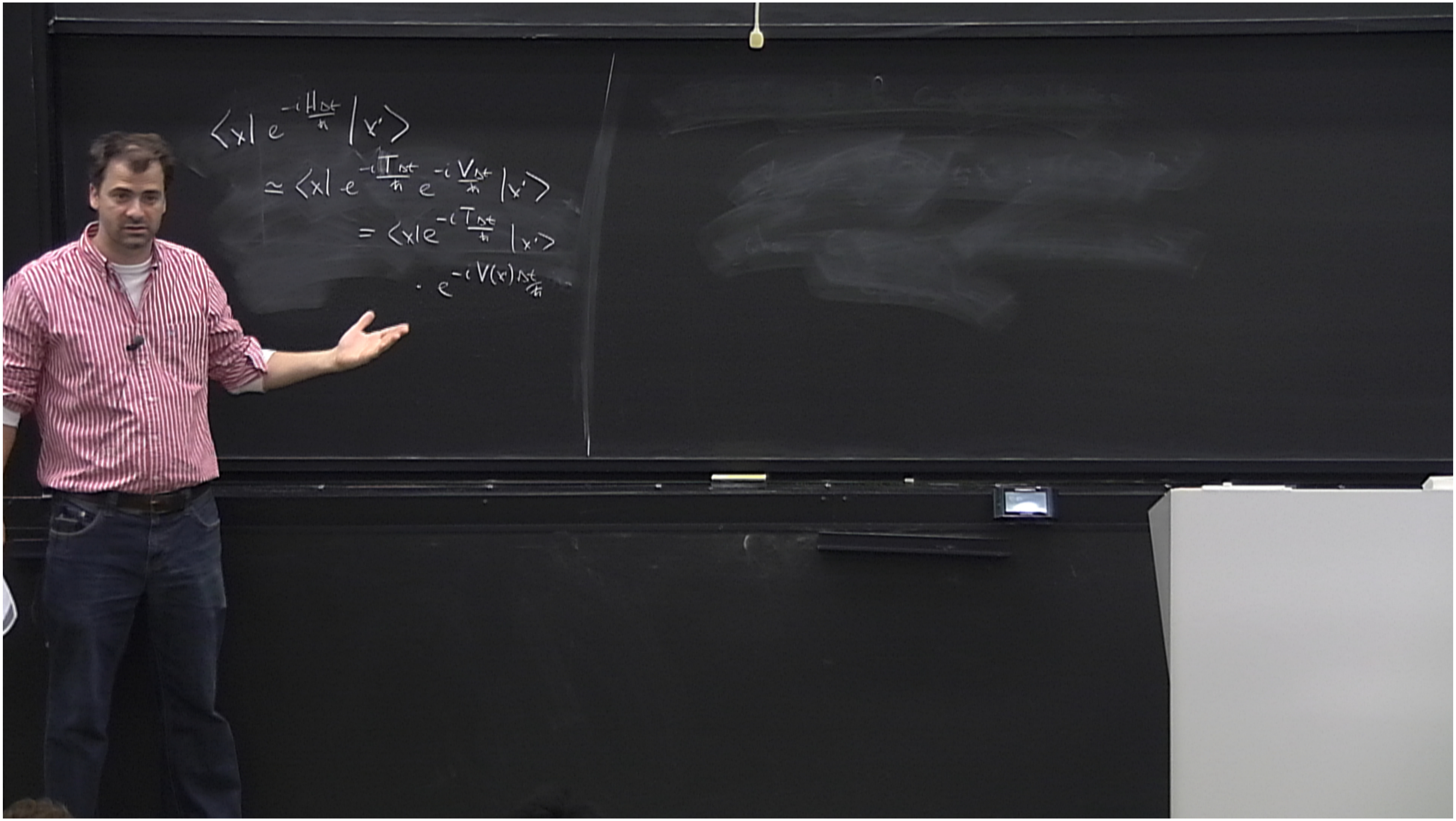
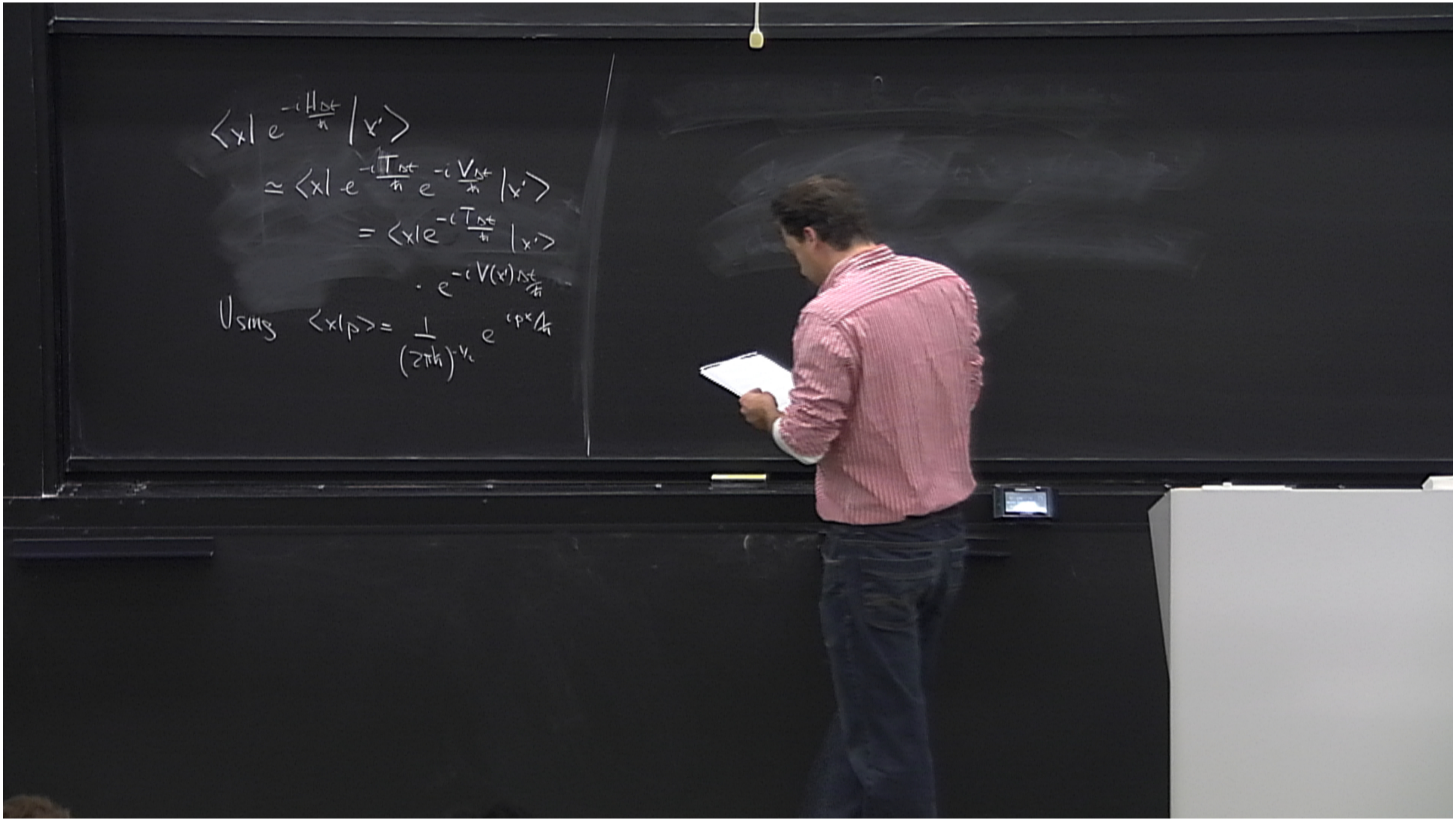
$$U(t_N, t_1) = U(t_N, t_{N-1})\, U(t_{N-1}, t_{N-2}) \cdots U(t_2, t_1)$$

$$G(x,t; x_0, t_0) = \int \cdots \int G(x,t; x_N, t_N) \cdot G(x_N, t_N; x_{N-1}, t_{N-1}) \cdots G(x_1, t_1; x_0, t_0)\, dx_1\, dx_2 \cdots dx_N$$

lim $N \to \infty$, $\Delta t \to 0$

$$G(x, t_i; x', t_{i-1}) = \langle x | \hat{U}(t_i, t_{i-1}) | x' \rangle = \langle x | \exp\left(-i \frac{\hat{H} \Delta t}{\hbar}\right) | x' \rangle$$


$\langle x| e^{-i\frac{H\Delta t}{\hbar}} |x'\rangle$
$\simeq \langle x| e^{-i\frac{T\Delta t}{\hbar}} e^{-i\frac{V\Delta t}{\hbar}} |x'\rangle$
$= \langle x| e^{-i\frac{T\Delta t}{\hbar}} |x'\rangle$
$\cdot\, e^{-iV(x')\frac{\Delta t}{\hbar}}$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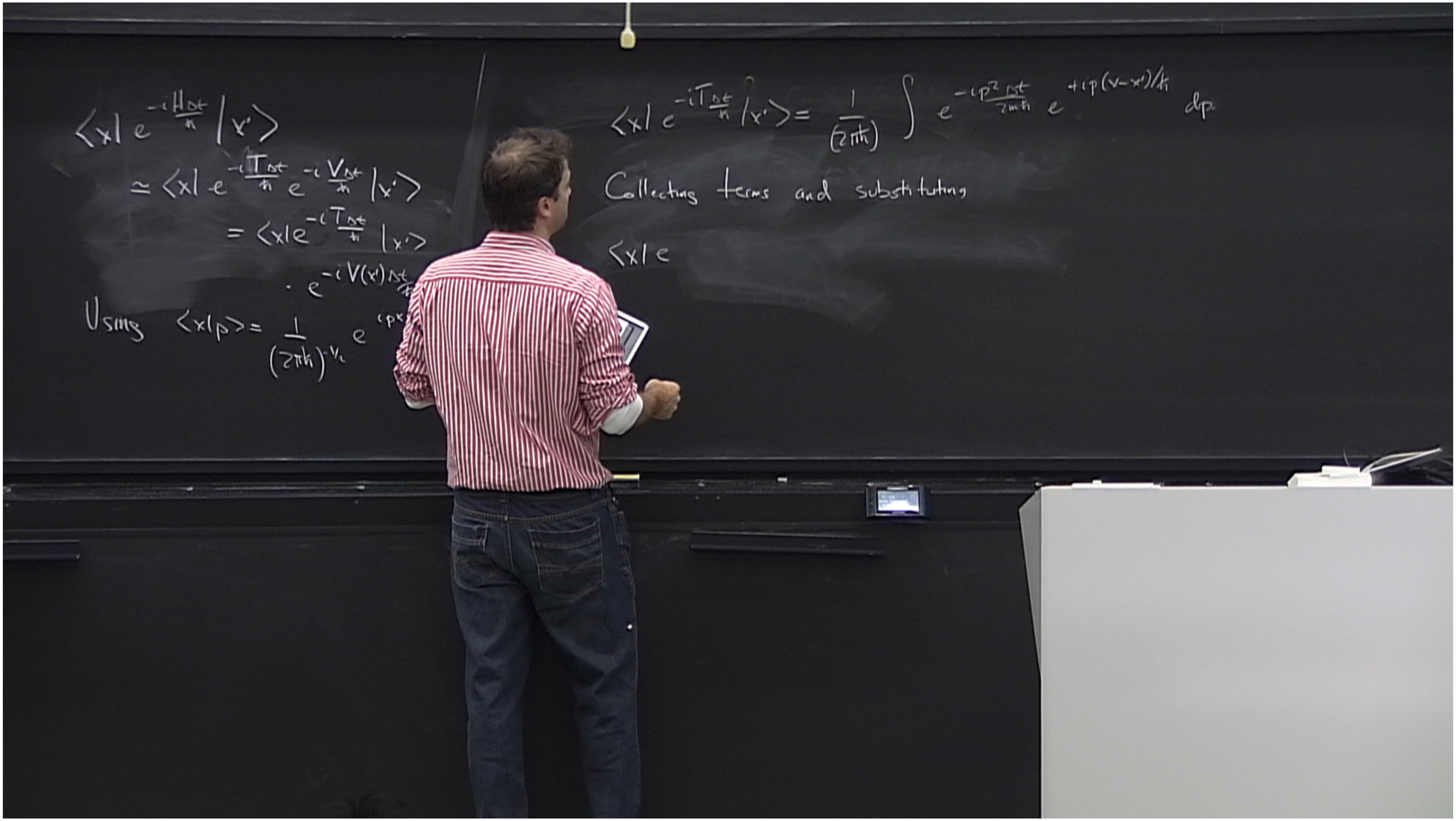
$\cdot e^{-iV(x')\Delta t/\hbar}$
Using $\langle x|p\rangle = \frac{1}{(2\pi\hbar)^{-1/2}} e^{ipx/\hbar}$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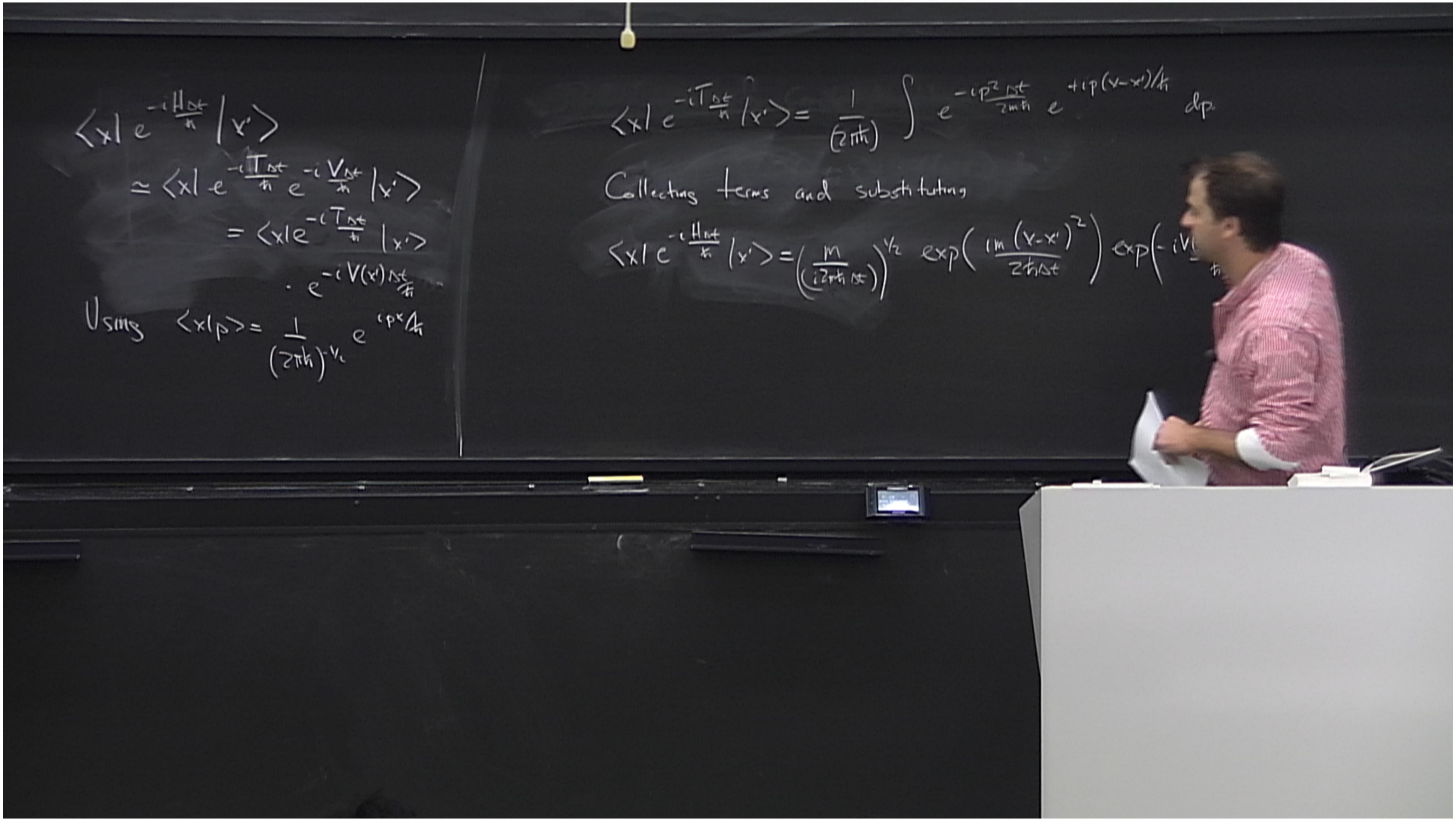
Using $\langle x|p\rangle = \frac{1}{(2\pi\hbar)^{1/2}} e^{ipx}$
$\langle x| e^{-i\frac{T\Delta t}{\hbar}} |x'\rangle = \frac{1}{(2\pi\hbar)} \int e^{-i\frac{p^2\Delta t}{2m\hbar}} e^{+ip(x-x')/\hbar} dp$
Collecting terms and substituting
$\langle x| e$


$\langle x| e^{-iH\Delta t/\hbar} |x'\rangle$
$\simeq \langle x| e^{-iT\Delta t/\hbar} e^{-iV\Delta t/\hbar} |x'\rangle$
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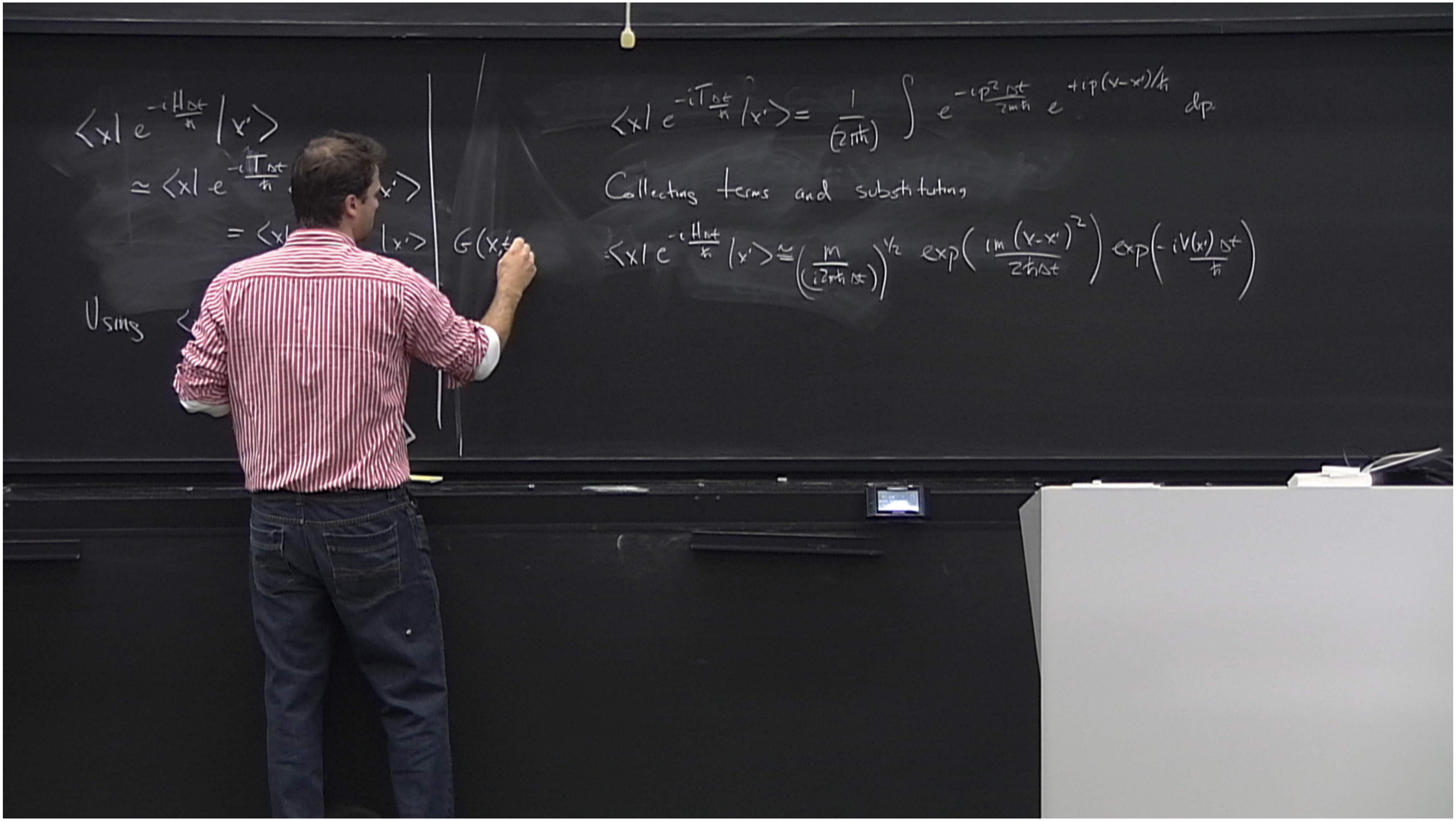
$\langle x| e^{-iT\Delta t/\hbar} |x'\rangle = \frac{1}{(2\pi\hbar)} \int e^{-ip^2\Delta t/2m\hbar} e^{+ip(x-x')/\hbar} dp$
Collecting terms and substituting
$\langle x| e^{-iH\Delta t/\hbar} |x'\rangle = \left(\frac{m}{i2\pi\hbar\Delta t}\right)^{1/2} \exp\left(\frac{im(x-x')^2}{2\hbar\Delta t}\right) \exp\left(-iV\right.$



$\langle x| e^{-iH\Delta t/\hbar} |x'\rangle$
$\simeq \langle x| e^{-iT\Delta t/\hbar}$
Using
$\langle x| e^{-iT\Delta t/\hbar} |x'\rangle = \frac{1}{(2\pi\hbar)} \int e^{-ip^2\Delta t/2m\hbar} e^{+ip(x-x')/\hbar} dp$
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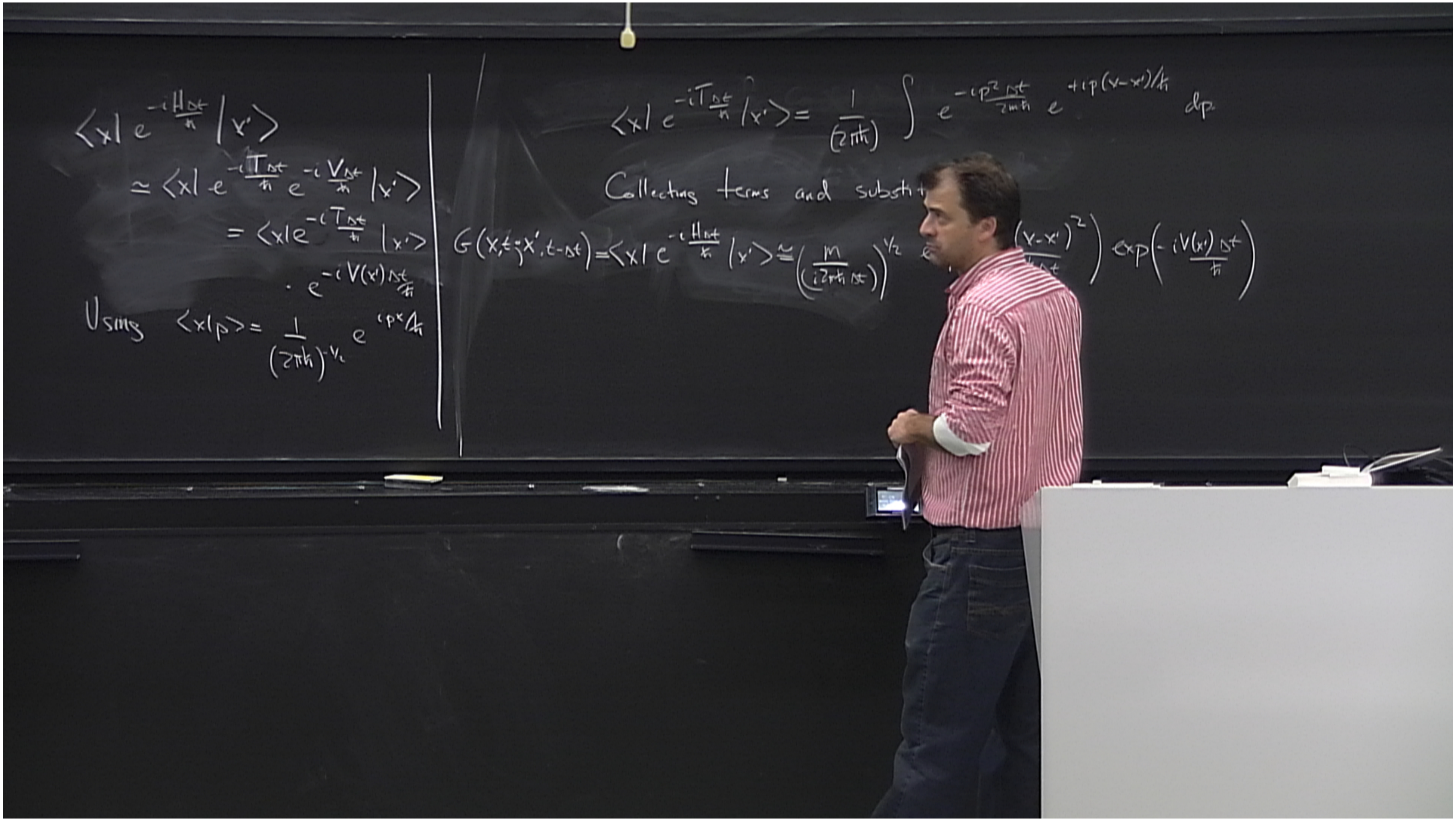
$\langle x| e^{-iH\Delta t/\hbar} |x'\rangle \cong \left(\frac{m}{(i2\pi\hbar\Delta t)}\right)^{1/2} \exp\left(\frac{im(x-x')^2}{2\hbar\Delta t}\right) \exp\left(-\frac{iV(x')\Delta t}{\hbar}\right)$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