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Eventually we try to explain some of the open questions for two other assumptions and future directions.

Perimeter Institute, Waterloo – Canada
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Critical Collapse in the Axion-Dilaton System in Diverse Dimensions

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based on arXiv:**1108.0078** , appeared in **Class. Quantum Grav. 29(2012) 025006**

and some unpublished works , collaboration with **Prof. L. Alvarez-Gaume**

Outline

- Introductory remarks
- Discrete self similarity, some remarks on the subject
- Axion-Dilaton system
- 3 different assumptions
- E.o.m's , symmetries and boundary conditions
- Choptuik exponent
- 1- Hamade-Horne-Stewart method
- 2- New method
- Open questions &Conclusion

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We study the gravitational collapse of the axion-dilaton system suggested by type IIB string theory in dimensions ranging from four to ten.

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Introductory remarks

Classical general relativity predicts that we can make black holes. If we have enough mass in a given region of space then we have a process of 'gravitational collapse'; the matter is squeezed towards an infinite curvature singularity at r = 0 and the space-time outside r = 0 is described by the vacuum solution.

Gravitational collapse is one of the most interesting topics in TH physics .

There are important problems at classical level and even more fascinating ones in Quantum level

Progress has taken place over the last two decades

Classical	Quantum
Numerical relativity	Zoology of black objects
Critical phenomena	Ads/CFT a revolutionary

Scaling in gravitational collapse

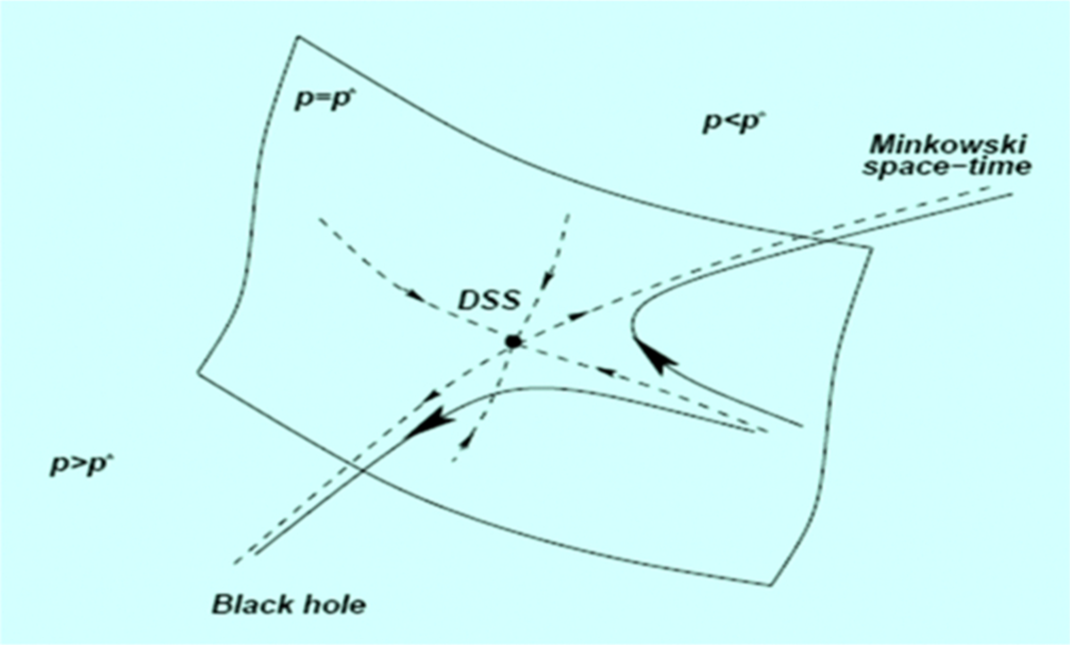
Choptuik studied gravitational collapse for a m=0 scalar field coupled to gravity in order to answer questions of black hole formation from regular initial conditions.

Christodoulou asked:

Is it possible to create black holes with arbitrarily small mass?

The answer is yes: Type I,II collapse

Critical behavior in phase space



Choptuik's (93) showed the existence of a co-dimension one critical surface.

For generic one parameter families of initial data, parameterized by p, there is a critical value p* where it crosses the critical surface.

There are two possible large time evolutions, or fixed points:

A BH forms with arbitrarily small mass

Or the system bounces and it is radiated away to infinity leaving behind M4

The critical solution has an unstable mode, or relevant direction.

The eigenvalue of the relevant direction leads to the BH critical exponent.

Basic results

For the spherical collapse of the massless scalar field, the metric takes the form:

$$ds^2 = -\alpha^2(t,r)dt^2 + a^2(t,r)dr^2 + r^2 d\Omega^2_{\mathrm{d}-2}$$

By looking at one-parameter families of initial conditions, Choptuik found the existence of a critical solution, there are two basic properties:

The critical solution is independent of the initial conditions. On the supercritical side, the size of the small BH satisfies a universal scaling law. The critical solution exhibits DSS:

The self-similarity is manifested by a periodicity of the critical solution for the scalar field $\chi(t, r)$ *of the form* (with the echoing period)

$$r_{BH} \sim (p - p^*)^\gamma \quad ; \quad Z_*(e^{n\Delta}\, t, e^{n\Delta}\, r) = Z_*(t,r)$$

The numerical results of Choptuik on the spherically symmetric gravitational collapse of a real scalar field, ϕ, have inspired interest s in gravitational collapse just at the threshold for formation of black holes.

Further numerical results for perfect fluids suggest that the phenomena discovered by Choptuik are not just restricted to spherically symmetric real scalar fields.

The thought experiment employed by Choptuik, in the context of numerical relativity, is to "tune" across the critical threshold in the space of initial conditions that separates the non-black hole endstate **from the black hole endstate**

Two distinct kinds of scaling appear. First, just above the black hole threshold, the black hole mass is

$$M_{\rm bh}(p) \propto (p - p^{*})^{\gamma}, \; \gamma \approx 0.37,$$

where p is some tuning parameter of the initial conditions.

Second, exactly at the threshold, there appears a unique critical solution acting as an attractor for all nearby initial conditions on threshold.

This critical solution exhibits a striking, recurrent "echoing" behavior:

Asymptotically at small time-scales t and length-scales r near the collapse, it repeats itself at ever decreasing scales

$$t' = e^{-n\Delta}t,\ r' = e^{-n\Delta}r\ (\text{for } n \in \mathbb{Z}_+)$$

Here Δ is either a fixed constant of the solution ($\Delta \approx \log 30$ for a real scalar field) **demonstrating a discrete self-similarity**; **or an arbitrary constant, demonstrating a continuous self-similarity .**

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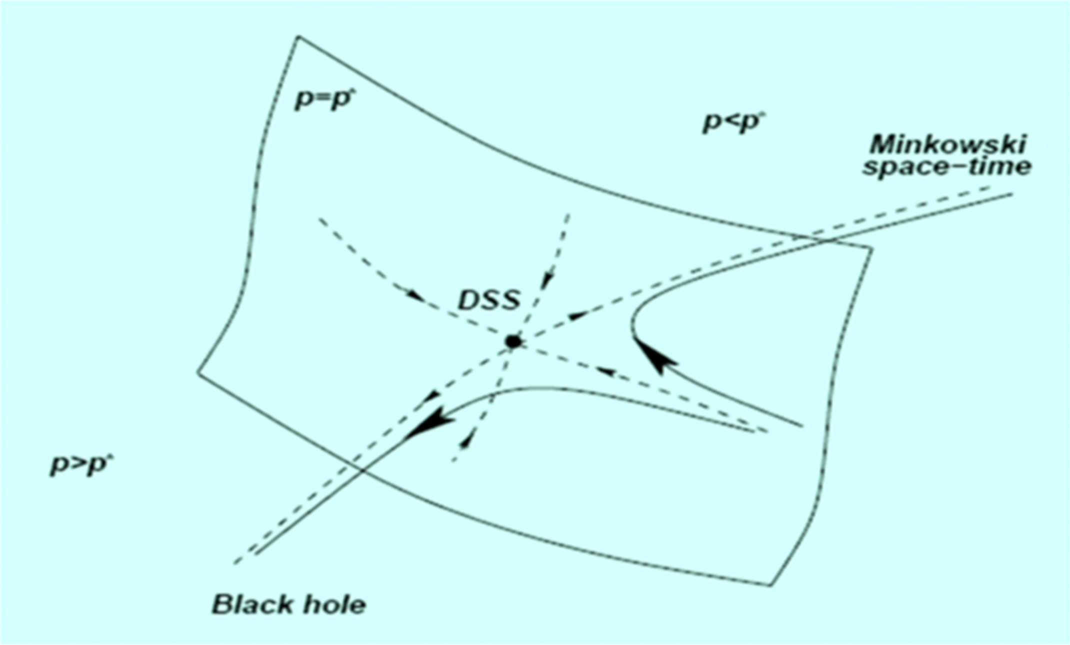

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Choptuik Vs Liapounov,

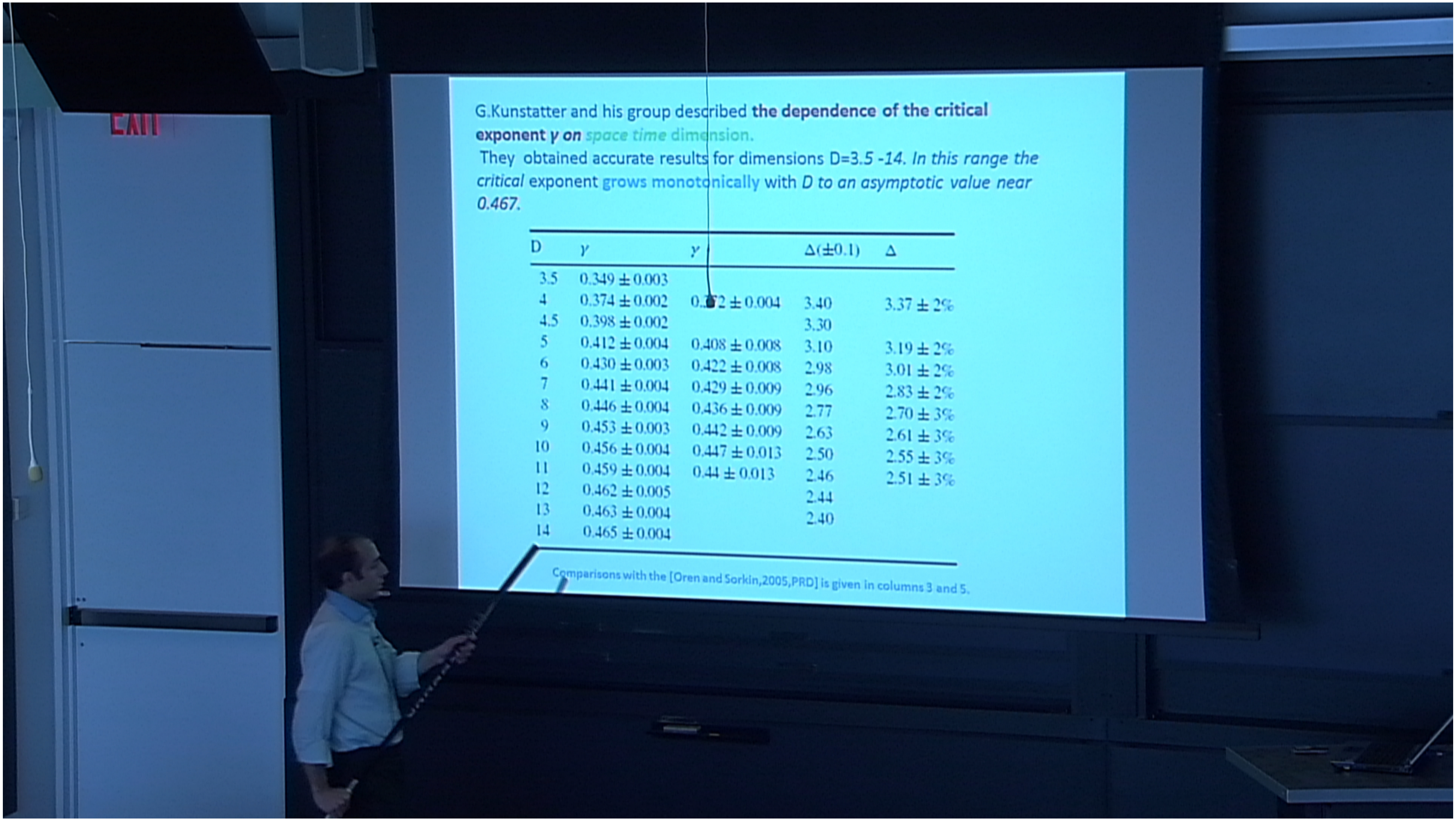
The numerical values obtained **[Choptuik, Gundlach,J.Bland, G.Kunstatter et all ,CQG,2005,dimension dependence...]** depend **only on the type of matter considered**.

They do not depend on initial conditions. The solution is characterized by having a single unstable direction.

Thus computing the Choptuik exponent is related to computing the Liapounov exponent of the small perturbations around the critical solution

$$Z_p(\tau,\zeta) \approx Z_*(\tau,\zeta) + \sum_{k=1}^{\infty} C_k(p) e^{\lambda_k \tau} \delta_k Z(\tau,\zeta)$$

$$\gamma = -\frac{1}{\lambda_1}$$


G.Kunstatter and his group described the dependence of the critical exponent γ on space time dimension.
They obtained accurate results for dimensions D=3.5 -14. In this range the critical exponent grows monotonically with D to an asymptotic value near 0.467.
D | γ | γ | Δ(±0.1) | Δ
3.5 | 0.349 ± 0.003
4 | 0.374 ± 0.002 | ± 0.004 | 3.40 | 3.37 ± 2%
4.5 | 0.398 ± 0.002 | | 3.30
5 | 0.412 ± 0.004 | 0.408 ± 0.008 | 3.10 | 3.19 ± 2%
6 | 0.430 ± 0.003 | 0.422 ± 0.008 | 2.98 | 3.01 ± 2%
7 | 0.441 ± 0.004 | 0.429 ± 0.009 | 2.96 | 2.83 ± 2%
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9 | 0.453 ± 0.003 | 0.442 ± 0.009 | 2.63 | 2.61 ± 3%
10 | 0.456 ± 0.004 | 0.447 ± 0.013 | 2.50 | 2.55 ± 3%
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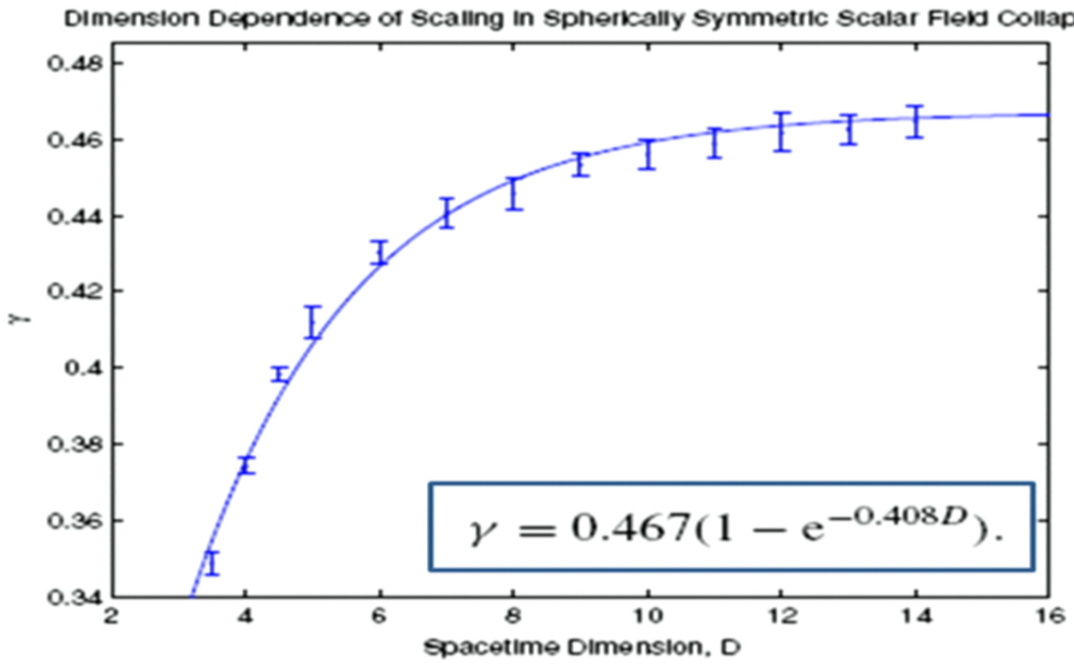
Remarkably, the data are well fit by a simple exponential



Plot of results for γ as a function of spacetime dimension.

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Now we revisit the self-similar, spherical gravitational collapse studied in the past in four-dimensions and extend their analysis to higher dimensions and to other implementations of self similarity.

Our work is motivated by the AdS/CFT correspondence .

We are still far from a holographic description of black hole formation in type IIB string theory. In the context of type IIB and AdS/CFT one would like to consider collapse on spaces which approach asymptotically $AdS5\times S5$.

With the bosonic fields in the theory, a natural system to consider involves the axion-dilaton and the self-dual 5-form field. One can show that in five dimensions the simplest dynamical set to study involves just the Einstein-axion-dilaton system with a cosmological constant.

This poses an apparent problem in considering self similar collapse, because Einstein spaces do not admit homothetic vector fields.

However, in the context of critical gravitational collapse we are considering the collapse of matter to form small mass black holes and we only need to consider a small space-time region close to where the singularity forms.

This should be independent of the asymptotic structure of the space-time where the collapse takes place.

There is numerical evidence in asymptotically AdS space-times showing that this is the case [arxiv: gr-qc/0201026,G.Kunstatter et al ,PRD].

Hence we will eliminate the cosmological constant and analyze self-similar critical collapse in dimensions 4–10.

The axion/dilaton system

We report on threshold behavior of gravitational collapse, and Choptuik scaling, in classical 3+1-dimensional low-energy effective string theory: i.e., in general relativity coupled to a dilaton Φ and an axion a.
We will consider the d-dimensional spherical collapse of an

axion-dilaton (a, Φ) system. Both field can be combined into a single complex scalar field

$$\tau \equiv a + ie^{-\phi}$$

The effective action for the model is:

$$S = \frac{1}{16\pi G}\int d^d x\sqrt{-g}\left(R - \frac{1}{2}\frac{\partial_a \tau \partial^a \bar{\tau}}{(\operatorname{Im}\tau)^2}\right)$$

where R is the scalar curvature. The equations of motion are:

$$R_{ab} - \frac{1}{4(\operatorname{Im}\tau)^2}(\partial_a \tau \partial_b \bar{\tau} + \partial_a \bar{\tau} \partial_b \tau) = 0$$

$$\nabla^a \nabla_a \tau + \frac{i\nabla^a \tau \nabla_a \tau}{\operatorname{Im}\tau} = 0.$$

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The action has an extra global symmetry not present in general relativity, an SL(2, R) symmetry that acts on Φ and a, but leaves the space-time metric invariant; this is a classical version of the conjectured SL(2, Z) symmetry .

The theory is classically invariant under SL(2, R) transformations:

$$\tau \to \frac{a\tau + b}{c\tau + d} ,$$

where $(a, b, c, d) \in R$, $ad - bc = 1$ and g_{ab} does not transform.

if we put a=0, the action reduces to choptuik model which has discrete self similarity the spherically symmetric metric can be written as:

$$ds^2 = (1 + u(t,r))\left(-b(t,r)^2 dt^2 + dr^2\right) + r^2 d\Omega_{d-2}^2 \, .$$

The time coordinate is chosen so that spherical collapse on the time axis first occurs at t = 0, hence the collapse takes place for t < 0.

We can still implement time re-definitions
for the metric , hence ,we set b(t, 0) = 1 (time scaling), t < 0 and regularity for t < 0 implies that u(t, 0) = 0, t < 0.

Continuous self-similarity means we work exactly at the threshold for black hole formation and construct a critical solution ,it also means the existence of a homothetic Killing vector ξ generating global scale transformations

$$\mathcal{L}_\xi g_{ab} = 2 g_{ab} \, .$$

Thus a region of the space time before the singularity forms has homothety , i.e. a conformal Killing vector of weight 2.

In spherical coordinates $\xi = t\,\partial/\partial t + r\,\partial/\partial r$. Defining the scale invariant variable
$z = -r/t$
self-similarity of the metric means that the unknown functions u(t, r), b(t, r) are just functions of z so .

$$b(t,r) = b(z), \quad \text{and} \quad u(t,r) = u(z),$$

The next question to address is the transformation of τ (t, r) under scale transformations.

Since the action is SL(2, R)-invariant, we can compensate a scale transformation of the coordinates (t, r) by an SL(2, R) transformation. Thus if we change variables to (t, z), we obtain a differential condition for τ (t, z):

$$t\,\frac{\partial}{\partial t}\,\tau(t,z) \;=\; \alpha_0 \,+\, \alpha_1\,\tau \,+\, \alpha_2\,\tau^2$$

Some remarks about ω

Due to

$$t\frac{\partial}{\partial t}\tau(t,z) = \alpha_0 + \alpha_1\tau + \alpha_2\tau^2$$

the following ansatze is inconsistent.

$$\tau(t,r) = \frac{1-(-t)^{i\omega}f(z)}{1+(-t)^{i\omega}f(z)},$$

Because if we use this ansatze in the above eq. we will gain the values

$$\alpha_1 = 0, \alpha_0 = -\alpha_2 = -\frac{iw}{2}$$

while all of them, have to be real numbers. Therefore the above ansatze is inconsistent unless ω be purely imaginary which can not be .

For this particular ansatze e.o.m's are not scale invariant either.

In the hyperbolic case:

$$\tau(t,r) = \frac{1-(-t)^{\omega} f(z)}{1+(-t)^{\omega} f(z)},$$

and under a scaling transformation $t \to \lambda t$, $\tau(t, r)$ changes by an SL(2, R) boost. Using an SL(2, R)-transformation we can transform

$$\tau(t,r) \to -(-t)^{\omega} f(z)$$

Finally, in the parabolic case a scaling transformation can be compensated by a translation.

After some simple SL(2, R) transformations of $\tau(t, r)$ the parabolic ansatz becomes:

$$\tau(t,r) = f(z) + \omega \log(-t) \quad ; \quad w \in \mathbb{R}$$

We have written $-t$ throughout the previous equations because the collapse will take place for $t < 0$, we can also extend the solutions for $t > 0$, hence the correct ansatze for any value of t is to replace $-t \to |t|$.

In all these expressions, $f(z)$ is an arbitrary complex function and ω an arbitrary real parameter to be fixed by requiring the critical solution to be regular.

Equations of Motion

We collect in this section the equations of motion for the three ansatze, although later we will only present results for the elliptic case.

Since we are implementing spherical symmetry, the gravitational degrees of freedom do not propagate, there are no gravitational waves.

Hence $u(z)$, $b(z)$ should be expressed in terms of $f(z)$. Common to all three cases, it is simple to show that

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with α_0,1,2 real numbers. The quadratic polynomial on the right hand side has two roots which can be two complex conjugate numbers, two distinct real numbers or a double real root.

They correspond to compensating the scaling transformation with respectively an elliptic, a hyperbolic, or a parabolic transformation in SL(2, R).

By straightforward manipulations and redefinitions the three cases yield the following ansatze for τ (t, z). **In the elliptic case:**

$$\tau(t,r) = i\frac{1-(-t)^{i\omega}f(z)}{1+(-t)^{i\omega}f(z)},$$

ω is a real constant but not arbitrary so under a scaling transformation $t \to \lambda t$, τ (t, r) changes by an SL(2, R) rotation

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$\omega = 0$ case

The only consistent values for this particular case is

$$\alpha_0 = \alpha_1 = \alpha_2 = 0$$

This is strong constraint shows that we should not consider $\omega = 0$ case in the equations of motion.

On the other hand applying $\omega = 0$ case , will provide a trivial solution which is $f(z) = \text{constant}$; $b(z) = 1$.

To make sense, we have to find the leading behavior and next to the leading behavior of the functions in the presence of **non vanishing ω** .

We have written $-t$ throughout the previous equations because the collapse will take place for $t < 0$, we can also extend the solutions for $t > 0$, hence the correct ansatze for any value of t is to replace $-t \to |t|$.

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Hence $u(z)$, $b(z)$ should be expressed in terms of $f(z)$. Common to all three cases, it is simple to show that

$$u(z) = -\frac{z\, b'(z)}{(d-3)\, b(z)}.$$

This follows from the Einstein equation for the angular variables.

The other equations of motion for the self-similar solution involve b(z), f(z).

In fact the equation for b is a first order linear inhomogeneous equation for b(z) whose initial condition is determined by b(0) = 1 together with the initial conditions for f(z), f'(z), that are determined by requiring smoothness of the critical solution.

In fact initial conditions assumed smooth, may be given on the hyper surface t=-1.

This determines not only the initial conditions but also the possible value of ω .
The equations of motion are quite complicated, and their derivation is rather tedious but straightforward.

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The equations in the elliptic case in 4 dims are given by the system of ODEs:

$$0 = b' + \frac{z(b^2 - z^2)}{b(1-|f|^2)^2} f'\bar{f}' - \frac{i\omega(b^2 - z^2)}{b(1-|f|^2)^2}(f\bar{f}' - \bar{f}f') - \frac{\omega^2 z|f|^2}{b(1-|f|^2)^2} ,$$

$$\begin{aligned} 0 = f'' &- \frac{z(b^2+z^2)}{b^2(1-|f|^2)^2} f'^2\bar{f}' + \frac{2}{(1-|f|^2)}\left(1 - \frac{i\omega(b^2+z^2)}{2b^2(1-|f|^2)}\right)\bar{f}f'^2 + \frac{i\omega(b^2+2z^2)}{b^2(1-|f|^2)^2} f f'\bar{f}' \\ &+ \frac{2}{z}\left(1 + \frac{i\omega z^2(1+|f|^2)}{(b^2-z^2)(1-|f|^2)} + \frac{\omega^2 z^4|f|^2}{b^2(b^2-z^2)(1-|f|^2)^2}\right) f' + \frac{\omega^2 z}{b^2(1-|f|^2)^2} f^2\bar{f}' \\ &+ \frac{i\omega}{(b^2-z^2)}\left(1 - \frac{i\omega(1+|f|^2)}{(1-|f|^2)} - \frac{\omega^2 z^2|f|^2}{b^2(1-|f|^2)^2}\right) f . \end{aligned}$$

The equations in the elliptic case in d dims are given by the system of ODEs:

$$
\begin{aligned}
0 &= b' + \frac{2z(b^2 - z^2)}{(d-2)b(-1+|f|^2)^2} f'\bar{f}' - \frac{2i\omega(b^2 - z^2)}{(d-2)b(-1+|f|^2)^2}(f\bar{f}' - \bar{f}f') - \frac{2\omega^2 z|f|^2}{(d-2)b(-1+|f|^2)^2}, \\
0 &= f'' - \frac{2z(b^2+z^2)}{(d-2)b^2(-1+|f|^2)^2} f'^2\bar{f}' + \frac{2}{(1-|f|^2)}\left(1 - \frac{i\omega(b^2+z^2)}{(d-2)b^2(1-|f|^2)}\right)\bar{f}f'^2 \\
&\quad + \frac{2i\omega(b^2+2z^2)}{(d-2)b^2(-1+|f|^2)^2} f f'\bar{f}' + \frac{2}{z}\left(\frac{(z^2 - \frac{(d-2)b^2}{2})}{(z^2-b^2)} + \frac{i\omega z^2(1+|f|^2)}{(b^2-z^2)(1-|f|^2)}\right. \\
&\quad \left. + \frac{2\omega^2 z^4|f|^2}{(d-2)b^2(b^2-z^2)(1-|f|^2)^2}\right) f' + \frac{2\omega^2 z}{(d-2)b^2(-1+|f|^2)^2} f^2\bar{f}' + \\
&\quad \frac{2i\omega}{(b^2-z^2)}\left(\frac{1}{2} - \frac{i\omega(1+|f|^2)}{2(1-|f|^2)} - \frac{\omega^2 z^2|f|^2}{(d-2)b^2(-1+|f|^2)^2}\right) f.
\end{aligned}
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It is useful to note that the equations are invariant under a global re-definition of the phase of f(z). We can choose the phase of f(z) to any convenient value at a particular z that we will choose to be the origin.

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&\left. + \frac{2\omega^2 z^4|f|^2}{(d-2)b^2(b^2-z^2)(1-|f|^2)^2}\right) f' + \frac{2\omega^2 z}{(d-2)b^2(-1+|f|^2)^2} f^2\bar{f}' + \\
&\frac{2i\omega}{(b^2 - z^2)}\left(\frac{1}{2} - \frac{i\omega(1+|f|^2)}{2(1-|f|^2)} - \frac{\omega^2 z^2|f|^2}{(d-2)b^2(-1+|f|^2)^2}\right) f.
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In the hyperbolic case, the equations of motion are quite similar, but now they are invariant under a constant scaling $f \to \lambda f$, hence we can choose $|f(z)|$ or its real or imaginary part as we wish at a particular value of z:

$$
\begin{aligned}
0 &= b' - \frac{2z(b^2 - z^2)}{(d-2)b(f-\bar{f})^2} f'\bar{f}' + \frac{2\omega(b^2 - z^2)}{(d-2)b(f-\bar{f})^2}(f\bar{f}' + \bar{f}f') + \frac{2\omega^2 z|f|^2}{(d-2)b(f-\bar{f})^2} \\
0 &= -f'' - \frac{2z(b^2 + z^2)}{(d-2)b^2(f-\bar{f})^2} f'^2\bar{f}' + \frac{2}{(f-\bar{f})}\left(\frac{1}{\bar{f}} + \frac{\omega(b^2+z^2)}{(d-2)b^2(f-\bar{f})}\right)\bar{f}f'^2, \\
&\quad + \frac{2\omega(b^2 + 2z^2)}{(d-2)b^2(f-\bar{f})^2} ff'\bar{f}' + \frac{2}{z}\left(-\frac{(z^2 - \frac{(d-2)b^2}{2})}{(z^2 - b^2)} + \frac{\omega z^2(f+\bar{f})}{(b^2 - z^2)(f-\bar{f})}\right. \\
&\quad \left. + \frac{2\omega^2 z^4|f|^2}{(d-2)b^2(b^2-z^2)(f-\bar{f})^2}\right) f' - \frac{2\omega^2 z}{(d-2)b^2(f-\bar{f})^2} f^2\bar{f}' + \\
&\quad \frac{2\omega}{(b^2 - z^2)}\left(-\frac{1}{2} - \frac{\omega(f+\bar{f})}{2(f-\bar{f})} - \frac{\omega^2 z^2|f|^2}{(d-2)b^2(f-\bar{f})^2}\right) f.
\end{aligned}
$$

Properties of the critical solutions in the elliptic case

We briefly present some of our results in the study of the critical solution in the elliptic case.

In all three non-linear systems we have five singular points, $z = \pm 0$ represents the axis $r = 0$ and regularity is easily imposed.
The point $z = \infty$ represents the hyper surface $t = 0$.

However, space-time should be smooth on this hyper surface, except at the axis, since it lies in the Cauchy development of the initial conditions.

Away from $r = 0$ there is nothing special in this surface, hence regularity is also imposed.

The equations in the elliptic case in 4 dims are given by the system of ODEs:

$$0 = b' + \frac{z(b^2 - z^2)}{b(1-|f|^2)^2} f'\bar{f}' - \frac{i\omega(b^2 - z^2)}{b(1-|f|^2)^2}(f\bar{f}' - \bar{f}f') - \frac{\omega^2 z|f|^2}{b(1-|f|^2)^2} ,$$

$$0 = f'' - \frac{z(b^2 + z^2)}{b^2(1-|f|^2)^2} f'^2\bar{f}' + \frac{2}{(1-|f|^2)}\left(1 - \frac{i\omega(b^2+z^2)}{2b^2(1-|f|^2)}\right)\bar{f}f'^2 + \frac{i\omega(b^2+2z^2)}{b^2(1-|f|^2)^2} f f'\bar{f}'$$
$$+\frac{2}{z}\left(1 + \frac{i\omega z^2(1+|f|^2)}{(b^2-z^2)(1-|f|^2)} + \frac{\omega^2 z^4|f|^2}{b^2(b^2-z^2)(1-|f|^2)^2}\right) f' + \frac{\omega^2 z}{b^2(1-|f|^2)^2} f^2\bar{f}'$$
$$+\frac{i\omega}{(b^2-z^2)}\left(1 - \frac{i\omega(1+|f|^2)}{(1-|f|^2)} - \frac{\omega^2 z^2|f|^2}{b^2(1-|f|^2)^2}\right) f .$$

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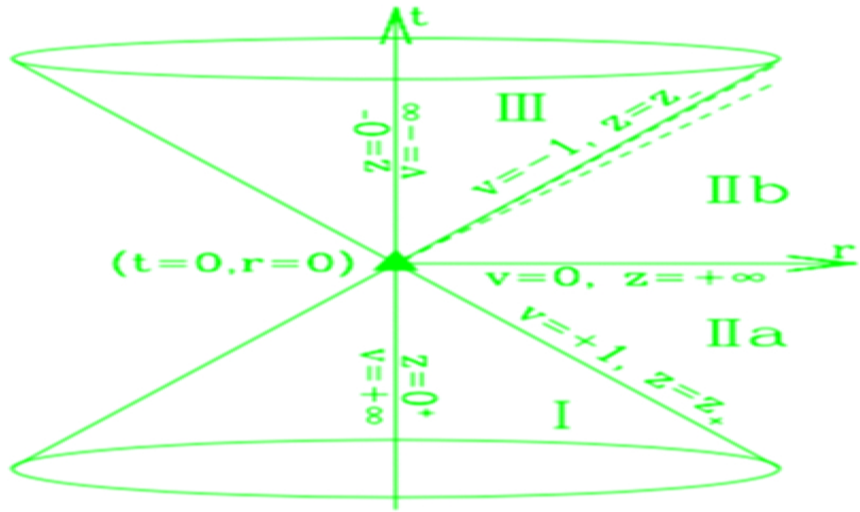

FIG. 1. Interpretation of the axion/dilaton choptuon. There are horizons at v = +1 (characteristic hyper surface) and v = −1 (Cauchy horizon). There is a space time singularity at (t = 0,r = 0) which is a single point in the causal structure.

Region I is a collapsing sphere of gravitationally bound field; in Regions IIa and IIb this blends smoothly into an outgoing wave. The outgoing wave oscillates infinitely many times approaching the Cauchy horizon at v = −1.

The dashed lines indicate the first two oscillations.
Region III is nearly but not precisely flat.

The simplest way to see that there is no problem at z = ∞ is through change of variables and a re-definition of the fields f(z), b(z) .

Using the function of $f(z) = f_m(z)e^{if_a(z)}$ we can divide (13) to the modulous and phase parts . We are going to apply Frobenious method to find the leading behavior of $f_m(z)$, $f_a(z)$ at infinity. By replacing $f_m(z) = uz^k$ and $f_a(z) = vz^s$ into equations of motion we will obtain the behavior of the function at $z = \infty$ as, $f_m(z) = z^{\mp iw}$, $f_a(z) = f_0$ where f_0 is a constant. But because of finding asymptotic solution the only acceptable solution is z^{-iw} meanwhile ω must be positive number. Thus at infinity solution is $f(z) = z^{-iw}e^{if_0}$.

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Indeed, equations of motion may be rendered regular at $z = \infty$ by the following change of variables :

, requiring regularity both at the origin and at z+ leaves only four parameters to be determined: |f(0)|, w, z+, |f(z+)|.

To find them we proceed by integrating from close to the origin towards z+ and also from a little below z+ towards the origin, and match the results at an intermediate point, say z = 1.

By matching the functions and the relevant first derivatives we can determine the unknown parameters in the critical solution. We find that if

$\omega = 1.176$, $z+ = 2.609$, $|f(0)| = 0.892$, $|f(z+)| = 0.364$

then the equations are smooth at z = 0 and z = z+.

We do this for every dimension between d = 4 and d = 10.

Choptuik exponents

For each such D we can uniquely determine the Choptuik exponent by analyzing the perturbations. **Here we have gone beyond what is in the literature.**

Hamade-Horne-Stewart Method

Given the critical solution, we can perturb it to find the critical exponent γ as follows. Let h be any function determining the critical solution: b or f. Next consider small perturbations around the critical solution:

$$h(z,t) = h_{ss}(z) + \epsilon |t|^{-\kappa} h_{pert}(z)$$

where hss(z) is the critical solution, ϵ is a small number, κ is a constant, and hpert(z) depends only on z. Substituting h(z, t) in the full equations of motion of motion and keeping first order terms in ϵ, gives a set of linear equations for the perturbation and an eigen value equation for κ.

This eigen value equation can in principle have a number of possible solutions for κ. The solution with the largest value of Re(κ) will be responsible for the fastest growing perturbation in the above , and is called the "most relevant mode." The critical exponent is given by

$$\gamma = \frac{1}{Re(\kappa)}$$

The numerical accuracy decreases as the dimension increases. The third significant digit in Table 3 should not be trusted, specially in dimensions 9 and 10.

At higher dimensions it is more difficult to obtain numerical stability near criticality.

Interesting thing is that similar thing happens in scaling phenomenon found in general relativistic critical gravitational collapse of a massless scalar field to higher dimensions in (D=11) [E.Sorkin, hep-th/0502034].

So far three is no strong evidence about the violation of universality .Our computations for all 10 dimensions and for elliptic case quite well matched .

The results in the hyperbolic and parabolic cases in four dimensions indicating that there are also critical solutions with Choptuik exponents different from those found for the elliptic case [L.A.Gaume and E.H,work in progress].

New method of the perturbations for Choptuik exponent

In order to proceed first we keep general form of the metric in terms of the functions b(t, r) and u(t,r) in the equations of motion

$$ds^2 = (1 + u(t,r))(-b(t,r)^2 dt^2 + dr^2) + r^2(d\theta^2 + sin\theta^2 d\phi^2)$$

$$ds^2 = (1 + u(t,r))(-b(t,r)^2 dt^2 + dr^2) + r^2(d\theta_1^2 + sin\theta_1^2(d\theta_2^2 + sin\theta_2^2 d\theta_3^2))$$

$$\text{t, r}, \theta_1, \theta_2, \theta_3 = \phi$$

To find perturbed equation we consider the general form of the ansatze

$$\tau(t,r) = i\frac{1 - f(t,r)}{1 + f(t,r)}$$

$$b(t,r)=b(-r/t), \qquad u(t,r)=u(-r/t)$$

$$du(t,r)=(-t)^{-k}u_1(-r/t), \qquad db(t,r)=(-t)^{-k}b_1(-r/t)$$

$$\tau(t,r)=i\frac{1-(-t)^{i\omega}f(-r/t)}{1+(-t)^{i\omega}f(-r/t)}, \qquad d\tau(t,r)=-2i\frac{(-t)^{i\omega-k}f_1(-r/t)}{(1+(-t)^{i\omega}f(-r/t))^2}$$

we can investigate that even after taking variations from the fields the e.o.m's are explicitly scale invariant .

Using the equations that related to R_{01}, R_{22} , using δR_{22}(the equation after variation of R_{22}) and using δR_{01} we could remove u_1, u_1', u, u', from δR_{00} .Finally after some computations from δR_{00} equation we will gain an equation for $b_1'(-r/t)$ as

Open Question

The SL(2, R) symmetry of the classical type IIB string theory is supposed to break to SL(2, Z) once quantum effects are taken into account.

This raises the very interesting possibility that the critical solution in the quantum case will not be continuous self-similar but rather it will have discrete self-similarity as for the massless scalar field first analyzed by Choptuik [Choptuik,1993,universality...,PRL].

In other words, we want to know if there are elements Γ ∈ SL(2, Z), such that:

$$\tau(e^{\Delta_\Gamma}\, t, e^{\Delta_\Gamma}\, r) = \frac{a\tau + b}{c\tau + d}; \qquad \Gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}),$$

with Δ the corresponding echo parameter. To settle this question one would have to do the full numerical integration of Einstein's equation without assuming continuous self-similarity.

In conclusion, full scale numerical work on this model is highly desirable, for several reasons.

The solution we have given is, by assumption, continuously self-similar. It is important to see if discretely self-similar solutions also exist.

Discretely self-similar solutions may manifest themselves by a form of Choptuik "echoing" in which the system is invariant after a finite scale transformation by some scale factor $\exp(-\Delta)$, up to an $SL(2, R)$ transformation.

Conclusion

So we found $\gamma = .2616$ which is very close to the value

that Hamade-Horne-Stewart have already found it . $\gamma = .2641$

We found choptuik exponent in 4 dims for hyperbolic ansatz $\gamma = .363$,

We also obtained choptuik exponent in 4 dims for parabolic ansatz $\gamma = .324$

Any way this is strong constraint shows that choptuik exponent has some differences which depends strongly on the related ansatze and also it depends on the dimensions and matter content.

Of course here matter content for all 3 different assumptions is the same .

Note that people already have found that for different theory it depends on matter content.