

Title: Integral Transforms & Green's Functions - Lecture 2

Date: Aug 28, 2012 09:00 AM

URL: <http://pirsa.org/12080029>

Abstract:

REVIEW:

$$\begin{cases} \Delta \varphi = -g & \text{in } \Omega \\ \varphi = \varphi_0 & \text{on } \partial\Omega \end{cases}$$

$$\varphi(u) = - \int_{\Omega} [Gg + \int_{\partial\Omega} \varphi \frac{\partial G}{\partial n} - G \frac{\partial \varphi}{\partial n}] dS$$

REVIEW:

$$\begin{array}{ll} \Delta \varphi = -g & \text{in } \Omega \\ \varphi = \varphi_0 & \text{on } \partial\Omega \end{array}$$

$$- \int_{\Omega} G g + \int_{\partial\Omega} \varphi \frac{\partial G}{\partial n} - G \frac{\partial \varphi}{\partial n} dS$$

REVIEW:

$$\begin{array}{ll} \Delta \varphi = -g & \text{in } \Omega \\ \varphi = \varphi_0 & \text{on } \partial\Omega \end{array}$$

$$\begin{array}{l} \Delta G = -\delta \\ G|_{\partial\Omega} = 0 \end{array}$$

DIRICHLET G.F.

$$1 = - \int_{\Omega} [Gg + \int_{\partial\Omega} \varphi \frac{\partial G}{\partial n} - \cancel{G \frac{\partial \varphi}{\partial n}}] dS$$

REVIEW:

DIRICHLET

$$\begin{aligned} \Delta \varphi &= -g & \text{in } \Omega \\ \varphi &= \varphi_0 & \text{on } \partial\Omega \end{aligned}$$

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DIRICHLET G.F.

NEUMANN

$$\begin{aligned} \Delta \varphi &= -g \\ \frac{\partial \varphi}{\partial n} &= 0 & \text{on } \partial\Omega \end{aligned}$$

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REVIEW:

DIRICHLET

$$\boxed{\begin{array}{ll} \Delta \varphi = -g & \text{in } \Omega \\ \varphi = \varphi_0 & \text{on } \partial\Omega \end{array}}$$

$$\boxed{\begin{array}{l} \Delta G_D = -\delta \\ G_D|_{\partial\Omega} = 0 \end{array}}$$

DIRICHLET G.F.

NEUMANN

$$\boxed{\begin{array}{l} \Delta \varphi = -g \\ \frac{\partial \varphi}{\partial n} = 0 \text{ on } \partial\Omega \end{array}}$$

$$\boxed{\begin{array}{l} \Delta G_N = -\delta \\ \frac{\partial G_N}{\partial n}|_{\partial\Omega} = 0 \end{array}}$$

$$\varphi(u) = - \int_{\Omega} g u \, dx - \int_{\partial\Omega} \left[G \frac{\partial \varphi}{\partial n} \right] dS$$

REVIEW:

DIRICHLET

$$\begin{cases} \Delta \psi = -g & \text{in } \Omega \\ \psi = \psi_0 & \text{on } \partial\Omega \end{cases}$$

$$\Delta G_D = -\delta$$

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DIRICHLET G.F.

NEUMANN

$$\Delta \psi = -g$$

$$\frac{\partial \psi}{\partial n} = 0 \quad \text{on } \partial\Omega$$

$$\Delta G_N = -\delta$$

$$\frac{\partial G_N}{\partial n}|_{\partial\Omega} = 0$$

$$\psi(n) \left[G_D g + \int_{\partial\Omega} \psi \frac{\partial G_D}{\partial n} - G_D \frac{\partial \psi}{\partial n} \right] dS$$



REVIEW:

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$$= - \int_{\Omega} G g + \int_{\partial\Omega} \psi \frac{\partial G}{\partial n} - G \frac{\partial \psi}{\partial n} dS$$




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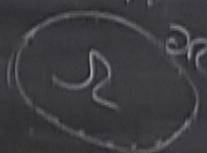
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DIRICHLET G.F.



$$\psi(x) = - \int_{\Omega} G_D g + \int_{\partial\Omega} \psi \frac{\partial G_D}{\partial n} - G_D \frac{\partial \psi}{\partial n}$$

FOURIER TRANSFORM

INTEGRAL TRANSFORM

$$\hat{f}(k) = \int_a^b K(t, k) f(t) dt$$

FOURIER TRANSFORM

INTEGRAL TRANSFORM

$$\hat{f}(k) = \int_a^b \underbrace{K(t, k)}_{\text{KERNEL}} f(t) dt$$

FOURIER TRANSFORM

INTEGRAL TRANSFORM

EXs:

$$\hat{f}(k) = \int_a^b \underbrace{K(t, k)}_{\text{KERNEL}} f(t) dt$$

FOURIER TRANSFORM

INTEGRAL TRANSFORM

$$\hat{f}(k) = \int_a^b \underbrace{K(t, k)}_{\text{KERNEL}} f(t) dt$$

Xs: 1) FOURIER SERIES

$$S_m[f](k) = \int_{-\pi}^{\pi} f(t) D_m(t-k) dt$$

FOURIER TRANSFORM

INTEGRAL TRANSFORM

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FOURIER TRANSFORM

INTEGRAL TRANSFORM

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EXS: 1) FOURIER SERIES

$$S_m[f](k) = \int_{-\pi}^{\pi} f(t) \underbrace{D_m(t-k)}_{\text{DIRICHLET KERNEL}} dt$$

$$D_m = \frac{1}{2\pi} \frac{\sin[(m+\frac{1}{2})(t-k)]}{\sin(\frac{t-k}{2})}$$

FOURIER TRANSFORM

INTEGRAL TRANSFORM

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EXS: 1) FOURIER SERIES

$$S_m[f](k) = \int_{-\pi}^{\pi} f(t) \underbrace{1}_{\text{KERNEL}} dt$$

$$D_n = \frac{1}{2\pi} \frac{\sin\left[\left(n+\frac{1}{2}\right)\left(t-\frac{k}{2}\right)\right]}{\sin\left(\frac{t-k}{2}\right)}$$

INTEGRAL TRANSFORM

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2) LAPLACE TRANSFORM

$$a=0, b=\infty, K(t, k) = e^{-kt}$$

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2) LAPLACE TRANSFORM $a=0, b=\infty, K(t, k) = e^{-kt}$

FF: FOURIER INTEGRAL FORMULA

$$\hat{f}(k) = \int_{\mathbb{R}} f(x) e^{-i x \cdot k} dx = F[f](k)$$

INTEGRAL TRANSFORM

$$\hat{f}(k) = \int_a^b \underbrace{K(t, k)}_{\text{KERNEL}} f(t) dt$$

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2) LAPLACE TRANSFORM

$$a=0, b=\infty, K(t, k) = e^{-kt}$$

DEF. FOURIER INTEGRAL FORMULA

$$\hat{f}(k) = \int_{\mathbb{R}} f(x) e^{-ix \cdot k} dx = F(f)(k)$$

$$\check{f}(k) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(k) e^{ix \cdot k} dk = F^{-1}(f)$$

2) LAPLACE TRANSFORM

$$a=0, b=\infty, K(t, k) = e^{-kt}$$

DEF. FOURIER INTEGRAL FORMULA

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx = F(f)(k)$$

$$\check{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} dk = F^{-1}(f)(k)$$

DEF. FOURIER INTEGRAL FORMULA

$$x \cdot k = (x, k) = \langle x | k \rangle$$

$$\begin{aligned} \hat{f}(k) &= \int_{\mathbb{R}^d} f(x) e^{-i x \cdot k} dx = F[f](k) \\ \check{f}(x) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(k) e^{i x \cdot k} dk = F^{-1}[f](k) \end{aligned}$$

DEF. FOURIER INTEGRAL FORMULA

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$$x \cdot k = (x, k) = \langle x | k \rangle$$

$$\stackrel{\text{TODAY}}{=} x_1 k_1 + x_2 k_2 + \dots - x_1 k_1 + x_2 k_2 + \dots$$

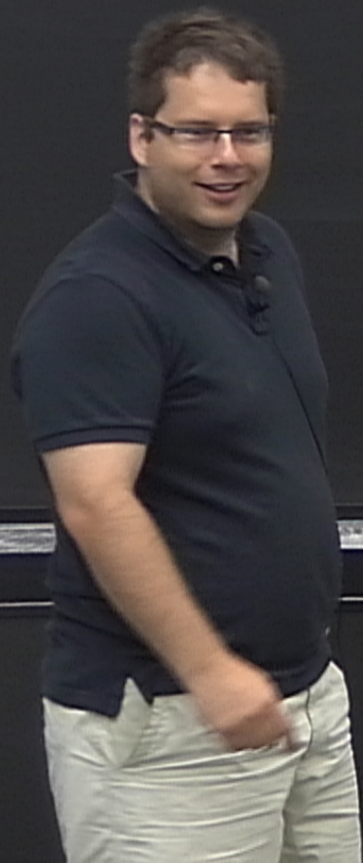
DEF. FOURIER INTEGRAL FORMULA

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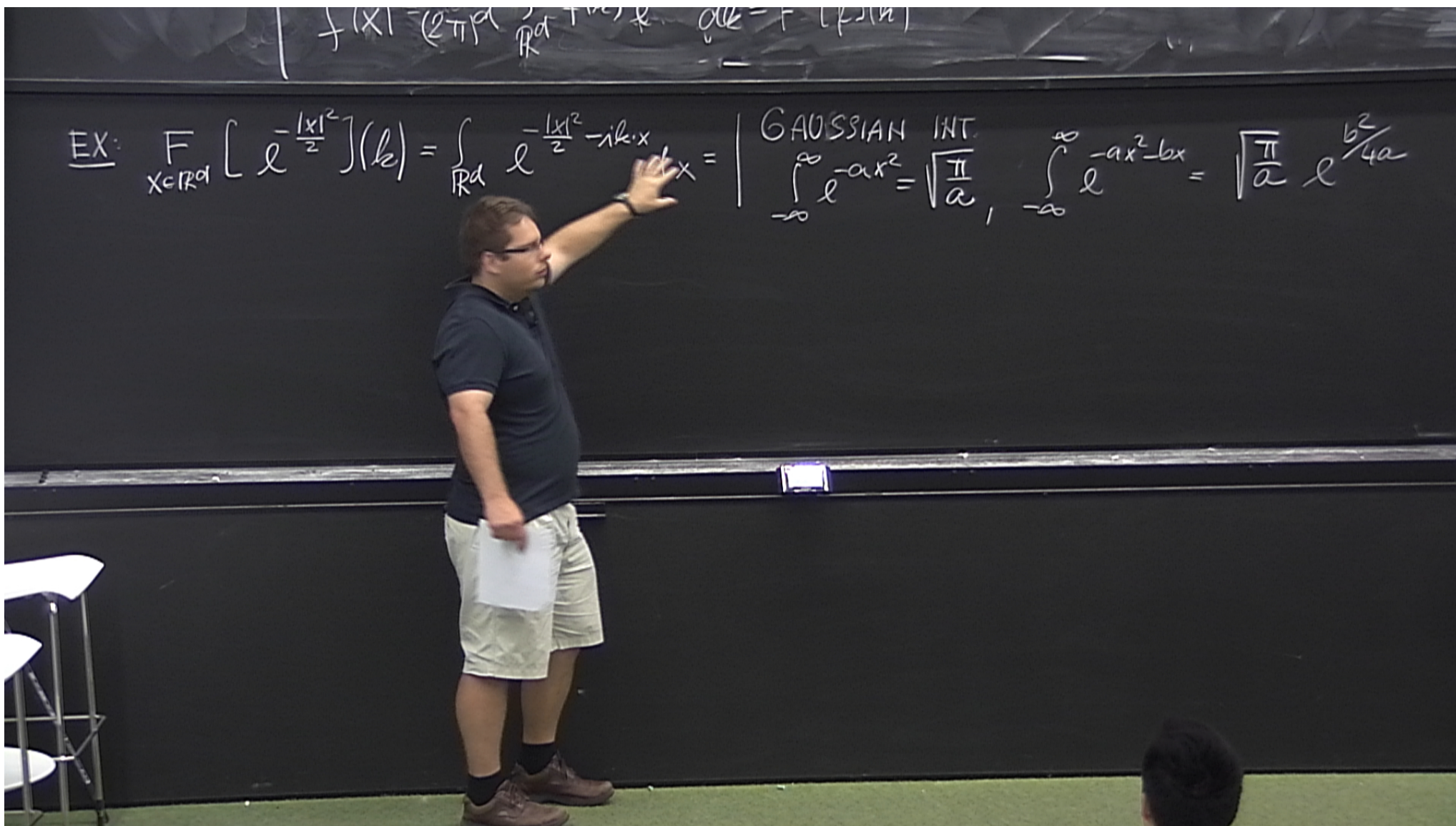
EX:
$$\mathcal{F}_{x \in \mathbb{R}^d} \left[e^{-\frac{|x|^2}{2}} \right](k) = \int_{\mathbb{R}^d} e^{-\frac{|x|^2}{2} - ik \cdot x} dx$$



EX: $F_{x \in \mathbb{R}^d} \left[e^{-\frac{|x|^2}{2}} \right](k) = \int_{\mathbb{R}^d} e^{-\frac{|x|^2}{2} - ik \cdot x} dx = \left| \begin{array}{l} \text{GAUSSIAN INT.} \\ \int_{-\infty}^{\infty} \end{array} \right.$

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 | GAUSSIAN INT.

$$\int_{-\infty}^{\infty} e^{-ax^2} = \sqrt{\frac{\pi}{a}}$$



$$\text{EX: } \mathcal{F}_{x \in \mathbb{R}^d} \left[e^{-\frac{|x|^2}{2}} \right](k) = \int_{\mathbb{R}^d} e^{-\frac{|x|^2}{2} - ik \cdot x} dx = \left| \begin{array}{l} \text{GAUSSIAN INT.} \\ \int_{-\infty}^{\infty} e^{-ax^2} = \sqrt{\frac{\pi}{a}}, \quad \int_{-\infty}^{\infty} e^{-ax^2 - bx} = \sqrt{\frac{\pi}{a}} e^{-\frac{b^2}{4a}} \\ a = \frac{1}{2}, \quad b = \end{array} \right.$$

$$\begin{aligned}
 \text{EX: } \mathcal{F}_{x \in \mathbb{R}^d} \left[e^{-\frac{|x|^2}{2}} \right](k) &= \int_{\mathbb{R}^d} e^{-\frac{|x|^2}{2} - ik \cdot x} dx = \left(\begin{array}{l} \text{GAUSSIAN INT} \\ \int_{-\infty}^{\infty} e^{-ax^2} = \sqrt{\frac{\pi}{a}}, \quad \int_{-\infty}^{\infty} e^{-ax^2 - bx} = \sqrt{\frac{\pi}{a}} e^{-\frac{b^2}{4a}} \\ a = \frac{1}{2}, \quad b = ik \end{array} \right) \\
 &= (2\pi)^{d/2} e^{-|k|^2/2}
 \end{aligned}$$

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$$\int_{-\infty}^{\infty} e^{-ax^2} = \sqrt{\frac{\pi}{a}}, \quad \int_{-\infty}^{\infty} e^{-ax^2 - bx} = \sqrt{\frac{\pi}{a}} e^{-\frac{b^2}{4a}}$$

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 \text{EX: } F_{x \in \mathbb{R}^d} \left[e^{-\frac{|x|^2}{2}} \right](k) &= \int_{\mathbb{R}^d} e^{-\frac{|x|^2}{2} - ik \cdot x} dx = \left(\begin{array}{l} \text{GAUSSIAN INT.} \\ \int_{-\infty}^{\infty} e^{-ax^2} = \sqrt{\frac{\pi}{a}}, \quad \int_{-\infty}^{\infty} e^{-ax^2 - bx} = \sqrt{\frac{\pi}{a}} e^{-\frac{b^2}{4a}} \\ a = \frac{1}{2}, \quad b = ik \end{array} \right) \\
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 \text{[NOTE. } F^{-1} \left[e^{-|k|^2/2} (2\pi)^{d/2} \right](x) &= e^{-|x|^2/2}
 \end{aligned}$$

$$\mathcal{F}(x) = (2\pi)^{d/2} \int_{\mathbb{R}^d} f(x) e^{-ik \cdot x} dx = \mathcal{F}[f](k)$$

EX: $\mathcal{F}_{x \in \mathbb{R}^d} \left[e^{-\frac{|x|^2}{2}} \right](k) = \int_{\mathbb{R}^d} e^{-\frac{|x|^2}{2} - ik \cdot x} dx =$

GAUSSIAN INT. $\int_{-\infty}^{\infty} e^{-ax^2} = \sqrt{\frac{\pi}{a}}, \quad \int_{-\infty}^{\infty} e^{-ax^2 - bx} = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}}$
 $a = \frac{1}{2}, \quad b = ik$

$= (2\pi)^{d/2} e^{-|k|^2/2}$

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 \end{aligned}$$

$$\hat{f}(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(k) e^{ix \cdot k} dk = F^{-1}(f)(k)$$

$$-x_1 k_1 + x_2 k_2 + \dots$$

$$f \in L^1 \quad \int f < \infty, \quad \int f^2 < \infty$$

EX: $F \left[e^{-\frac{|x|^2}{2}} \right](k) = \int_{\mathbb{R}^d} e^{-\frac{|x|^2}{2} - ik \cdot x} dx =$

GAUSSIAN INT. $\int_{-\infty}^{\infty} e^{-ax^2} = \sqrt{\frac{\pi}{a}}, \quad \int_{-\infty}^{\infty} e^{-ax^2 - bx} = \sqrt{\frac{\pi}{a}} e^{-\frac{b^2}{4a}}$

$b = ik$

$= (2\pi)^{d/2} e^{-|k|^2/2}$

[NOTE: $F^{-1} \left[e^{-|k|^2/2} (2\pi)^{d/2} \right](x) = e^{-|x|^2/2}$]

BASIC PROPERTIES LET $f, g \in L^1 \cap L^2 \rightarrow$

1) $\hat{f} \in C$

2) $\lim_{|k| \rightarrow \infty} \hat{f}(k) = 0$ (R-L)

3) CONVOLUTION: $F[f * g] = \hat{f}(k) \hat{g}(k)$

8) F AND F^{-1} ARE INVERSIONS

4) SHIFT: $f_a(x) = f(x+a) \Rightarrow \hat{f}_a(k) = e^{ik \cdot a} \hat{f}(k)$

5) SCALE: $a \in \mathbb{R} \Rightarrow F[f(ax)](k) = \frac{1}{|a|^d} \hat{f}\left(\frac{k}{a}\right)$

6) DERIVATIVES: $F[D^k f](k) = (ik)^k \hat{f}(k)$
 $F[(ix)^k f](k) = (-1)^{|k|} D^k \hat{f}(k)$

7) PARSEVAL: $(f, g)_{L^2} = \frac{1}{(2\pi)^d} (\hat{f}, \hat{g})_{L^2}$

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EXAMPLE. SOLVE THE CAUCHY PROBLEM FOR THE
DIFFUSION EQUATION.

$$\frac{\partial u}{\partial t} - \Delta u = f \quad \text{on } \mathbb{R}^n \times \mathbb{R}_+$$

EXAMPLE. SOLVE THE CAUCHY PROBLEM FOR THE
DIFFUSION EQUATION.

$$\frac{\partial u}{\partial t} - \Delta u = f \quad \text{ON } (0, T) \times \mathbb{R}^n$$

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$$\frac{\partial u}{\partial t} - \Delta u = f \quad \text{ON } (0, T) \times \mathbb{R}^d$$
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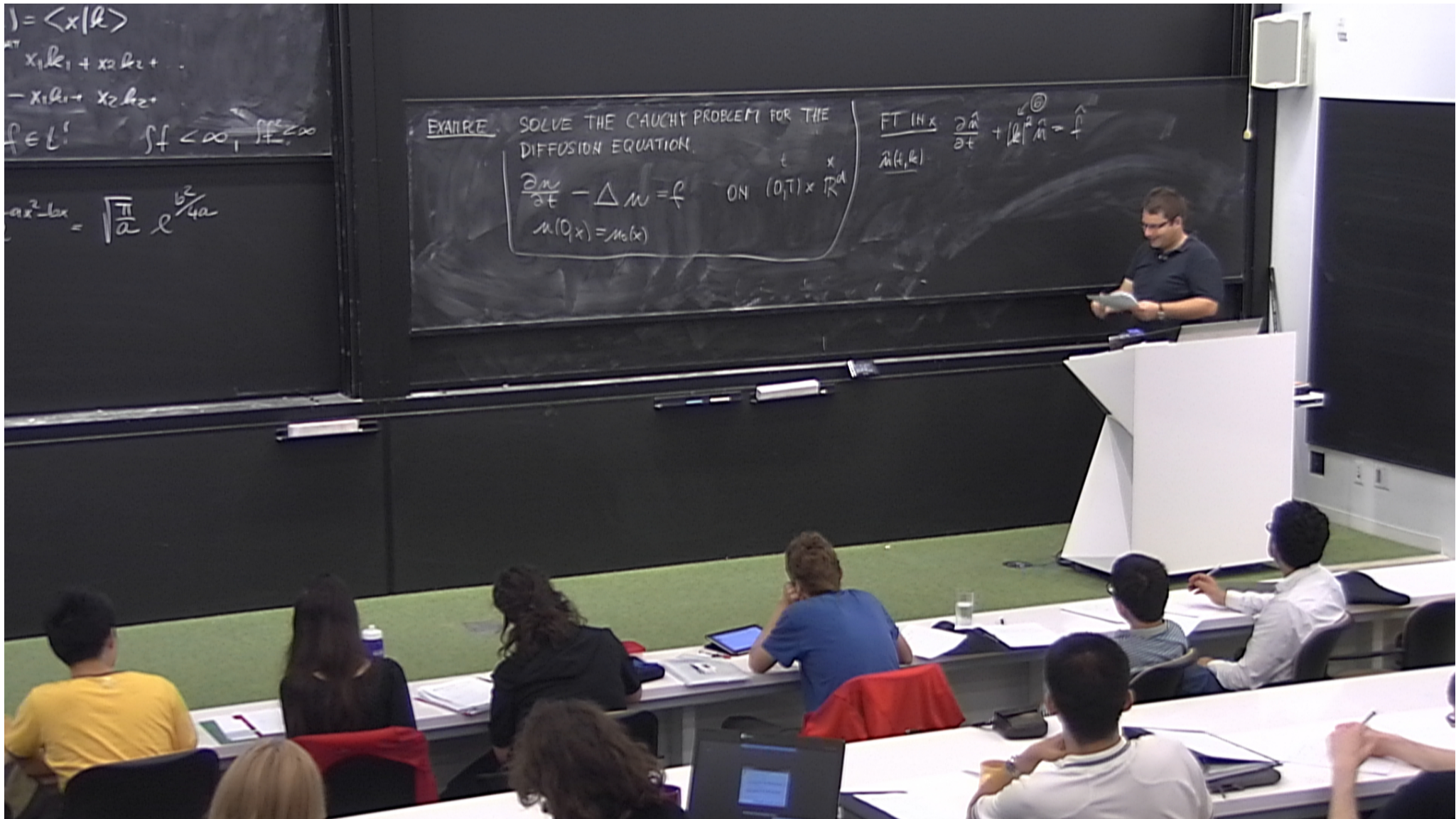
$$\frac{\partial u}{\partial t} - \Delta u = f \quad \text{ON } (0, T) \times \mathbb{R}^d$$
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FT IN x $\frac{\partial u}{\partial t}$

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FT IN x $\frac{\partial \hat{u}}{\partial t}$
 $\hat{u}(t, k)$



$$f = \langle x | k \rangle$$
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$$f \in L^1 \quad \int f < \infty, \quad \int f^2 < \infty$$

$$e^{-ax^2 - bx} = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}}$$

EXAMPLE. SOLVE THE CAUCHY PROBLEM FOR THE DIFFUSION EQUATION.

$$\frac{\partial w}{\partial t} - \Delta w = f \quad \text{ON } (0, T) \times \mathbb{R}^d$$
$$w(0, x) = w_0(x)$$

FT IN x

$$\hat{w}(t, k) \quad \frac{\partial \hat{w}}{\partial t} + |k|^2 \hat{w} = \hat{f}$$

EXAMPLE. SOLVE THE CAUCHY PROBLEM FOR THE
DIFFUSION EQUATION.

$$\frac{\partial u}{\partial t} - \Delta u = f \quad \text{ON } (0, T) \times \mathbb{R}^d$$
$$u(0, x) = u_0(x)$$

FT IN x : $\frac{\partial \hat{u}}{\partial t} + |\mathbf{k}|^2 \hat{u} = \hat{f}$

$\hat{u}(t, \mathbf{k})$

METHOD OF INTE



METHOD OF INTEGRATION FACTOR

$$\frac{dy}{dx}$$

METHOD OF INTEGRATION FACTOR

$$\frac{dy}{dx} + ay = b \quad / \quad e^{ax}$$



METHOD OF INTEGRATION FACTOR

$$\frac{dy}{dx} + ay = b \quad / \quad e^{ax}$$

$$\frac{d}{dx}(ye^{ax}) = be^{ax}$$

METHOD OF INTEGRATION FACTOR

$$\frac{dy}{dt} + ay = b \quad / \quad e^{at}$$

$$\frac{d}{dt}(ye^{at}) = be^{at} \quad / \quad \int_0^t dt$$

METHOD OF INTEGRATION FACTOR

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EXERCISE. SOLVE THE CAUCHY PROBLEM FOR THE
DIFFUSION EQUATION.

$$\frac{\partial u}{\partial t} - \Delta u = f \quad \text{ON } (0, T) \times \mathbb{R}^d$$

$$u(0, x) = u_0(x)$$

FT IN x $\frac{\partial \hat{u}}{\partial t} + |k|^2 \hat{u} = \hat{f}$ ^⑥

$\hat{u}(t, k)$

$$\hat{u}(t, k) = \hat{u}_0(k) e^{-|k|^2 t} + \int_0^t \hat{f}(\tau, k) e^{-|k|^2 (t-\tau)} d\tau$$

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 8) F AND F^{-1} ARE INVERSIONS $(f, g)_{L^2} = \frac{1}{(2\pi)^d} (\hat{f}, \hat{g})_{L^2}$

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WE KNOW: $e^{-|k|^2 t/2} = \frac{1}{(2\pi)^{d/2}} e^{-|x|^2/2}$

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WE KNOW: $e^{-|k|^2 t} = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-|x|^2/4t} e^{-ik \cdot x} dx$

⑤ $a = \frac{1}{4t} \Rightarrow e^{-|k|^2 t} = \frac{1}{(2t)^{d/2}} e^{-\frac{|x|^2}{4t}} \frac{1}{(2\pi)^{d/2}}$

$$\textcircled{5} \quad a = \frac{1}{4t} \Rightarrow e^{-|x|^2/t} = \frac{1}{(4t)^{d/2}} e^{-\frac{|x|^2}{4t}} \frac{1}{(2\pi)^{d/2}}$$

$$g(t, x) = \frac{1}{(4t)^{d/2}} e^{-\frac{|x|^2}{4t}}$$

$$\textcircled{5} \quad a = \frac{1}{4t} \Rightarrow e^{-|x|^2 t} = \frac{1}{(2t)^{d/2}} e^{-\frac{|x|^2}{4t}} \frac{1}{(2\pi)^{d/2}}$$

$$\boxed{g(t, x) = \frac{1}{(2t)^{d/2}} e^{-\frac{|x|^2}{4t}}} \quad \text{HEAT KERNEL}$$

$$\hat{u}(t, x) = \frac{1}{(2\pi)^{d/2}} \left[u_0 * g(t, \cdot) + \int_0^t \right]$$

$$\textcircled{5} \quad a = \frac{1}{4t} \Rightarrow e^{-|x|^2 t} = \frac{1}{(2t)^{d/2}} e^{-\frac{|x|^2}{4t}} \frac{1}{(2\pi)^{d/2}}$$

$$\boxed{g(t, x) = \frac{1}{(2t)^{d/2}} e^{-\frac{|x|^2}{4t}}} \quad \text{HEAT KERNEL}$$

$$\hat{u}(t, x) = \frac{1}{(2\pi)^{d/2}} \left[u_0 * g(t, \cdot) + \int_0^t f(\tau, \cdot) * g(t-\tau)(\cdot) \right]$$

$$\textcircled{5} \quad a = \frac{1}{4t} \Rightarrow e^{-|x|^2/a} = \frac{1}{(2t)^{d/2}} e^{-\frac{|x|^2}{4t}} \frac{1}{(2\pi)^{d/2}}$$

$$\boxed{g(t, x) = \frac{1}{(2t)^{d/2}} e^{-\frac{|x|^2}{4t}}} \quad \text{HEAT KERNEL}$$

$$\hat{u}(t, x) = \frac{1}{(2\pi)^{d/2}} \left[u_0 * g(t, \cdot) + \int_0^t f(\tau, \cdot) * g(t-\tau, \cdot)(x) d\tau \right]$$

$$\hat{h}(t, x) = \frac{1}{(2\pi)^{d/2}} \left[\widehat{m_0 * g}(x, t) + \int_0^t \widehat{f(\tau, \cdot) * g(t-\tau, \cdot)}(x) d\tau \right]$$

LOOKING BENEATH HATS.

$m(t, x)$

$$\hat{h}(t, x) = \frac{1}{(2\pi)^{d/2}} \left[\widehat{m_0 * g}(x, t) + \int_0^t \widehat{f(\tau, \cdot) * g(t-\tau, \cdot)}(x) d\tau \right]$$

LOOKING BENEATH HATS.

$$M(t, x) = \frac{1}{(2\pi)^{d/2}} \left(\int_{\mathbb{R}^d} \frac{e^{-\frac{|x-y|^2}{4t}}}{(2t)^{d/2}} m_0(y) dy \right)$$

$$\hat{h}(t, x) = \frac{1}{(2\pi)^{d/2}} \left[\widehat{m_0 * g}(x, t) + \int_0^t \widehat{f(\tau, \cdot) * g(t-\tau, \cdot)}(x) d\tau \right]$$

LOOKING BENEATH HATS.

$$M(t, x) = \frac{1}{(2\pi)^{d/2}} \left[\int_{\mathbb{R}^d} \frac{e^{-\frac{|x-y|^2}{4t}}}{(2t)^{d/2}} m_0(y) dy + \int_0^t \int_{\mathbb{R}^d} \frac{e^{-\frac{|x-y|^2}{4(t-\tau)}}}{[2(t-\tau)]^{d/2}} f(\tau, y) dy d\tau \right]$$

FT IN QM:



FT IN QM: • STATE $|\psi\rangle$
• SPEC. EIGENSTATES



FT IN QM:

- STATE $|\psi\rangle$
- SPEC. EIGENSTATES OF OPS. $\hat{X}$ AND $\hat{P}$

FT IN QM:

• STATE $|\psi\rangle$

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$$|x\rangle = X| \quad \text{[unclear]}$$

FT IN QM:

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$$\hat{X}|x\rangle = x|x\rangle$$

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FT IN QM:

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FORM A COMPLETE ORTHOGONAL BASIS

$$\mathbb{I} = \sum |x\rangle\langle x| =$$



FT IN QM:

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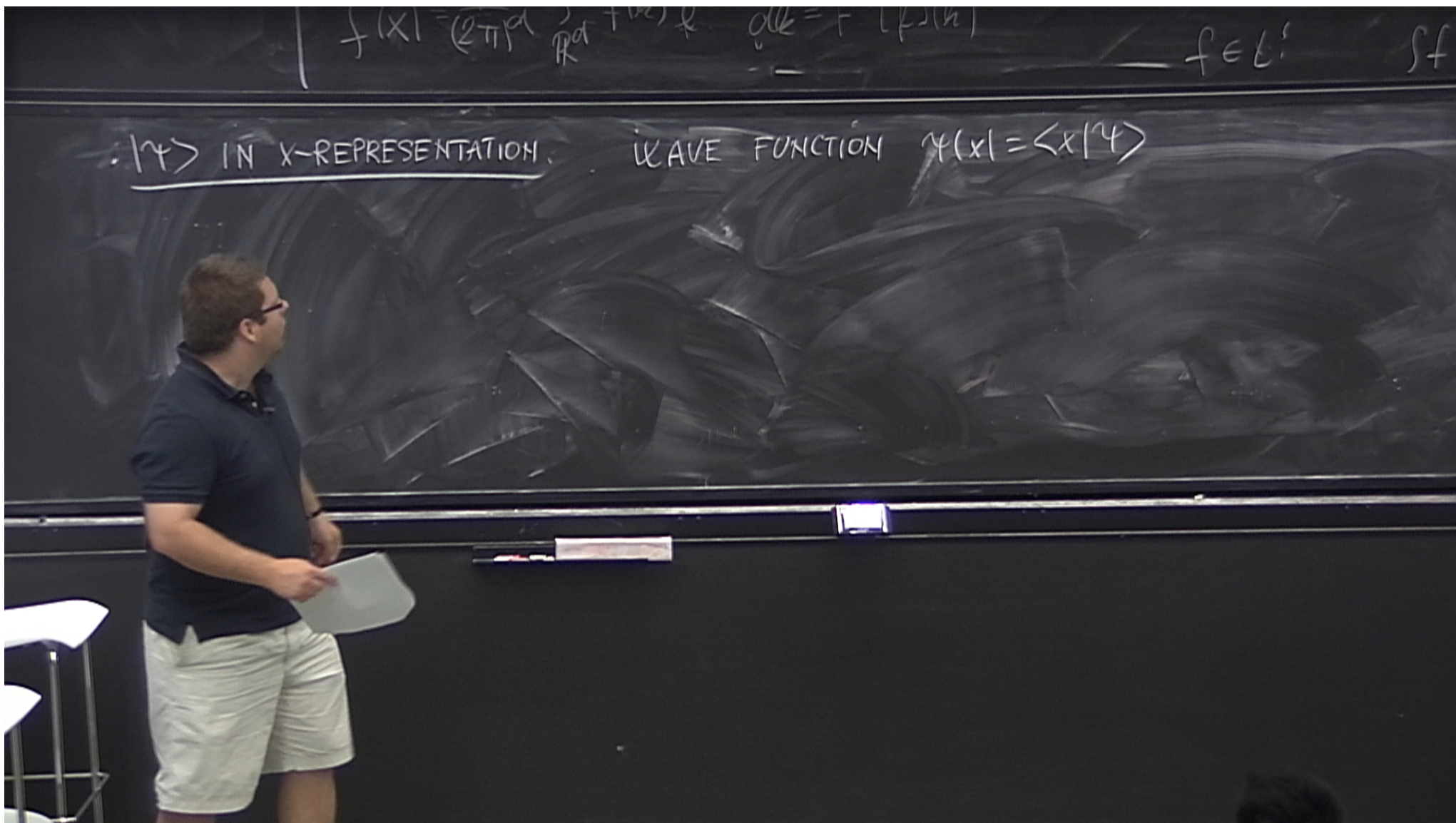
$$\hat{P}|p\rangle = p|p\rangle$$

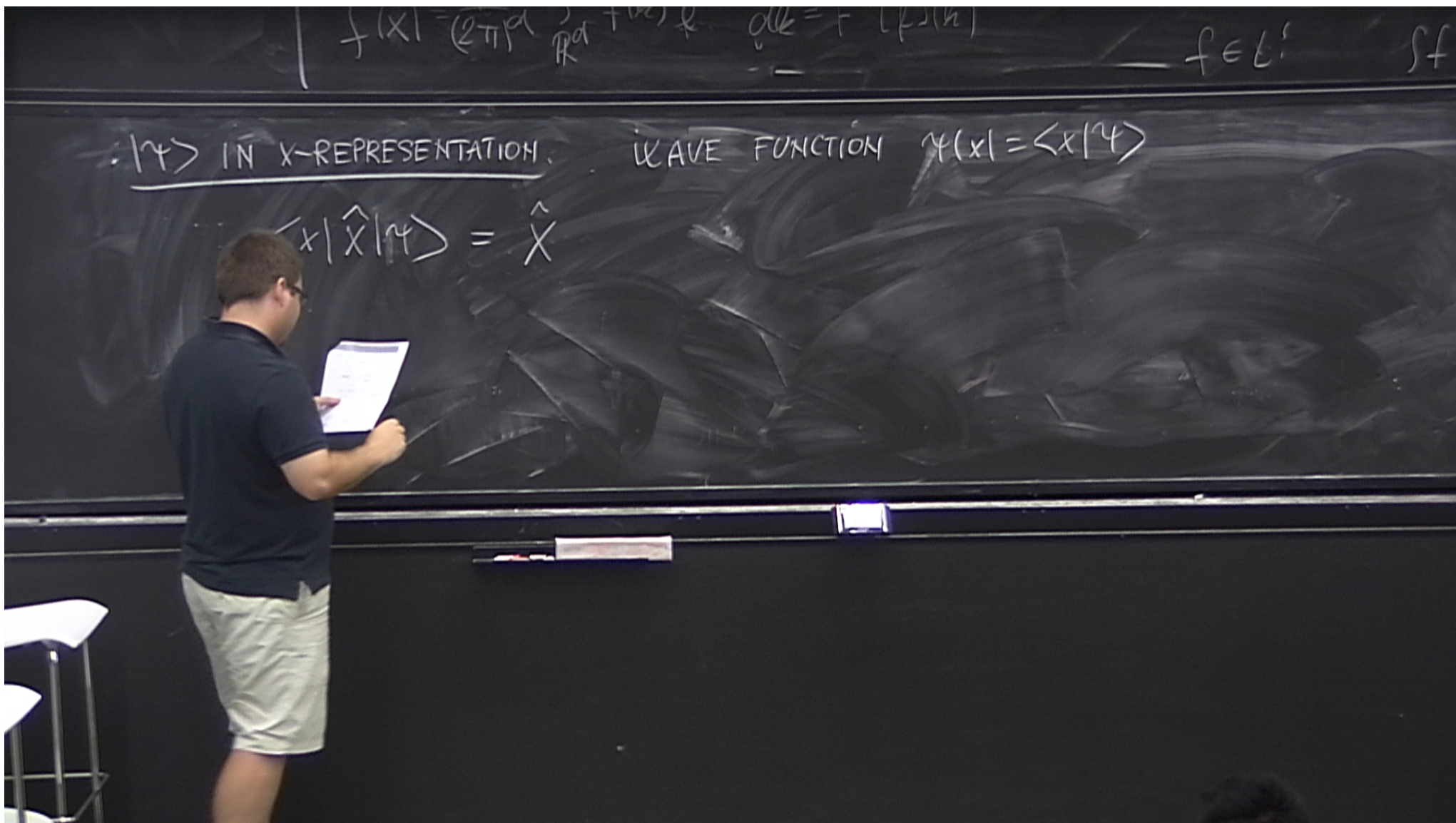
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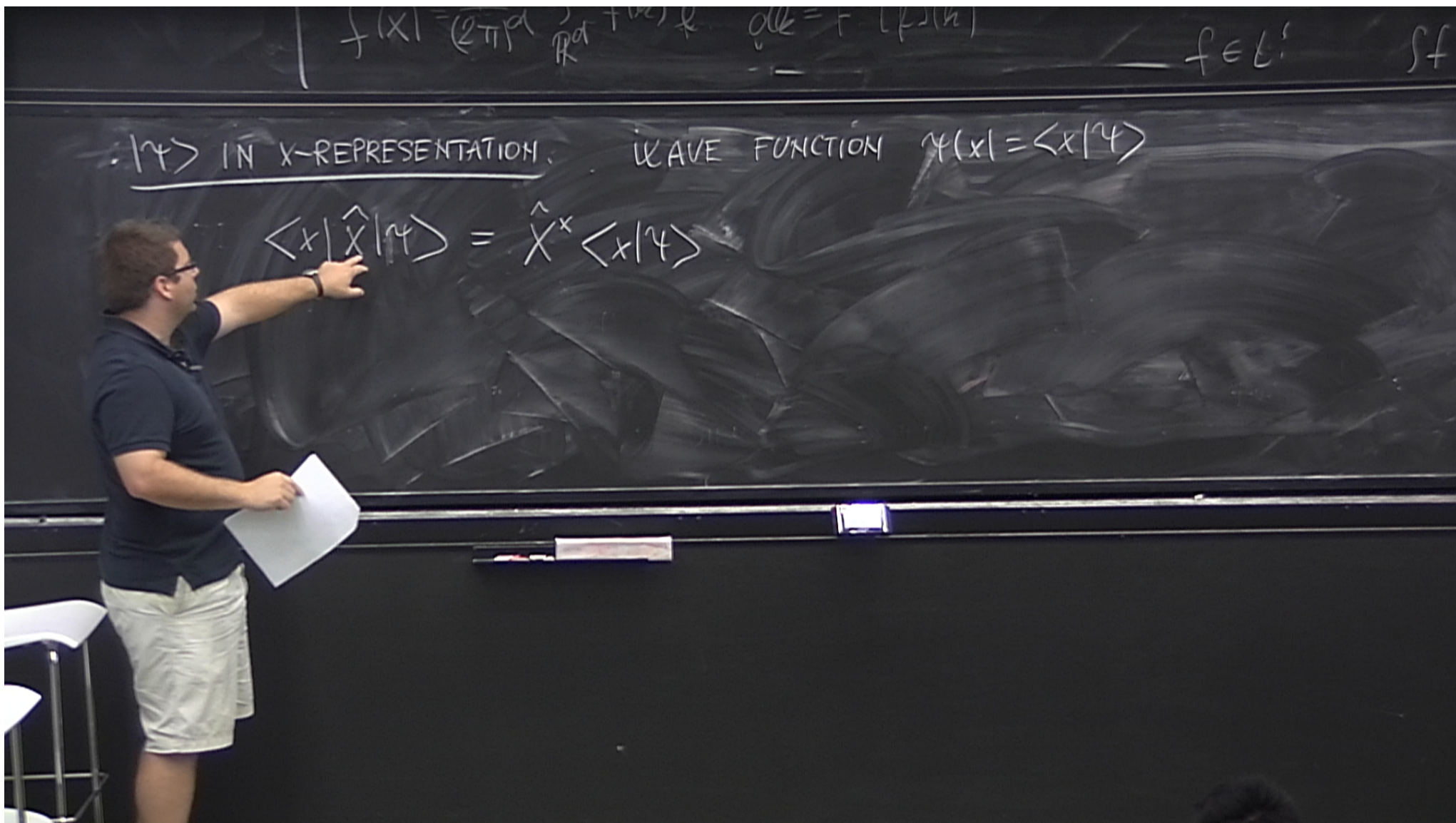
$$\mathbb{I} = \sum |x\rangle\langle x| = \int |x\rangle\langle x| dx$$

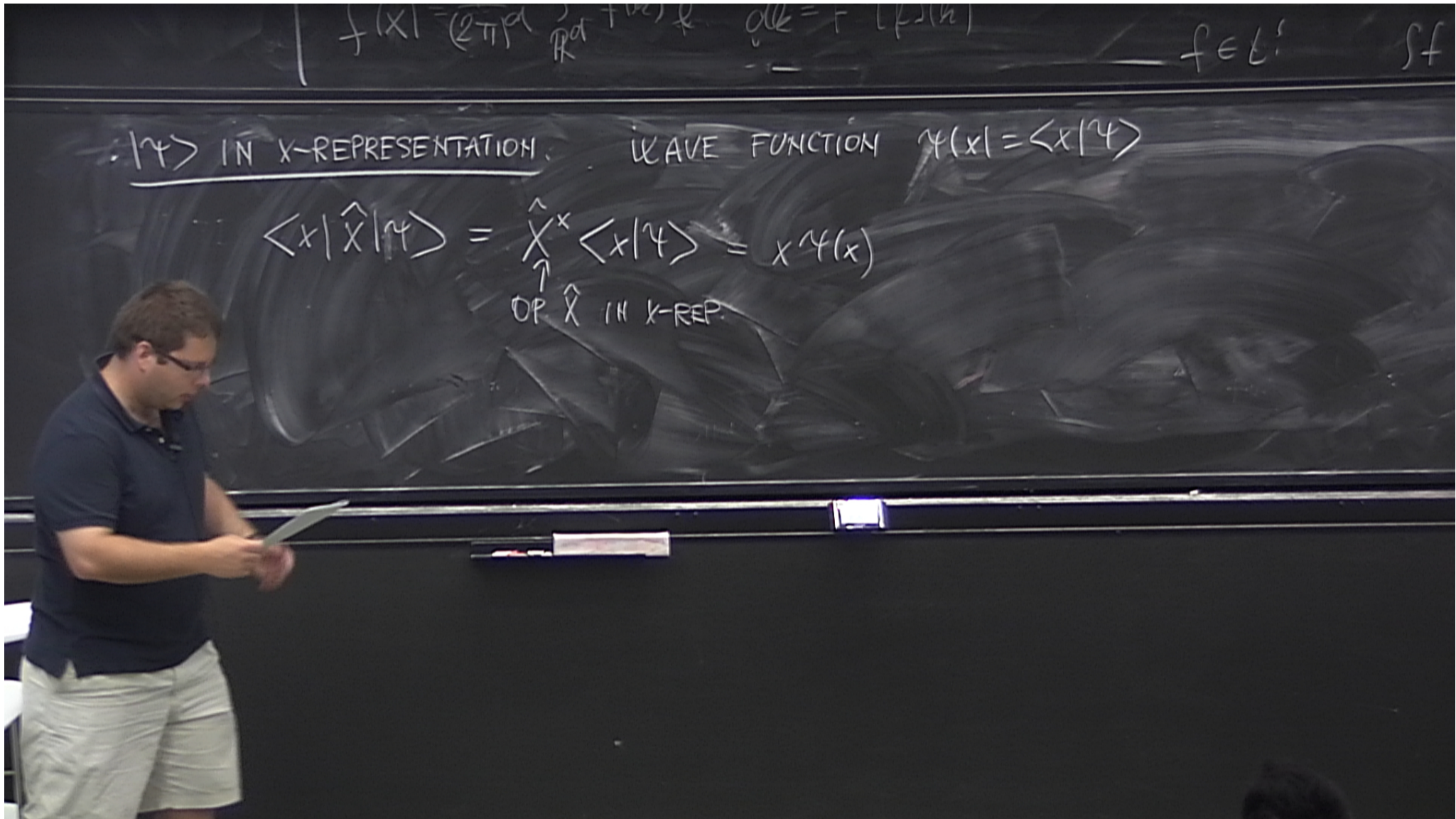
$$\mathbb{I} = \int |p\rangle\langle p| = \frac{1}{2\pi\hbar}$$

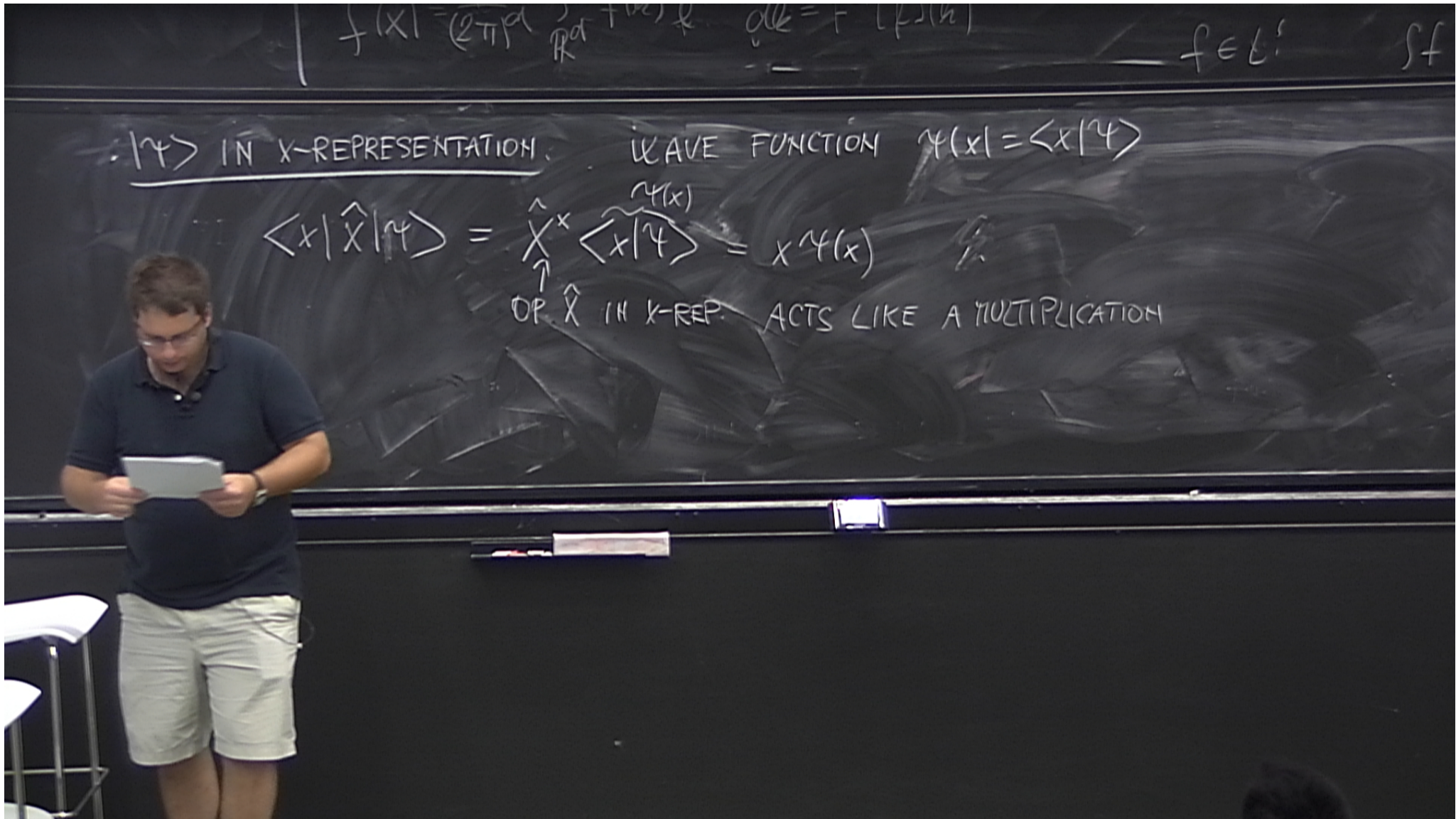
$$y(x) = y(0) = \int_0^x b(t) e^{at} dt$$











$|\psi\rangle$ IN X-REPRESENTATION WAVE FUNCTION $\psi(x) = \langle x|\psi\rangle$

$$\langle x|\hat{X}|\psi\rangle = \hat{X}^x \langle x|\psi\rangle = x\psi(x)$$

OP. $\hat{X}$ IN X-REP. ACTS LIKE A MULTIPLICATION

• I WANT TO SATISFY $[\hat{X}, \hat{P}] = i$

$$y(t)e^{at} - y(0) = \int_0^t dt' b(t')e^{at'}$$

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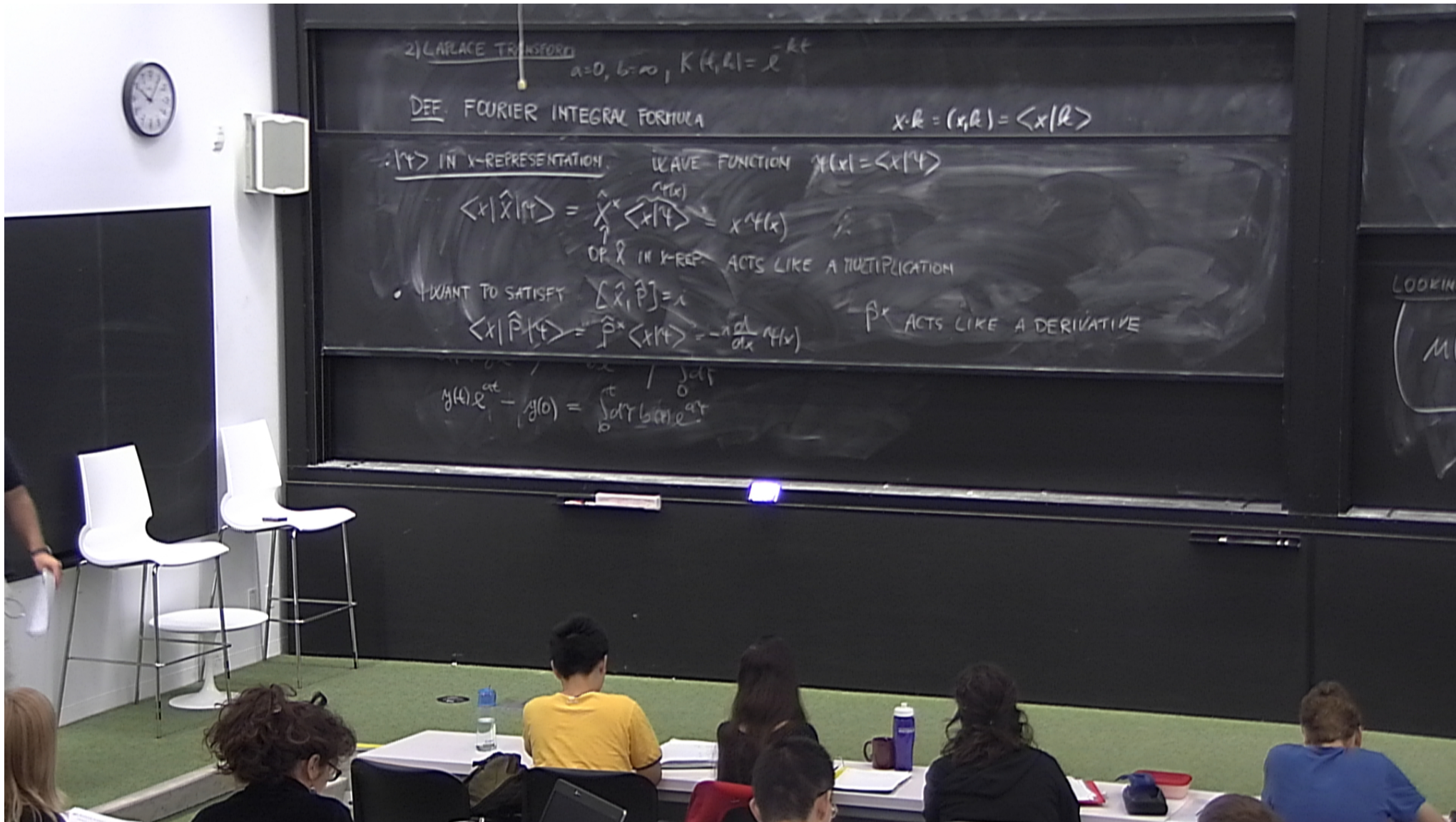
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$$\langle x|\hat{P}|\psi\rangle = \hat{P}^x \langle x|\psi\rangle = -i\frac{d}{dx}\psi(x)$$

$$y(t)e^{at} - y(0) = \int_0^t dt b(t)e^{at}$$



DEF: FOURIER INTEGRAL FORMULA

$$x \cdot k = (x, k) = \langle x | k \rangle$$

$|\psi\rangle$ IN X-REPRESENTATION

WAVE FUNCTION $\psi(x) = \langle x | \psi \rangle$

$$\langle x | \hat{X} | \psi \rangle = \hat{X}^x \langle x | \psi \rangle = x \psi(x)$$

OP $\hat{X}$ IN X-REP. ACTS LIKE A MULTIPLICATION

• I WANT TO SATISFY $[\hat{X}, \hat{P}] = i$

$\hat{P}$ ACTS LIKE A DERIVATIVE

$$\langle x | \hat{P} | \psi \rangle = \hat{P}^x \langle x | \psi \rangle = -i \frac{d}{dx} \psi(x)$$

$$y(t)e^{at} - y(0) = \int_0^t dt b(t)e^{at}$$



OP. $\hat{X}$ IN X-REP. ACTS LIKE A MULTIPLICATION

• I WANT TO SATISFY $[\hat{X}, \hat{P}] = i$

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$$\langle x | \hat{P} | \psi \rangle = \hat{P}^* \langle x | \psi \rangle = -i \frac{d}{dx} \psi(x)$$

$$\langle x | \hat{P} | p \rangle = \underbrace{p \langle x | p \rangle} = \underbrace{P^* \langle x | p \rangle} = -i \frac{d}{dx} \langle x | p \rangle$$

• I WANT TO SATISFY $[\hat{x}, \hat{p}] = i$

$\hat{p}$ ACTS LIKE A DERIVATIVE

$$\langle x | \hat{p} | \psi \rangle = \hat{p}^* \langle x | \psi \rangle = -i \frac{d}{dx} \psi(x)$$

$$\langle \psi | = \langle p | : \quad \langle x | \hat{p} | p \rangle = p \langle x | p \rangle = \hat{p}^* \langle x | p \rangle = -i \frac{d}{dx} \langle x | p \rangle$$

$$\Rightarrow \langle x | p \rangle = e^{ipx}$$

• I WANT TO SATISFY $[\hat{x}, \hat{p}] = i$

$\hat{p}$ ACTS LIKE A DERIVATIVE

$$\langle x | \hat{p} | \psi \rangle = \hat{p}^* \langle x | \psi \rangle = -i \frac{d}{dx} \psi(x)$$

$\psi(x) = 1/p$: $\langle x | \hat{p} | p \rangle = p \langle x | p \rangle = p^* \langle x | p \rangle = -i \frac{d}{dx} \langle x | p \rangle$

$$\Rightarrow \langle x | p \rangle = e^{i p x}$$

• P-REPRESENT

$$\psi(p) = \langle p | \psi \rangle = \sum_x \underbrace{\langle p | x \rangle}_{=1} \langle x | \psi \rangle =$$

• I WANT TO SATISFY $[\hat{x}, \hat{p}] = i$

$\hat{p}$ ACTS LIKE A DERIVATIVE

$$\langle x | \hat{p} | \psi \rangle = \hat{p}^* \langle x | \psi \rangle = -i \frac{d}{dx} \psi(x)$$

$$\psi(x) = \langle x | \psi \rangle : \langle x | \hat{p} | p \rangle = p \langle x | p \rangle = p^* \langle x | p \rangle = -i \frac{d}{dx} \langle x | p \rangle$$

$$\Rightarrow \langle x | p \rangle = e^{i p \cdot x}$$

• P-REPRESENT

$$\psi(p) = \langle p | \psi \rangle = \sum_x \underbrace{\langle p | x \rangle}_{=1} \langle x | \psi \rangle = \int \psi(x) e^{i p \cdot x}$$

• I WANT TO SATISFY $[\hat{x}, \hat{p}] = i$

$\hat{p}$ ACTS LIKE A DERIVATIVE

$$\langle x | \hat{p} | \psi \rangle = \hat{p}^* \langle x | \psi \rangle = -i \frac{d}{dx} \psi(x)$$

$\psi(x) = \langle p | \psi \rangle$: $\langle x | \hat{p} | p \rangle = p \langle x | p \rangle = p^* \langle x | p \rangle = -i \frac{d}{dx} \langle x | p \rangle$

$$\Rightarrow \langle x | p \rangle = e^{i p x}$$

• P-REPRESENT

$$\psi(p) = \langle p | \psi \rangle = \sum_x \underbrace{\langle p | x \rangle}_{=1} \langle x | \psi \rangle = \int \psi(x) e^{-i p \cdot x} dx$$

- I WANT TO SATISFY $[\hat{x}, \hat{p}] = i$

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$$\langle x | \hat{p} | \psi \rangle = \hat{p}^* \langle x | \psi \rangle = -i \frac{d}{dx} \psi(x)$$

$$\psi(x) = \langle x | \psi \rangle : \langle x | \hat{p} | p \rangle = p \langle x | p \rangle = p^* \langle x | p \rangle = -i \frac{d}{dx} \langle x | p \rangle$$

$$\Rightarrow \langle x | p \rangle = e^{i p x}$$

P AND X REPS.
CONNECTED

- P-REPRESENT.

$$\psi(p) = \langle p | \psi \rangle = \sum_x \underbrace{\langle p | x \rangle}_{=1} \langle x | \psi \rangle = \int \psi(x) e^{-i p x} dx$$

• I WANT TO SATISFY $[\hat{x}, \hat{p}] = i$

$$\langle x | \hat{p} | \psi \rangle = \hat{p}^x \langle x | \psi \rangle = -i \frac{d}{dx} \psi(x)$$

$\hat{p}^x$ ACTS LIKE A DERIVATIVE

$$\langle \psi | \hat{p} | \psi \rangle = \langle x | \hat{p} | \psi \rangle = p \langle x | \psi \rangle = \hat{p}^x \langle x | \psi \rangle = -i \frac{d}{dx} \langle x | \psi \rangle$$

$$\Rightarrow \langle x | \psi \rangle = e^{i p \cdot x}$$

P AND X REPS.
CONNECTED BY F.T.

• P-REPRESENT.

$$\psi(p) = \langle p | \psi \rangle = \sum_x \langle p | x \rangle \langle x | \psi \rangle = \int \psi(x) e^{-i p \cdot x} dx$$

FT OF δ -FUNCTION

$$f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} dk e^{ik \cdot x} \hat{f}(k)$$

FT OF $\mathcal{J}$ -FUNCTION

$$f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} dk e^{ik \cdot x} \underbrace{f(k)}_{\int_{\mathbb{R}^d} f(y) e^{-ik \cdot y} dy} =$$

FT OF δ -FUNCTION:

$$f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} dk e^{ik \cdot x} \underbrace{f(k)}_{\int_{\mathbb{R}^d} f(y) e^{-ik \cdot y} dy} = \int_{\mathbb{R}^d} dy f(y) \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} dk e^{ik \cdot (x-y)}$$

$$\delta(x-y) = \frac{1}{(2\pi)\alpha} \int_{\mathbb{R}} e^{ik(x-y)} dk$$

NOTE

$$\int \gamma d\gamma$$

$$\delta(x-y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ik(x-y)} dk$$

NOTE: NOTHING ELSE THAN

$$\langle x|y \rangle = \delta(x-y) = \sum_{\substack{\underbrace{\langle x|p \rangle}_{e^{ipx}} \underbrace{\langle p|y \rangle}_{e^{-ipy}}}} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ip \cdot (x-y)} dp$$