

Title: Integral Transforms & Green's Functions - Lecture 1

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Abstract:

INTEGRAL TRANSFORMS & GREEN'S FUNCTIONS

PLAN:

- 1) GREEN'S F
- 2) FOURIER T
- 3)

INTEGRAL TRANSFORMS & GREEN'S FUNCTIONS

PLAN:

- 1) GREEN'S F
- 2) FOURIER T
- 3) CLASS. ELECTRODYNAMICS
- 4)

CONSIDER BVP:

$$L y = \alpha y'' + \beta y' + \gamma y = f$$

$$y(a) = y(b) = 0$$

$$x \in \langle a, b \rangle$$

CONSIDER BVP:

$$L y = \alpha y'' + \beta y' + \gamma y = f$$

$$y(a) = y(b) = 0$$

(BVP)
 $x \in (a, b)$

$$\mathcal{L} \mathcal{E} \mathcal{L} \quad G(x, \xi). \quad \mathcal{L} G = \overline{\partial}(x)$$

LET: $G(x, \xi)$.

$$\boxed{\begin{aligned} LG &= \delta(x - \xi) \\ G(a, \xi) &= G(b, \xi) = 0 \end{aligned}}$$

G ... GREEN'S FUNCTION

(6)

$\Rightarrow$

$$y(x) = \int_a^b G(x, \xi) f(\xi) d\xi$$

HOW TO FIND MIRACULOUS G ?

LET: $G(x, \xi)$.

$$\boxed{\begin{aligned} LG &= \delta(x - \xi) \\ G(a, \xi) &= G(L, \xi) = 0 \end{aligned}} \quad (6)$$

G ... GREEN'S FUNCT

PROOF:

$$\Rightarrow y(x) = \int_a^L G(x, \xi) f(\xi) d\xi \quad \text{IS THE SOL OF (BVP)}$$
$$Ly =$$

HOW TO FIND MIRACULOUS G

LET: $G(x, \xi)$.

$$\boxed{\begin{aligned} L G &= \delta(x - \xi) \\ G(a, \xi) &= G(b, \xi) = 0 \end{aligned}}$$

G ... GREEN'S FUNCTION

HOW TO FIND MIRACULOUS G ?

(6) $\Rightarrow u(x) = \int_a^b G(x, \xi) f(\xi) d\xi$ IS THE SOL OF (BVP)

PROOF

LET: $G(x, \xi)$.

$$LG = \delta(x - \xi)$$

$$G(a, \xi) = G(b, \xi) = 0$$

G ... GREEN'S FUNCTION

TO FIND MIRACULOUS G ?

$$(6) \Rightarrow y(x) = \int_a^b G(x, \xi) f(\xi) d\xi, \text{ IS THE SOL OF (BVP)}$$

PROOF: $Ly = \int_a^b (LG) f(\xi) d\xi = \int_a^b \delta(x - \xi) f(\xi) d\xi = f(x)$

$$y(a) = \int_a^b G(a, \xi) f(\xi) d\xi = 0 = y(b)$$

RECIPE: NOTE THAT EXCEPT $x=\xi$ G SOLVES THE HOMOGENEOUS EQ. ($Ly=0$)

1) $x < \xi$: $G(x, \xi) = A(\xi) y_1(x) + B(\xi) y_2(x)$

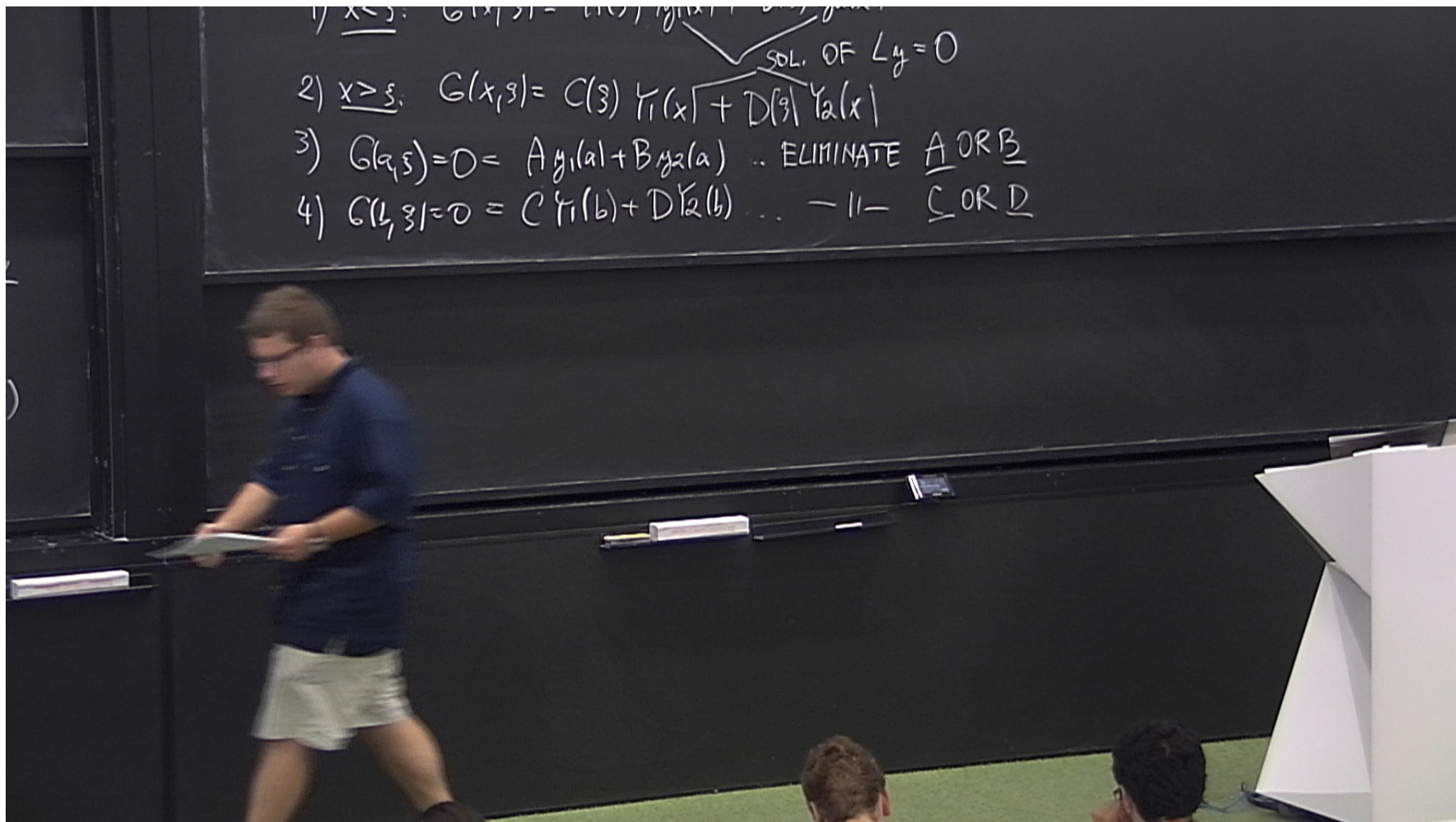
RECIPE: NOTE THAT EXCEPT $x=5$ G SOLVES THE HOMOGENEOUS EQ. ($L_y = 0$)

1) $x < \xi$: $G(x, \xi) = A(\xi) y_1(x) + B(\xi) y_2(x)$
 $\underbrace{\hspace{10em}}_{\text{sol. of } L_y = 0}$

RECIPE: NOTE THAT EXCEPT $x=5$ G SOLVES THE HOMOGENEOUS EQ. ($L_y = 0$)

1) $x < \xi$: $G(x, \xi) = A(\xi) y_1(x) + B(\xi) y_2(x)$
SOL. OF $L_y = 0$

2) $x > 3$.



$$3) G(a, \xi) = 0 = Ay_1(a) + By_2(a) \quad \dots \text{ELIMINATE } \underline{A} \text{ OR } \underline{B}$$

$$4) G(b, \xi) = 0 = Cy_1(b) + Dy_2(b) \quad \dots - || - \quad \underline{C} \text{ OR } \underline{D}$$

$$5) G \text{ CONTINUOUS AT } x = \xi$$

$$Ay_1(\xi) + By_2(\xi) = Cy_1(\xi) + Dy_2(\xi)$$

$$6) \underline{\text{JUMP CONDITION}} \quad [G']$$

5) G CONTINUOUS AT $x = \xi$

$$A y_1(\xi) + B y_2(\xi) = C y_1(\xi) + D y_2(\xi)$$

6) JUMP CONDITION $[G]_{x=\xi^-}^{x=\xi^+} = \frac{1}{\alpha(\xi)}$

$\Rightarrow$ ALL "CONSTANTS" ARE FIXED

$$y(x) = \int_a^b G(x, \xi) f(\xi) d\xi$$
$$= y_1(x) \int_x^b A(\xi) f(\xi) d\xi + y_2(x) \int_x^a$$

5) G CONTINUOUS AT $x = \xi$

$$A y_1(\xi) + B y_2(\xi) = C Y_1(\xi) + D Y_2(\xi)$$

6) JUMP CONDITION $[G]_{x=\xi^-}^{x=\xi^+} = \frac{1}{\alpha(\xi)}$

$\Rightarrow$ ALL "CONSTANTS" ARE FIXED

$$\begin{aligned} y(x) &= \int_a^b G(x, \xi) f(\xi) d\xi \\ &= y_1(x) \int_a^b A(\xi) f(\xi) d\xi + y_2(x) \int_x^b B(\xi) f(\xi) d\xi \\ &\quad + Y_1(x) \int_a^x C(\xi) f(\xi) d\xi + Y_2(x) \int_a^x D(\xi) f(\xi) d\xi \end{aligned}$$

REMARKS • G CONTINUOUS

IF DISCONTINUOUS $\Rightarrow G \sim \delta, G' \sim \delta',$
 $LG \sim \cancel{G'} + \dots \sim \delta \quad \downarrow$

• $\delta(x-s)$

$\Rightarrow$ ALL "CONSTANTS" ARE FIXED

REMARKS . • G CONTINUOUS

IF DISCONTINUOUS $\Rightarrow G \sim \delta, G' \sim \delta', LG \sim G'' + \dots \sim \delta$

$$\bullet LG = \delta(x-s) \quad / \quad \int_{s-}^{s+} dx$$

$$I = \int_{s-}^{s+} (\alpha G'' + \beta G' + \gamma G) dx \simeq$$

LET: $G(x, \xi)$.

$$L G = \delta(x - \xi)$$

$$G(a, \xi) = G(b, \xi) = 0$$

G ... GREEN'S FUNCTION

(6)

$\Rightarrow$

PROOF

HOW TO FIND MIRACULOUS G ?

• WE HAVE CONSTRUCTED y_s .. SPECIAL SOLUTION

OF

$$L y_s = f$$

$$y(a) = y(b) = 0$$

$$y = y_s + y_h$$

$$L y_h = 0$$

TAKE

• WE HAVE CONSTRUCTED y_s .. SPECIAL SOLUTION

OF

$$L y_s = f$$

$$y(a) = y(b) = 0$$

$$y = y_s + y_H$$

$$L y_H = 0$$

• NOW TAKE A SOL

$$L y_H = 0$$

$$y_H(a) =$$

PDE'S

1) POISSON'S EQUATION

PDE'S

1) POISSON'S EQUATION ('FORCED LAPLACE EQ')

DIRICHLET PROBLEM.

$$u = -g(x) \quad x \in \mathcal{D}$$

ON

ED LAPLACE EQ)

$$\Delta = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \dots$$

$$\begin{aligned} \Delta \varphi &= -\rho(x) & x \in \Omega \\ \varphi &= \varphi_0 & \text{ON } \partial\Omega \end{aligned} \quad (\text{DP})$$

• WE H
OF

• NOW T

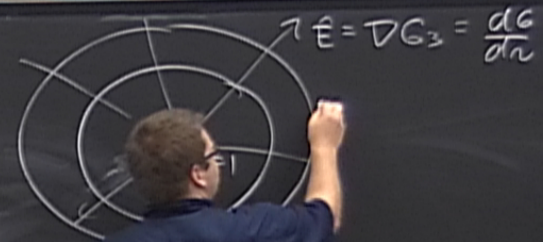
STEP 1: FREE SPACE GREEN'S FUNCTION.

STEP 1: FREE SPACE GREEN'S FUNCTION (3D)

$$\Delta G_3 = \delta(\mathbf{r} - \mathbf{r}_0)$$

STEP 1: FREE SPACE GREEN'S FUNCTION (3D)

$$\Delta G_3 = \delta(\mathbf{r} - \mathbf{r}_0)$$



$$\hat{\mathbf{r}} \cdot \hat{\mathbf{e}} = \nabla G_3 = \frac{dG}{dr}$$

STEP 1: FREE SPACE GREEN'S FUNCTION (3D)

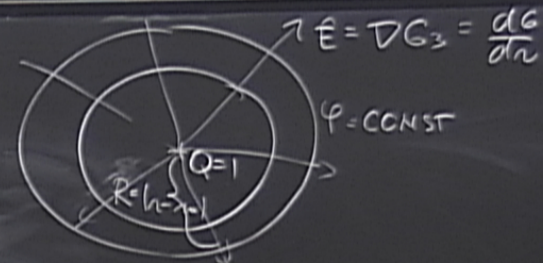
$$\Delta G_3 = \delta(\mathbf{r}-\mathbf{r}_0)$$

STOKE'S THEOREM

$$\oint_{\partial R} \omega = \int_R d\omega$$

$$\int_V \Delta G_3 dV = \int_V (\nabla G_3) dV$$

HOW TO FIND



$$y(a) = \int_a^b G(a, s) f(s) ds = 0 = y(b)$$

STEP 1: FREE SPACE GREEN'S FUNCTION (3D)

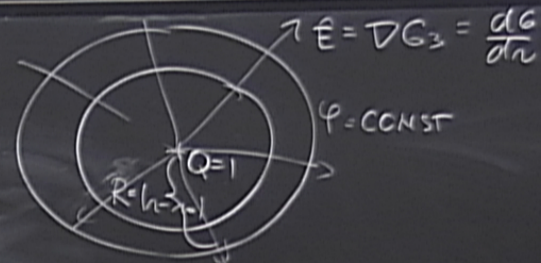
$$\Delta G_3 = \delta(\mathbf{r}-\mathbf{r}_0)$$

STOKE'S THEOREM

$$\oint_{\partial \mathcal{R}} \omega = \int_{\mathcal{R}} d\omega$$

$$\int_{\mathcal{R}} \Delta G_3 dV = \int_{\mathcal{R}} \nabla \cdot (\nabla G_3) dV = \int_{\partial \mathcal{R}} \nabla G_3 \cdot d\mathbf{S} =$$

HOW TO FIND MIRACULOUS G ?



$$y(r) = \int_a^b G(r, s) f(s) ds = 0 = y(l)$$

STEP 1: FREE SPACE GREEN'S FUNCTION (3D)

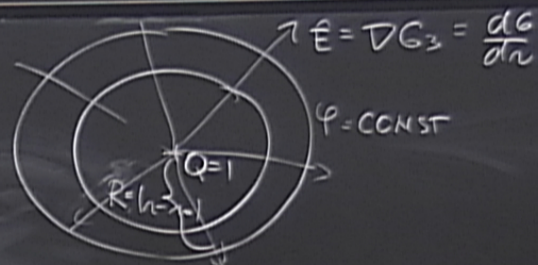
$$\Delta G_3 = \delta(\mathbf{r}-\mathbf{r}_0)$$

STOKE'S THEOREM

$$\int_{\partial R} \omega = \int_R d\omega$$

$$\int_V \Delta G_3 dV = \int_V \nabla \cdot (\nabla G_3) dV = \int_{\partial V} \nabla G_3 \cdot d\mathbf{S} = \int_{\partial V} \left(\frac{dG_3}{dn}, 0, 0 \right) dS$$

HOW TO FIND MIRACULOUS G ?



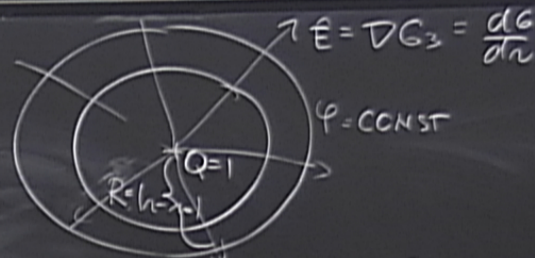
$$y(x) = \int_a^b G(x, s) f(s) ds \quad (b)$$

STEP 1: FREE SPACE GREEN'S FUNCTION (3D)

$$\Delta G_3 = \delta(\mathbf{r}-\mathbf{r}_0)$$

STOKE'S THEOREM

$$\int_{\partial \mathcal{R}} \omega = \int_{\mathcal{R}} d\omega$$



$$\int_{\mathcal{R}} \Delta G_3 dV = \int_{\mathcal{R}} \nabla \cdot (\nabla G_3) dV = \int_{\partial \mathcal{R}} \nabla G_3 \cdot d\mathbf{S} = \int_{\partial \mathcal{R}} \left(\frac{dG_3}{dr} \right)_{r=R} dS = 4\pi R^2 \frac{dG_3}{dR} = \int \delta(\mathbf{r}-\mathbf{r}_0) dV$$

HOW TO FIND MIRACULOUS G ?

$$y(a) = \int_a^b G(a, s) f(s) ds = 0 = y(b)$$

STOKE'S THEOREM

$$\int_{\partial R} \omega = \int_R d\omega$$

$$\int \Delta G_3 dV = \int \nabla \cdot (\nabla G_3) dV = \int \nabla G_3 dS = \int_{\partial R} \left(\frac{dG_3}{dr}, 0, 0 \right) dS = 4\pi R^2 \frac{dG_3}{dR} = \int \delta(r-r_0) dV = 1$$

$$\frac{dG_3}{dR} = \frac{1}{4\pi R^2}$$

STOKE'S THEOREM

$$\int_{\partial R} \omega = \int_R d\omega$$

$$\int_V \Delta G_3 dV = \int_V \nabla \cdot (\nabla G_3) dV = \int_S \nabla G_3 \cdot d\mathbf{S} = \int_{\partial R} \left(\frac{dG_3}{dr}, 0, 0 \right) \cdot d\mathbf{S} = 4\pi R^2 \frac{dG_3}{dR} = \int_V \delta(r-r_0) dV = 1$$

$$\frac{dG_3}{dR} = \frac{1}{4\pi R^2} \Rightarrow G_3 = -\frac{1}{4\pi R}$$

STOKE'S THEOREM

$$\oint_{\mathcal{C}} \omega = \int_{\mathcal{R}} d\omega$$

$$\int_{\mathcal{R}} \Delta G_3 dV = \int_{\mathcal{R}} \nabla \cdot (\nabla G_3) dV = \int_{\partial \mathcal{R}} \nabla G_3 \cdot d\mathbf{S} = \int_{\partial \mathcal{R}} \left(\frac{dG_3}{dr}, 0, 0 \right) dS = 4\pi R^2 \frac{dG_3}{dR} = \int \nabla(n-n_0) dV = 1$$

$$\frac{dG_3}{dR} = \frac{1}{4\pi R^2} \Rightarrow \boxed{G_3 = -\frac{1}{4\pi R}} \quad R = |n - n_0|$$

COULOMB'S LAW

$$\oint_{\Sigma} \vec{E} \cdot d\vec{A} = \oint_{\Sigma} V(\vec{r}) dV = \oint_{\Sigma} V(\vec{r}) dS = \oint_{\Sigma} \frac{\partial V}{\partial r} (dr, d\theta, d\phi) = \dots$$

$$\frac{dG_3}{dR} = \frac{1}{4\pi R^2} \Rightarrow \boxed{G_3 = -\frac{1}{4\pi R}} \quad R = |r - r_0|$$

COULOMB'S LAW

$$G_3 \xrightarrow{R \rightarrow \infty} 0$$

STEP 2. GREEN'S IDENTITIES

$$\oint_{\mathcal{L}} \vec{G}_3 \cdot d\vec{l} = \oint_{\mathcal{L}} \nabla \cdot (\vec{G}_3) dV = \oint_{\partial \mathcal{L}} \vec{G}_3 \cdot d\vec{S} = \oint_{\partial \mathcal{L}} (d\vec{r}_1 \cdot \vec{G}_3) dS = \oint_{\partial \mathcal{L}} \frac{1}{R} d\vec{r}_1 \cdot \vec{G}_3 dS = \oint_{\partial \mathcal{L}} \frac{1}{R} d\vec{r}_1 \cdot \vec{G}_3 dS$$

$$\frac{dG_3}{dR} = \frac{1}{4\pi R^2} \Rightarrow \boxed{G_3 = -\frac{1}{4\pi R}} \quad R = |\vec{r} - \vec{r}_0|$$

COULOMB'S LAW

$$G_3 \xrightarrow{R \rightarrow \infty} 0$$

STEP 2. GREEN'S IDENTITIES

$$\oint_{\mathcal{L}} \nabla \cdot (\phi \nabla \chi) = \oint_{\partial \mathcal{L}} \phi \nabla \chi \cdot d\vec{S}$$

$$\oint_{\partial V} \vec{G}_3 \cdot d\vec{A} = \oint_{\partial V} \vec{G}_3 \cdot \vec{n} dA = \oint_{\partial V} G_3 dA = \oint_{\partial V} \frac{1}{R} dA = \oint_{\partial V} \frac{1}{|r - r_0|} dA$$

$$\frac{dG_3}{dR} = \frac{1}{4\pi R^2} \Rightarrow \boxed{G_3 = -\frac{1}{4\pi R}} \quad R = |r - r_0|$$

COULOMB'S LAW

$$G_3 \xrightarrow{R \rightarrow \infty} 0$$

STEP 2. GREEN'S IDENTITIES

$$\oint_{\partial V} \nabla \cdot (\phi \nabla \chi) dV = \oint_{\partial V} (\phi \nabla \chi) \cdot d\vec{S} =$$

$$\int_V \Delta G_3 dV = \int_V \nabla \cdot (\nabla G_3) dV = \int_{\partial V} \nabla G_3 \cdot d\mathbf{S} = \int_{\partial V} \left(\frac{\partial G_3}{\partial r_1} \frac{\partial V}{\partial r_1} + \dots \right) dS$$

$$\frac{dG_3}{dR} = \frac{1}{4\pi R^2} \Rightarrow \boxed{G_3 = -\frac{1}{4\pi R}} \quad R = |\mathbf{r} - \mathbf{r}_0|$$

COULOMB'S LAW

$$G_3 \xrightarrow{R \rightarrow \infty} 0$$

STEP 2. GREEN'S IDENTITIES

$$\int_V \nabla \cdot (\phi \nabla \chi) dV = \int_{\partial V} \phi \nabla \chi \cdot d\mathbf{S} = \int_V [\phi \Delta \chi + \nabla \phi \cdot \nabla \chi] dV$$

$$\int_V \nabla \cdot (\phi \nabla \chi) dV = \int_V \phi (\nabla \cdot \nabla \chi) + \nabla \phi \cdot \nabla \chi dV = \int_V \phi \Delta \chi + \nabla \phi \cdot \nabla \chi dV$$

$$\frac{dG_3}{dR} = \frac{1}{4\pi R^2} \Rightarrow \boxed{G_3 = -\frac{1}{4\pi R}} \quad R = |\mathbf{r} - \mathbf{r}_0|$$

COULOMB'S LAW

$$G_3 \xrightarrow{R \rightarrow \infty} 0$$

STEP 2. GREEN'S IDENTITIES

$$\int_V \nabla \cdot (\phi \nabla \chi) dV = \boxed{\int_{\partial V} (\phi \nabla \chi) \cdot d\mathbf{S} = \int_V [\phi \Delta \chi + \nabla \phi \cdot \nabla \chi] dV} \quad \text{1ST GREEN'S IDENTITY}$$

$$\pi R^2 \Rightarrow \boxed{G_3 = -\frac{1}{4\pi R}} \quad R = |r - r_0|$$

COULOMB'S LAW

$$G_3 \xrightarrow{R \rightarrow \infty} 0$$

$$\vec{dS} = \hat{n} \cdot dS$$

IDENTITIES

$$\int_V (\nabla \cdot \vec{F}) dV = \int_S (\vec{F} \cdot \vec{dS}) = \int_V (\nabla \cdot \vec{F}) dV = \int_V (\nabla \phi \cdot \nabla \chi + \phi \Delta \chi) dV$$

$\hat{n}$

GRE

$$\pi R^2 \Rightarrow \boxed{G_3 = -\frac{1}{4\pi R}} \quad R = |r - r_0|$$

COULOMB'S LAW

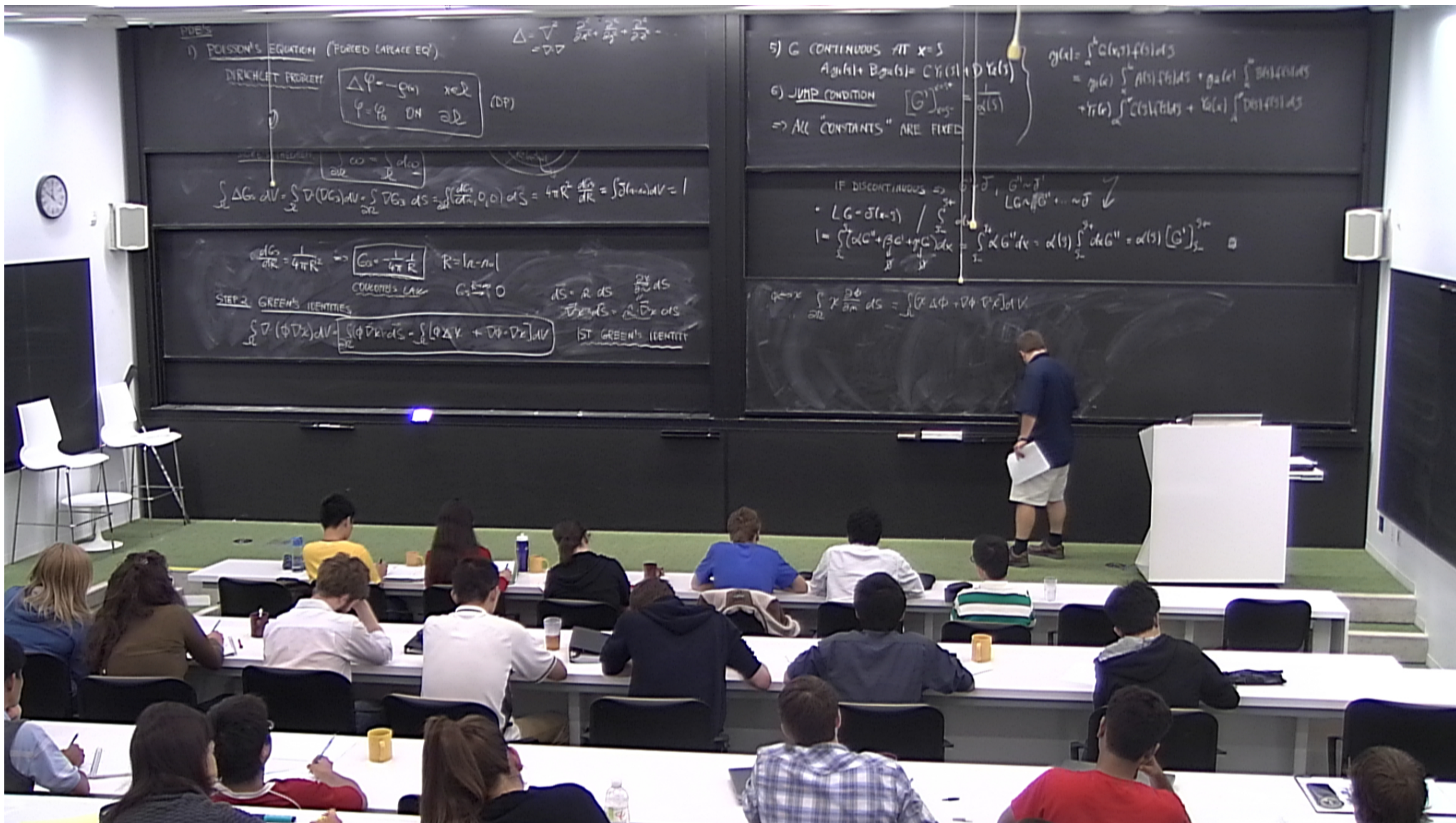
$$G_3 \xrightarrow{R \rightarrow \infty} 0$$

$$\vec{dS} = \hat{n} \cdot dS$$

IDENTITIES

$$\int_V (\nabla \cdot \vec{F}) dV = \int_{\partial V} (\vec{F} \cdot \vec{dS}) = \int_V [\nabla \cdot \vec{F}] dV$$

ST GREEN



$$\phi \leftrightarrow \chi \quad \int_{\partial \Omega} \chi \frac{\partial \phi}{\partial n} dS = \int_{\Omega} [\chi \Delta \phi + \nabla \phi \cdot \nabla \chi] dV$$

$$\phi \leftrightarrow \chi \quad \int_{\partial R} \chi \frac{\partial \phi}{\partial n} dS = \int_V [\chi \Delta \phi + \nabla \phi \cdot \nabla \chi] dV \quad (2)$$

SUBTRACTING

$$\phi \leftrightarrow \chi \quad \int_{\partial R} \chi \frac{\partial \phi}{\partial n} dS = \int_V [\chi \Delta \phi + \nabla \phi \cdot \nabla \chi] dV \quad (2)$$

SUBTRACTING

$$\int_{\partial R} \left(\phi \frac{\partial \chi}{\partial n} - \chi \frac{\partial \phi}{\partial n} \right) dS = \int_V [\phi \Delta \chi - \chi \Delta \phi] dV$$

$$\phi \leftrightarrow \chi \quad \int_{\partial R} \chi \frac{\partial \phi}{\partial n} dS = \int_V [\chi \Delta \phi + \nabla \phi \cdot \nabla \chi] dV \quad (2)$$

SUBTRACTING

$$\int_{\partial R} \left(\phi \frac{\partial \chi}{\partial n} - \chi \frac{\partial \phi}{\partial n} \right) dS = \int_V [\phi \Delta \chi - \chi \Delta \phi] dV$$

2ND-GREEN'S IDENTITY

$$L y_s = f$$

$$y(a) = y(b) = 0$$

$$y = y_s + y_H$$

$$L y_H = 0$$

• NOW TAKE A SOL

$$L y_H = 0$$

$$y_H(a) = P, \quad y_H(b) = Q$$

$$\Rightarrow y = y_s + y_H \dots \text{SOL OF}$$

$$\begin{aligned} y(a) &= P \\ y(b) &= Q \end{aligned}$$

$$\phi \leftrightarrow \chi \quad \int_{\partial R} \chi \frac{\partial \phi}{\partial n} dS = \int_V [\chi \Delta \phi + \nabla \phi \cdot \nabla \chi] dV \quad (2)$$

SUBTRACTING

$$\int_{\partial R} \left(\phi \frac{\partial \chi}{\partial n} - \chi \frac{\partial \phi}{\partial n} \right) dS = \int_V [\phi \Delta \chi - \chi \Delta \phi] dV$$

2ND-GREEN'S IDENTITY

- APPLY TO $\phi = \psi$ ψ OBEYS $\Delta \psi = -\rho$
 $\chi = G_3$ (UNSPECIFIED BC)

$$L y_s = f$$

$$y(a) = y(b) = 0$$

- NOW TAKE A SOL

$$L y_H = 0$$

$$y = y_s + y_H$$

$$L y_H = 0$$

$$y_H(b) = Q$$

$$\Rightarrow y = y_s + y_H \quad \text{SOL OF}$$

$$y(a) = P$$

$$y(b) = Q$$

• APPLY TO $\phi = \psi$ ψ OBEYS $\Delta\psi = -g$
 $\chi = G_3$ (UNSPECIFIED BC)

$$1 = \int_{\Sigma} (\alpha G'' + \beta G' + \gamma G) dx \approx \int_{\Sigma} \alpha G'' dx = \alpha(3) \int_{\Sigma} dx G'' = \alpha(3) [G']_{\Sigma}$$

$$\int_{\partial\Omega} \left(\psi \frac{\partial G_3}{\partial n} - G_3 \frac{\partial \psi}{\partial n} \right) dS = \int_{\Omega} \left[\psi \nabla^2 G_3 + G_3 g \right] dV = \psi(r_0) + \int_{\Omega} G_3 g$$

$\psi(r_0)$

• APPLY TO $\phi = \psi$ ψ OBEYS $\Delta\psi = -g$
 $\chi = G_3$ (UNSPECIFIED BC)

$$1 = \int_{\Sigma} (\alpha G'' + \beta G' + \gamma G) dx \approx \int_{\Sigma} \alpha G'' dx = \alpha(3) \int_{\Sigma} dx G'' = \alpha(3) [G']_{\Sigma}$$

$$\int_{\partial\Omega} \left(\psi \frac{\partial G_3}{\partial n} - G_3 \frac{\partial \psi}{\partial n} \right) dS = \int_{\Omega} \left[\psi \nabla^2 G_3 + G_3 g \right] dV = \psi(r_0) + \int_{\Omega} G_3 g$$

$$\boxed{\psi(r_0) = - \int_{\Omega} G_3 g + \int_{\partial\Omega} \left(\psi \frac{\partial G_3}{\partial n} - G_3 \frac{\partial \psi}{\partial n} \right) dS}$$

• APPLY TO $\phi = \psi$ ψ OBEYS $\Delta\psi = -g$
 $\chi = G_3$ (UNSPECIFIED BC)

$$\int_{\partial R} \left(\psi \frac{\partial G_3}{\partial n} - G_3 \frac{\partial \psi}{\partial n} \right) dS = \int_V \left[\psi \delta(r-r_0) + G_3 g \right] dV = \psi(r_0) + \int_V G_3 g$$

$$\boxed{\psi(r_0) = - \int_V G_3 g + \int_{\partial R} \left(\psi \frac{\partial G_3}{\partial n} - G_3 \frac{\partial \psi}{\partial n} \right) dS} \quad \text{3RD GREEN'S IDENTITY}$$

• APPLY TO $\phi = \psi$ ψ OBEYS $\Delta\psi = -g$
 $\chi = G_3$ (UNSPECIFIED BC)

$$\int_{\partial R} \left(\psi \frac{\partial G_3}{\partial n} - G_3 \frac{\partial \psi}{\partial n} \right) dS = \int_R \left[\psi \Delta(G_3) + G_3 \Delta(\psi) \right] dV = \psi(r_0) + \int_R G_3 g$$

$$\boxed{\psi(r_0) = - \int_R G_3 g + \int_{\partial R} \left(\psi \frac{\partial G_3}{\partial n} - G_3 \frac{\partial \psi}{\partial n} \right) dS} \quad \text{3RD GREEN'S IDENTITY}$$

$$y_H = 0 \quad y_H(a) = P, \quad y_H(b) = Q$$

• APPLY TO $\phi = \psi$ ψ OBEYS $\Delta\psi = -g$
 $\chi = G_3$ (UNSPECIFIED BC)

$$\int_{\partial R} \left(\psi \frac{\partial G_3}{\partial n} - G_3 \frac{\partial \psi}{\partial n} \right) dS = \int_R \left[\psi \nabla^2 (G_3) + G_3 \nabla^2 \psi \right] dV = \psi(r_0) + \int_R G_3 g$$

$$\boxed{\psi(r_0) = - \int_R G_3 g + \int_{\partial R} \left(\psi \frac{\partial G_3}{\partial n} - G_3 \frac{\partial \psi}{\partial n} \right) dS} \quad \text{3RD GREEN'S IDENTITY}$$

PROBLEM: CANNOT SPECIFY BOTH ψ AND $\frac{\partial \psi}{\partial n}$ ($\rightarrow$ N.B.)

$$y_H = 0 \quad y_H(a) = P, \quad y_H(b) = Q$$

• APPLY TO $\phi = \psi$ ψ OBEYS $\Delta\psi = -g$
 $\chi = G_3$ (UNSPECIFIED BC)

$$\int_{\partial R} \left(\psi \frac{\partial G_3}{\partial n} - G_3 \frac{\partial \psi}{\partial n} \right) dS = \int_R \left[\psi \Delta(\chi - \psi) + G_3 g \right] dV = \psi(\rho_0) + \int_R G_3 g$$

$$\boxed{\psi(\rho_0) = - \int_R G_3 g + \int_{\partial R} \left(\psi \frac{\partial G_3}{\partial n} - G_3 \frac{\partial \psi}{\partial n} \right) dS} \quad \text{3RD GREEN'S IDENTITY}$$

PROBLEM: CANNOT SPECIFY BOTH ψ AND $\frac{\partial \psi}{\partial n}$ ($\rightarrow$ NOT WELL POSED)
 (3RD) BEAUTIFUL BUT HIDES A DEVIL!

$$L y_H = 0 \quad y_H(a) = P, \quad y_H(b) = Q$$

$$P(\rho_0) = - \int_{\Sigma} G_3 g + \int_{\Sigma} \left(\varphi \frac{\partial G_3}{\partial n} - G_3 \frac{\partial \varphi}{\partial n} \right) dS \quad \text{3RD GREEN'S IDENTITY}$$

PROBLEM: CANNOT SPECIFY BOTH φ AND $\frac{\partial \varphi}{\partial n}$ ($\rightarrow$ NOT WELL POSED)
 (3RD) BEAUTIFUL BUT HIDES A DEVIL!

STEP 3. DIRICHLET GREEN'S FUNCTION ON Ω .

PROBLEM: CANNOT SPECIFY BOTH φ AND $\frac{\partial \varphi}{\partial n}$ ($\rightarrow$ NOT WELL POSED)
(3RD) BEAUTIFUL BUT HIDES A DEVIL?

STEP 3. DIRICHLET GREEN'S FUNCTION ON Ω .

- 1) $\Delta G(x, x_0) = 0$ EXCEPT x_0
- 2) $G(x, x_0)|_{\partial\Omega} = 0$
- 3) $G = G_3 + H$, H FINITE AT x_0
 $\Delta H = 0$ ON Ω

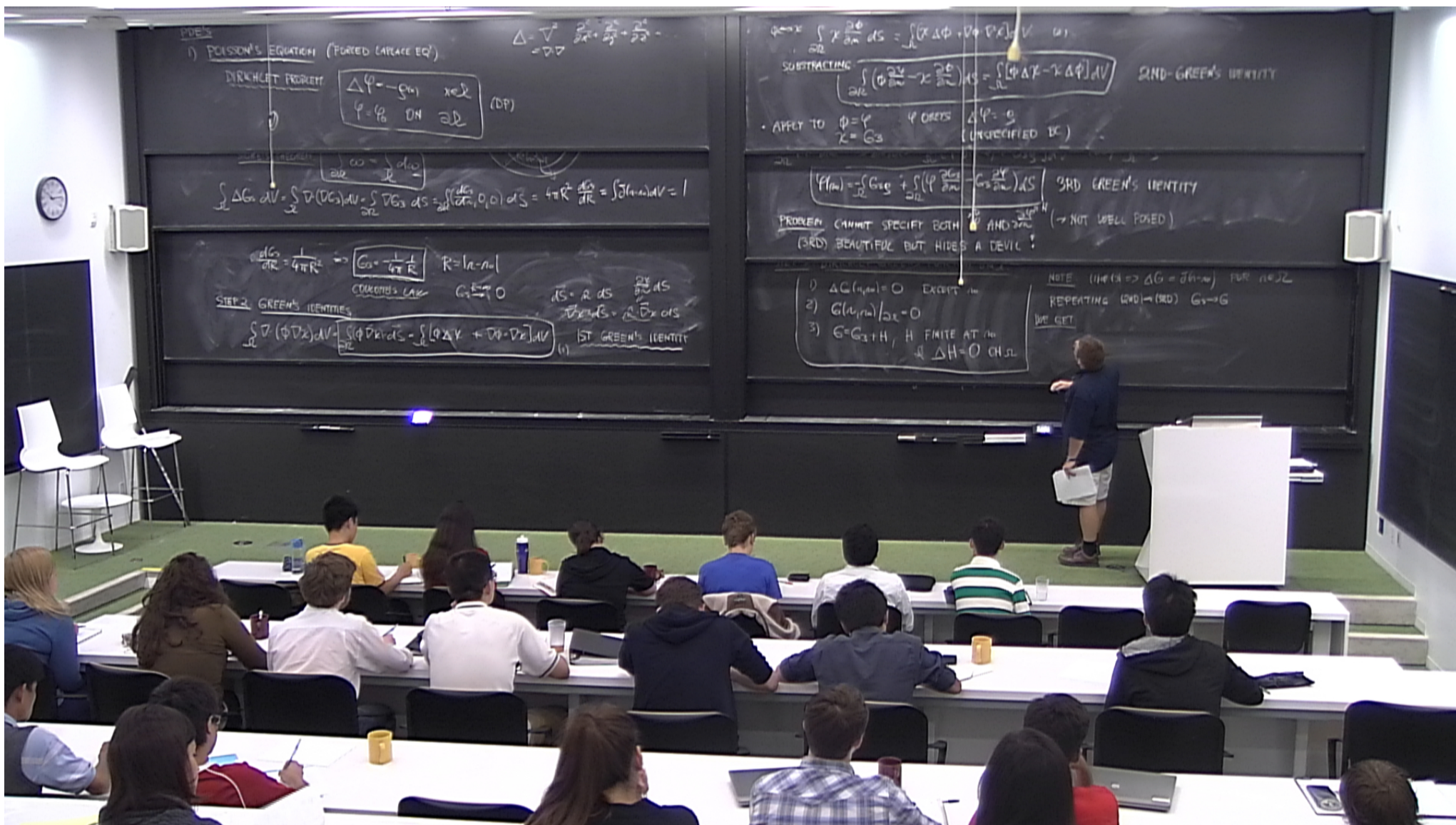
$$y_H(a) = P, \quad y_H(b) = Q$$

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NOTE: (1) & (2) $\Rightarrow \Delta G = \delta(x - x_0)$ FOR $x \in \Omega$



$$\varphi|_{\partial\Omega} = -\int_{\Omega} G_3 g + \int_{\partial\Omega} \left(\varphi \frac{\partial G_3}{\partial n} - G_3 \frac{\partial \varphi}{\partial n} \right) dS \quad \text{3RD GREEN'S IDENTITY}$$

PROBLEM: CANNOT SPECIFY BOTH φ AND $\frac{\partial \varphi}{\partial n}$ ($\rightarrow$ NOT WELL POSED)
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NOTE: (1) $\Rightarrow \Delta G = \delta(n - n_0)$ FOR $n \in \Omega$

REPEATING (2ND) $\rightarrow$ (3RD) $G_3 \rightarrow G$

WE GET: $\varphi|_{\partial\Omega} = -\int_{\Omega} G g$

$$\frac{1}{R} \quad R = |r - r_0|$$

$$\left. \begin{array}{l} (r - r_0) \\ \lim_{r \rightarrow \infty} \rightarrow 0 \end{array} \right\} \varphi(r) = - \int G(r', r) \rho(r') dr' = Q G(d)$$

