

Title: Quantum Theory - Lecture 8

Date: Sep 19, 2012 09:00 AM

URL: <http://www.pirsa.org/12090009>

Abstract:

Today

- Abstract Formulation
of Quantum & Classical
Mechanics
- Bell's Theorem

Recall Hamilton's Mechanics

Hamiltonian fn $H = H(\vec{q}, \vec{p}, t)$

Eq^s of Motion

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}$$

$$\frac{dp}{dt} = - \frac{\partial H}{\partial q}$$

Recall Hamilton's Mechanics

$$\text{Hamiltonian fn } H = H(\vec{q}, \vec{p}, t)$$

Eqs of Motion

$$\vec{q} = (q_1, \dots, q_n)$$

$$\vec{p} = (p_1, \dots, p_n)$$

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}$$

$$\frac{dp_i}{dt} = - \frac{\partial H}{\partial q_i}$$

Recall Hamilton's Mechanics

Hamiltonian fn $H = H(\vec{q}, \vec{p}, t)$

Eq^s of Motion

$$\vec{q} = (q_1, \dots, q_n)$$

$$\vec{p} = (p_1, \dots, p_n)$$

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}$$

$$\frac{dp_i}{dt} = - \frac{\partial H}{\partial q_i}$$

In general we consider

functions $f = f(q, p, t)$

$g = g(q, p, t)$

& we can define a Poisson
bracket $\{ \cdot, \cdot \}$

Recall Hamilton's Mechanics

Hamiltonian fn $H = H(\vec{q}, \vec{p}, t)$

Eqs of M

$$\vec{q} = (q_1, \dots, q_n)$$

$$\vec{p} = (p_1, \dots, p_n)$$

do

In general we consider

functions $f = f(q, p, t)$

$g = g(q, p, t)$

& we can define a Poisson

bracket $\{ \cdot, \cdot \}$ satisfying

① $\{f, g\} = -\{g, f\}$ antisymmetry

② $\{af + bg, h\} = a\{f, h\} + b\{g, h\}$

Hamilton's Mechanics

Hamiltonian $H = H(\vec{q}, \vec{p}, t)$

of Motion

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}$$

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$$

In general we consider

functions $f = f(q, p, t)$

$g = g(q, p, t)$

& we can define a Poisson

bracket $\{ \cdot, \cdot \}$ satisfying

$$\textcircled{1} \{f, g\} = -\{g, f\} \quad \text{antisymmetry}$$

$$\textcircled{2} \{af + bg, h\} = a\{f, h\} + b\{g, h\} \quad \text{where } a, b \in \mathbb{C}$$

linearity

Hamilton's Mechanics

Hamiltonian $H = H(\vec{q}, \vec{p}, t)$

Equations of Motion

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}$$

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$$

In general we consider

$$\text{functions } f = f(q, p, t), \quad h = h(q, p, t) \\ g = g(q, p, t)$$

& we can define a Poisson

bracket $\{ \cdot, \cdot \}$ satisfying

$$\textcircled{1} \{f, g\} = -\{g, f\} \quad \text{antisymmetry}$$

$$\textcircled{2} \{af + bg, h\} = a\{f, h\} + b\{g, h\} \quad \text{where } a, b \in \mathbb{C}$$

linearity

$$\begin{aligned} \textcircled{3} \{f, gh\} &= \{f, g\}h + g\{f, h\} && \text{Leibniz's Law} \\ &= \{f, \{g, h\}\} + \{h, \{f, g\}\} \\ &+ \{ \end{aligned}$$

$$(3) \{f, gh\} = \{f, g\}h + g\{f, h\}$$

Liebniz's Law

$$(4) \{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0$$

} Jacobi identity

$$(3) \{f, gh\} = \{f, g\}h + g\{f, h\}$$

Leibniz's Law

$$(4) \left\{ f, \{g, h\} \right\} + \left\{ h, \{f, g\} \right\} + \left\{ g, \{h, f\} \right\} = 0$$

} Jacobi identity

$$\{q_i, p_j\} = \delta_{ij}$$

$$\{q_i, q_j\} = 0$$

$$\{p_i, p_j\} = 0$$

$$(3) \{f, gh\} = \{f, g\}h + g\{f, h\}$$

Liebniz's Law

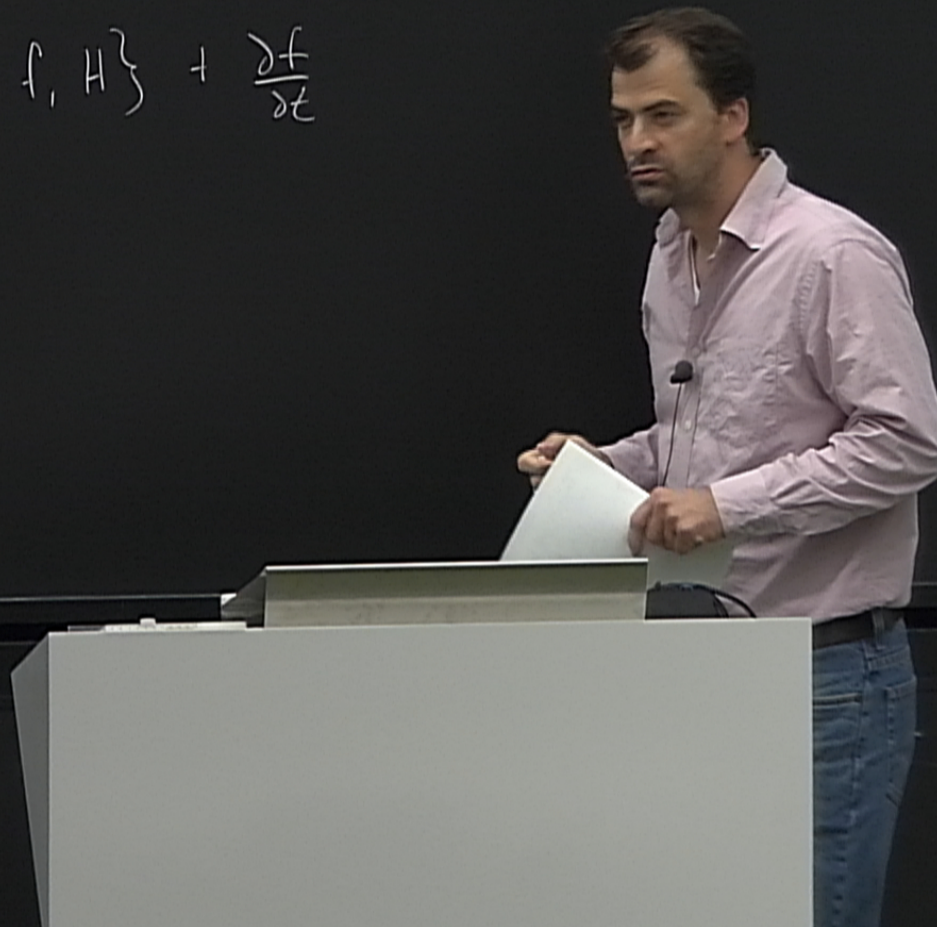
$$(4) \{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0$$

} Jacobi identity

$$(5) \begin{aligned} \{q_i, p_j\} &= \delta_{ij} \\ \{q_i, q_j\} &= 0 \\ \{p_i, p_j\} &= 0 \end{aligned}$$

Law
Postulate
~~Specify~~ a dynamical law

$$\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}$$



Law
Postulate
~~Specify~~ a dynamical law

$$\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}$$

Then a conservation principle:
any function with no explicit
time-dependence, $\frac{\partial f}{\partial t} = 0$,

Postulate
~~Specify~~ a dynamical law

$$\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}$$

Then a conservation principle:
any function with no explicit
time-dependence, $\frac{\partial f}{\partial t} = 0$,

then $\{f, H\} = 0$ "PB commutes"
implies f is a conserved quantity.

Postulate
~~Specify~~ a dynamical law

$$\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}$$

Then a conservation principle:
any function with no explicit
time-dependence, $\frac{\partial f}{\partial t} = 0$,

then $\{f, H\} = 0$ "PB commutes"
implies f is a conserved quantity.

Dirac observed that if
we assume

Postulate
~~Specify~~ a dynamical law

$$\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}$$

Then a conservation principle:
any function with no explicit
time-dependence, $\frac{\partial f}{\partial t} = 0$,

and $\{f, H\} = 0$ "P.B. commutes"
implies f is a conserved quantity.

Dirac observed that if
we assume the algebra
elements $\{q_i\}$ and $\{p_i\}$ are
commut

Dirac observed that if
we assume the algebra
elements $\{q_i\}$ and $\{p_i\}$ are
commutative, $q_i p_j = p_j q_i$

Dirac observed that if
we assume the algebra
elements $\{q_i\}$ and $\{p_i\}$ are
commutative, $q_i p_j = p_j q_i$,
then $\exists$ a representation
as a differentiable oper

$$\{f, g\} = \sum_{\alpha \in \{1, \dots, n\}} \sum_{i=1}^n \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i}$$

Dirac observed that if
 we assume the algebra
 elements $\{q_i\}$ and $\{p_i\}$
 commutative, $q_i q_j = q_j q_i$
 then $\exists$ a relation of the P.B.
 as a differential operator

$$\{f, g\} = \sum_{\alpha \in \{1, \dots, n\}} \sum_{i=1}^n \left[\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right]$$

Dirac observed that if
 we assume the alge-
 bra elements $\{q_i\}$ and $\{p_i\}$
 commutative,
 then $\exists$ a function H the P.B.
 as a different

$$\{f, g\} = \sum_{\alpha \in \{1, \dots, n\}} \sum_{i=1}^n \left[\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right]$$

$$\text{Then } \frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}$$

Dirac observed that if we assume the algebra elements $\{q_i\}$ and $\{p_i\}$ are commutative, $q_i p_i = p_i q_i$, then $\exists$ a representation as a differentiable

$$\{f, g\} = \sum_{\alpha \in \{1, \dots, n\}} \sum_{i=1}^n \left[\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right]$$

$$\text{Then } \frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}$$

$$\begin{aligned} \text{In 1D} &= \frac{\partial f}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial H}{\partial q} + \frac{\partial f}{\partial t} \\ \text{for 1 particle} &= \frac{\partial f}{\partial q} \frac{\partial q}{\partial t} + \frac{\partial f}{\partial p} \frac{\partial p}{\partial t} + \frac{\partial f}{\partial t} \end{aligned}$$

$$\{f, g\} = \sum_{\alpha \in \{1, \dots, n\}} \sum_{i=1}^n \left[\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right]$$

$$\text{Then } \frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}$$

$$\begin{aligned} \text{In 1D} \\ \text{for 1 particle} \end{aligned} \quad = \quad \frac{\partial f}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial H}{\partial q}$$

$$= \frac{\partial f}{\partial q} \frac{\partial g}{\partial t} + \frac{\partial f}{\partial p} \frac{\partial g}{\partial t}$$

If we let $f = \rho$, a probability density on phase space (spanned by $(q_1, \dots, q_n)$ and $(p_1, \dots, p_n)$)

$$\{f, g\} = \sum_{\alpha \in \{1, \dots, n\}} \sum_{i=1}^n \left[\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right]$$

$$\text{Then } \frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}$$

$$\begin{aligned} \text{In 1D} &= \frac{\partial f}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial H}{\partial q} + \frac{\partial f}{\partial t} \\ \text{for 1 particle} &= \frac{\partial f}{\partial q} \frac{\partial g}{\partial t} + \frac{\partial f}{\partial p} \frac{\partial p}{\partial t} + \frac{\partial f}{\partial t} \end{aligned}$$

If we let $f = \rho$, a probability density on phase space (spanned by $(q_1, \dots, q_n)$ and $(p_1, \dots, p_n)$)

then you can prove that

$$\frac{d\rho}{dt} = 0$$

$$\text{hence, } \frac{\partial \rho}{\partial t} = \{H, \rho\}$$

$$\{f, g\} = \sum_{\alpha \in \{1, \dots, n\}} \sum_{i=1}^n \left[\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right]$$

$$\text{Then } \frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}$$

$$\begin{aligned} \frac{1}{N} \frac{d}{dt} \ln D &= \frac{\partial f}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial H}{\partial q} + \frac{\partial f}{\partial t} \\ &= \frac{\partial f}{\partial q} \frac{\partial g}{\partial t} + \frac{\partial f}{\partial p} \frac{\partial p}{\partial t} + \frac{\partial f}{\partial t} \end{aligned}$$

If we let $f = \rho$, a probability density on phase space (spanned by $(q_1, \dots, q_n)$ and $(p_1, \dots, p_n)$)

then you can prove that

$$\frac{d\rho}{dt} = 0$$

$$\text{hence, } \frac{\partial \rho}{\partial t} = \{H, \rho\}$$

$$\{f, g\} = \sum_{\alpha \in \{1, \dots, n\}} \sum_{i=1}^n \left[\frac{\partial f}{\partial q_i^\alpha} \frac{\partial g}{\partial p_i^\alpha} - \frac{\partial f}{\partial p_i^\alpha} \frac{\partial g}{\partial q_i^\alpha} \right]$$

$$\text{Then } \frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}$$

$$\begin{aligned} \text{In 1D} \\ \text{for 1 particle} \end{aligned} \quad = \quad \frac{\partial f}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial H}{\partial q}$$

If we let $f = \rho$, a probability density on phase space (spanned by $(q_1, \dots, q_n)$ and $(p_1, \dots, p_n)$) then you can prove that

$$\frac{d\rho}{dt} = 0$$

$$\text{hence, } \frac{\partial \rho}{\partial t} = \{H, \rho\} \quad \leftarrow \text{Liouville equation}$$

$$\{f, g\} = \sum_{\alpha \in \{1, 2\}} \sum_{i=1}^n \left[\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right]$$

Then $\frac{d}{dt} \{f, H\} + \frac{\partial f}{\partial t}$

$$\begin{aligned} &= \frac{\partial f}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial H}{\partial q} + \frac{\partial f}{\partial t} \\ &= \frac{\partial f}{\partial q} \frac{\partial g}{\partial t} + \frac{\partial f}{\partial p} \frac{\partial g}{\partial t} + \frac{\partial f}{\partial t} \end{aligned}$$

If we let $f = \rho$, a probability density on phase space (spanned by $(q_1, \dots, q_n)$ and $(p_1, \dots, p_n)$) then you can prove that

$$\begin{aligned} \rho &= \rho(q, p) \\ \rho &\geq 0 \\ \int \rho \, dq \, dp &= 1 \end{aligned}$$

$$\frac{d\rho}{dt} = 0$$

hence, $\frac{\partial \rho}{\partial t} = \{H, \rho\}$

← Liouville equation

Motivation for Liouville eqⁿ

is it gives a description of
the realistic situation, which
is that one has imprecise
knowledge about $\bar{q}$ & $\bar{p}$ (or
some uncertainty) at any given time
due finite precision of measurements.

Motivation for Liouville eqⁿ

is it gives a description of the realistic situation, which is that one has imprecise knowledge about $\bar{q}$ & $\bar{p}$ (or some uncertainty) at any given time due finite precision of measurement.

Dirac realized

Dirac realized that simply dropping
the assumption that the algebra
elements $\vec{q}$ & $\vec{p}$ commute,

Dirac realized that simply dropping
the assumption that the algebra
elements $\vec{q}$ & $\vec{p}$ commute,
then the differential form
is no longer valid,

Dirac realized that simply dropping
the assumption that the algebra
elements $\vec{q}$ & $\vec{p}$ commute,
then the differential form
is no longer valid, & there
is a unique representation of

the Poisson bracket as
 $\{f, g\}$ [

Dirac realized that simply dropping the assumption that the algebra elements $\vec{q}$ & $\vec{p}$ commute, then the differential form is no longer valid, & there is a unique representation of

the Poisson bracket as

$$\{f, g\} = C (fg - gf)$$

for some scalar $C \in \mathbb{C}$

$$\{f, g\} = \sum_{\alpha \in \{1, \dots, n\}} \sum_{i=1}^n \left[\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right]$$

$$\text{Then } \frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}$$

$$\begin{aligned} \text{In 1D} \\ \text{for 1 particle} \end{aligned} \quad = \quad \frac{\partial f}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial H}{\partial q} + \frac{\partial f}{\partial t}$$

$$= \frac{\partial f}{\partial q} \frac{\partial g}{\partial t} + \frac{\partial f}{\partial p} \frac{\partial p}{\partial t} + \frac{\partial f}{\partial t}$$

If we let $f = \rho$, a probability density on phase space (spanned by $(q_1, \dots, q_n)$ and $(p_1, \dots, p_n)$) then you can prove that

$$\begin{aligned} \rho &= \rho(q, p, t) \\ \rho &\geq 0 \\ \int \rho dq dp &= 1 \end{aligned}$$

$$\frac{d\rho}{dt} = 0$$

hence, $\frac{\partial \rho}{\partial t} = \{H, \rho\}$ ← Liouville equation

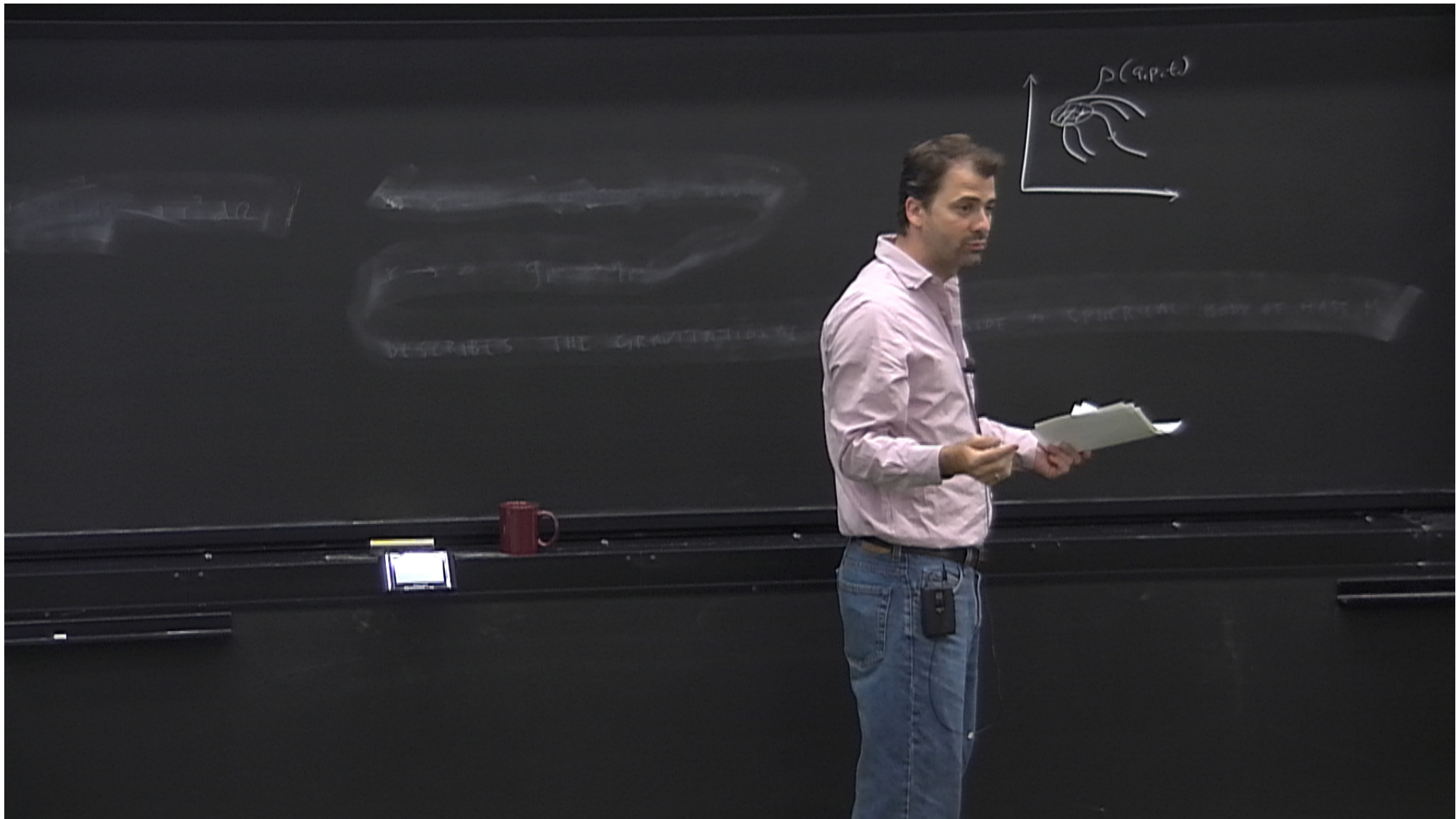
that simply dropping
that the algebra
& $\vec{p}$ commute,
ferential form
valid, & there
representation of

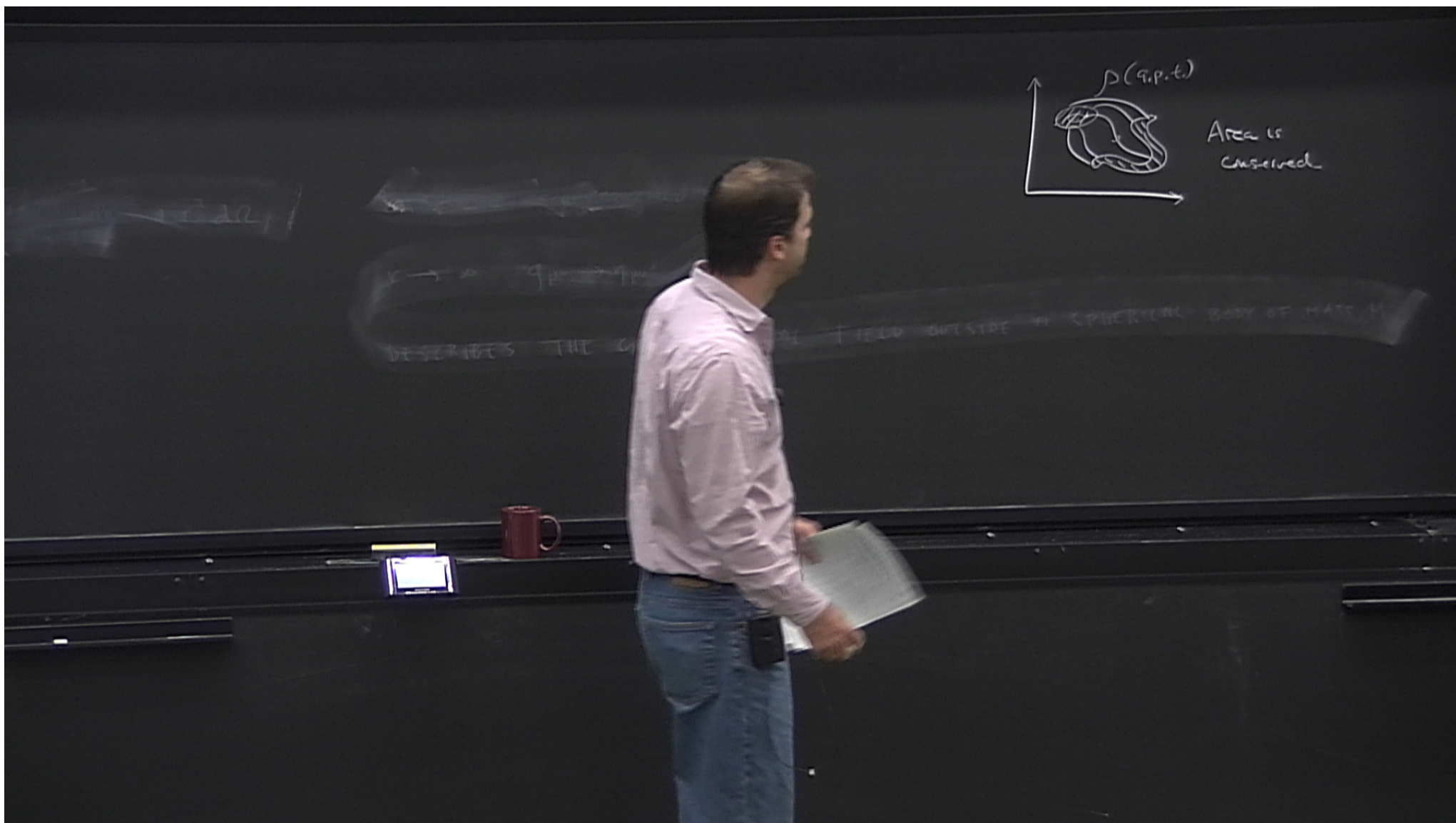
the Poisson bracket as

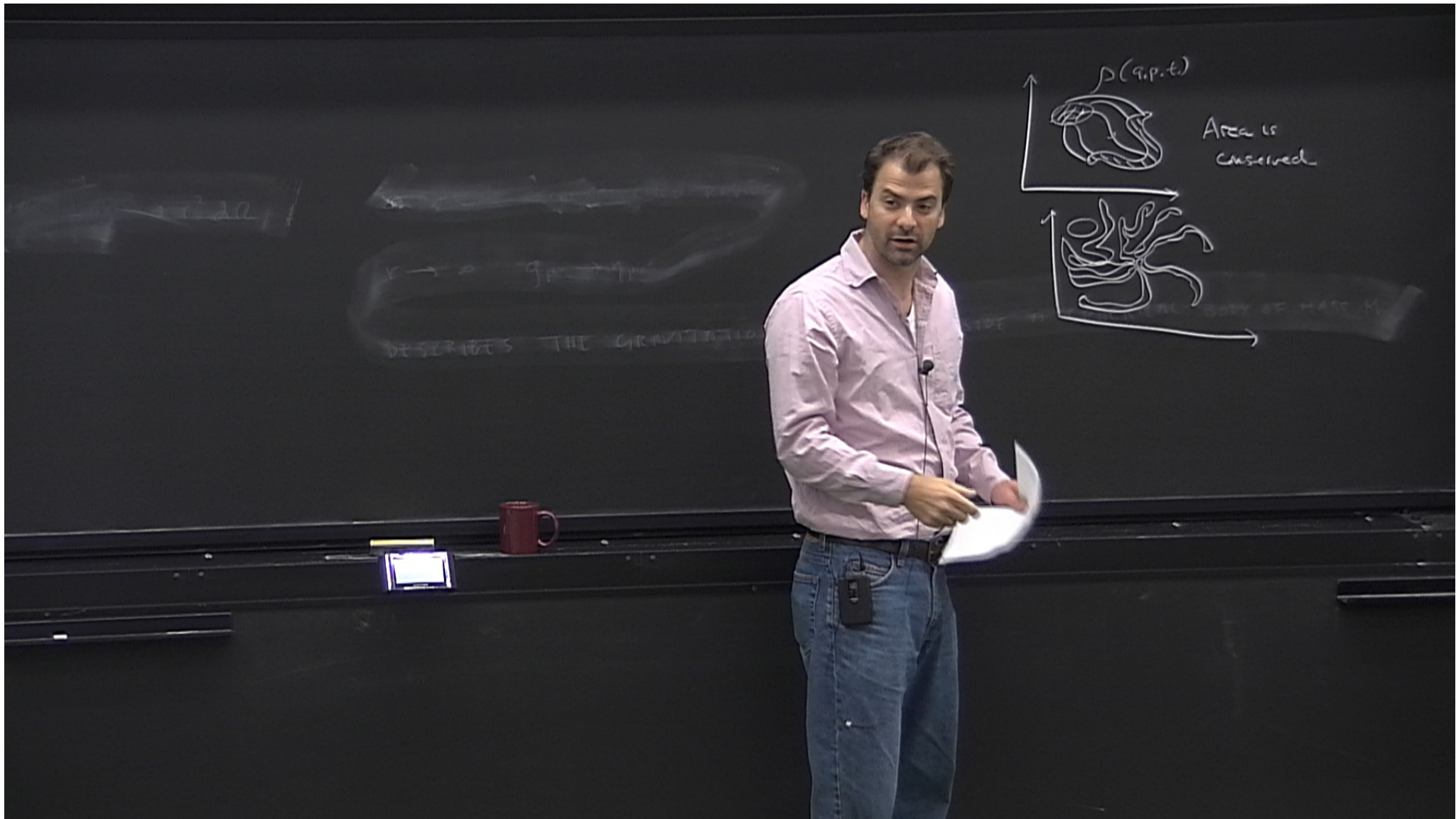
$$\{f, g\} = C (fg - gf) \equiv C [f, g]$$

for some scalar $C \in \mathbb{C}$, $C = 1/i\hbar$

defines commutator







$$\{f, g\} = \sum_{\alpha} \sum_{i=1}^n \left[\frac{\partial f}{\partial q_i^{\alpha}} \frac{\partial g}{\partial p_i^{\alpha}} - \frac{\partial f}{\partial p_i^{\alpha}} \frac{\partial g}{\partial q_i^{\alpha}} \right]$$

$$\text{Then } \frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}$$

$$\begin{aligned} \frac{1}{N} \frac{d}{dt} \int_{\Gamma} f \, d\Gamma &= \frac{\partial f}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial H}{\partial q} + \frac{\partial f}{\partial t} \\ &= \frac{\partial f}{\partial q} \frac{\partial g}{\partial t} + \frac{\partial f}{\partial p} \frac{\partial p}{\partial t} + \frac{\partial f}{\partial t} \end{aligned}$$

$q_i = q_i, p_i$
the P.B.

$$\begin{aligned} \rho &= \rho(q, p) \\ \rho &\geq 0 \\ \int_{\Gamma} \rho \, d\Gamma &= 1 \end{aligned}$$

If we let ρ be the probability density on phase space (spanned by $(q_1, \dots, q_n)$ and $(p_1, \dots, p_n)$) then you can prove that

$$\frac{d\rho}{dt} = 0$$

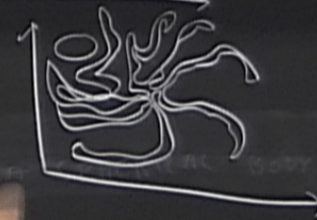
$$\text{hence, } \frac{\partial \rho}{\partial t} = \{H, \rho\}$$

• linear

← Liouville equation



Area is conserved



$$\{f, g\} = \sum_{\alpha \in \{1, \dots, n\}} \sum_{i=1}^n \left[\frac{\partial f}{\partial q_i^\alpha} \frac{\partial g}{\partial p_i^\alpha} - \frac{\partial f}{\partial p_i^\alpha} \frac{\partial g}{\partial q_i^\alpha} \right]$$

$$\text{Then } \frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}$$

$$\begin{aligned} \frac{1}{N} \frac{d}{dt} \int_{\text{for 1 particle}} &= \frac{\partial f}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial H}{\partial q} + \frac{\partial f}{\partial t} \\ &= \frac{\partial f}{\partial q} \frac{\partial g}{\partial t} + \frac{\partial f}{\partial p} \frac{\partial p}{\partial t} + \frac{\partial f}{\partial t} \end{aligned}$$

$q_i = q_i, p_i$
the P.B.

$p = p(q, p)$
 $p \geq 0$
 $\int_{\mathcal{M}} p = 1$

density on phase space (spanned by $(q_1, \dots, q_n)$ and $(p_1, \dots, p_n)$)

then you can prove that

$$\frac{dp}{dt} = 0$$

$$\text{hence, } \frac{\partial f}{\partial t} = \{H, p\}$$

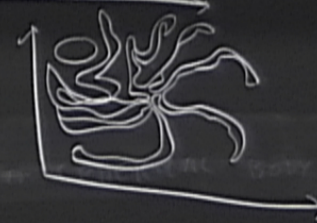
Liouville equation

- linear
- continuous
- deterministic

$\Rightarrow$ exactly analogous to von Neumann eqs

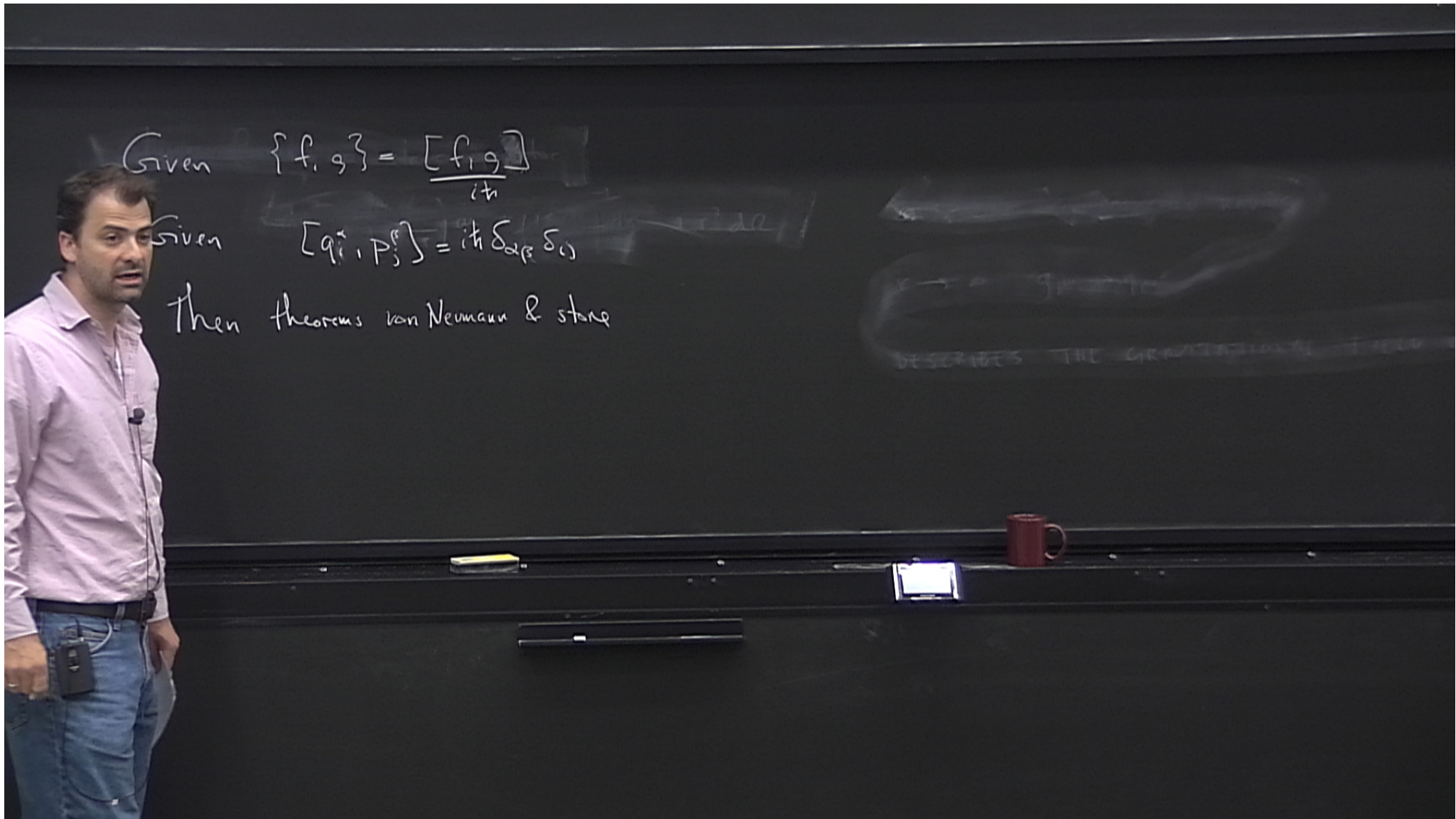


Area is conserved



Given $\{f, g\} = \frac{[f, g]}{i\hbar}$

$\{q_i^a, p_j^b\} = i\hbar \delta_{ij} \delta_{ab}$



Given $\{f, g\} = \frac{[f, g]}{i\hbar}$

Given $[q_i, p_j] = i\hbar \delta_{ij}$

Then theorems von Neumann & Stone
 imply that q & p admit an
 irrep as linear operators on $\mathcal{H} = L^2(\mathbb{R})$

Given $\{f, g\} = \frac{[f, g]}{i\hbar}$

Given $[q_i, p_j] = i\hbar \delta_{ij}$

Then theorems von Neumann & Stone
imply that q & p admit an
irrep as linear operators on $\mathcal{H} = L^2(\mathbb{R})$

$e^{iq} e^{ip}$

Given $\{f, g\} = \frac{[f, g]}{i\hbar}$

Given $[q_i, p_j] = i\hbar \delta_{ij}$

$e^{i q} e^{i p}$

Then theorems von Neumann & Stone
imply that q & p admit an
irrep as linear operators on $\mathcal{H} = L^2(\mathbb{R})$

Dirac realized that simply dropping
the assumption that the algebra
elements $\vec{q}$ & $\vec{p}$ commute,
then the differential form
is no longer valid, & there
is a unique representation of

the Poisson bracket as

$$\boxed{\{f, g\} = C (fg - gf)} \equiv C [f, g]$$

for some scalar $C \in \mathbb{C}$, $C = 1/i\hbar$

defines commutator

Given $\{f, g\} = \frac{[f, g]}{i\hbar}$

Given $[q_i, p_j] = i\hbar \delta_{ij}$

Then theorems von Neumann & Stone
imply that q & p admit an
irrep as linear operators on

For some $f = \hat{p}$

$$\frac{[f, g]}{i\hbar}$$

$$= i\hbar \delta_{\alpha\beta} \delta_{ij}$$

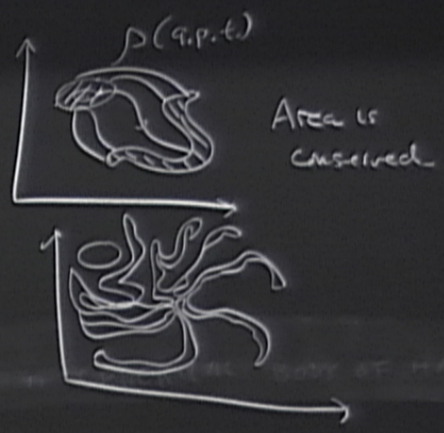
Neumann & Stone

admit an

linear operators on $\mathcal{H} = L^2(\mathbb{R})$

For some $f = \hat{f}$

$$\frac{d\hat{f}}{dt} = \frac{[\hat{f}, \hat{H}]}{i\hbar} + \frac{\partial \hat{f}}{\partial t}$$



$$\frac{[f, g]}{i\hbar}$$

$$= i\hbar \delta_{\alpha\beta} \delta_{ij}$$

Neumann & Stone

p admit an

linear operators on $H = L^2(\mathbb{R})$

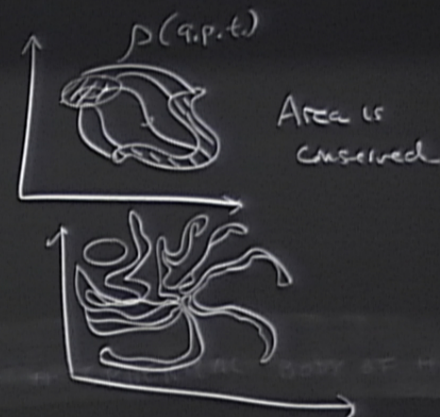
For some $f = \hat{f}$

$$\frac{d\hat{f}}{dt} = \frac{[\hat{f}, \hat{H}]}{i\hbar} + \frac{\partial \hat{f}}{\partial t}$$

⇒ Heisenberg eqⁿ of motion

Switch to Schrödinger picture

$$\frac{d\hat{p}}{dt} =$$



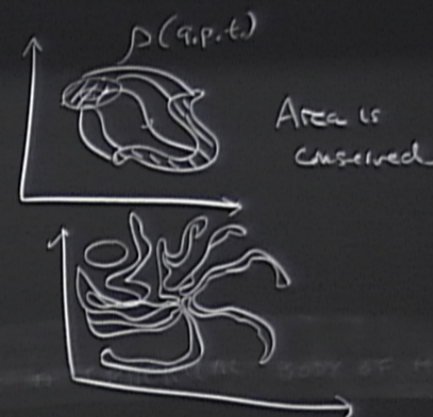
For some $f = \hat{f}$

$$\frac{d\hat{f}}{dt} = \frac{[\hat{f}, \hat{H}]}{i\hbar} + \frac{\partial \hat{f}}{\partial t}$$

$\Rightarrow$ Heisenberg eqⁿ of motion

Switch to Schrodinger picture

$$\frac{d\hat{p}}{dt} = \frac{[\hat{H}, \hat{p}]}{i\hbar}$$



For some $f = \hat{f}$

$$\frac{d\hat{f}}{dt} = \frac{[\hat{f}, \hat{H}]}{i\hbar} + \frac{\partial \hat{f}}{\partial t}$$

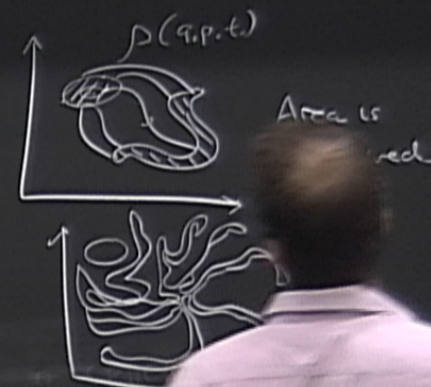
→ Heisenberg eqⁿ of motion

Switch to Schrödinger picture

②

van
Neumann
eqⁿ

$$\frac{d\hat{p}}{dt} = \frac{[\hat{H}, \hat{p}]}{i\hbar} + 0$$



at of
ebra
 $\{p_i\}$ are
 $p_i = p_{q_i}, q_i, q_i$
presentation of the P.B.
operator

$$\{f, g\} = \sum_{\alpha \in \{1, \dots, n\}} \sum_{i=1}^n \left[\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right]$$

$$\text{Then } \frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}$$

$$\begin{aligned} \text{In 1D} \\ \text{for 1 particle} \end{aligned} \quad \begin{aligned} &= \frac{\partial f}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial H}{\partial q} + \frac{\partial f}{\partial t} \\ &= \frac{\partial f}{\partial q} \frac{\partial g}{\partial t} + \frac{\partial f}{\partial p} \frac{\partial p}{\partial t} + \frac{\partial f}{\partial t} \end{aligned}$$

$$\begin{aligned} p &= p(q, t) \\ p &\geq 0 \\ \sum_{i=1}^n p_i &= 1 \\ \text{it} \end{aligned}$$

density on phase space (spanned by $(q_1, \dots, q_n)$ and $(p_1, \dots, p_n)$)

then you can prove that

$$\frac{dp}{dt} = 0$$

$$\text{hence, } \frac{\partial \rho}{\partial t} = \{H, \rho\}$$

← Liouville equation

- linear
- continuous
- deterministic

→ exactly analogous to von Neumann eqs

$$\frac{[f, g]}{i\hbar}$$

$$= i\hbar \delta_{q,p} \delta_{i,j}$$

Neumann & Stone
& p admit an
linear operators on $H = L^2(\mathbb{R})$

For some $f = \hat{f}$

$$\frac{d\hat{f}}{dt} = \frac{[\hat{f}, \hat{H}]}{i\hbar} + \frac{\partial \hat{f}}{\partial t}$$

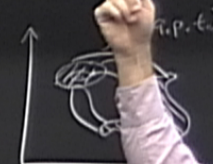
⇒ Heisenberg eqⁿ of motion

Switch to Schrödinger picture

②

von
Neumann
eqⁿ

$$\frac{d\hat{\rho}}{dt} = \frac{[\hat{H}, \hat{\rho}]}{i\hbar} + 0$$



Area is conserved



at of
ebra
 $\{p_i\}$ are
 $p_i = p_{q_i}, q_i = q_i$
presentation of the P.B.
operator

$$\{f, g\} = \sum_{\alpha \in \{1, \dots, n\}} \sum_{i=1}^n \left[\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right]$$

$$\text{Then } \frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}$$

$$\begin{aligned} \frac{1}{\text{for 1 particle}} &= \frac{\partial f}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial H}{\partial q} + \frac{\partial f}{\partial t} \\ &= \frac{\partial f}{\partial q} \frac{\partial g}{\partial t} + \frac{\partial f}{\partial p} \frac{\partial p}{\partial t} + \frac{\partial f}{\partial t} \end{aligned}$$

$$\begin{aligned} p &= p(q, p) \\ p &\geq 0 \\ \sum_{i=1}^n p_i &= 1 \end{aligned}$$

density on phase space (spanned by $(q_1, \dots, q_n)$ and $(p_1, \dots, p_n)$)

then you can prove that

$$\frac{dp}{dt} = 0$$

$$\text{hence, } \frac{\partial p}{\partial t} = \{H, p\}$$

← Liouville equation

- linear
- continuous
- deterministic

→ exactly analogous to von Neumann eqs

$$\frac{[f, g]}{i\hbar}$$

$$= i\hbar \delta_{q,p} \delta_{q,p}$$

mann & stone

first an

tors on

$$H = L^2(\mathbb{R})$$

For some $f = \hat{f}$

$$\frac{d\hat{f}}{dt} = \frac{[\hat{f}, \hat{H}]}{i\hbar} + \frac{\partial \hat{f}}{\partial t}$$

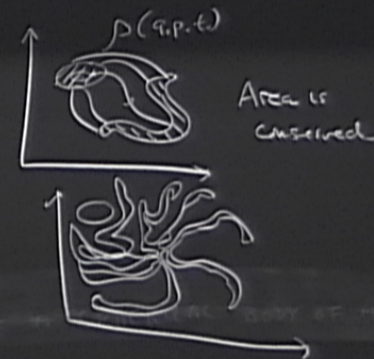
⇒ Heisenberg eqⁿ of motion

Switch to Schrodinger picture

②

von Neumann eqⁿ

$$\frac{d\hat{\rho}}{dt} = \frac{[\hat{H}, \hat{\rho}]}{i\hbar} + 0$$



operator

$$\frac{\partial q}{\partial t} = \frac{\partial H}{\partial p}, \quad \frac{\partial p}{\partial t} = -\frac{\partial H}{\partial q}$$

- linear
- continuous
- deterministic

$$\frac{\partial p}{\partial t} = \{H, p\}$$

→ exactly analogous to von Neumann eqs

$$\frac{[f, g]}{i\hbar}$$

$$= i\hbar \delta_{q,p} \delta_{i,j}$$

von Neumann & Stone

& p admit an

linear operators on

$$H = L^2(\mathbb{R})$$

For some $f = \hat{f}$

$$\frac{d\hat{f}}{dt} = \frac{[\hat{f}, \hat{H}]}{i\hbar} + \frac{\partial \hat{f}}{\partial t}$$

→ Heisenberg motion

Switch to Schrödinger picture

②
von Neumann eqs

Classical state update rule, after a measurement,

operator

$$\frac{\partial q}{\partial t} = \frac{\partial p}{\partial p} \frac{\partial p}{\partial t} + \frac{\partial f}{\partial t}$$

- linear
- continuous
- deterministic

$$\frac{\partial p}{\partial t} = \{H, p\}$$

→ exactly analogous to von Neumann eqs

$$\frac{[f, g]}{i\hbar}$$

$$= i\hbar \delta_{q,p} \delta_{i,j}$$

von Neumann & Stone

& p admit an

linear operators on

$$H = L^2(\mathbb{R})$$

For some $f = \hat{f}$

$$\frac{d\hat{f}}{dt} = \frac{[\hat{f}, \hat{H}]}{i\hbar} + \frac{\partial \hat{f}}{\partial t}$$

⇒ Heisenberg eqⁿ of motion

Switch to Schrödinger picture

②

von
Neumann
eqⁿ

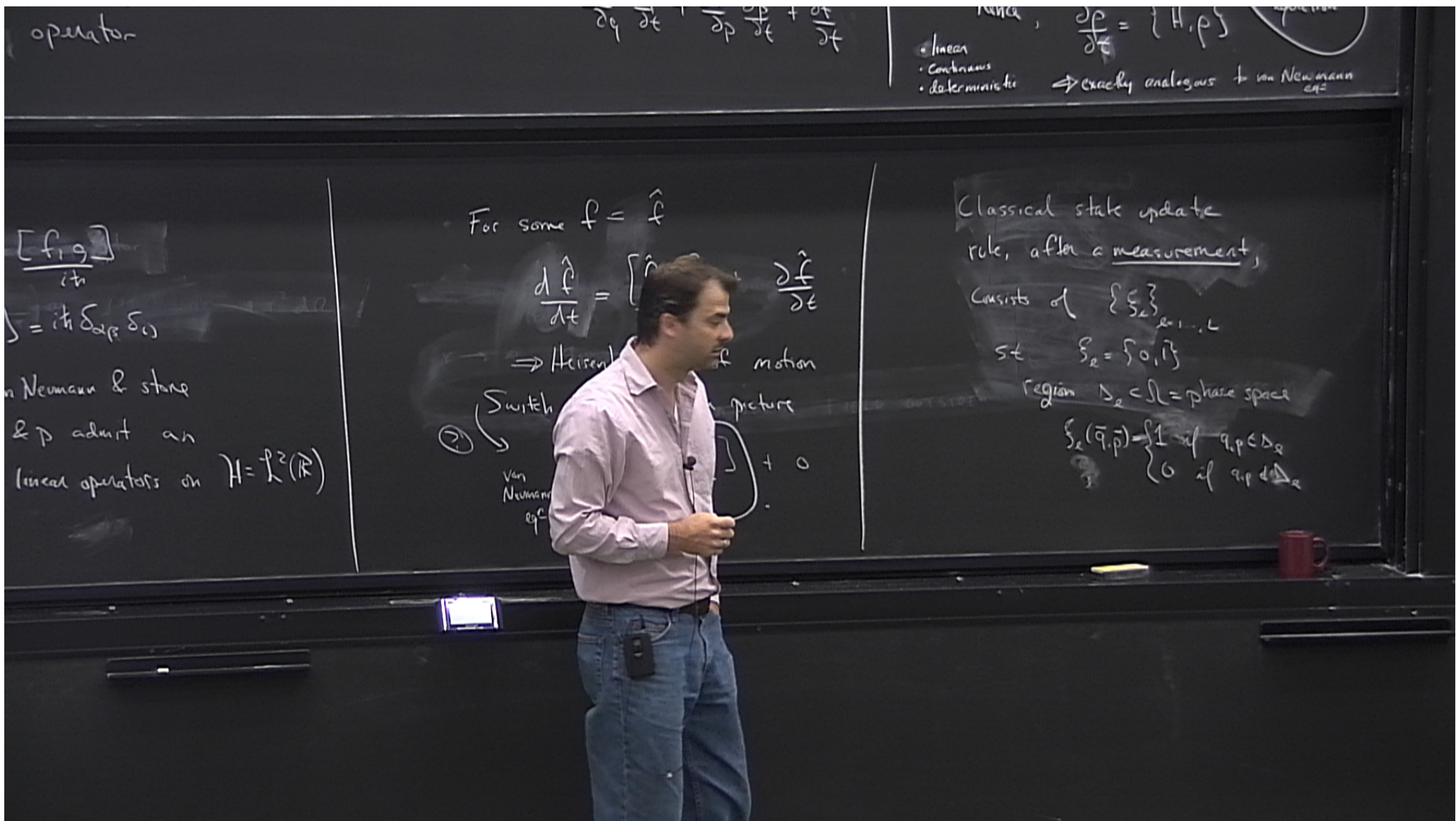
$$\frac{d\hat{f}}{dt} = \frac{[\hat{H}, \hat{f}]}{i\hbar} + \frac{\partial \hat{f}}{\partial t}$$

Classical state update

rule, after a measurement,

consists of $\{\xi_k\}_{k=1, \dots, L}$

$$\xi_k = \{0, 1\}$$



operator

$$\frac{\partial q}{\partial t} = \frac{\partial H}{\partial p}, \quad \frac{\partial p}{\partial t} = -\frac{\partial H}{\partial q}$$

- linear
- continuous
- deterministic

$$\frac{\partial f}{\partial t} = \{H, f\}$$

→ exactly analogous to von Neumann eq

$$\frac{[f, g]}{i\hbar}$$

$$= i\hbar \delta_{q,p} \delta_{q,p}$$

von Neumann & Stone

& p admit an

linear operators on $H = L^2(\mathbb{R})$

For some $f = \hat{f}$

$$\frac{d\hat{f}}{dt} = \left[\hat{H}, \hat{f} \right] + \frac{\partial \hat{f}}{\partial t}$$

→ Heisenberg picture

Switch

von Neumann eq

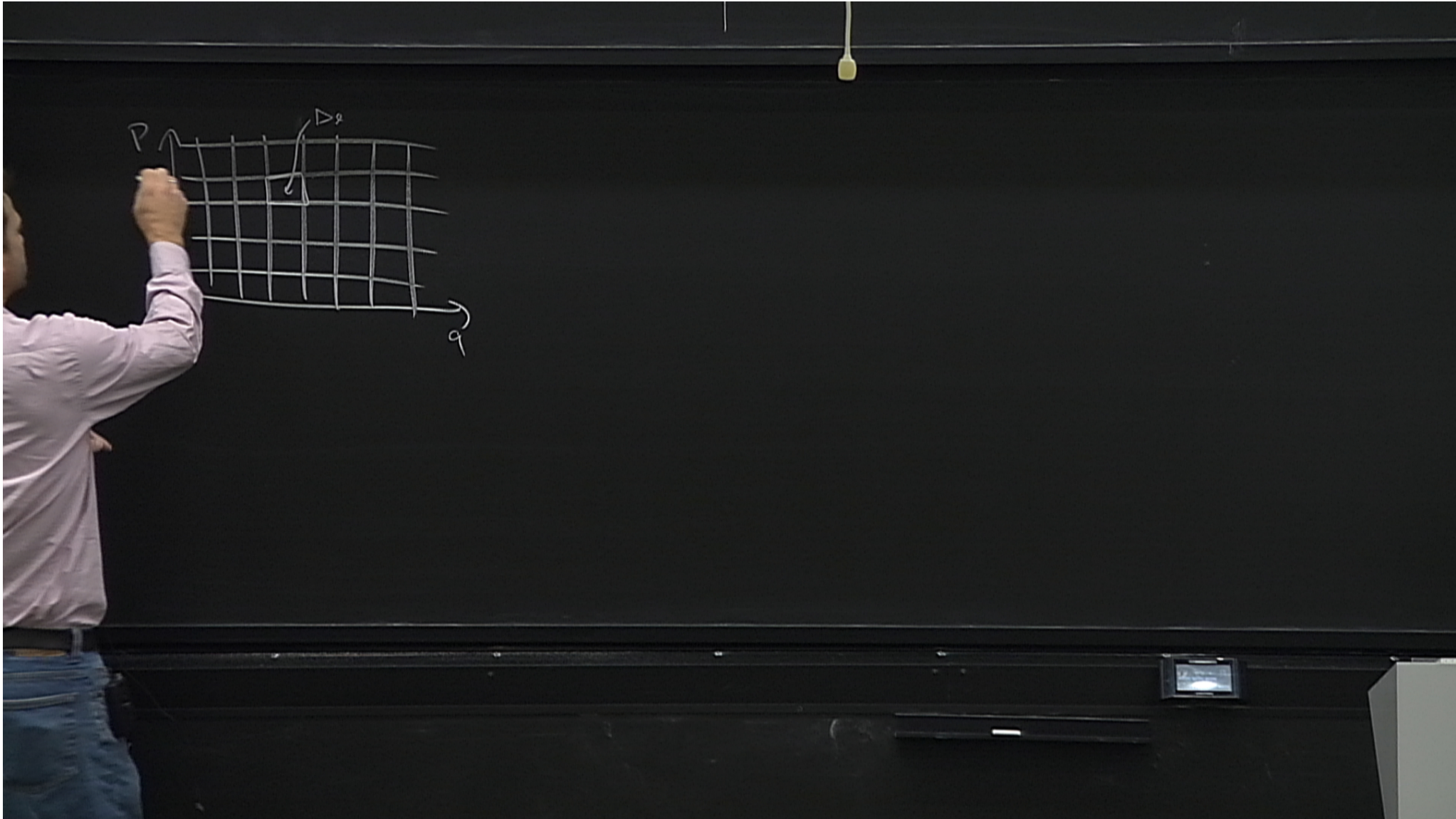
Classical state update rule, after a measurement,

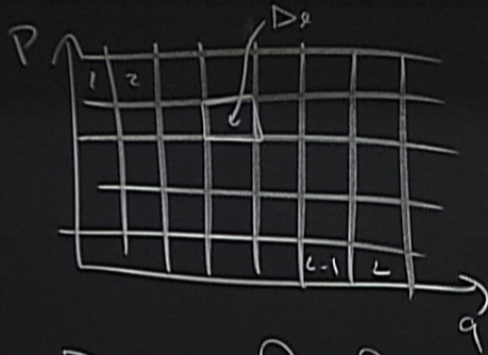
consists of $\{\xi_a\}_{a=1, \dots, L}$

$$\xi_a = \{0, 1\}$$

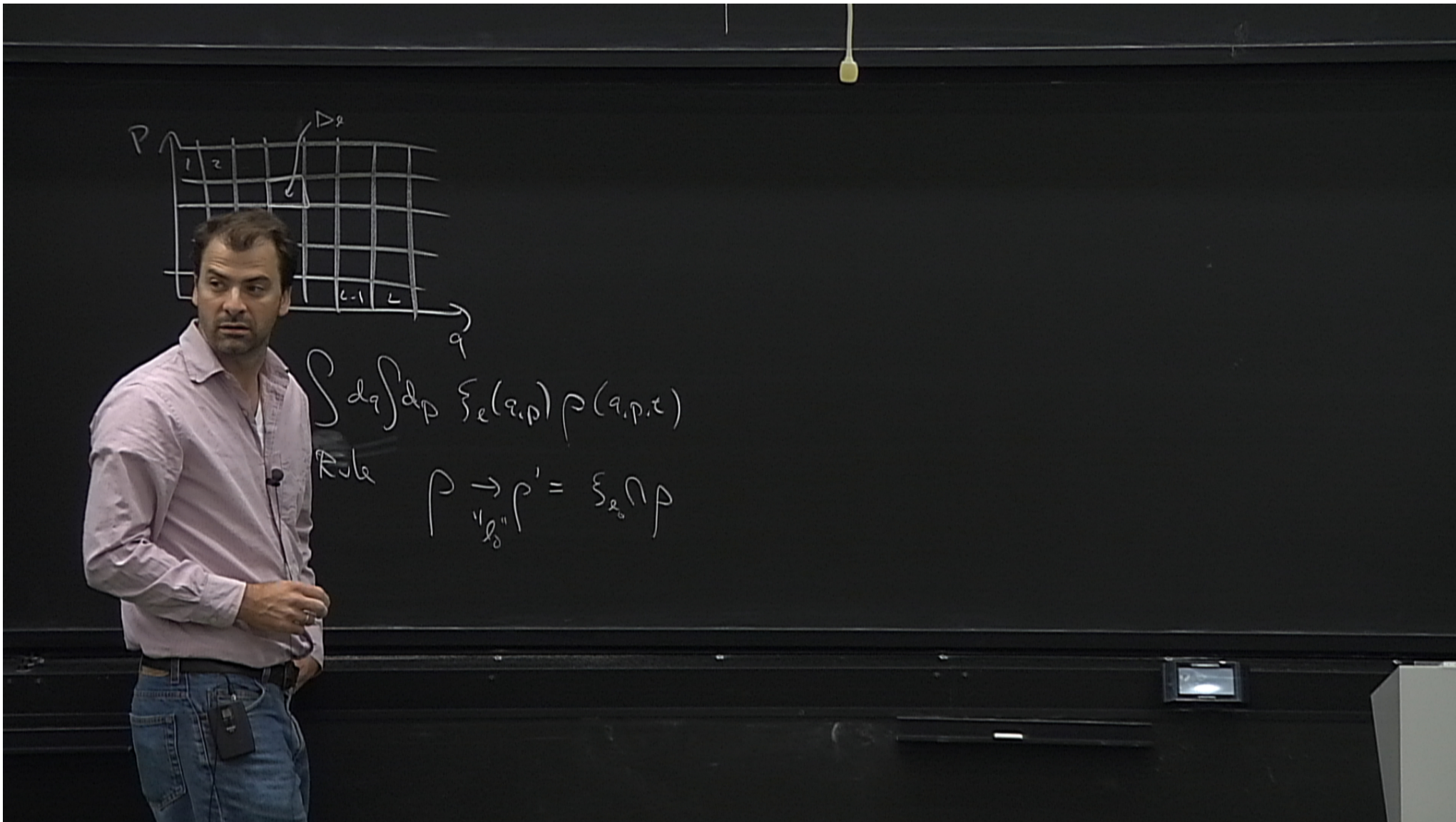
region $\Delta_a \subset \Omega = \text{phase space}$

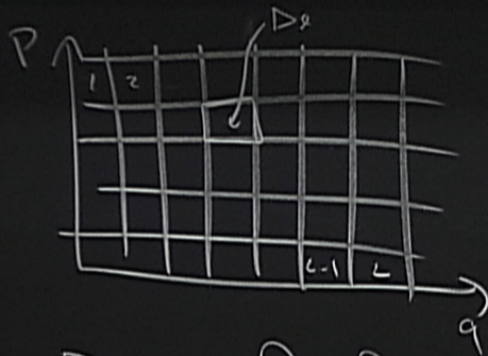
$$\xi_a(\bar{q}, \bar{p}) = \begin{cases} 1 & \text{if } \bar{q}, \bar{p} \in \Delta_a \\ 0 & \text{if } \bar{q}, \bar{p} \notin \Delta_a \end{cases}$$





$$P_c(x) = \int dq \int dp \xi_c(q, p) \rho(q, p, x)$$

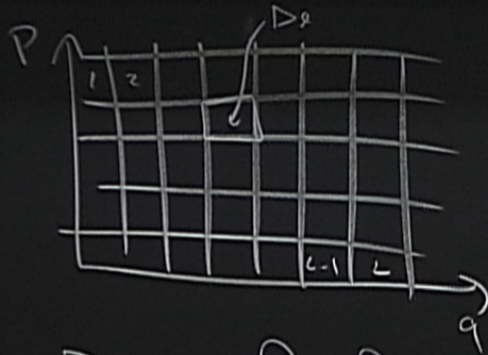




$$P_c(x) = \int dq \int dp \xi_c(q, p) \rho(q, p, x)$$

Update Rule

$$\rho \xrightarrow["\rho_0"]{} \rho' = \frac{\xi_c \cap \rho}{\int dq \int dp (\xi_c \cap \rho)}$$

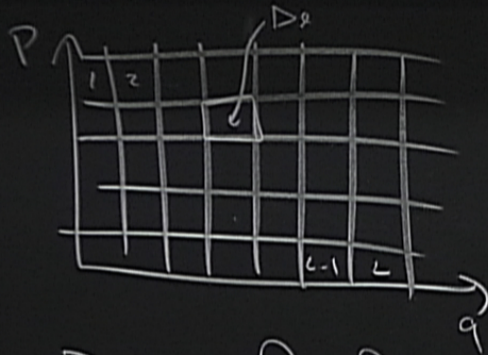


Classical update rule
is inconsistent with
the Liouville eqⁿ

$$P_c(x) = \int dq \int dp \xi_c(q,p) \rho(q,p,x)$$

Update Rule

$$\rho \xrightarrow["\delta"]{\rho} \rho' = \frac{\xi_c \cap \rho}{\int dq \int dp (\xi_c \cap \rho)}$$

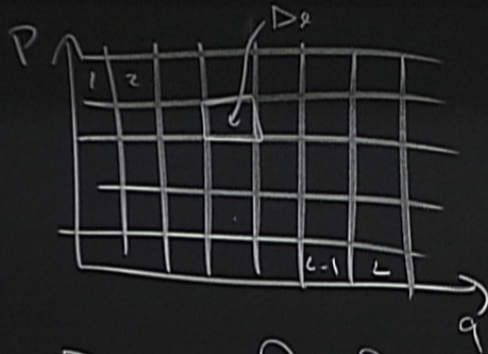


$$P_c(x) = \int dq \int dp \xi_c(q, p) \rho(q, p, x)$$

Update Rule

$$\rho \xrightarrow["\delta"]{\rho'} = \frac{\xi_c \cap \rho}{\int dq \int dp (\xi_c \cap \rho)}$$

Classical update rule
is inconsistent with
the Liouville eqⁿ



$$P_c(x) = \int dq \int dp \xi_c(q,p) \rho(q,p,x)$$

Update Rule

$$\rho \xrightarrow["\text{\"{e}}_0"]{\rho'} = \frac{\xi_c \cap \rho}{\int dq \int dp (\xi_c \cap \rho)}$$

Classical update rule
is inconsistent with
the Liouville eqⁿ

Non-conditioned state
after measurement

$$\rho \rightarrow \rho = \sum_c \xi_c \cap \rho$$

operator

$$\frac{\partial q}{\partial t} = \frac{\partial H}{\partial p}, \quad \frac{\partial p}{\partial t} = -\frac{\partial H}{\partial q}$$

- linear
- continuous
- deterministic

$$\frac{\partial f}{\partial t} = \{H, f\}$$

→ exactly analogous to von Neumann eq

$$\frac{[f, g]}{i\hbar}$$

$$= i\hbar \delta_{q,p} \delta_{i,j}$$

von Neumann & Stone

& p admit an

linear operators on $H = L^2(\mathbb{R})$

For some $f = \hat{f}$

$$\frac{d\hat{f}}{dt} = \frac{[\hat{f}, \hat{H}]}{i\hbar} + \frac{\partial \hat{f}}{\partial t}$$

→ Heisenberg eqⁿ of motion

Switch to Schrodinger picture

②

von
Neumann
eqⁿ

$$\frac{d\hat{f}}{dt} = \frac{[\hat{H}, \hat{f}]}{i\hbar} + 0$$

Classical state update

rule, after a measurement,

consists of $\{\xi_a\}_{a=1, \dots, L}$

$$\xi_a = \{0, 1\}$$

region $\Delta_a \subset \Omega = \text{phase space}$

$$\sum_a \xi_a = \Omega$$

$$\xi_a(\bar{q}, \bar{p}) = \begin{cases} 1 & \text{if } \bar{q}, \bar{p} \in \Delta_a \\ 0 & \text{if } \bar{q}, \bar{p} \notin \Delta_a \end{cases}$$

operator

$$\frac{\partial q}{\partial t} = \frac{\partial p}{\partial p} \frac{\partial f}{\partial t} + \frac{\partial f}{\partial t}$$

- linear
- continuous
- deterministic

$$\frac{\partial f}{\partial t} = \{H, f\}$$

→ exactly analogous to von Neumann eq

$$\frac{[f, g]}{i\hbar}$$

$$= i\hbar \delta_{q,p} \delta_{i,j}$$

von Neumann & Stone

& p admit an

linear operators on $H = L^2(\mathbb{R})$

For some $f = \hat{f}$

$$\frac{d\hat{f}}{dt} = \frac{[\hat{f}, \hat{H}]}{i\hbar} + \frac{\partial \hat{f}}{\partial t}$$

⇒ Heisenberg eqⁿ of motion

Switch to Schrödinger picture

②

von
Neumann
eqⁿ

$$\frac{d\hat{f}}{dt} = \frac{[\hat{H}, \hat{f}]}{i\hbar} + 0$$

Classical state update
rule, after a measurement,

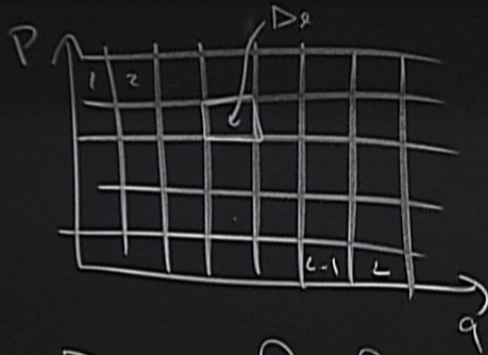
consists of $\{\xi_a\}_{a=1, \dots, L}$

$$\xi_a = \{0, 1\}$$

region $\Delta_a \subset \Omega = \text{phase space}$

$$\sum_a \Delta_a = \Omega$$

$$\xi_a(\bar{q}, \bar{p}) = \begin{cases} 1 & \text{if } \bar{q}, \bar{p} \in \Delta_a \\ 0 & \text{if } \bar{q}, \bar{p} \notin \Delta_a \end{cases}$$



$$P_c(x) = \int dq \int dp \xi_c(q,p) \rho(q,p,x)$$

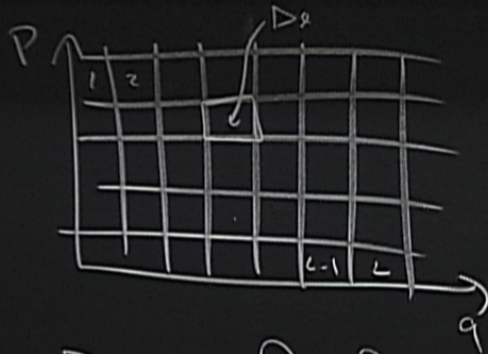
Update Rule

$$\rho \xrightarrow{\text{"b"}} \rho' = \frac{\xi_c \cdot \rho}{\int dq \int dp (\xi_c \cdot \rho)}$$

Classical update rule
is inconsistent with
the Liouville eqⁿ

Non-conditioned state
after measurement

$$\rho \rightarrow \rho = \sum_c \xi_c$$



$$P_c(z) = \int dq \int dp \delta_c(q, p) \rho(q, p, z)$$

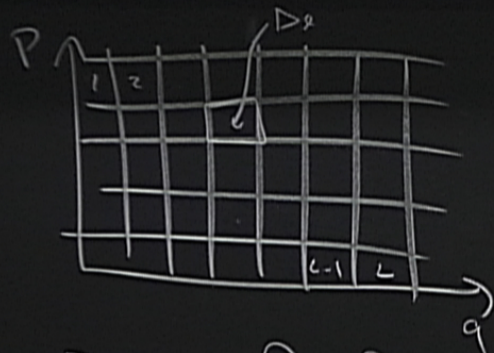
Update Rule

$$\rho \xrightarrow{\text{"L"}} \rho' = \frac{\Delta_z \cap \rho}{\int dq \int dp (\Delta_z \cap \rho)}$$

Classical update rule
is inconsistent with
the Liouville eqⁿ

Non-conditioned state
after measurement

$$\rho \rightarrow \rho = \sum_z \Delta_z \cap \rho$$



$$P_c(x) = \int dq \int dp \delta_c(q, p) \rho(q, p, t)$$

Update Rule

$\Rightarrow$ same as
QM

$$\rho \xrightarrow{\text{"} \delta \text{"}} \rho' = \frac{\Delta_c \cap \rho}{\int dq \int dp (\Delta_c \cap \rho)}$$

Classical update rule
is inconsistent with
the Liouville eqⁿ

Non-conditioned state
after measurement

$$\rho \rightarrow \rho = \sum_i \Delta_i \cap \rho$$

which is different than QM!

rule
with
 a_n

$$E = nh\nu, n=0,1,2,3,\dots$$

vs

$$E = a h \nu, a \geq 0.$$

red state
measurement

$$P = \sum_i D_i P_i$$

not than QM!

rule
with
 ω_n

red state
measurement

$$\rho = \sum_e D_e \rho$$

not than QM!

$$\underline{E = n(h\nu)}, n=0,1,2,3,\dots$$

vs

$$\underline{E = \underline{a} h\nu}, a \geq 0.$$

Given $\{f, g\} = \frac{[f, g]}{i\hbar}$

Given $[q_i^a, p_j^a] = i\hbar \delta_{ij} \delta_{ab}$

Then theorems von Neumann & Stone
imply that q & p admit an
irrep as linear operators on $\mathcal{H} = L^2(\mathbb{R})$
= irreducible
representation

For some $f = \hat{f}$

$$\frac{d\hat{f}}{dt} = \frac{[\hat{f}, \hat{H}]}{i\hbar} + \frac{\partial \hat{f}}{\partial t}$$

$\Rightarrow$ Heisenberg eqⁿ of motion

Switch to Schrödinger picture

② $\swarrow$
van Neumann eqⁿ

$$\frac{d\hat{p}}{dt} = \frac{[\hat{H}, \hat{p}]}{i\hbar} + 0$$

Classical state
rule, after a mea

Consists of $\{\xi_a$

s.t. $\xi_a = \xi_b$

region $\Delta_a \subset$

$$\sum_a \Delta_a = \Omega \quad \xi_a(\vec{q}, \vec{p}) =$$

