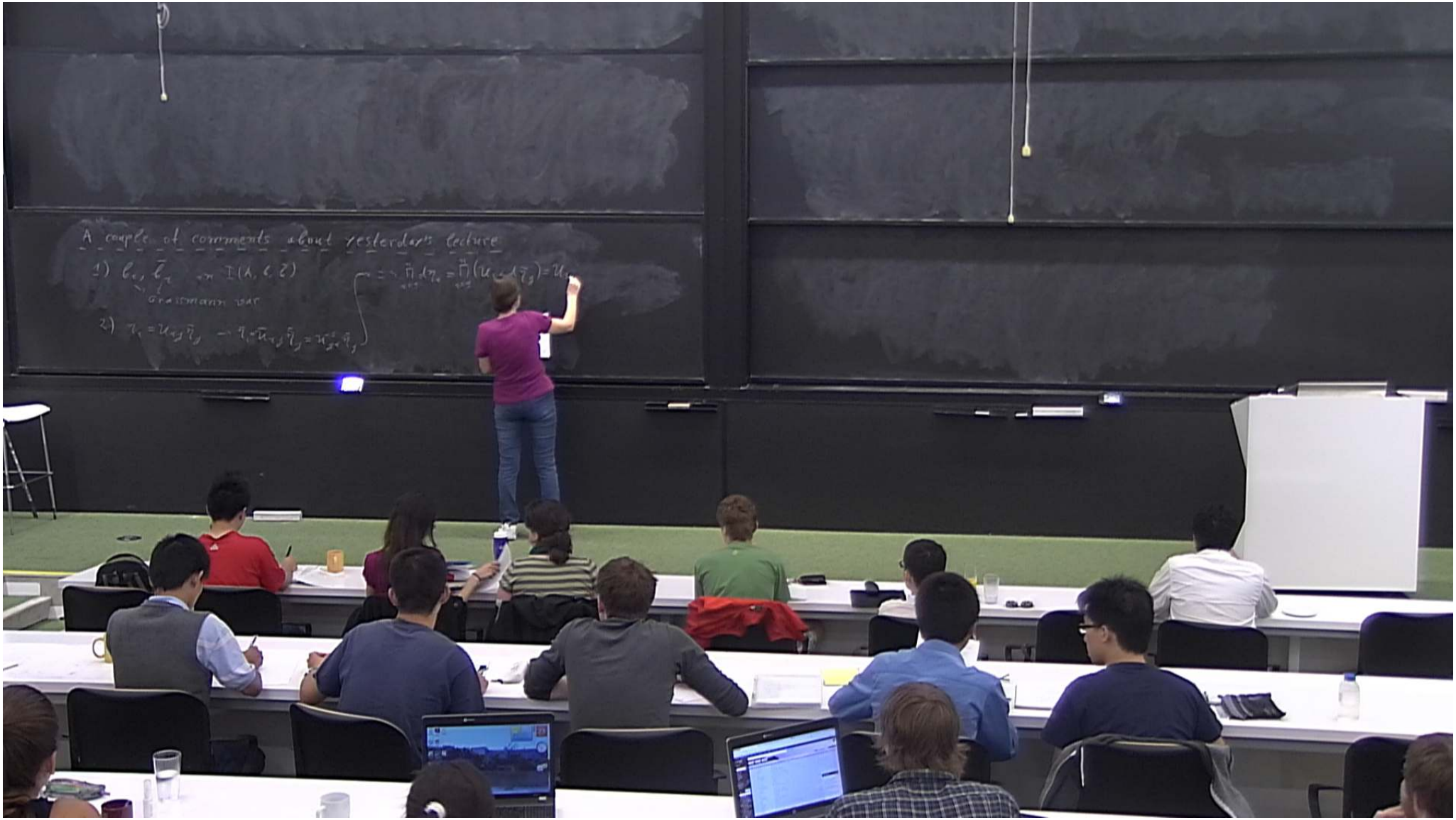
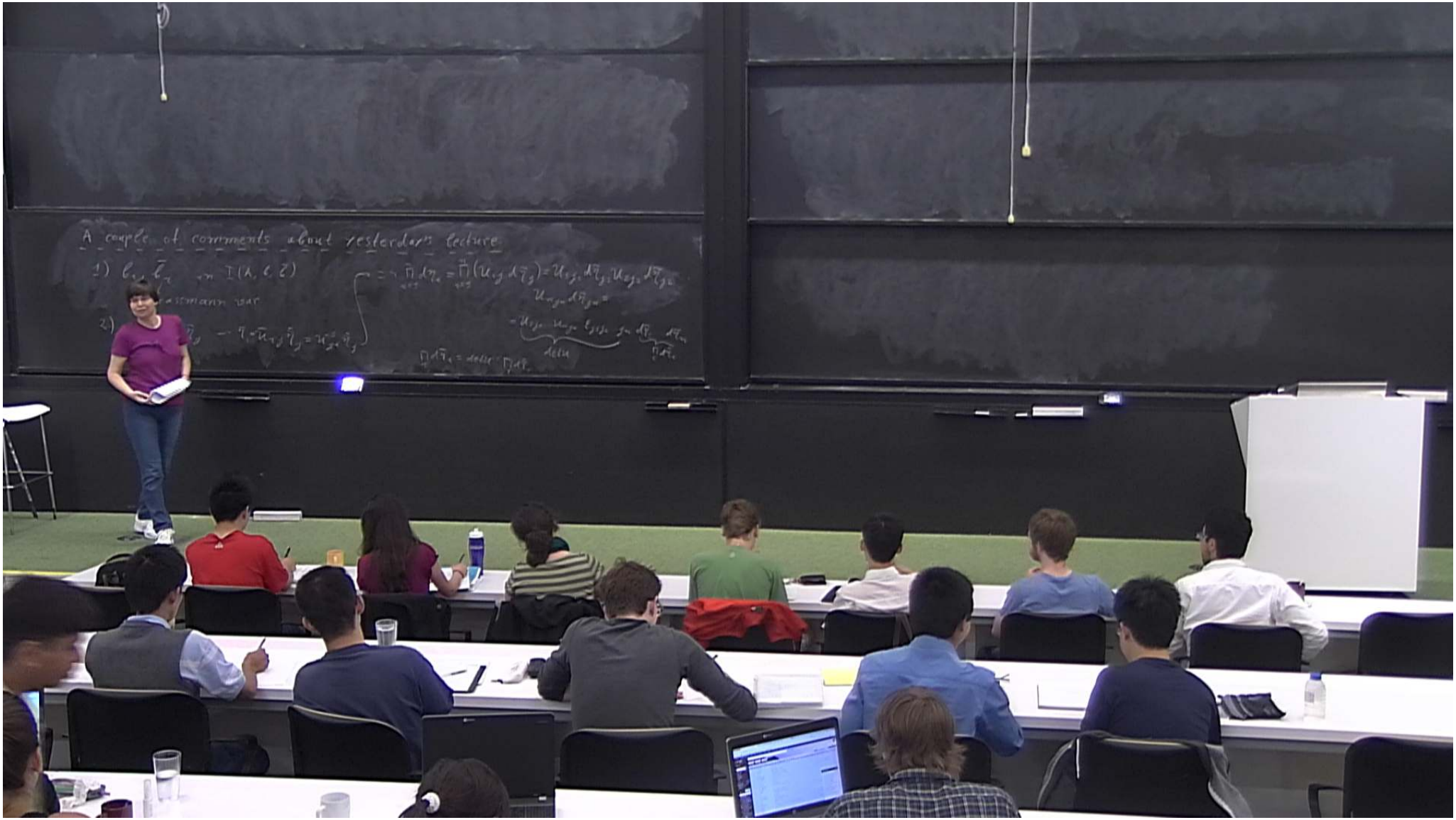
Title: Calculus of Variation and Gaussian Integrals - Lecture 3

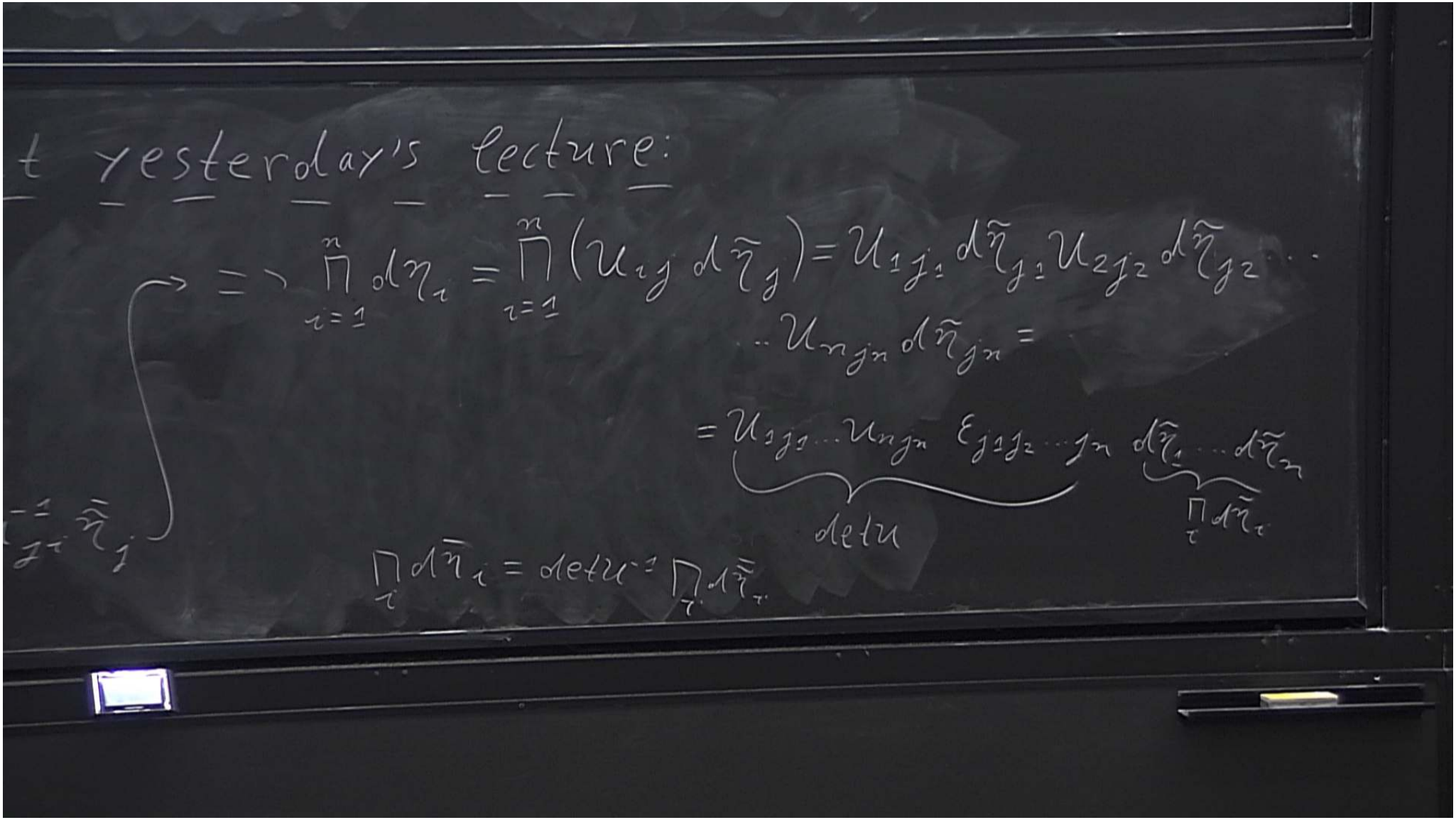
Date: Aug 23, 2012 10:30 AM

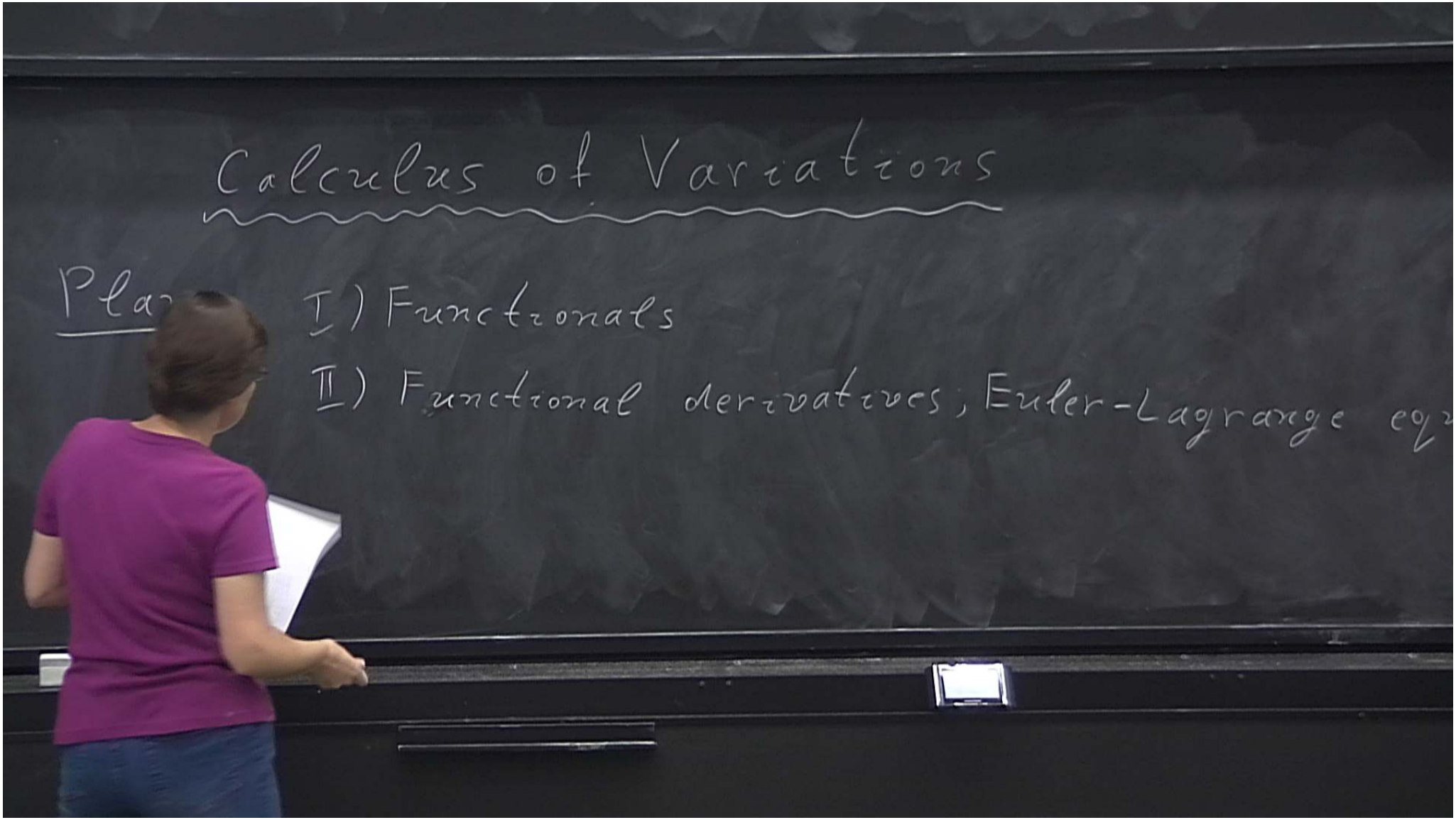
URL: http://pirsa.org/12080026

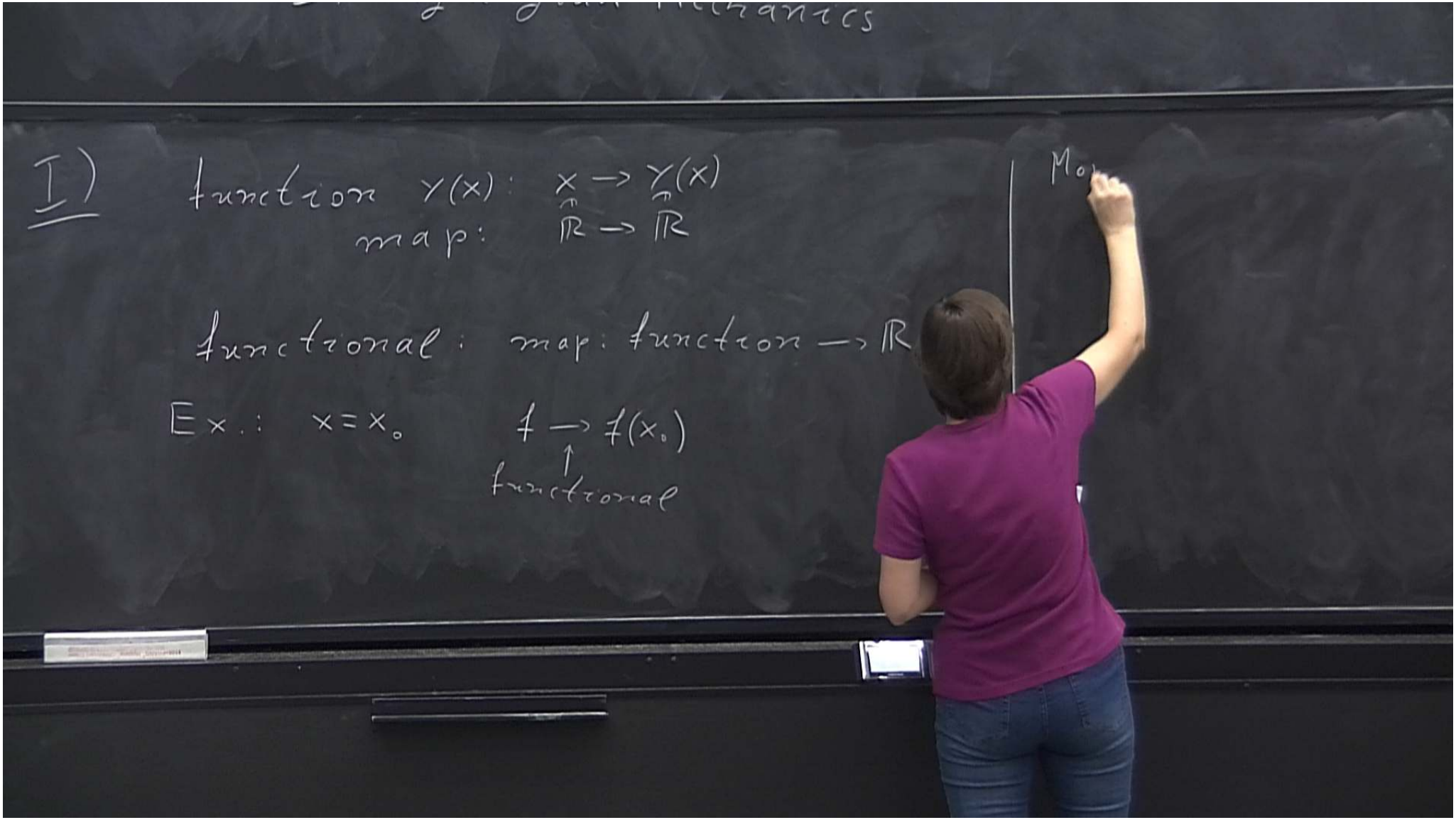
Abstract:

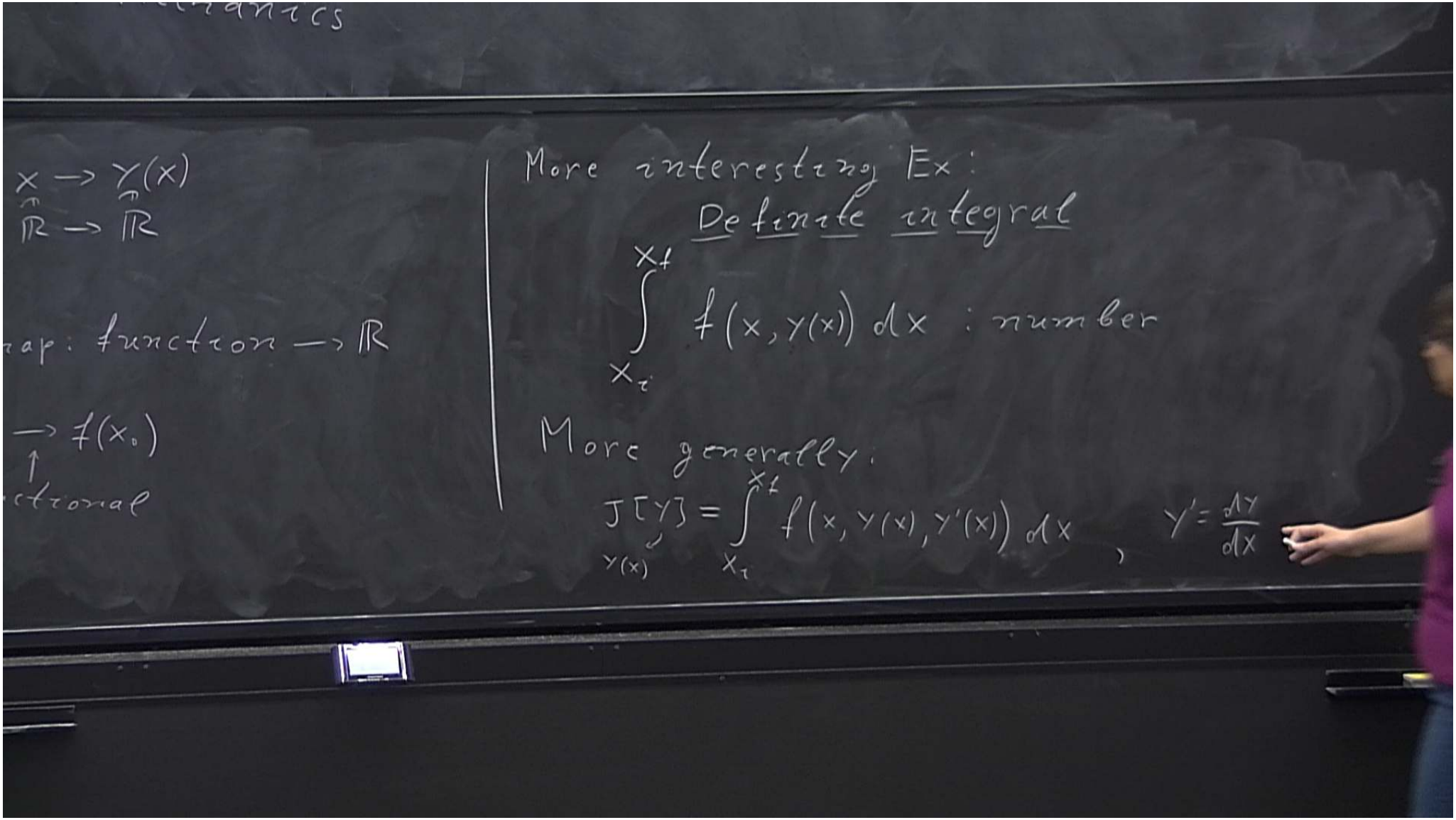

A couple of comments about yesterday's lecture
1)
Grassmann var
2)


A couple of comments about yesterdays lecture


t yesterday's lecture:
$\prod_{i=1}^{n} d\eta_i = \prod_{i=1}^{n} (U_{ij}\, d\tilde{\eta}_j) = U_{1j_1} d\tilde{\eta}_{j_1} U_{2j_2} d\tilde{\eta}_{j_2} \cdots$
$\cdots U_{nj_n} d\tilde{\eta}_{j_n} =$
$= U_{1j_1} \cdots U_{nj_n}\, \varepsilon_{j_1 j_2 \cdots j_n}\, d\tilde{\eta}_1 \cdots d\tilde{\eta}_n$
$\det U$
$\prod_i d\tilde{\eta}_i$
$\prod_i d\eta_i = \det U \cdot \prod_i d\tilde{\eta}_i$


Calculus of Variations
Pla
I) Functionals
II) Functional derivatives, Euler-Lagrange eq

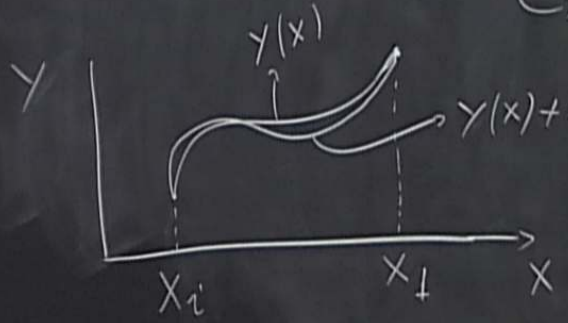

I)
functron $Y(x)$: $x \to Y(x)$
map: $\mathbb{R} \to \mathbb{R}$
functional: map: functron $\to \mathbb{R}$
Ex.: $x = x_0$
$f \to f(x_0)$
functional


$x \to y(x)$
$\mathbb{R} \to \mathbb{R}$
map: function $\to \mathbb{R}$
$\to f(x_0)$
ctional
More interesting Ex:
Definite integral
$\int_{x_i}^{x_f} f(x, y(x))\, dx$: number
More generally:
$J[y] = \int_{x_i}^{x_f} f(x, y(x), y'(x))\, dx$, $y' = \frac{dy}{dx}$
$y(x)$

II) Functional derivative:

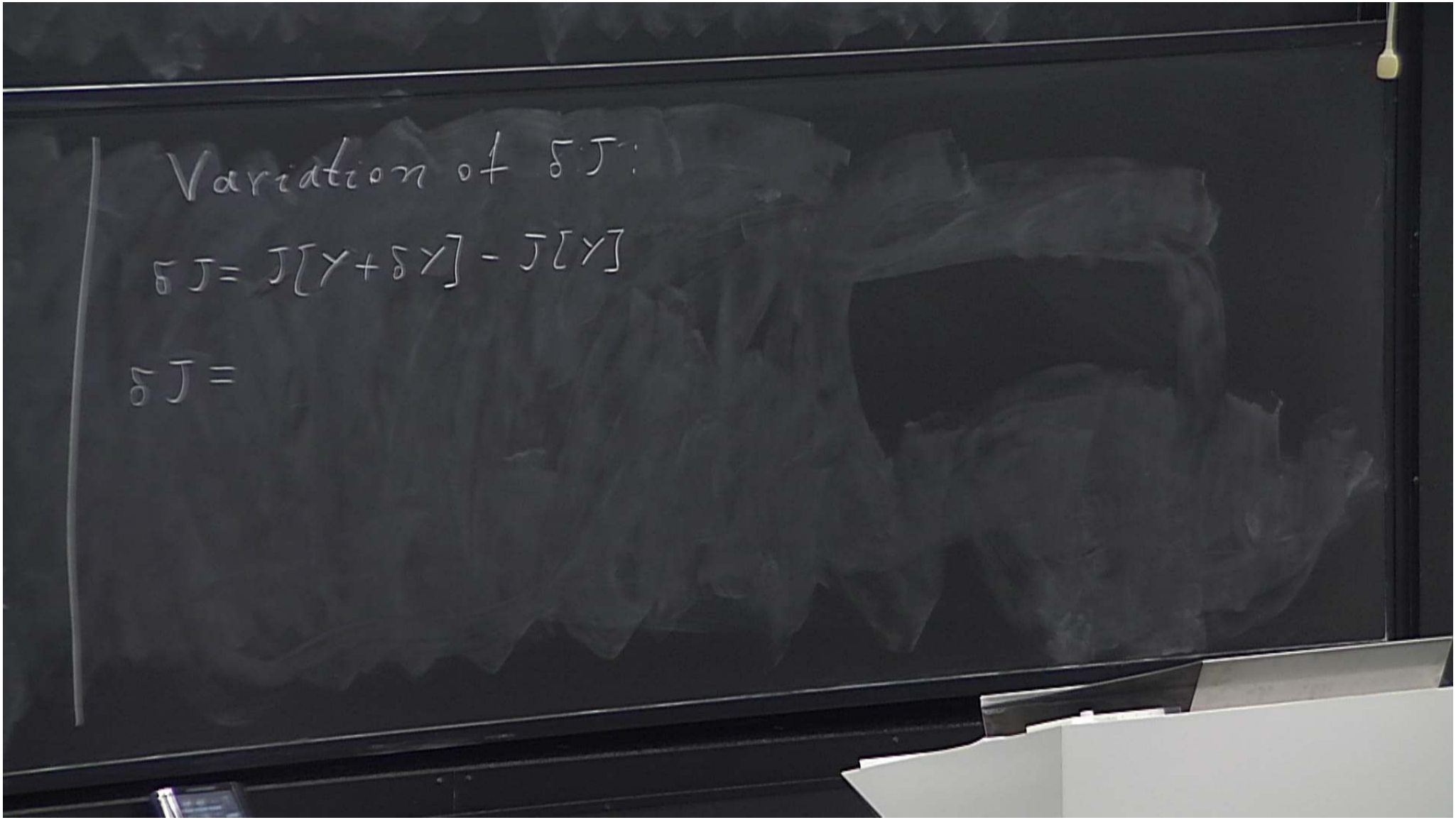
Consider the variation:

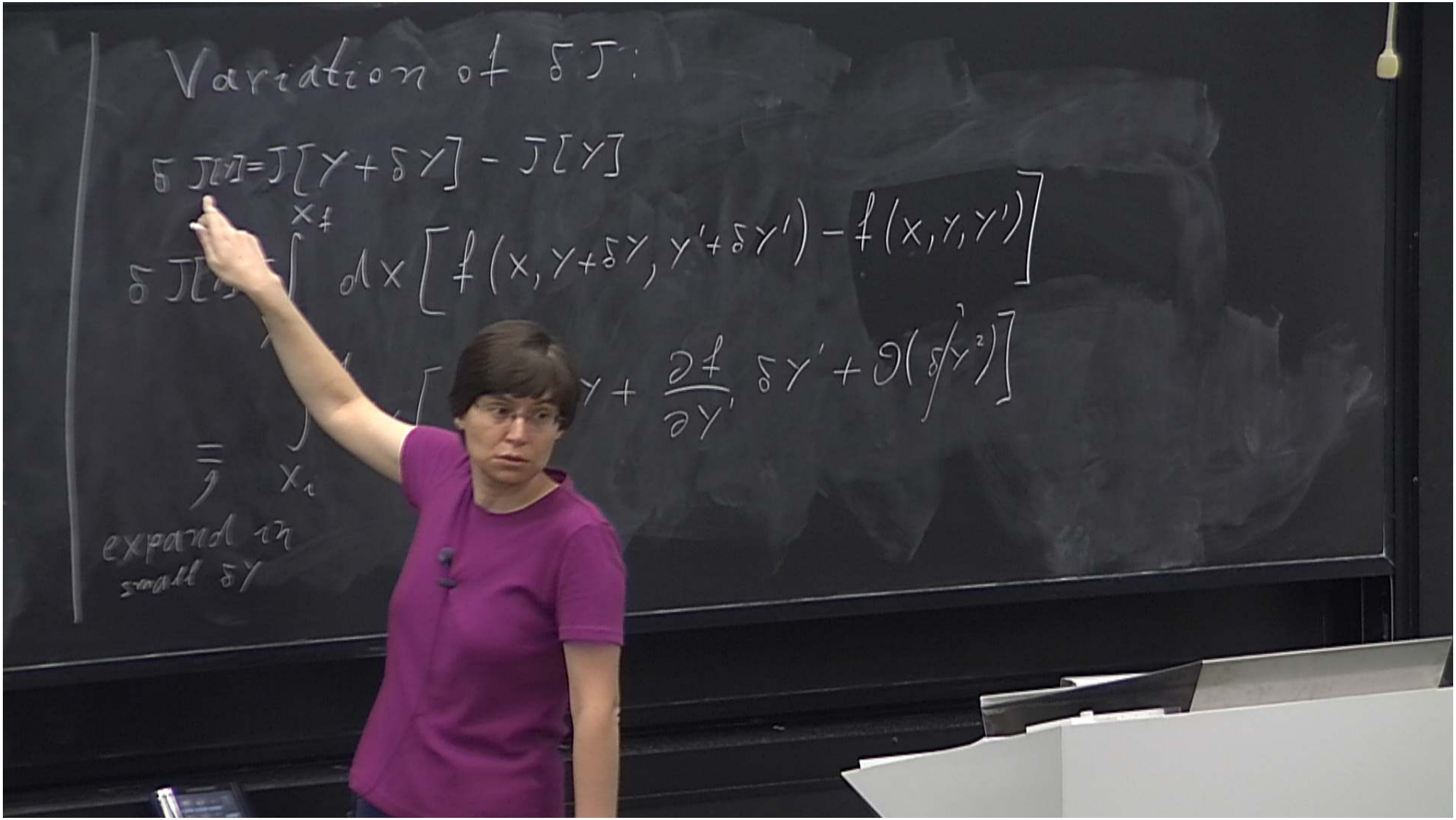
$$y(x) \to y(x) + \underbrace{\delta y}_{\text{small (infinitesimal)}}$$

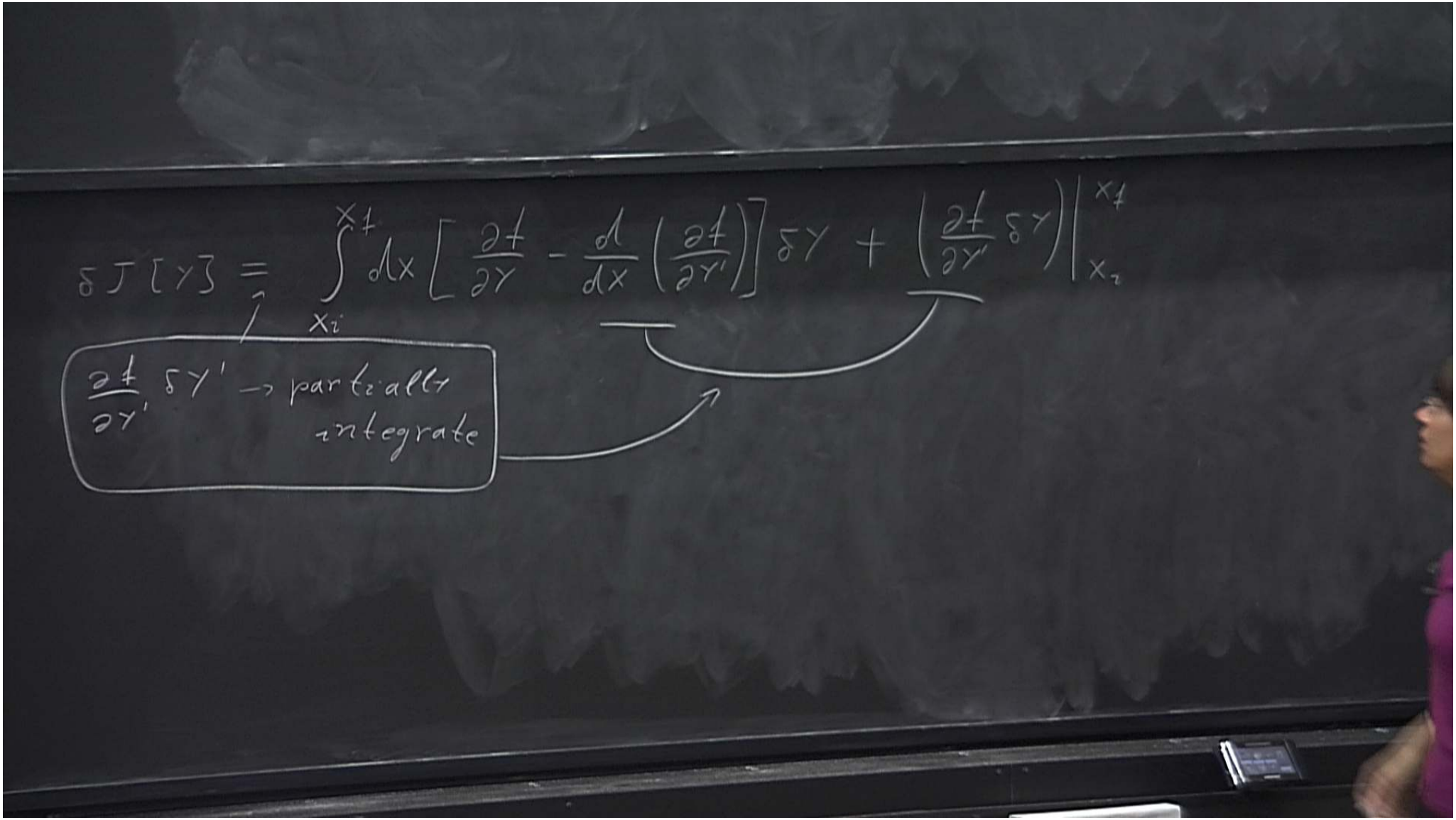


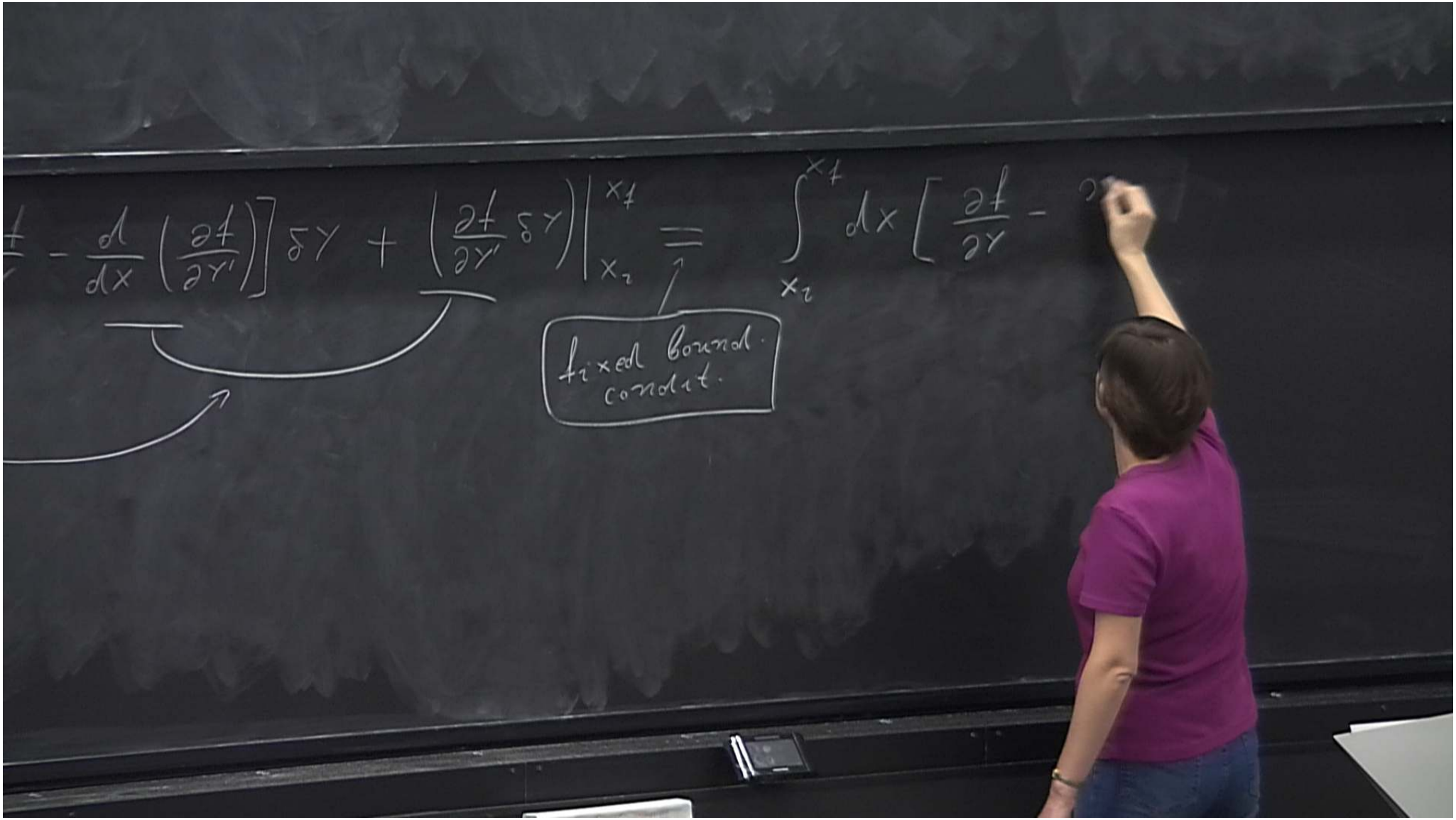
and also:

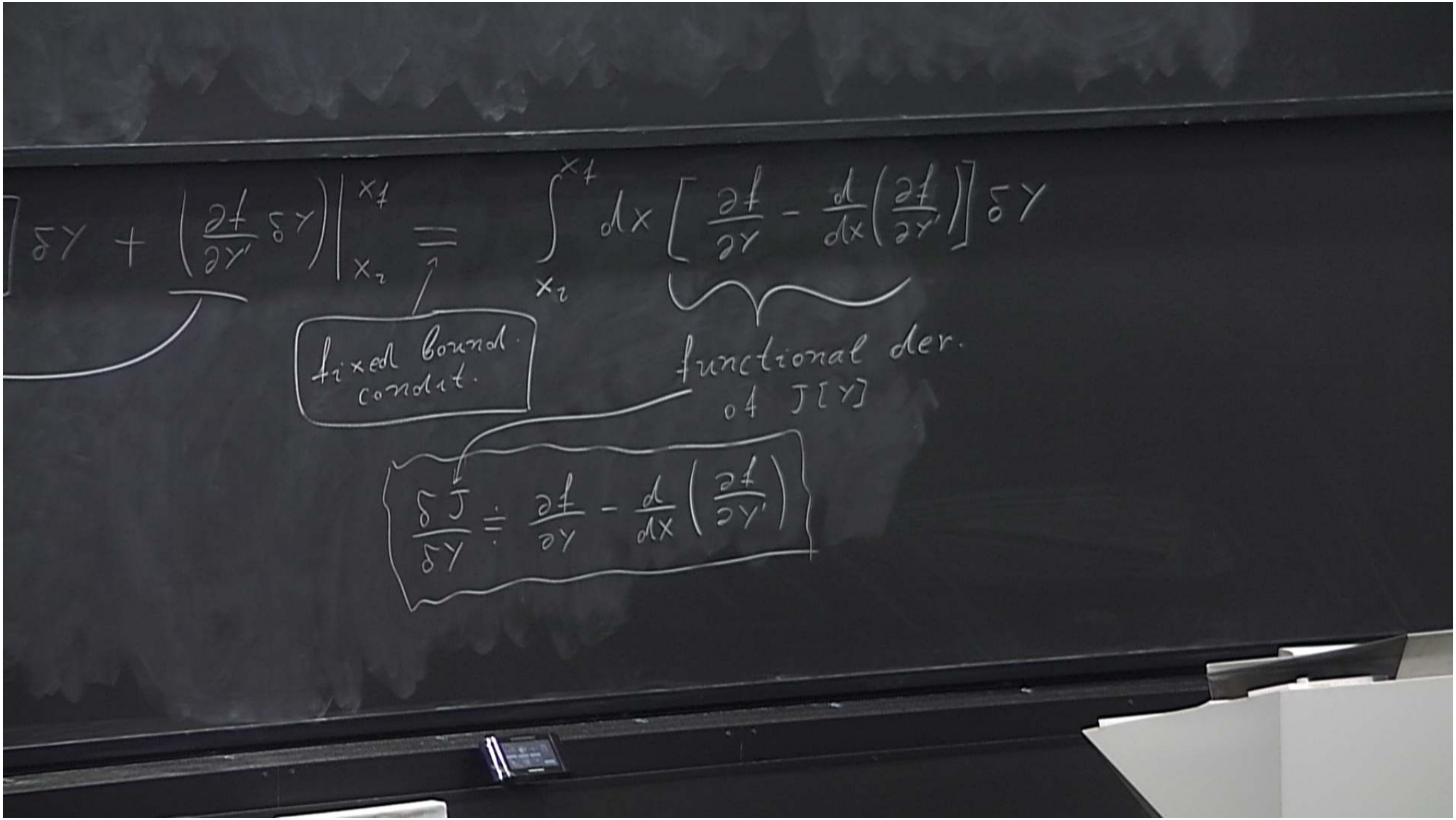
$$\left[\begin{array}{l} \delta y(x_i) = 0 \\ \delta y(x_f) = 0 \end{array}\right] \to \text{fixed b.c. cond.}$$

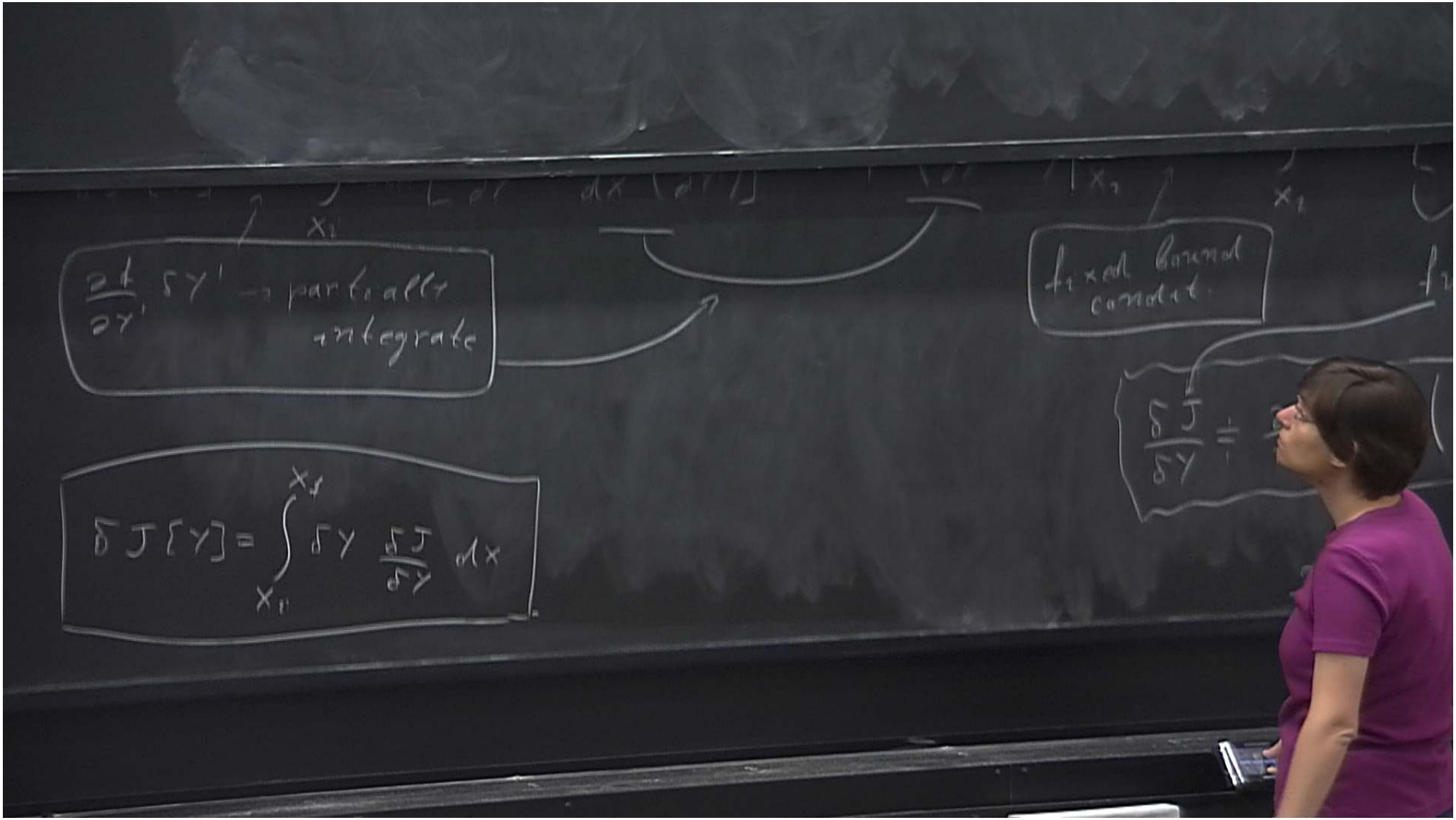

Variation of δJ:
δJ = J[y+δy] - J[y]
δJ =


Variation of δJ:
δJ[y] = J[y+δy] − J[y]
δJ[y] = ∫ dx [f(x, y+δy, y'+δy') − f(x, y, y')]
+ ∂f/∂y δy + ∂f/∂y' δy' + O(δy²)]
x_f
x_i
expand in small δy


$\delta J[y] = \int_{x_i}^{x_f} dx \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\delta y + \left(\frac{\partial f}{\partial y'}\delta y\right)\Big|_{x_1}^{x_f}$
$\frac{\partial f}{\partial y'}\delta y'$ → partially integrate

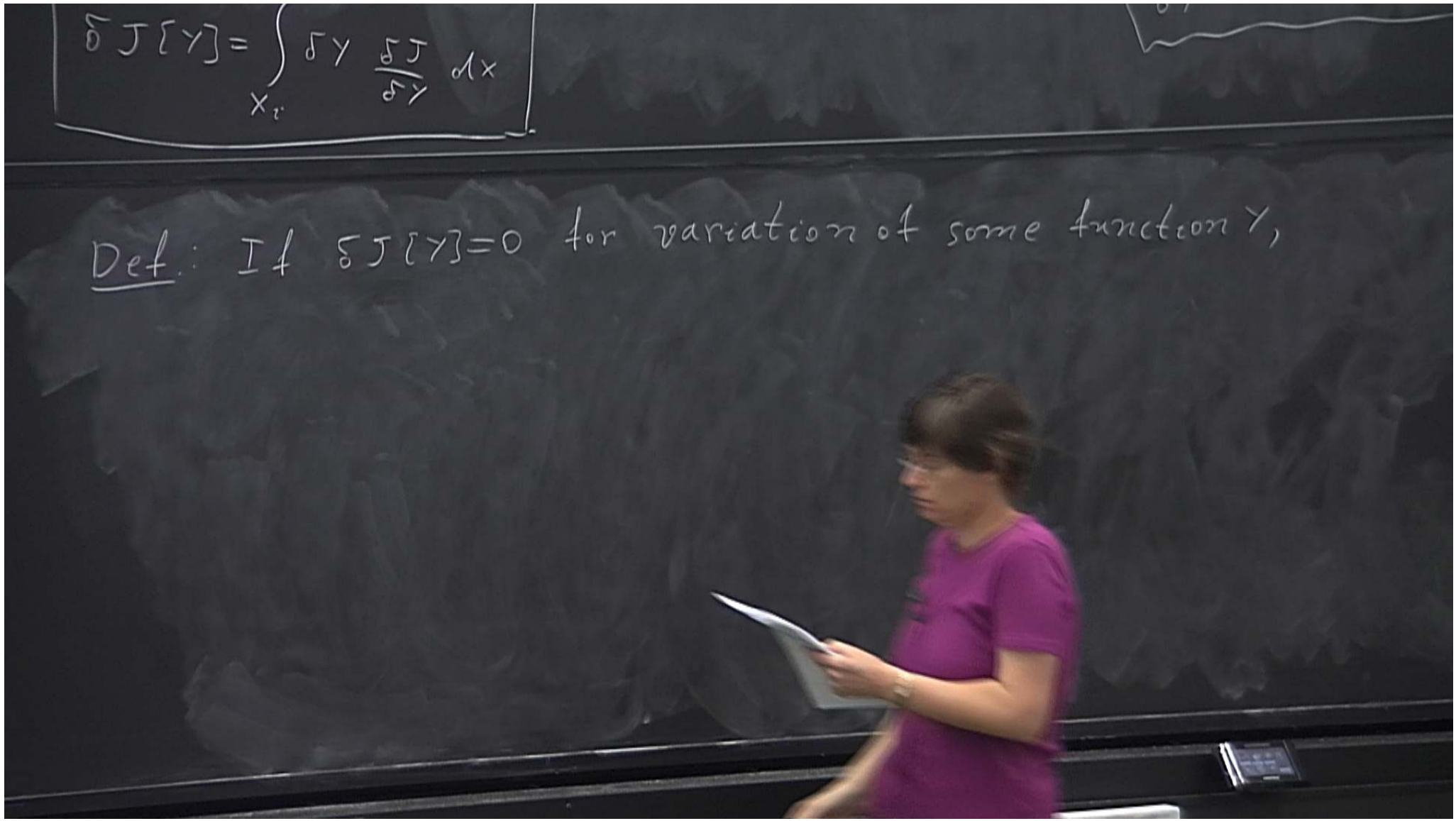

$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\Big]\delta y + \left(\frac{\partial f}{\partial y'}\delta y\right)\Big|_{x_1}^{x_2} = \int_{x_1}^{x_2} dx \left[\frac{\partial f}{\partial y} -$
fixed bound. condit.


$\delta Y + \left(\frac{\partial f}{\partial Y'}\delta Y\right)\Big|_{x_2}^{x_1} = \int_{x_2}^{x_1} dx \left[\frac{\partial f}{\partial Y} - \frac{d}{dx}\left(\frac{\partial f}{\partial Y'}\right)\right]\delta Y$
fixed Bound. condit.
functional der. of J[Y]
$\frac{\delta J}{\delta Y} = \frac{\partial f}{\partial Y} - \frac{d}{dx}\left(\frac{\partial f}{\partial Y'}\right)$


$\frac{\partial L}{\partial y'}\delta y'$ → partially integrate
fixed boundary condition
$\delta J[y] = \int_{x_1}^{x_2} \delta y \frac{\delta J}{\delta y} dx$

$$\delta J[y] = \int_{x_i} \delta y \, \frac{\delta J}{\delta y} \, dx$$

Def: If $\delta J[y] = 0$ for variation of some function y,

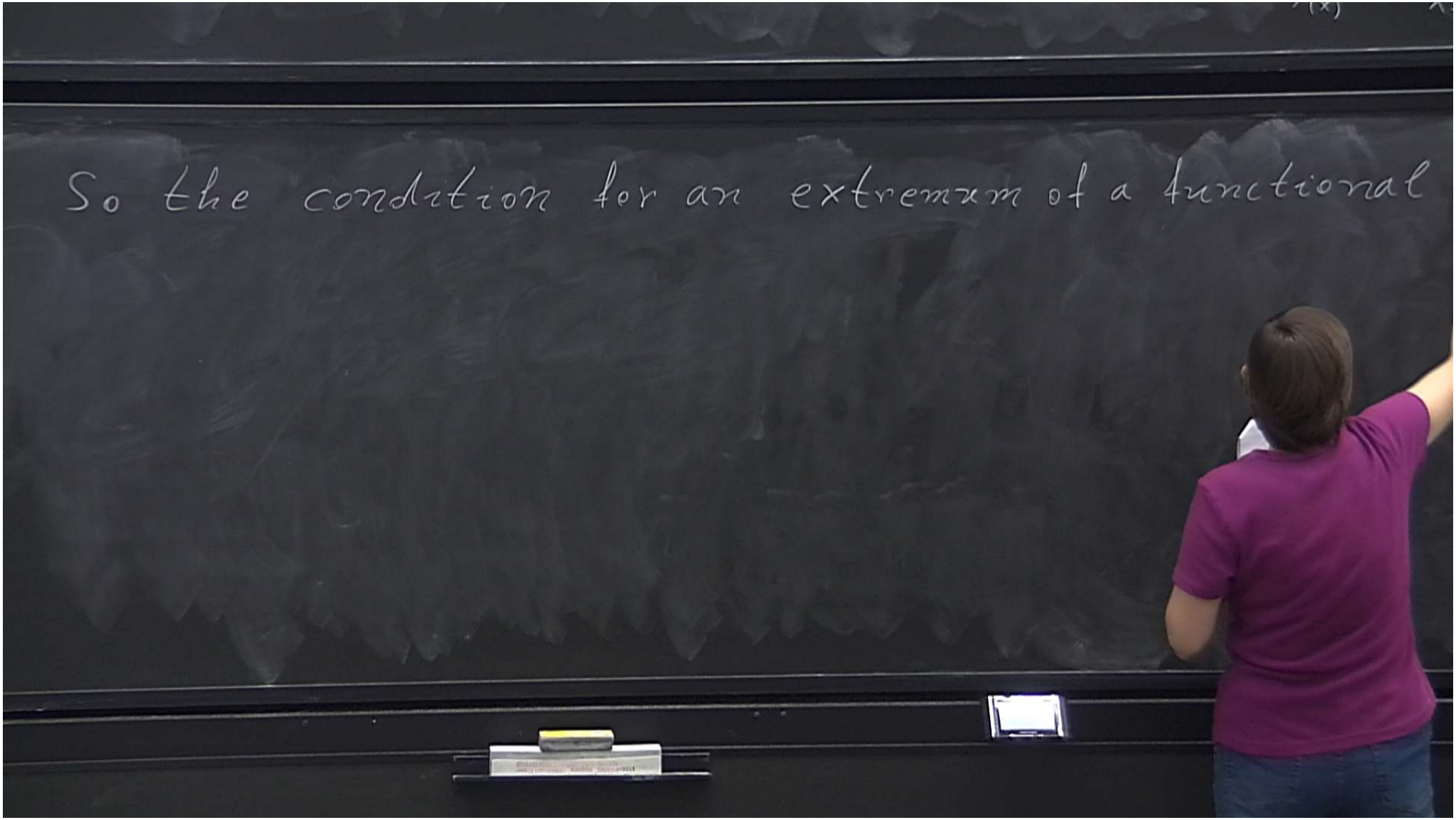


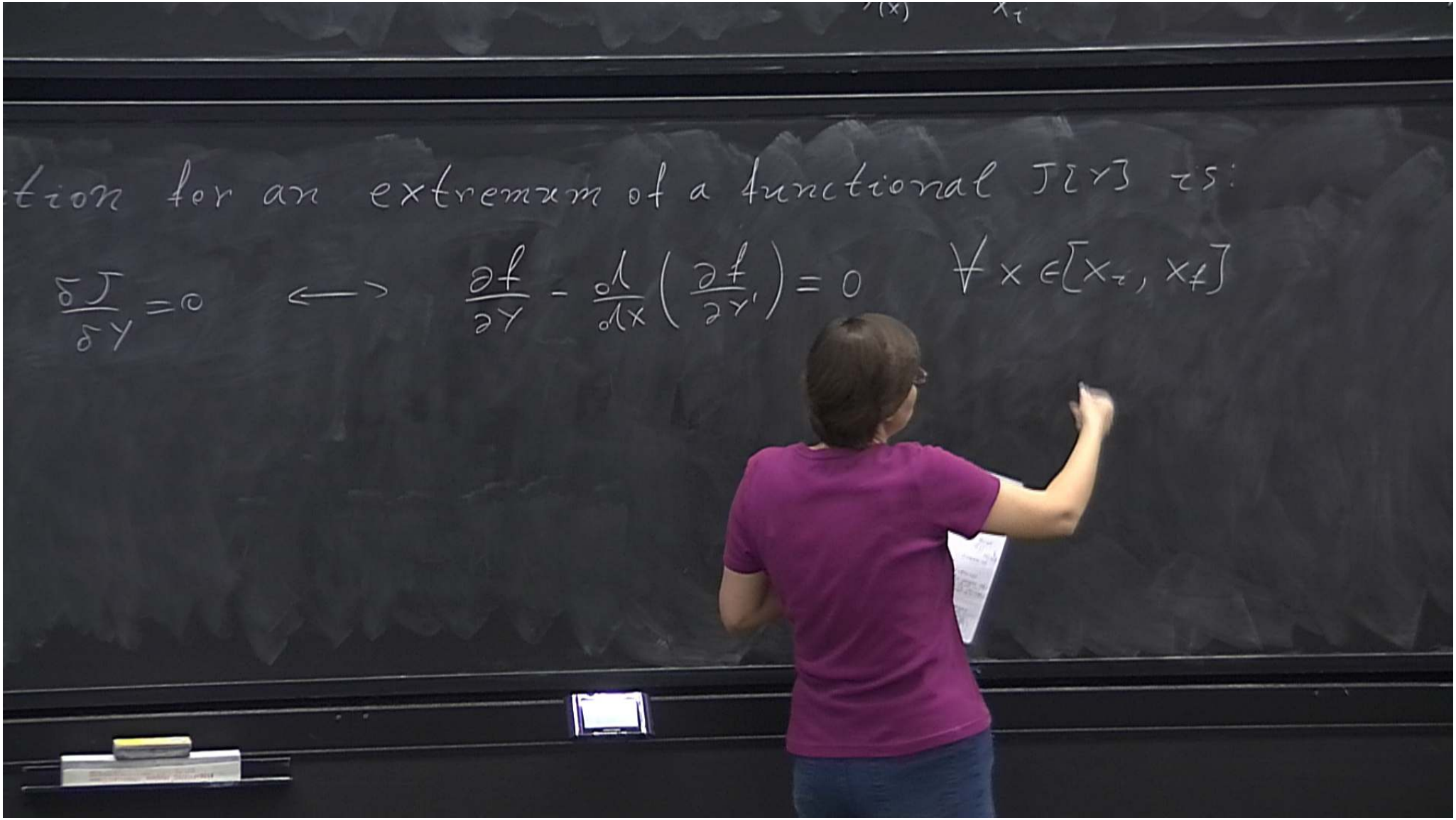
$$\delta J[y] = \int_{x_i} \delta y \, \frac{\delta J}{\delta y} \, dx$$

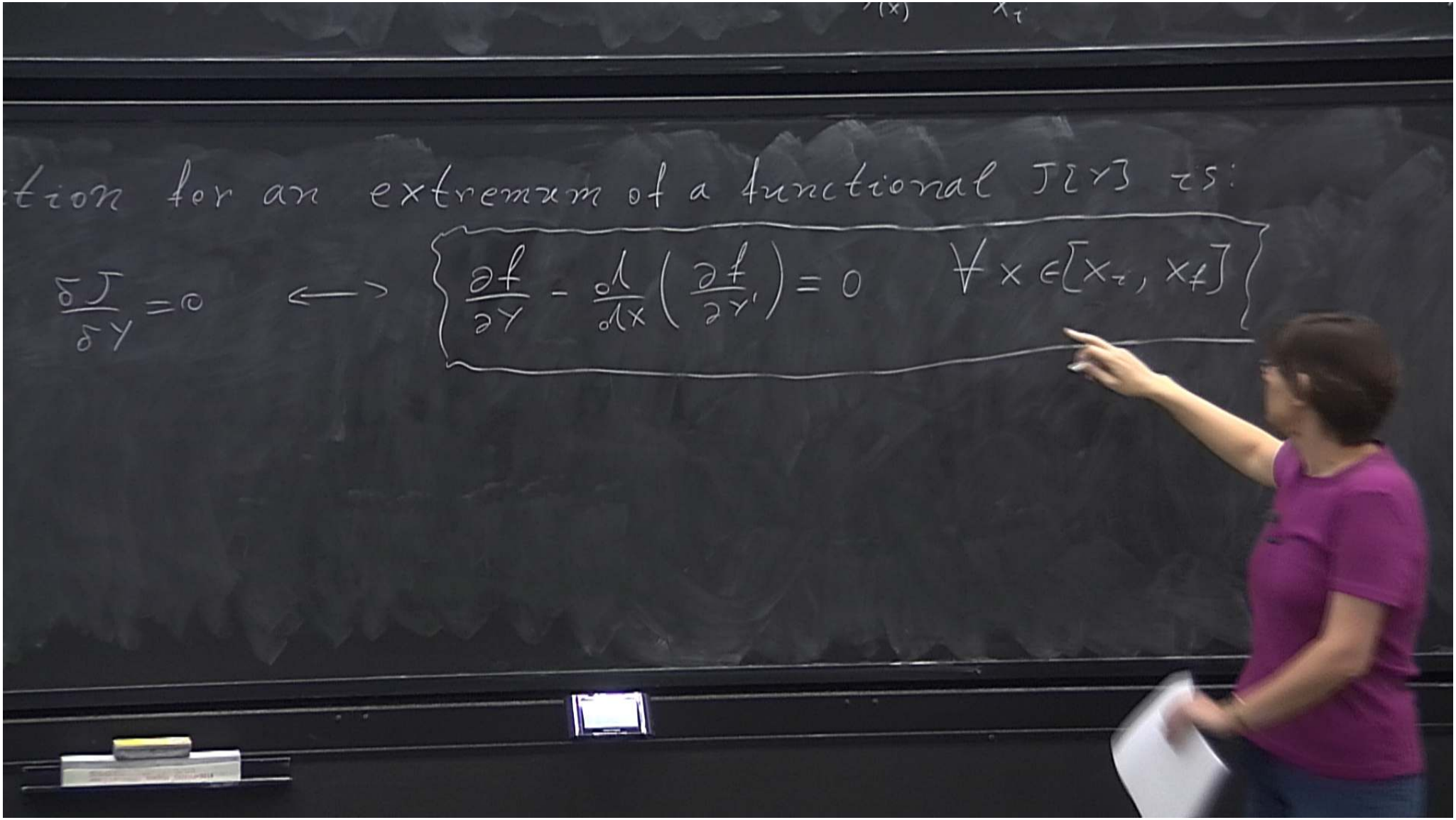
Def: If $\delta J[y] = 0$ for variation of some function y_1, then $y_1(x)$ is called stationary point of the functional $J[y]$.

Theorem: $J[y]$ is stationary under fixed boundary var. $y \to y + \delta y$ *if and only if*

$$\frac{\delta J}{\delta y} = 0 \quad \text{for every } x \in [x_i, x_f].$$


So the condition for an extremum of a functional


tion for an extremum of a functional J[y] is:
$\frac{\delta J}{\delta y} = 0 \quad \Longleftrightarrow \quad \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \qquad \forall x \in [x_i, x_f]$


tion for an extremum of a functional J[y] is:
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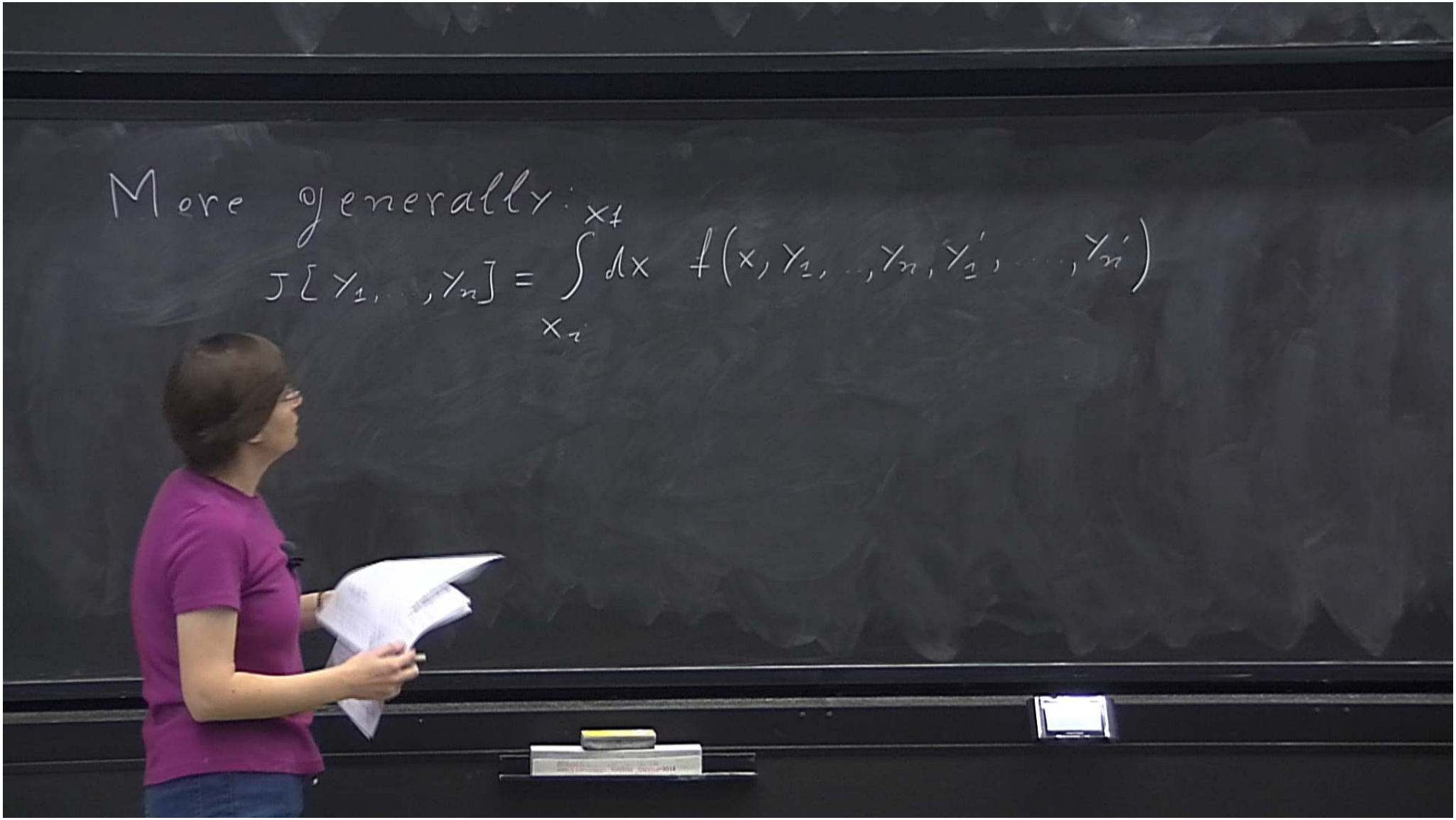
tion for an extremum of a functional $J[y]$ is:

$$\frac{\delta J}{\delta y} = 0 \quad \Longleftrightarrow \quad \left\{ \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \qquad \forall x \in [x_i, x_f] \right\}$$

solved for $y(x)$, not for x

Euler-Lagrange equation


More generally:
$J[y_1, \ldots, y_n] = \int_{x_1}^{x_t} dx \; f(x, y_1, \ldots, y_n, y_1', \ldots, y_n')$

More generally:

$$J[y_1, \ldots, y_n] = \int_{x_1}^{x_2} dx \; f(x, y_1, \ldots, y_n, y_1', \ldots, y_n')$$

Euler-Lagrange equations:

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_i'}\right) = 0 \quad , \quad i = 1, \ldots, n$$

More generally:

$$J[y_1, \ldots, y_n] = \int_{x_1}^{x_2} dx \; f(x, y_1, \ldots, y_n, y_1', \ldots, y_n')$$

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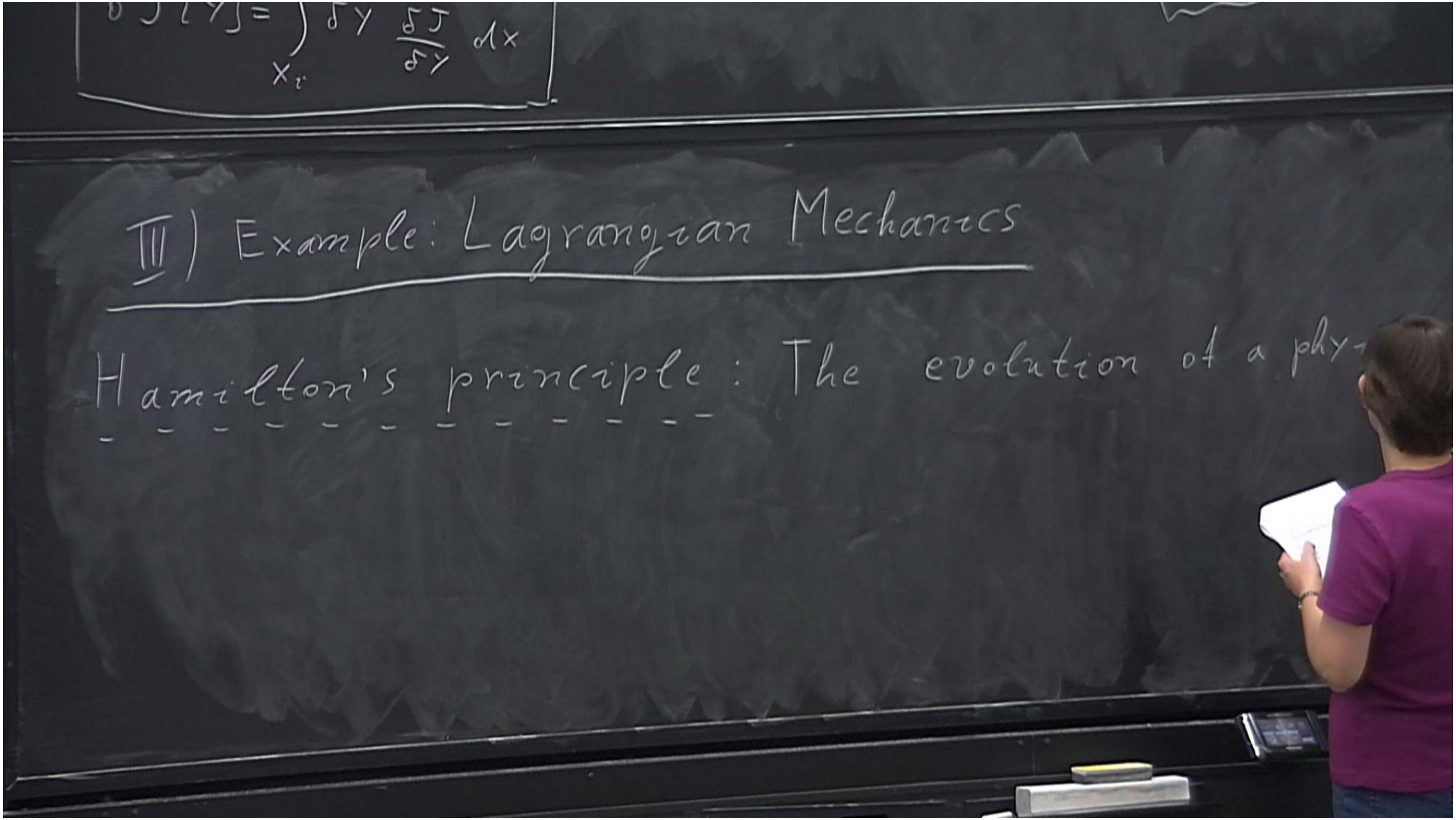
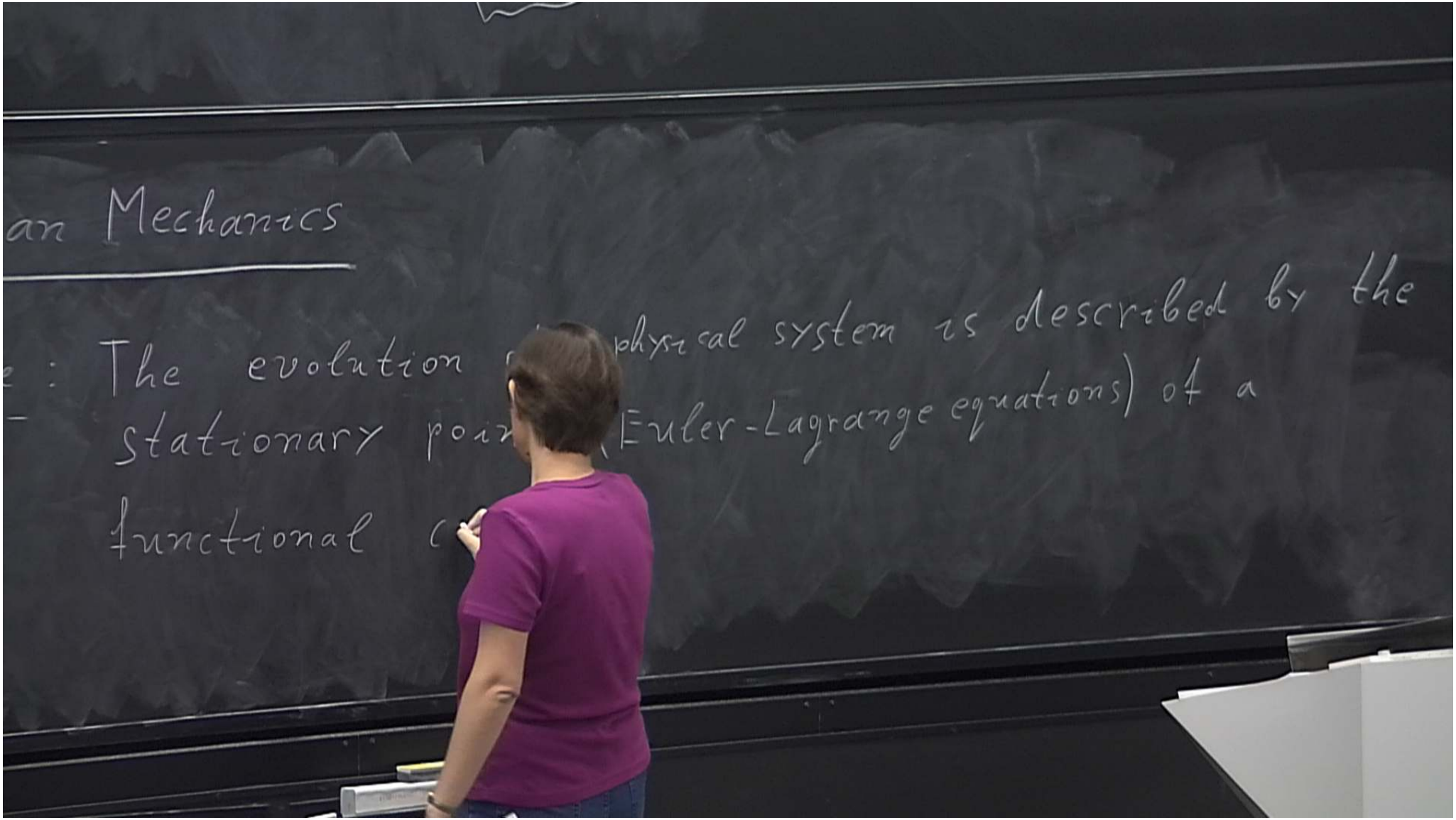
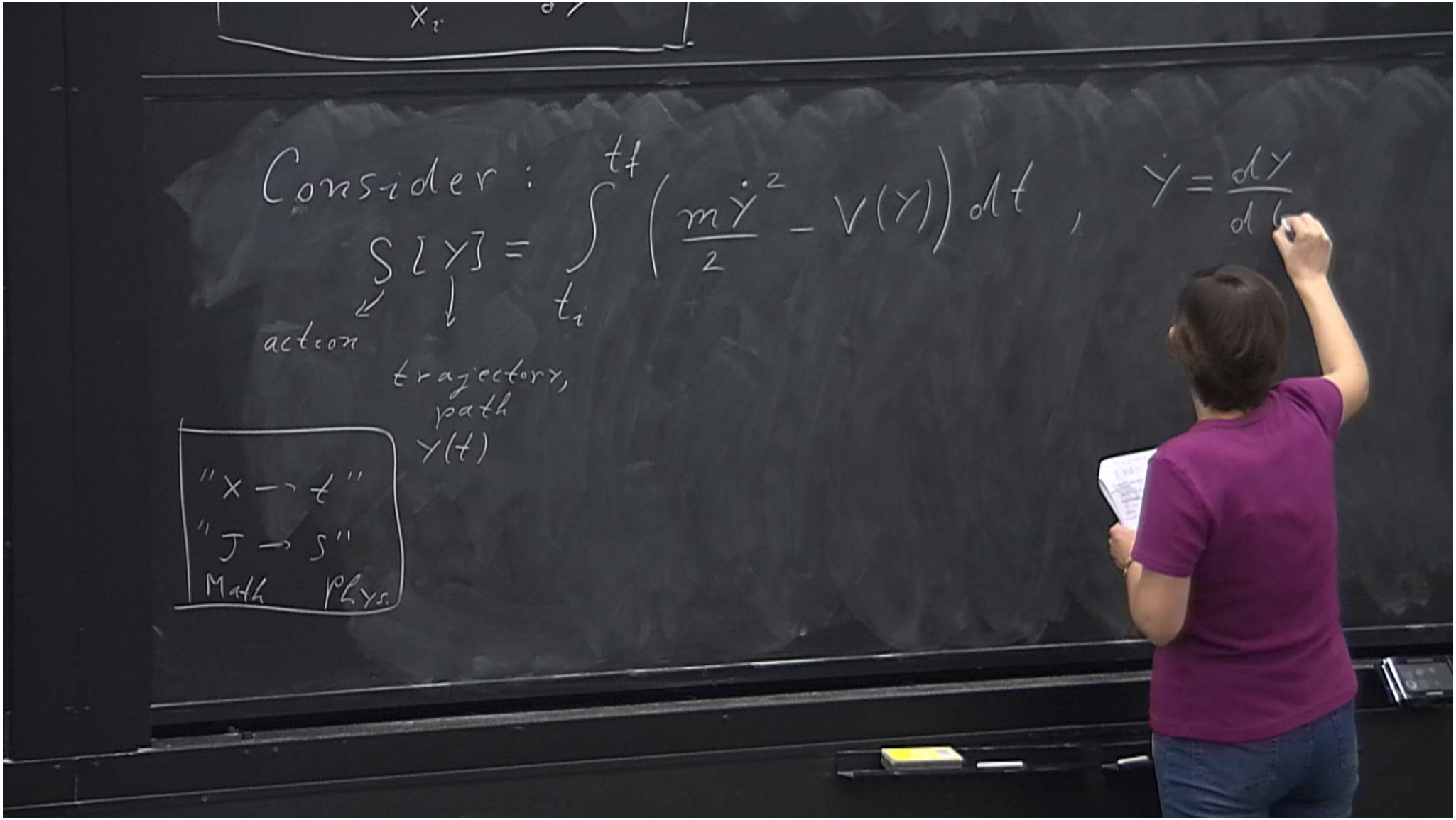
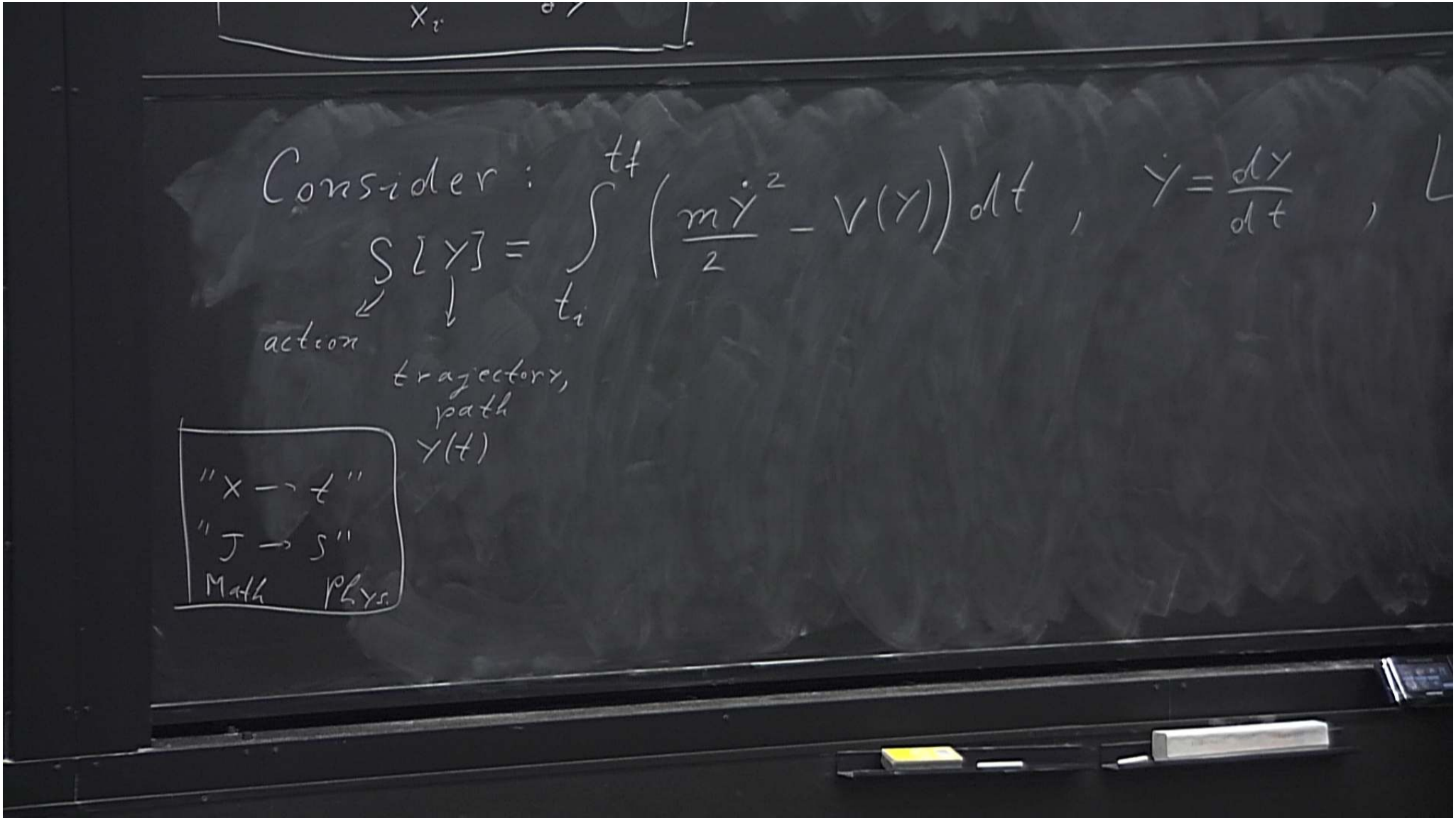

III) Example: Lagrangian Mechanics
Hamilton's principle: The evolution of a phy-


an Mechanics
The evolution
physical system is described by the
stationary
Euler-Lagrange equations) of a
functional


Consider:
$S[y] = \int_{t_i}^{t_f} \left(\frac{m\dot{y}^2}{2} - V(y) \right) dt$, $\dot{y} = \frac{dy}{dt}$
action
trajectory, path
$y(t)$
"$x \to t$"
"$J \to S$"
Math Phys.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$S[y] = \int_{t_i}^{t_f} \left(\frac{m\dot{y}^2}{2} - V(y) \right) dt$, $\dot{y} = \frac{dy}{dt}$,
action
trajectory, path
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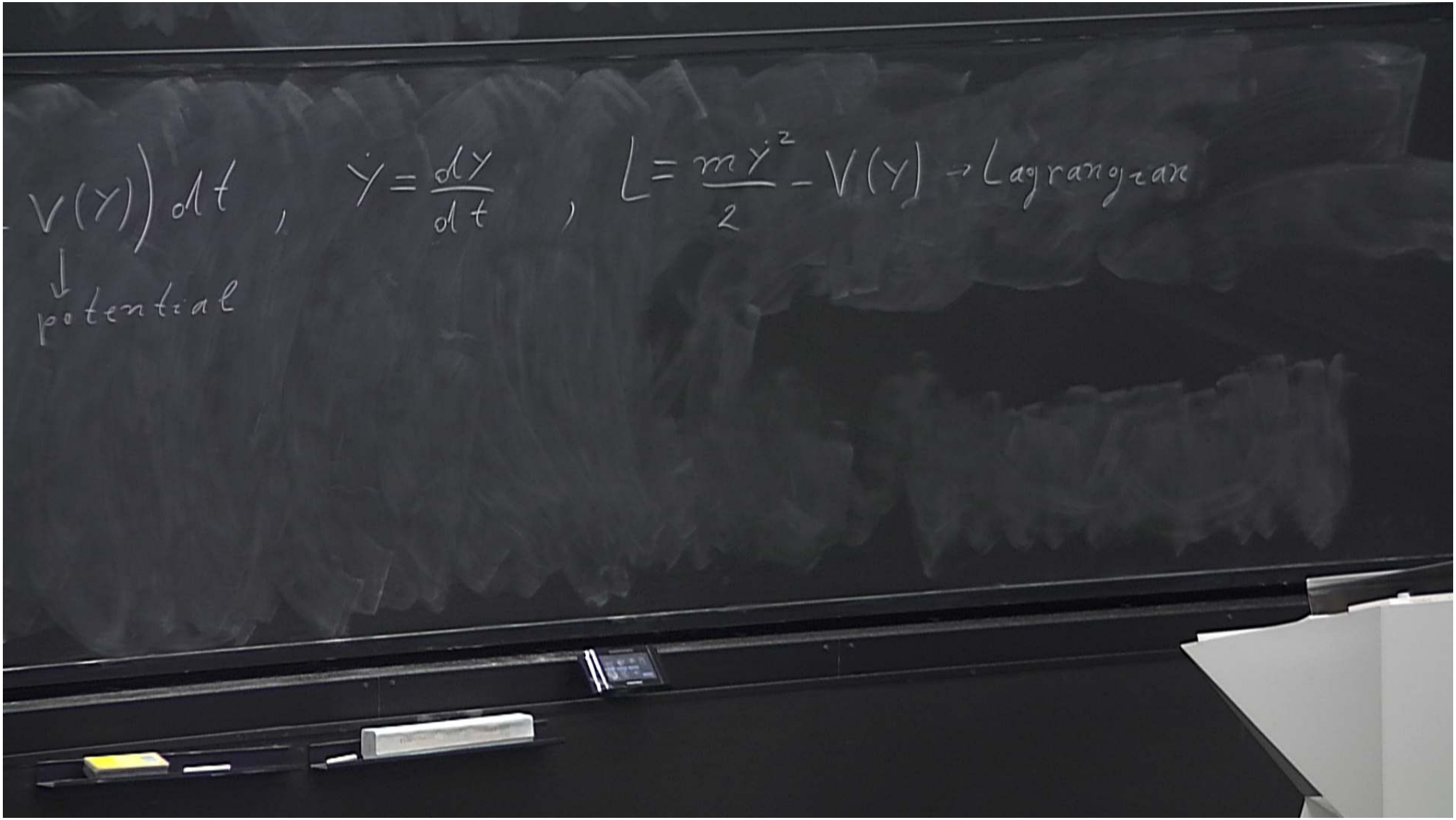
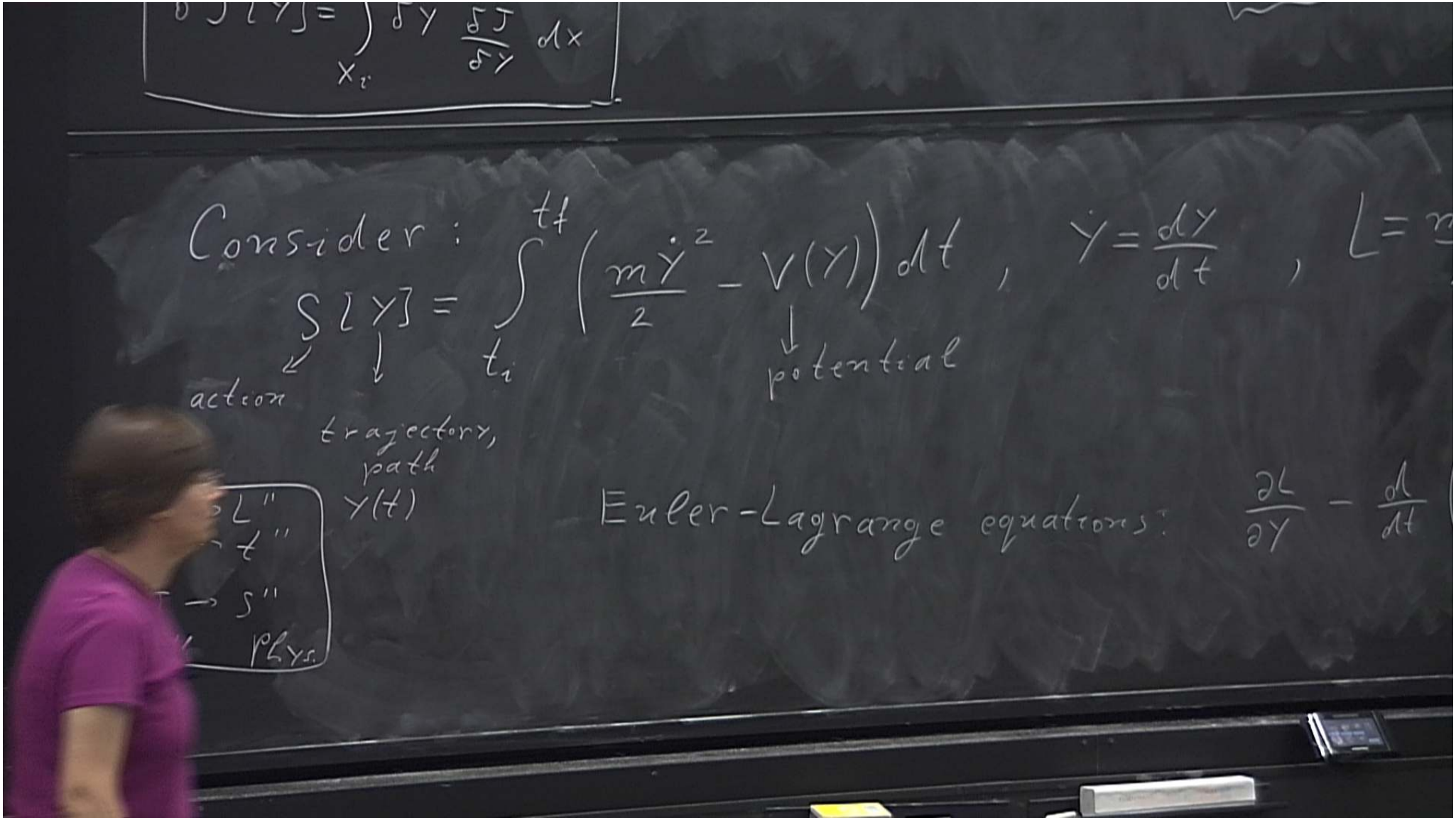
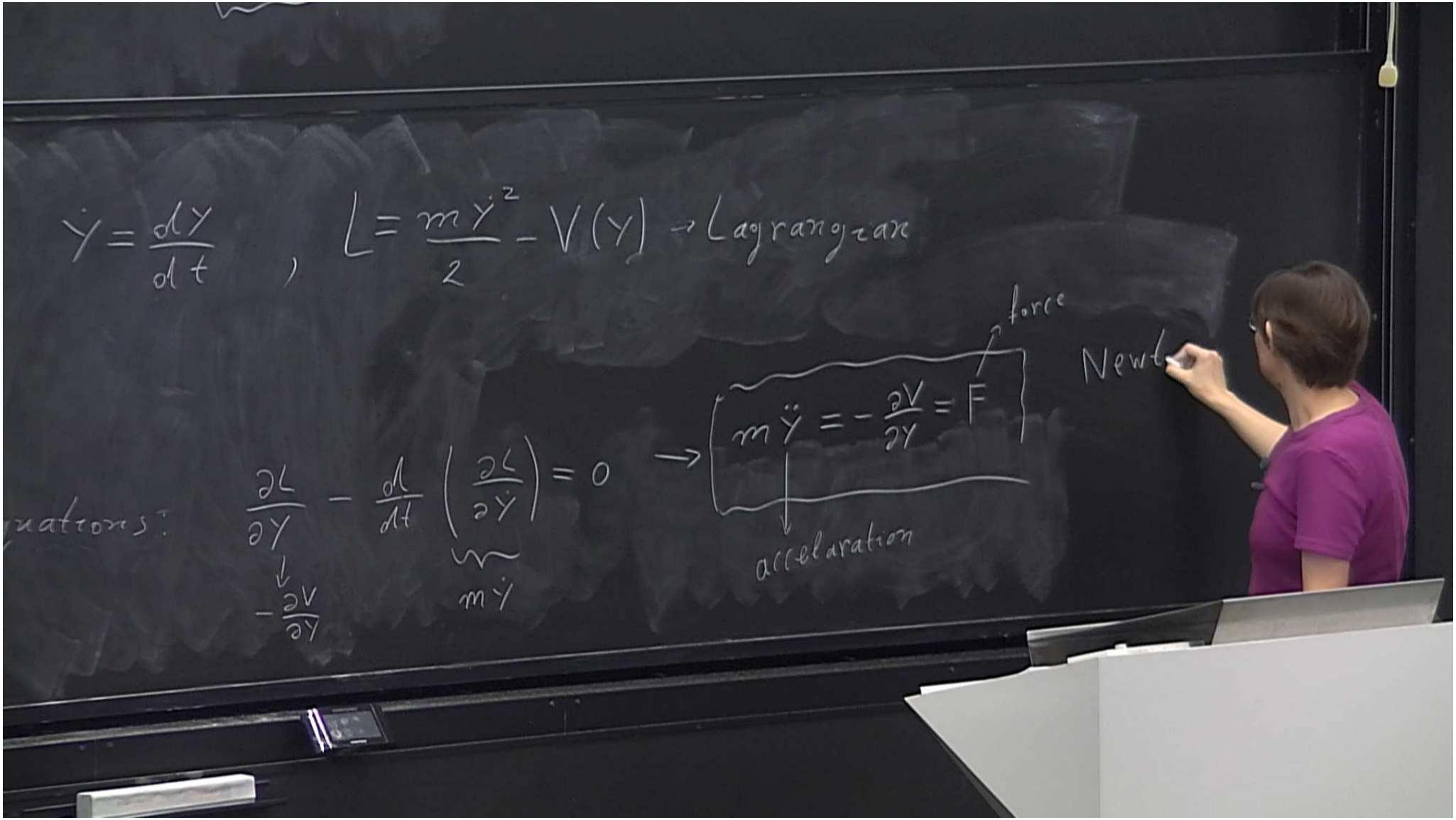
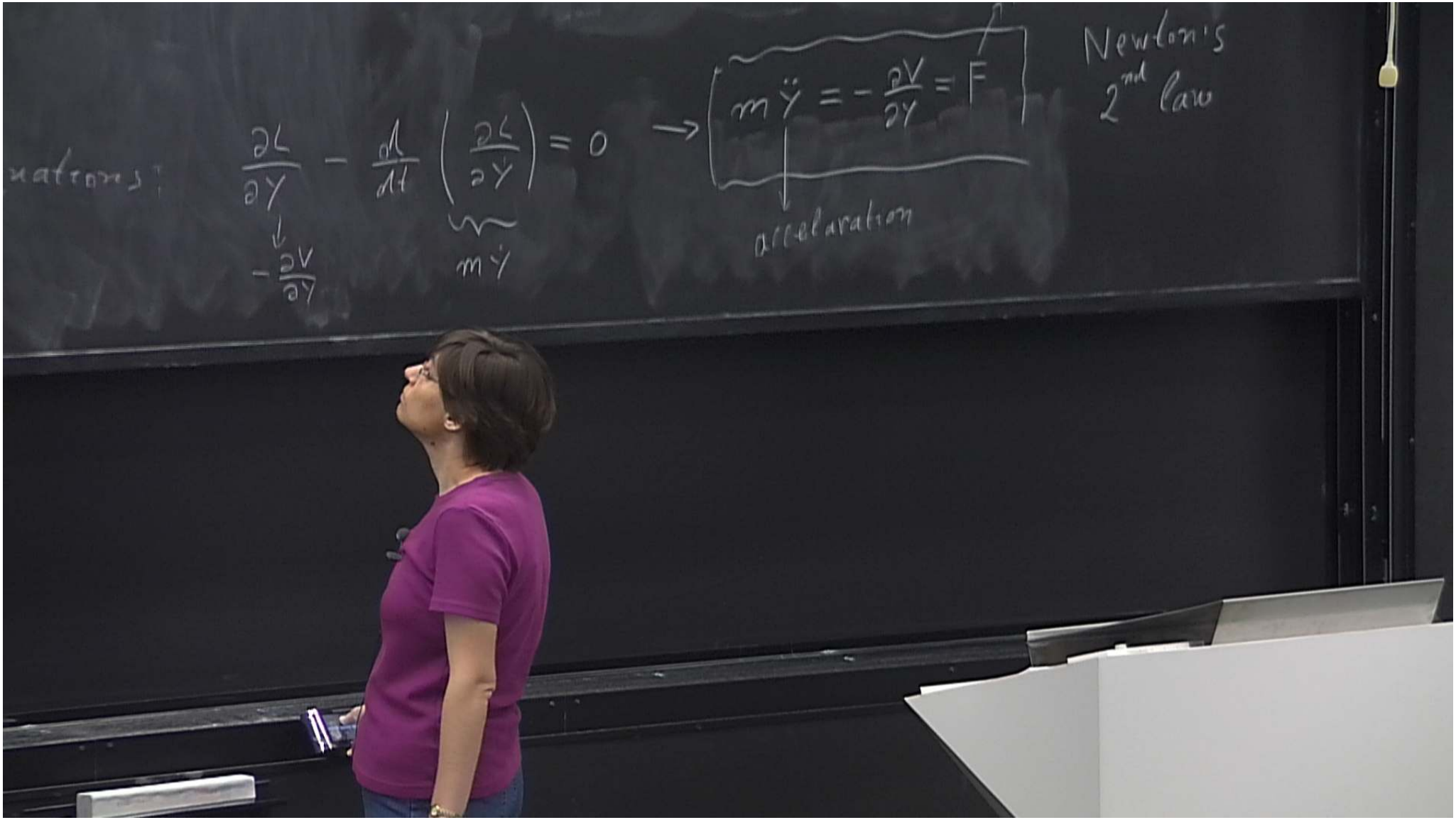
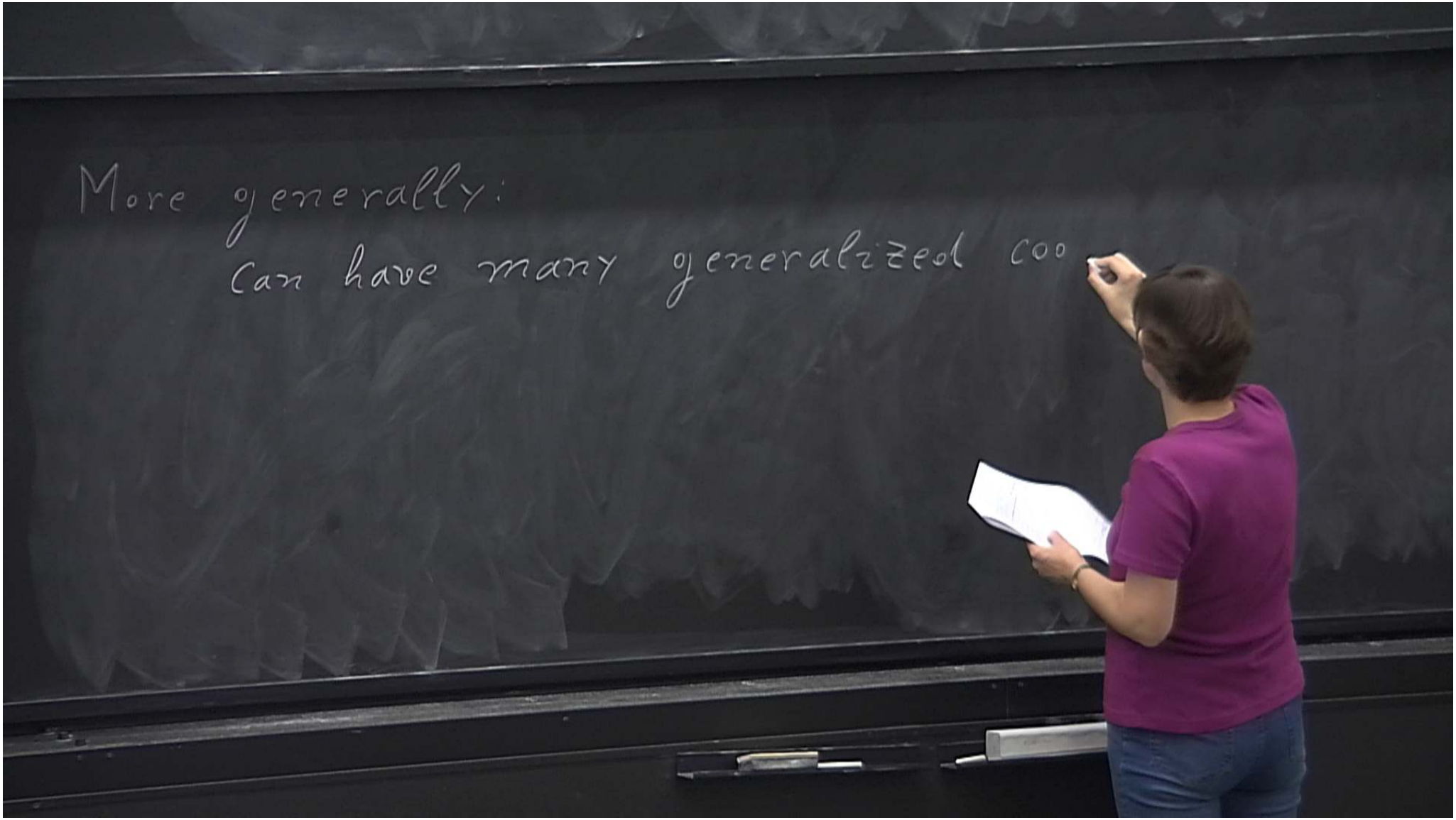
Math Phys.

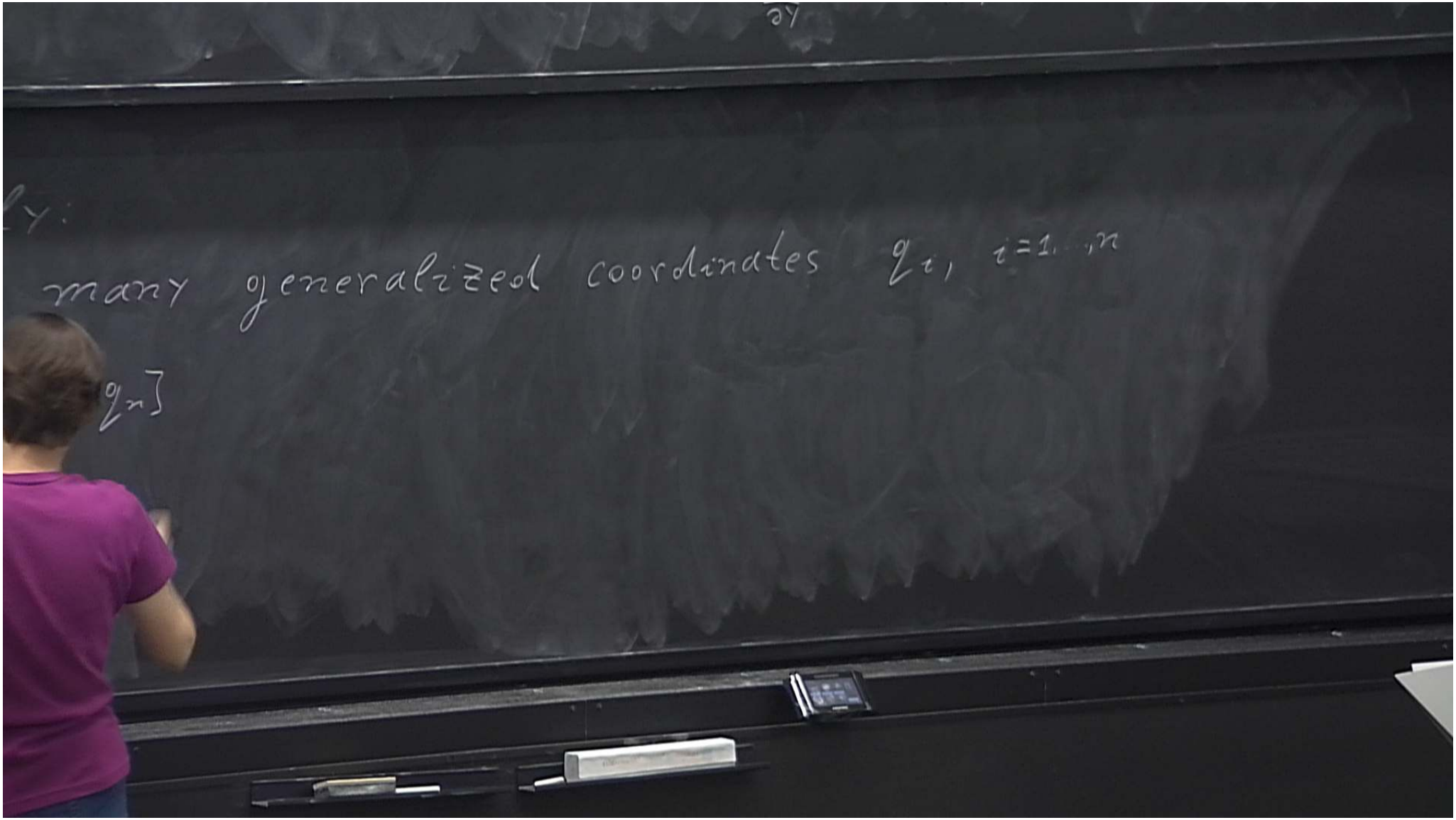

$V(Y)\,dt$, $\dot{Y} = \frac{dY}{dt}$, $L = \frac{m\dot{Y}^2}{2} - V(Y)$ → Lagrangean
↓
potential

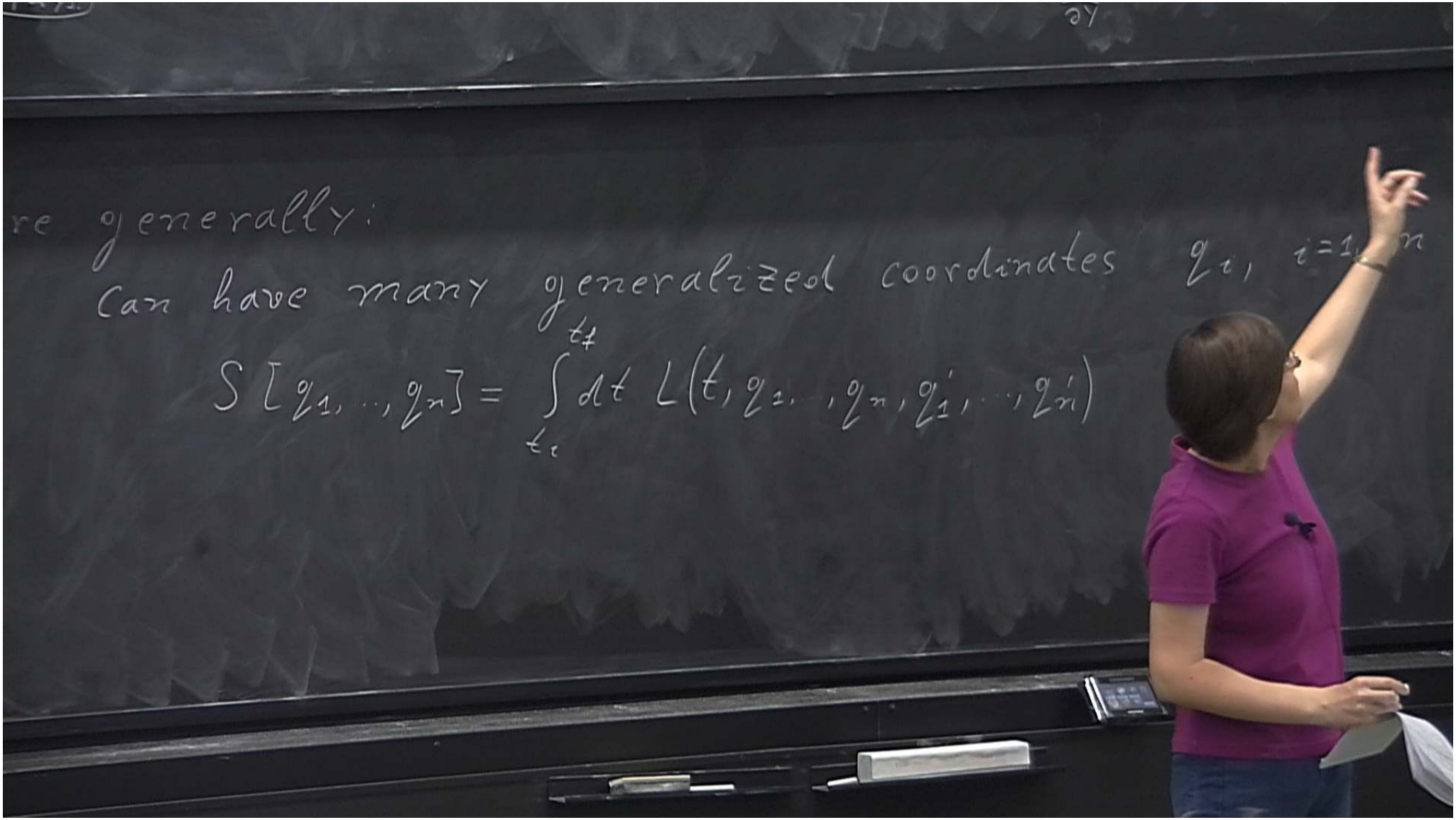

Consider:
$S[Y] = \int_{t_i}^{t_f} \left(\frac{m\dot{Y}^2}{2} - V(Y) \right) dt$, $\dot{Y} = \frac{dY}{dt}$,
action
trajectory, path
$Y(t)$
potential
Euler-Lagrange equations: $\frac{\partial L}{\partial Y} - \frac{d}{dt}$

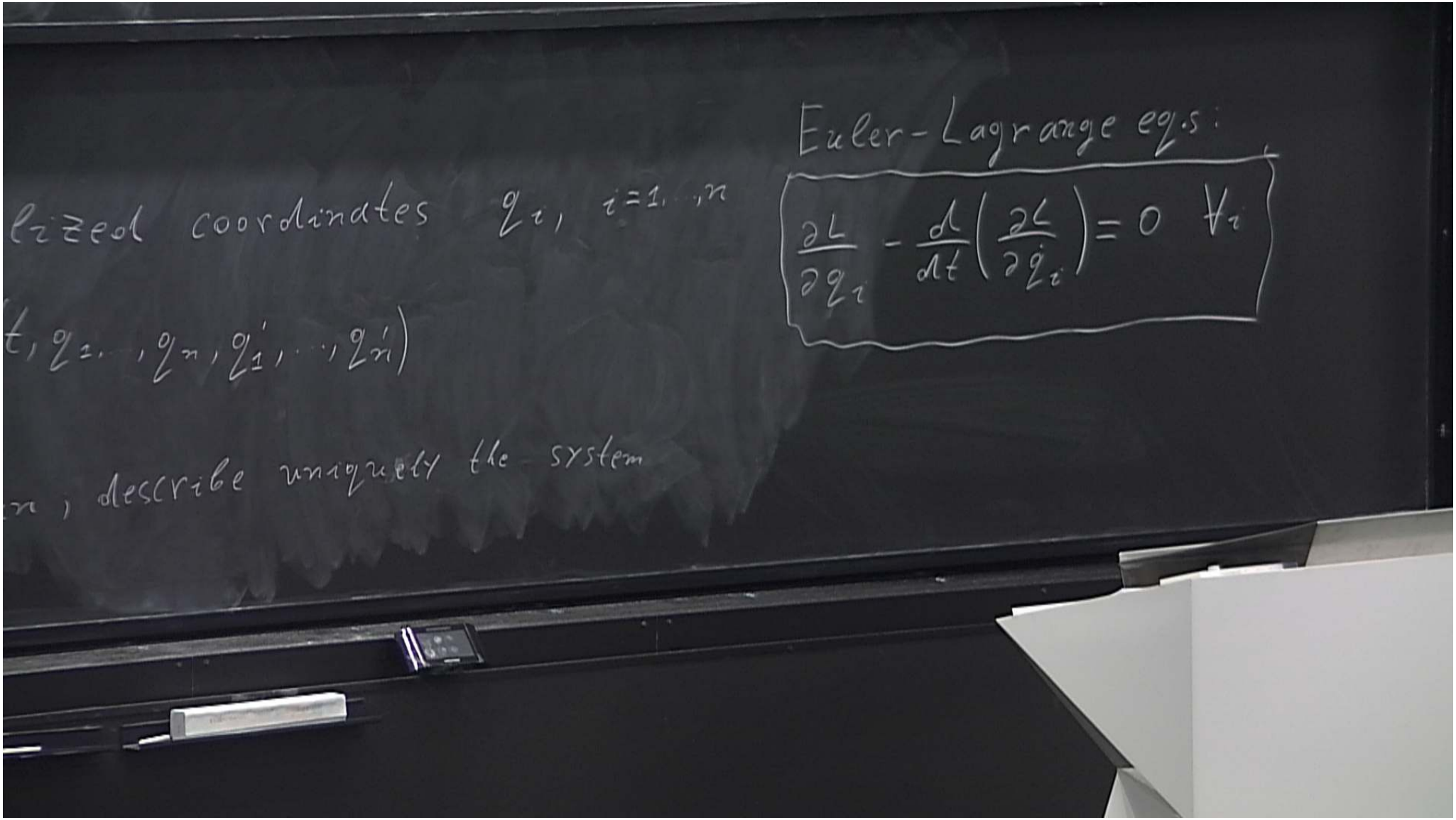

$\dot{y} = \frac{dy}{dt}$,
$L = \frac{m\dot{y}^2}{2} - V(y)$ → Lagrangian
$\frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = 0$ → $m\ddot{y} = -\frac{\partial V}{\partial y} = F$
force
acceleration
$-\frac{\partial V}{\partial y}$
$m\dot{y}$
Newt
uations:


$\frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = 0 \rightarrow \boxed{m\ddot{y} = -\frac{\partial V}{\partial y} = F}$
$-\frac{\partial V}{\partial y}$
$m\dot{y}$
acceleration
Newton's 2nd law

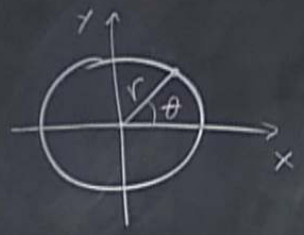

More generally:
Can have many generalized coo


many generalized coordinates q_i, $i = 1 \ldots n$
$q_n\}$


re generally:
Can have many generalized coordinates q_i, $i = 1, \ldots, n$
$S[q_1, \ldots, q_n] = \int_{t_i}^{t_f} dt\, L(t, q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n)$

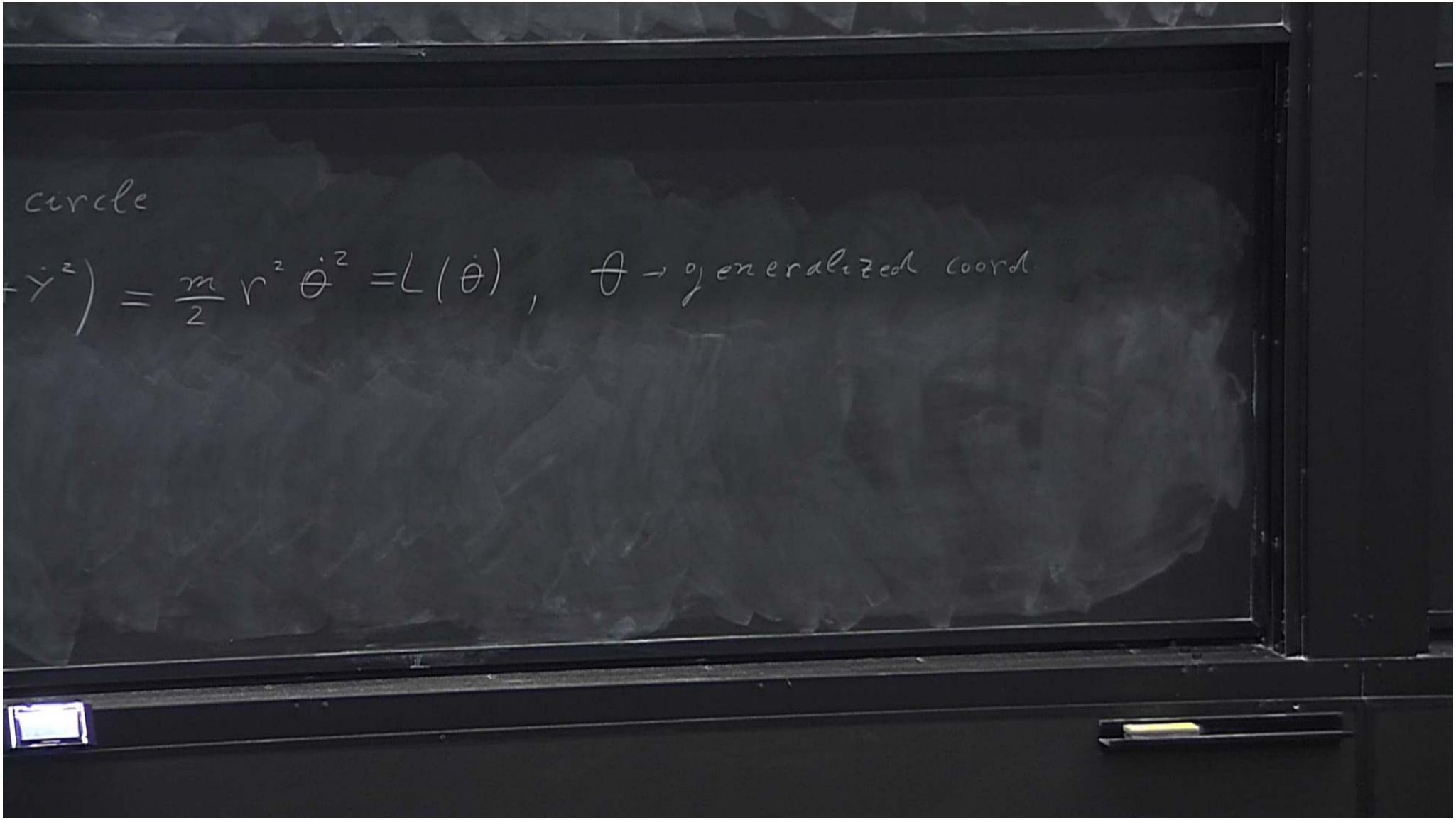

lized coordinates q_i, $i=1,\dots,n$
$t, q_1, \dots, q_n, q_1', \dots, q_n')$
n, describe uniquely the system
Euler-Lagrange eq.s:
$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0 \quad \forall i$

Example: Particle freely moving on a circle — no potential

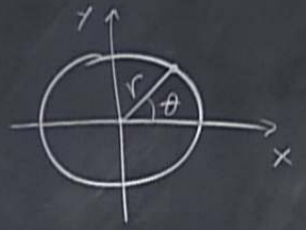


$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) = \frac{m}{2} r^2 \dot{\theta}$$


circle
$\dot{x}^2) = \frac{m}{2} r^2 \dot{\theta}^2 = L(\theta)$, θ → generalized coord

Example: Particle freely moving on a circle (no potential)

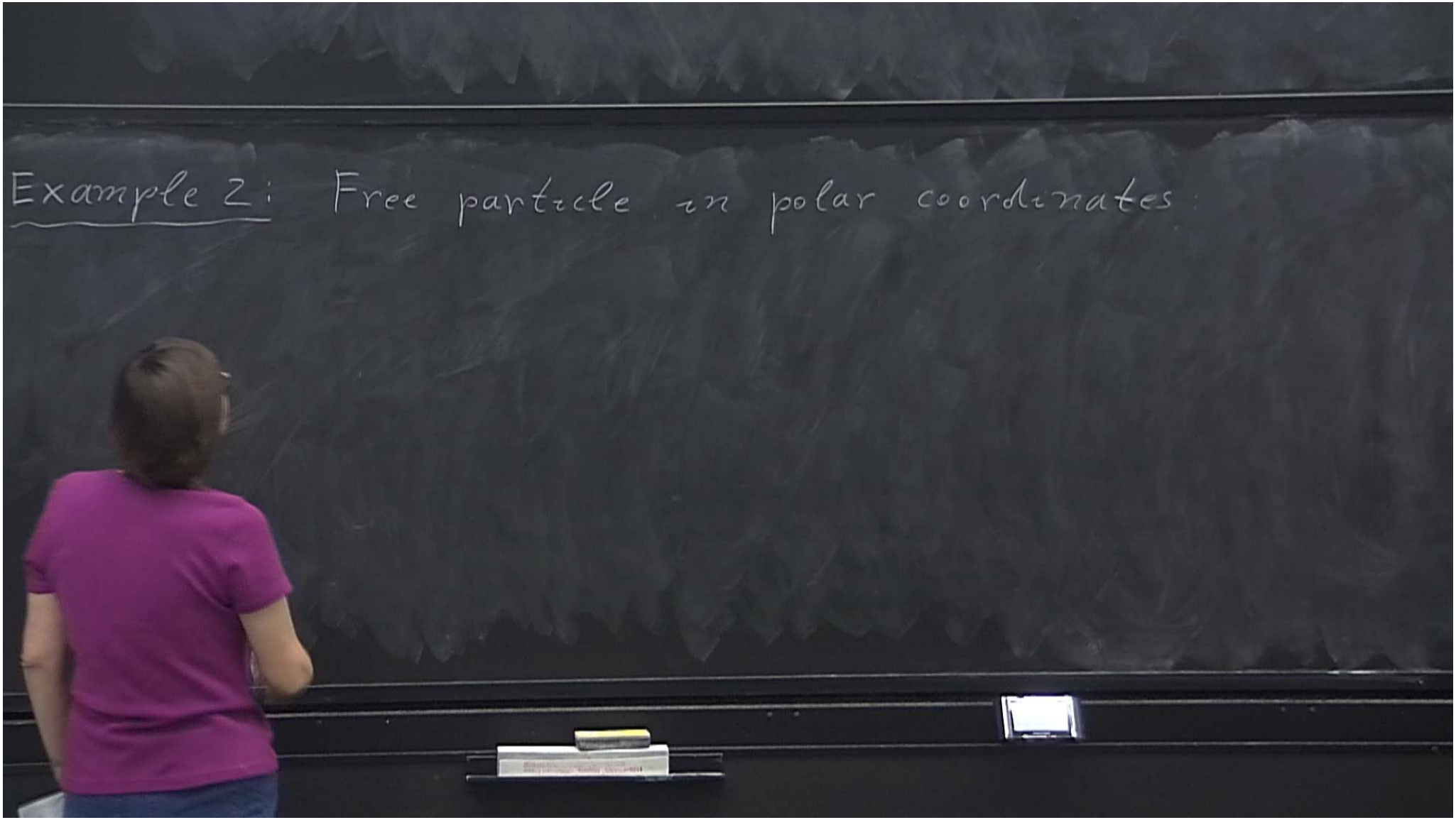


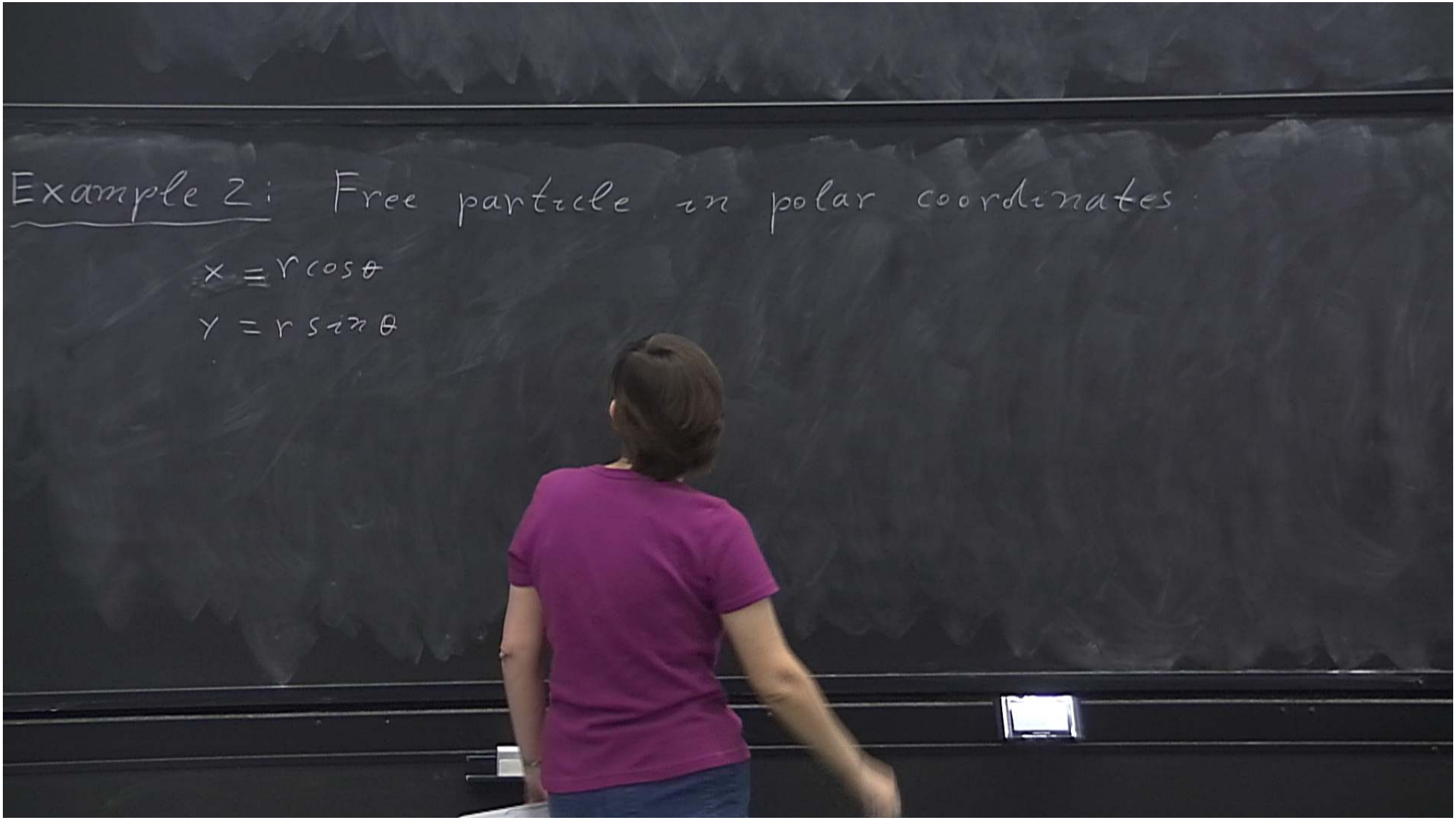
$x = r\cos\theta$

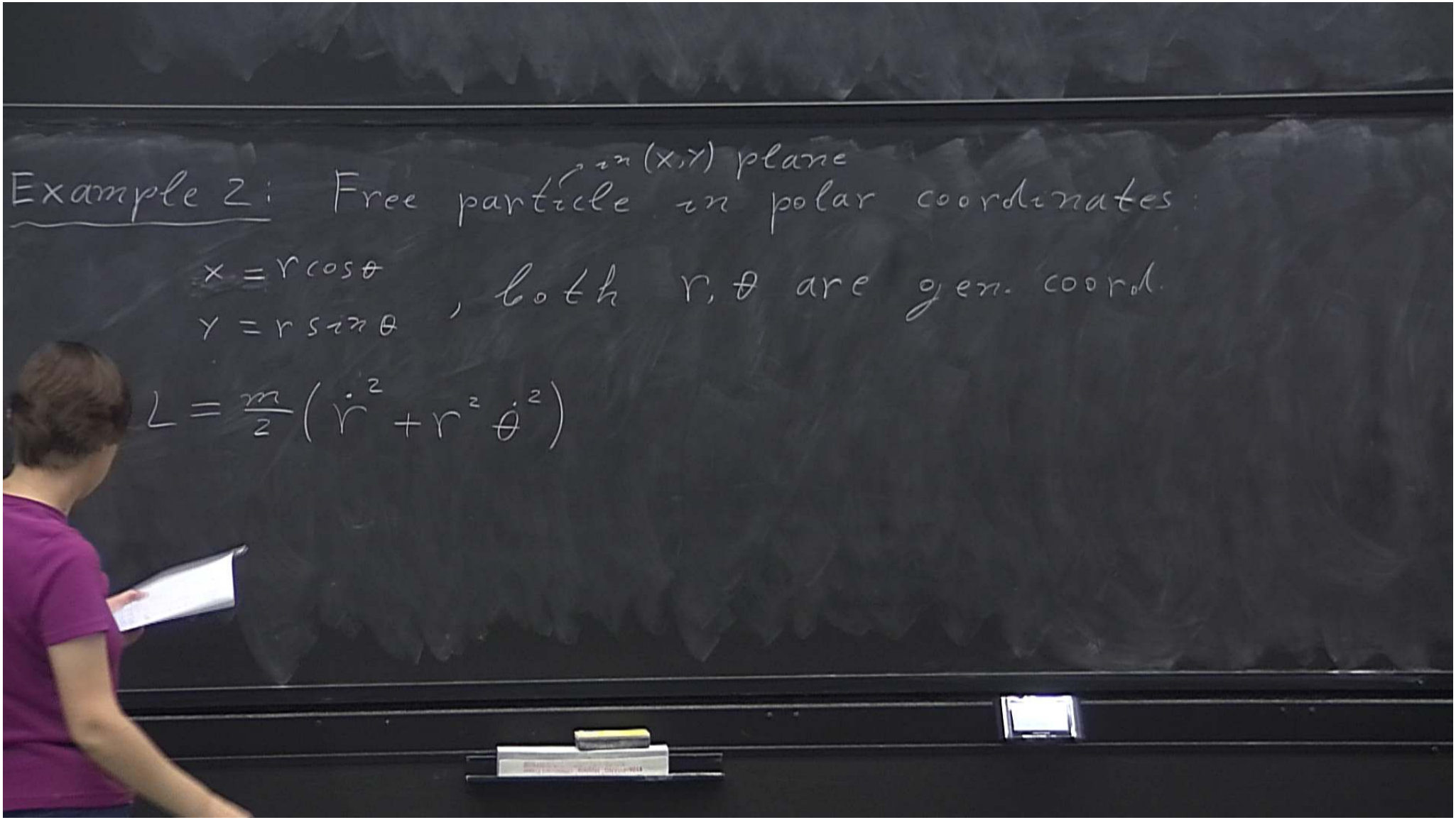
$y = r\sin\theta$

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) = \frac{m}{2} r^2 \dot{\theta}^2 = L$$

In general q_i: can be angles, radia, some small


Example 2: Free particle in polar coordinates:


Example 2: Free particle in polar coordinates
x = r cos θ
y = r sin θ


Example 2: Free particle in (x,y) plane in polar coordinates
x = r cos θ
y = r sin θ
, both r, θ are gen. coord.
L = m/2 (ṙ² + r² θ̇²)

Example 2: Free particle in (x,y) plane in polar coordinates:

$x = r\cos\theta$

$y = r\sin\theta$, both r, θ are gen. coord.

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

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$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \longrightarrow \ddot{\theta} - \frac{2}{r}\dot{r}\dot{\theta} = 0$$

] → solution is