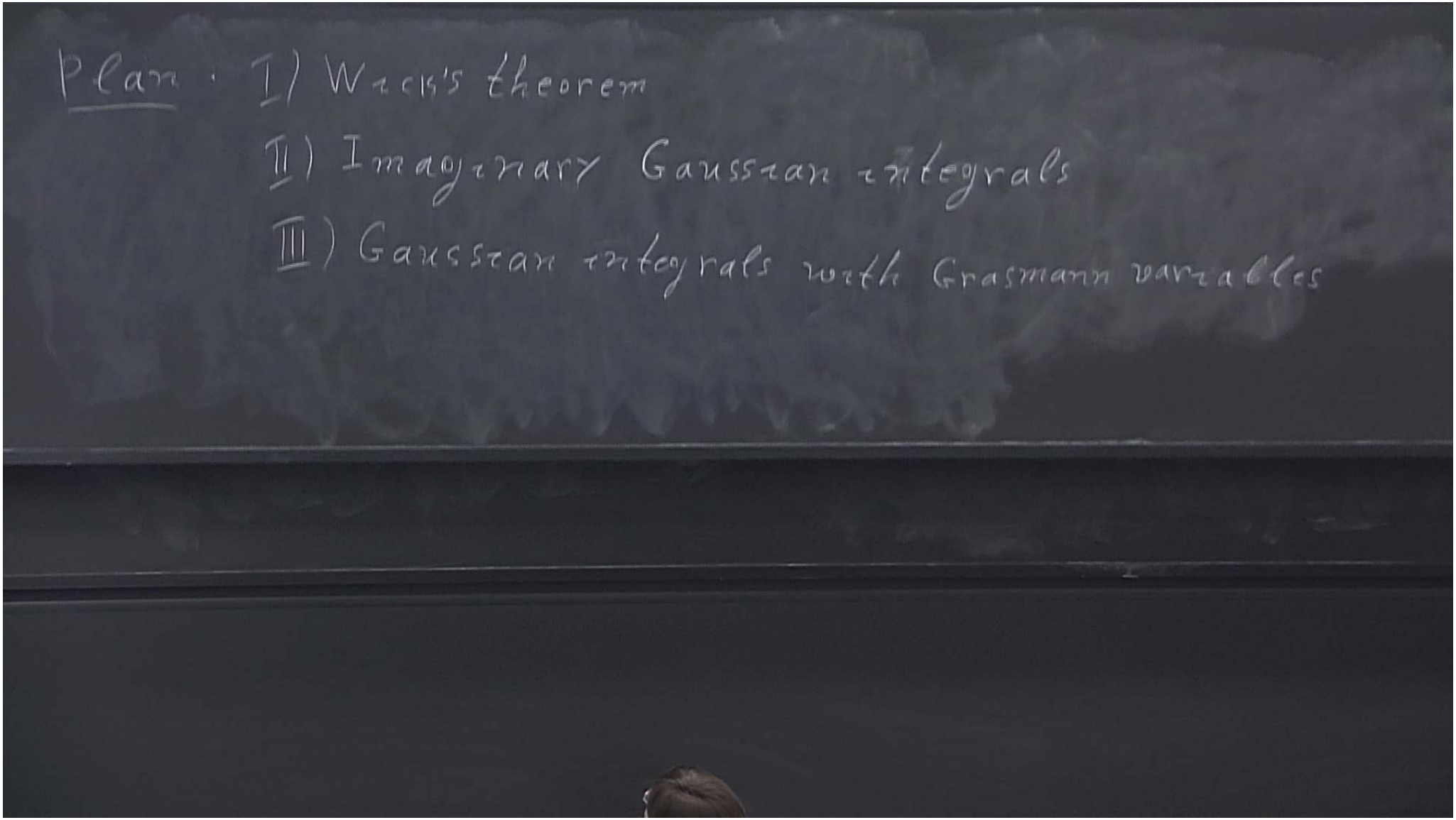
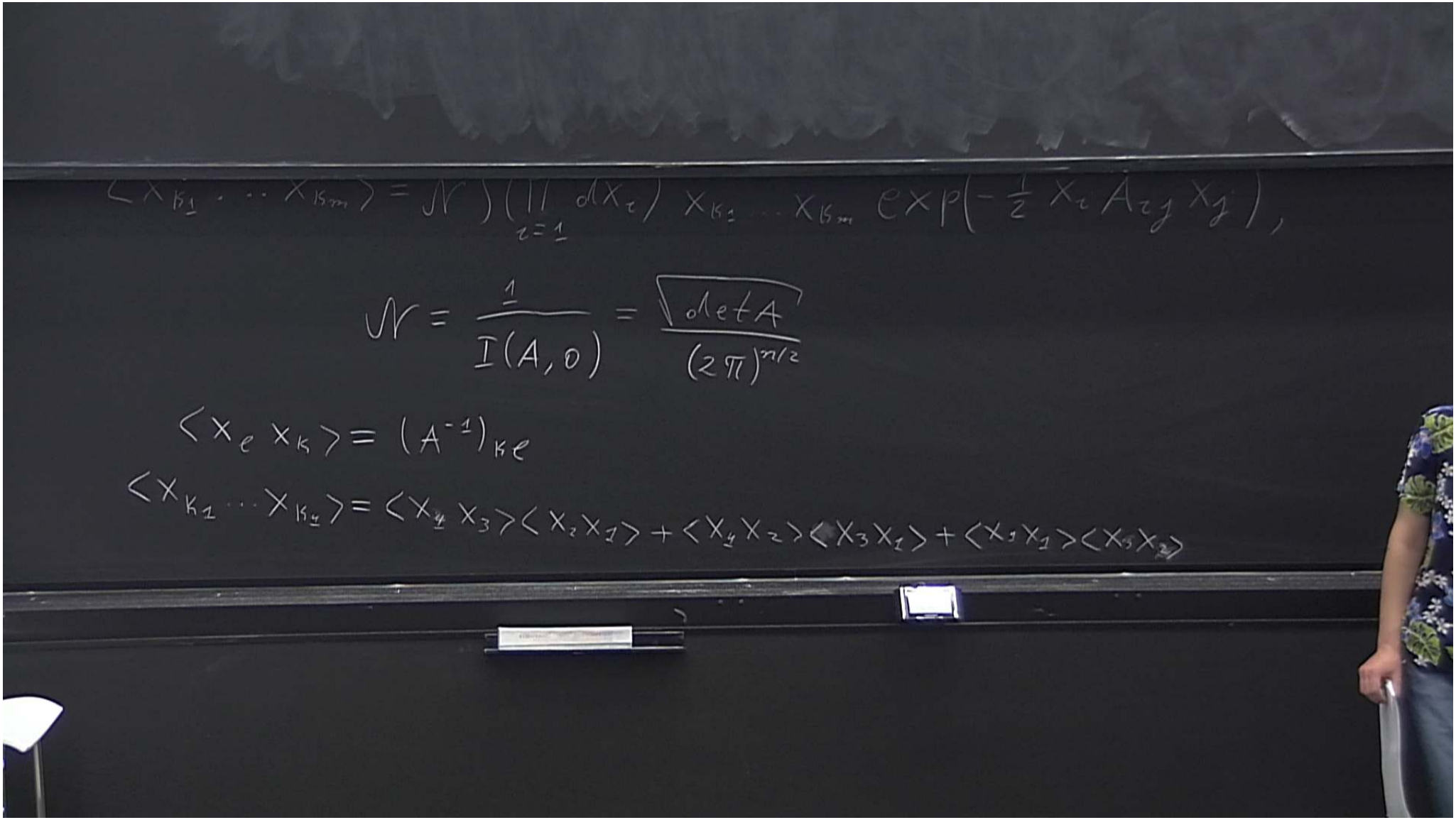
Title: Calculus of Variation and Gaussian Integrals - Lecture 2

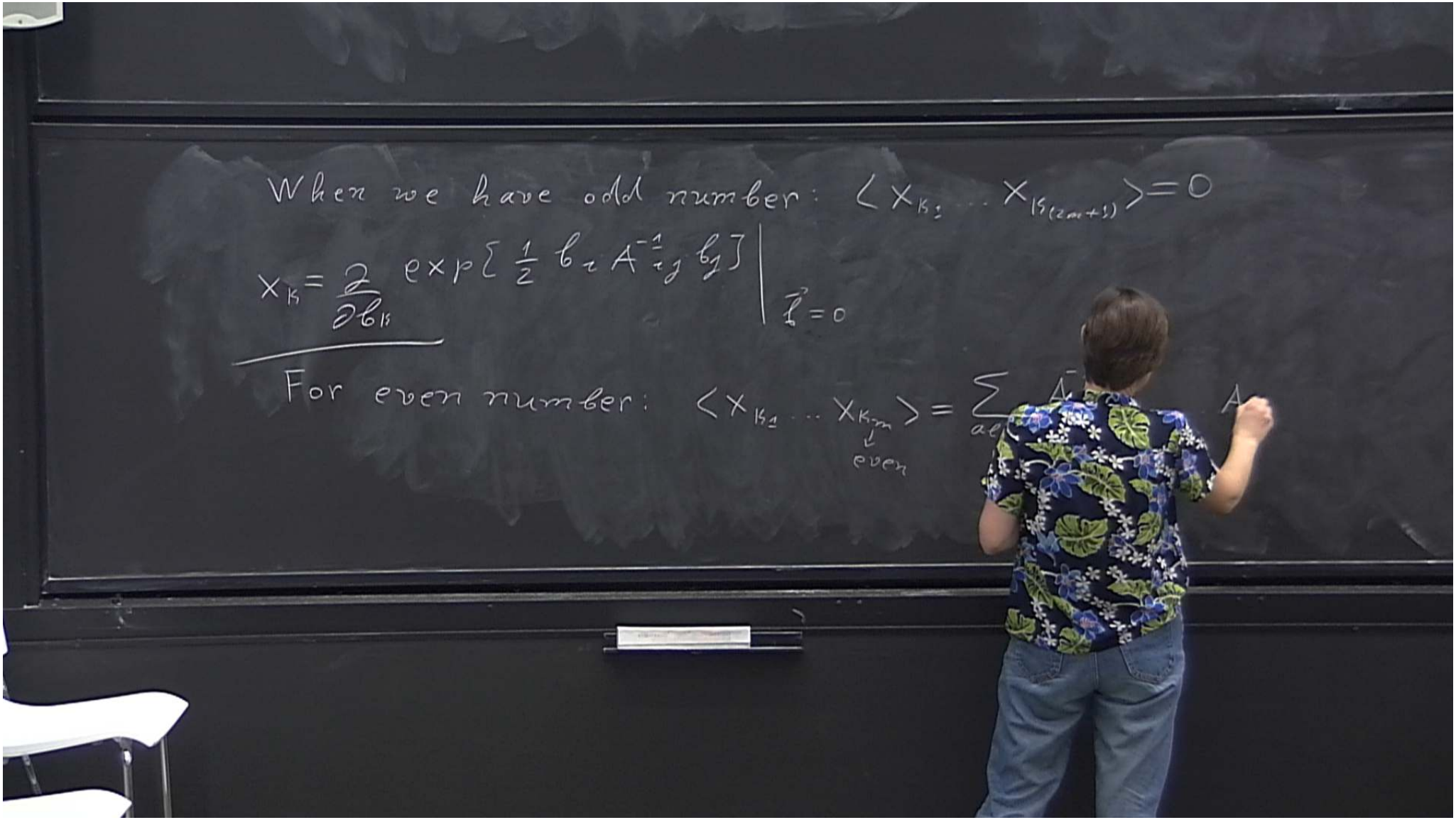
Date: Aug 22, 2012 10:30 AM

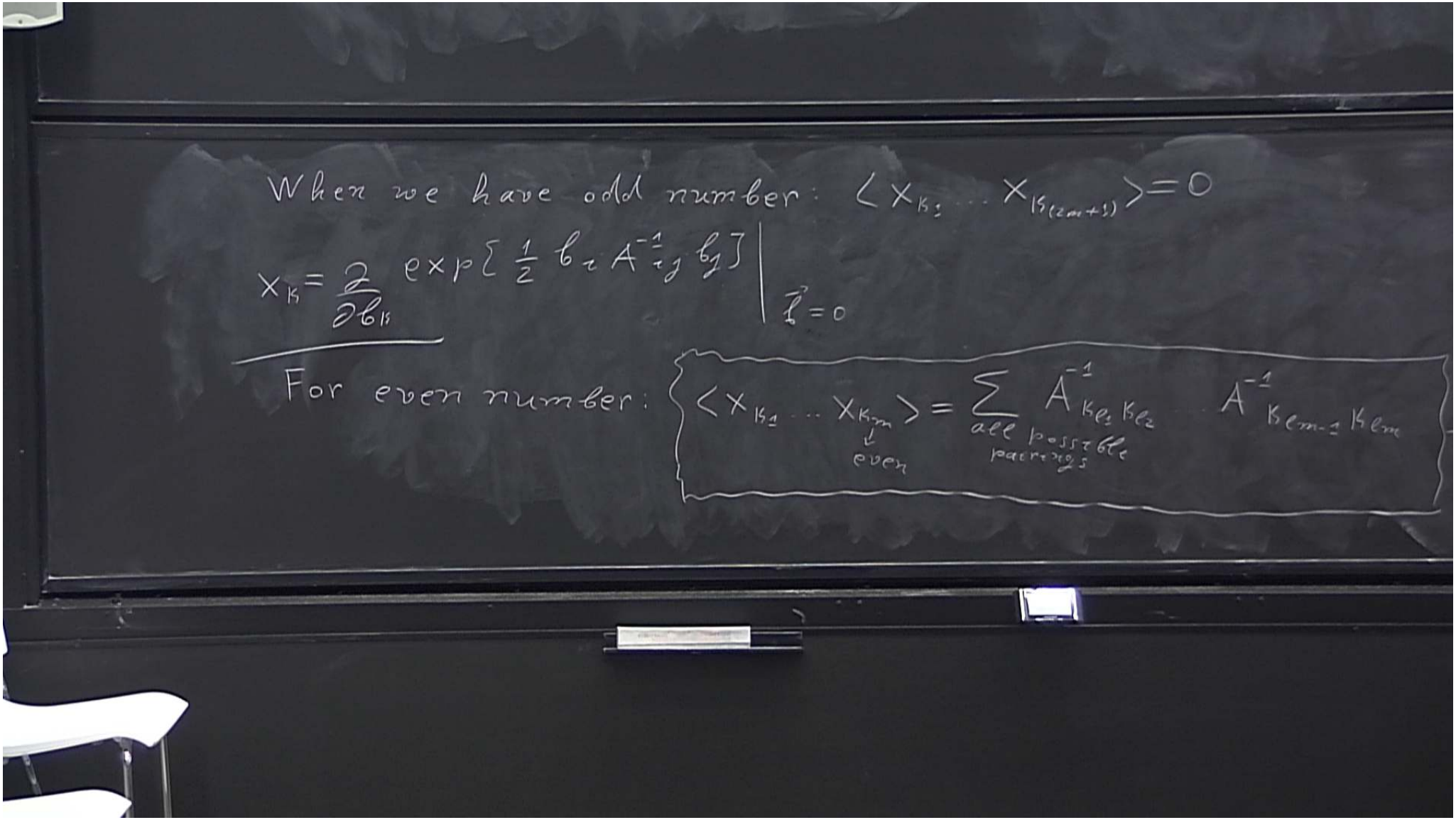
URL: http://pirsa.org/12080025

Abstract:


Plan · I) Wick's theorem
II) Imaginary Gaussian integrals
III) Gaussian integrals with Grasmann variables


$\langle x_{k_1} \cdots x_{k_{2m}} \rangle = \mathcal{N} \int \left(\prod_{i=1}^{n} dx_i \right) x_{k_1} \cdots x_{k_{2m}} \exp\left(-\frac{1}{2} x_i A_{ij} x_j \right),$
$\mathcal{N} = \frac{1}{I(A,0)} = \frac{\sqrt{\det A}}{(2\pi)^{n/2}}$
$\langle x_e x_k \rangle = (A^{-1})_{ke}$
$\langle x_{k_1} \cdots x_{k_4} \rangle = \langle x_4 x_3 \rangle \langle x_2 x_1 \rangle + \langle x_4 x_2 \rangle \langle x_3 x_1 \rangle + \langle x_4 x_1 \rangle \langle x_3 x_2 \rangle$


When we have odd number: $\langle X_{k_1} \dots X_{k_{(2m+1)}} \rangle = 0$
$X_{k_i} = \frac{2}{\partial b_{k_i}} \exp\{\frac{1}{2} b_i A^{-1}_{ij} b_j\} \Big|_{\vec{b}=0}$
For even number: $\langle X_{k_1} \dots X_{k_{2m}} \rangle = \sum_{\alpha\beta}$
even


When we have odd number: $\langle X_{k_1} \dots X_{k_{(2m+1)}} \rangle = 0$
$X_{k_i} = \frac{2}{\partial b_{k_i}} \exp\{\frac{1}{2} b_i A^{-1}_{ij} b_j\} \Big|_{\vec{b}=0}$
For even number:
$\langle X_{k_1} \dots X_{k_{2m}} \rangle = \sum_{\substack{\text{all possible} \\ \text{pairings}}} A^{-1}_{k_{e_1} k_{e_2}} \dots A^{-1}_{k_{e_{2m-1}} k_{e_{2m}}}$
$2m$ ↓ even

we have odd number: $\langle X_{k_1} \dots X_{k_{(2m+1)}} \rangle = 0$

$\exp\left\{ \frac{1}{2} b_i A^{-1}_{ij} b_j \right\} \Big|_{\vec{b}=0}$

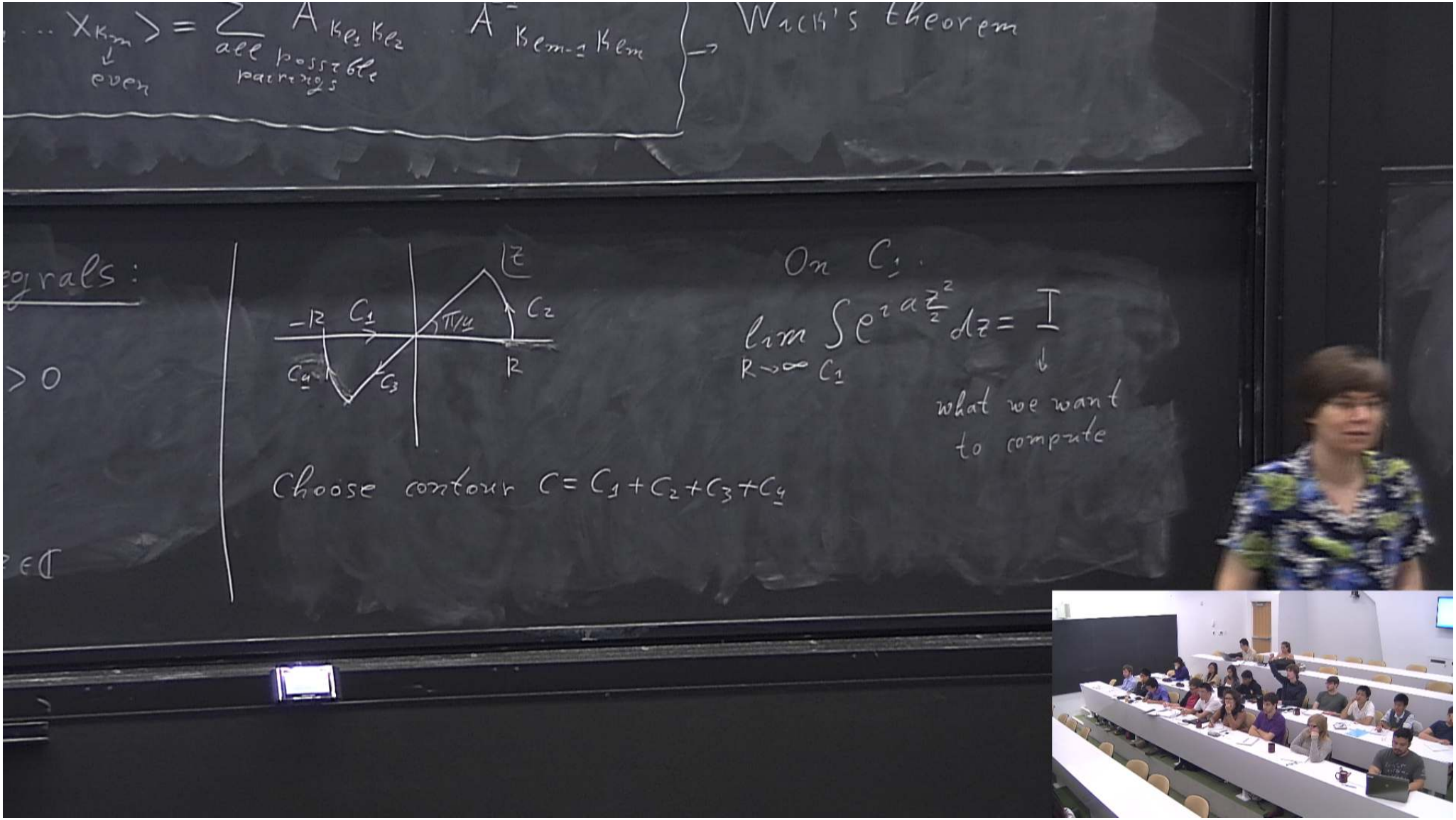
$A \rightarrow$ symmetric

even number:

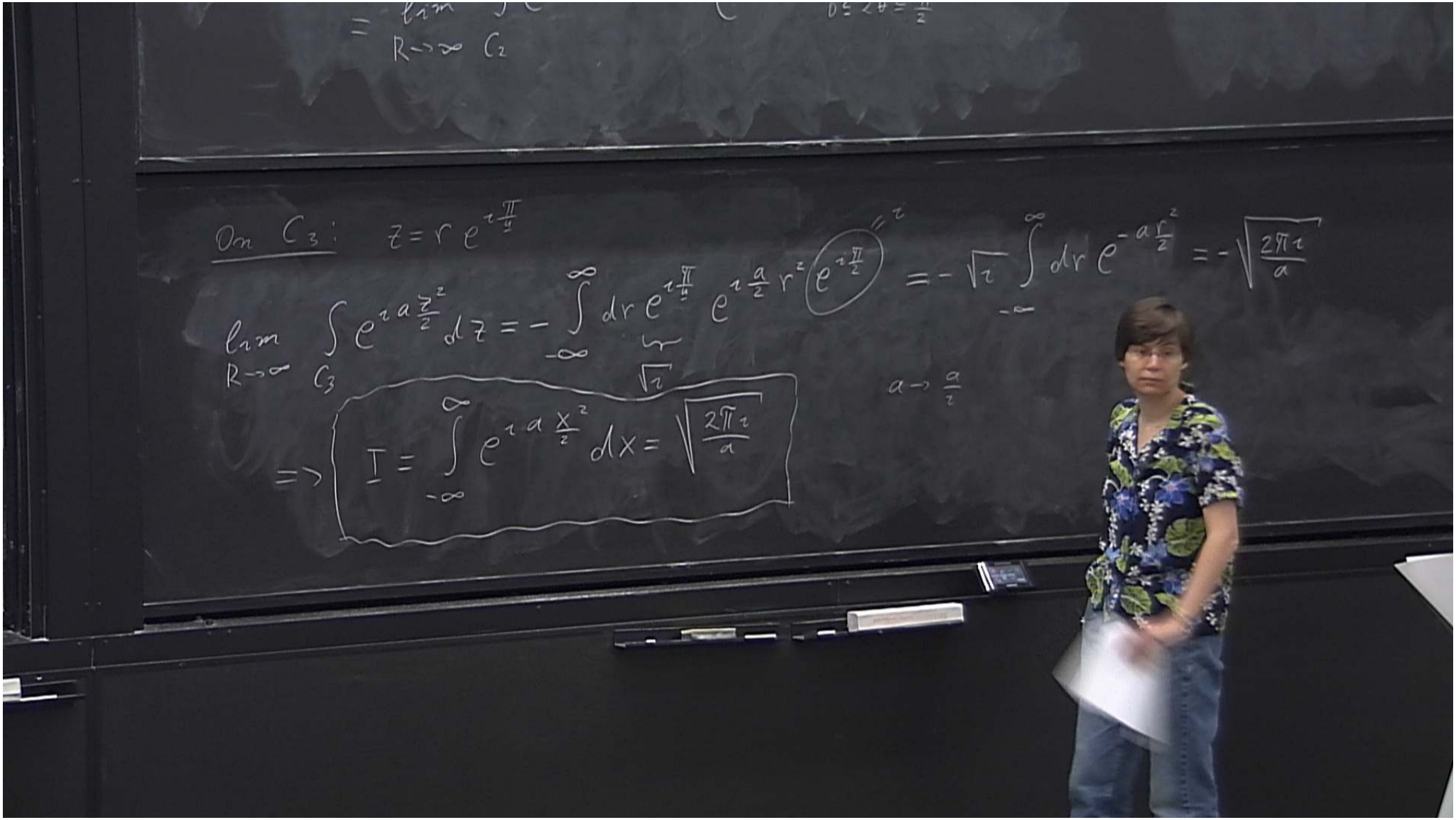
$$\langle X_{k_1} \dots X_{k_m} \rangle = \sum_{\substack{\text{all possible} \\ \text{pairings}}} A^{-1}_{k_1 k_2} \dots A^{-1}_{k_{m-1} k_m}$$

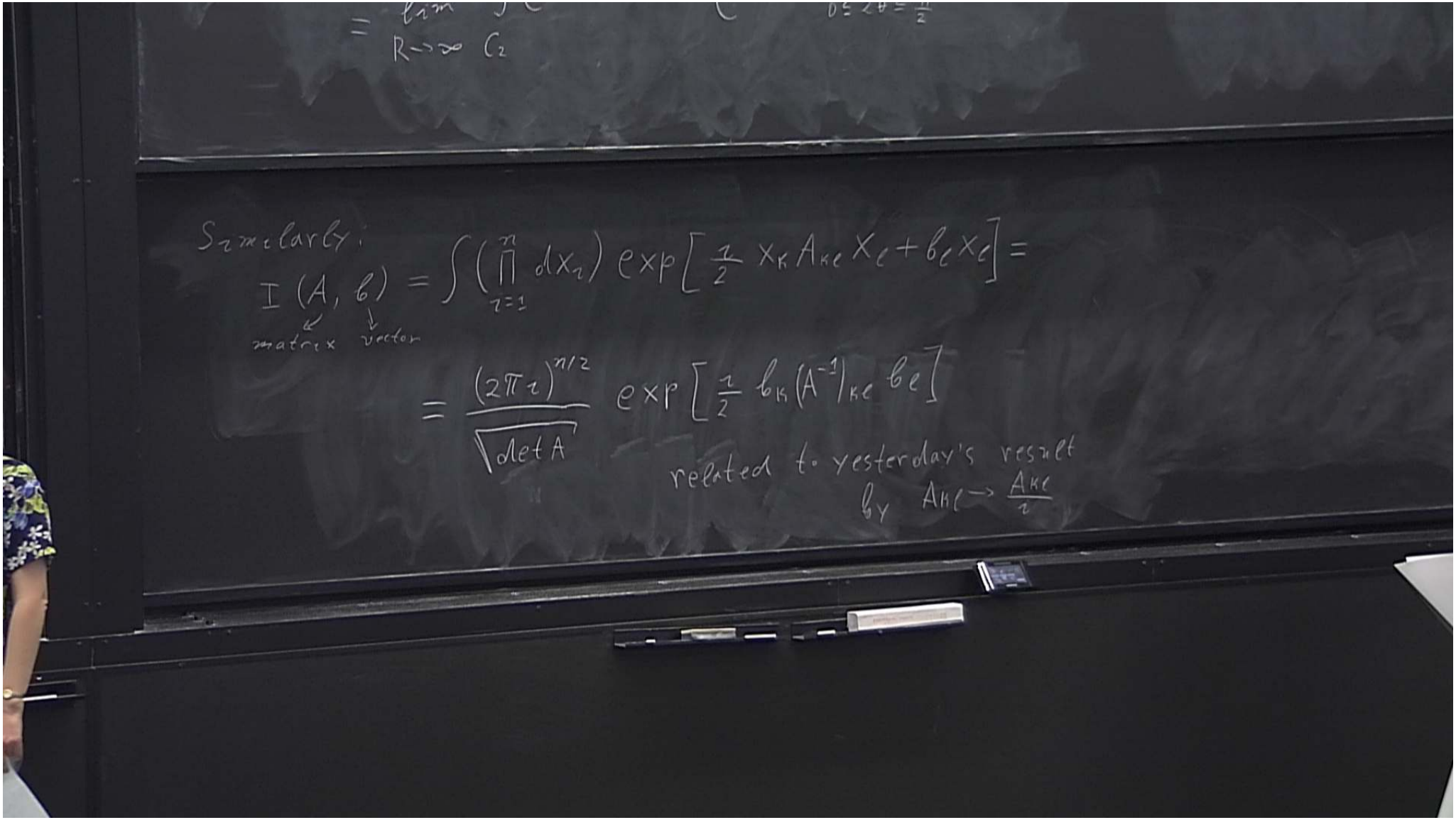
(X_{k_m}: m even)

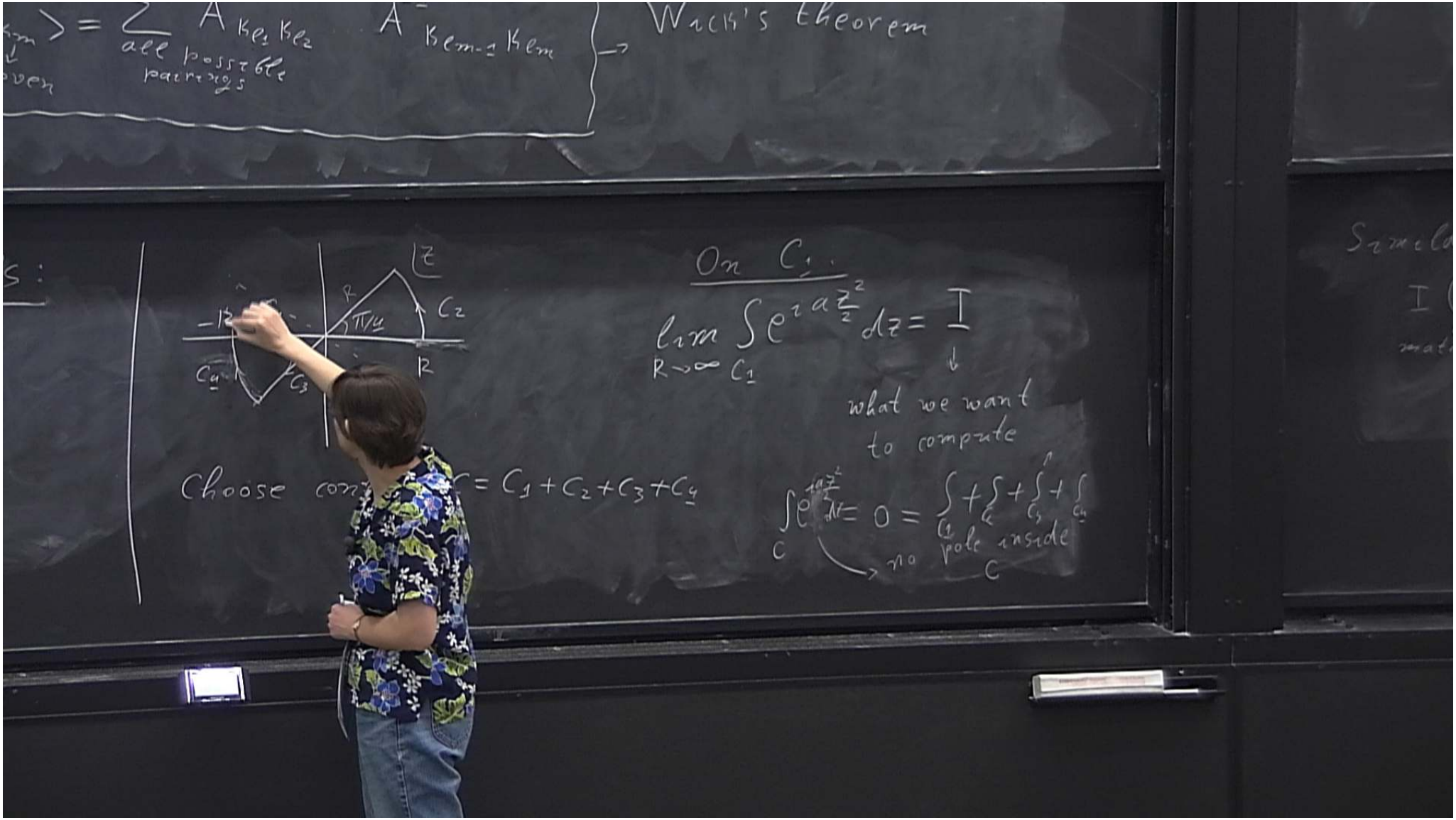
$\rightarrow$ Wick's theorem

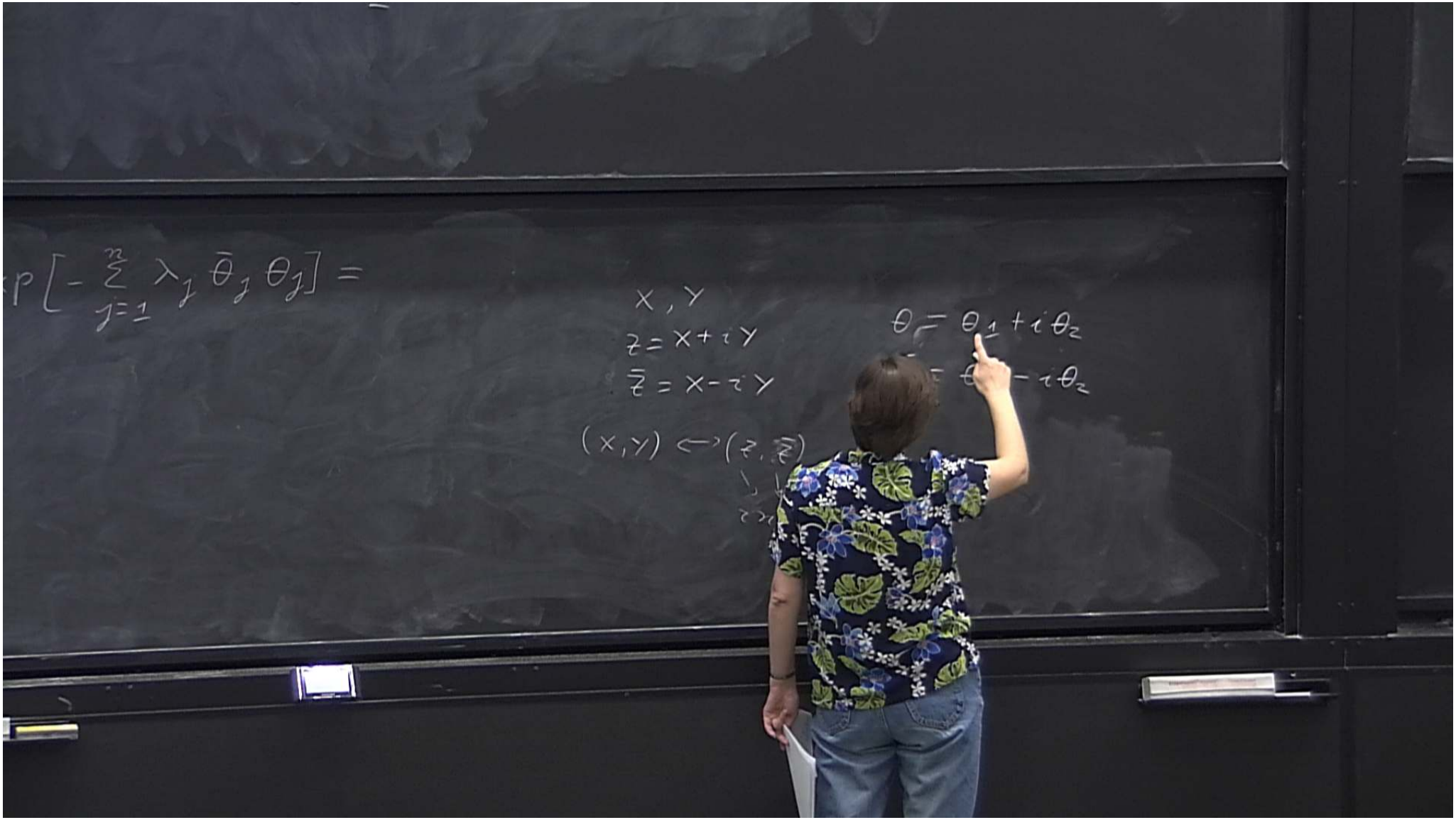


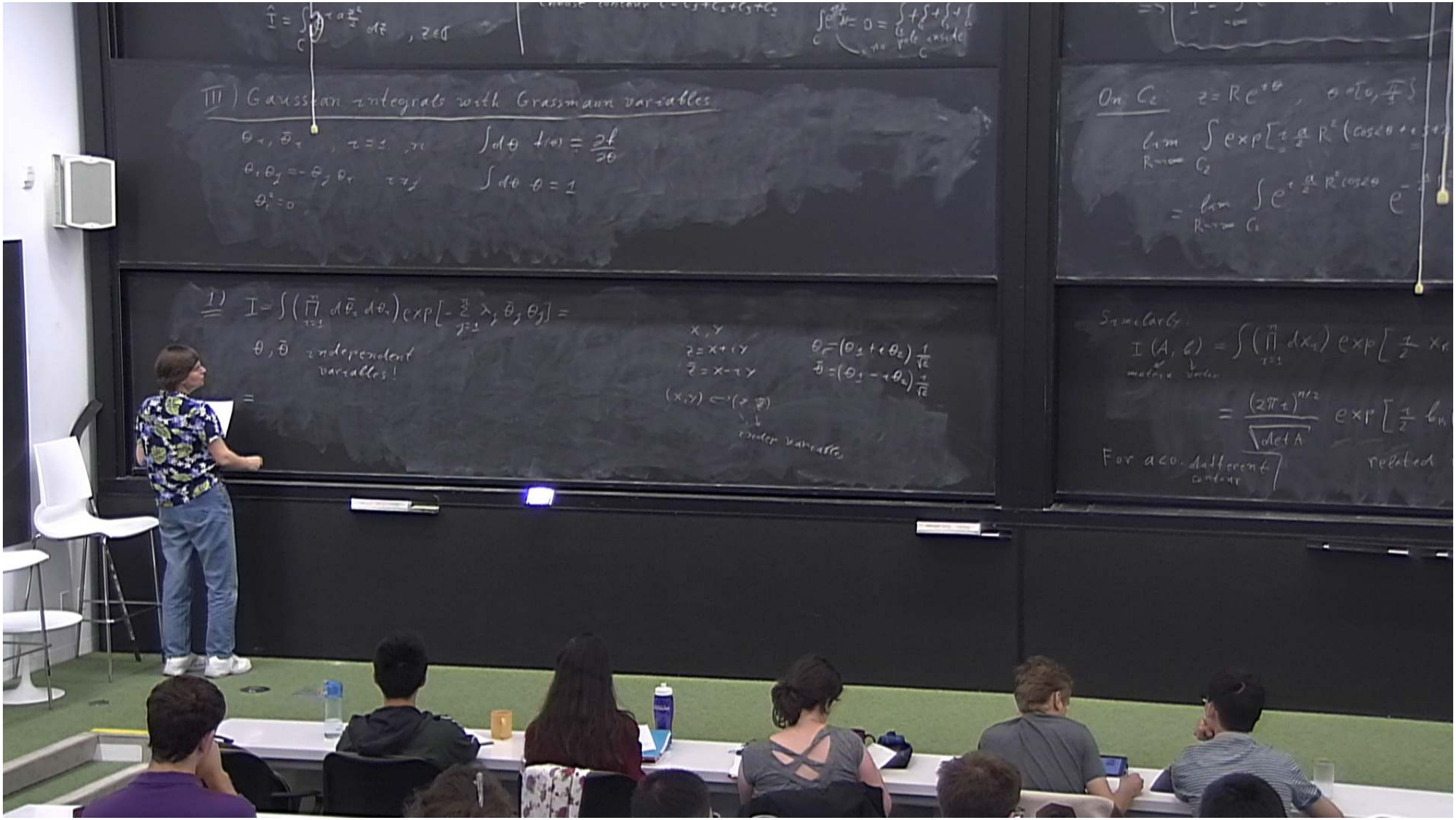
$\ldots X_{k_m}\rangle = \sum_{\text{all possible pairings}} A_{k_{\ell_1} k_{\ell_2}} \ldots A_{k_{\ell_{m-1}} k_{\ell_m}}$
even
Wick's theorem
egrals:
> 0
$\in \mathbb{C}$
$-R$
C_1
$\pi/4$
C_2
C_4
C_3
R
z
On C_1:
$\lim_{R\to\infty} \int_{C_1} e^{i a \frac{z^2}{2}} dz = I$
what we want to compute
Choose contour $C = C_1 + C_2 + C_3 + C_4$

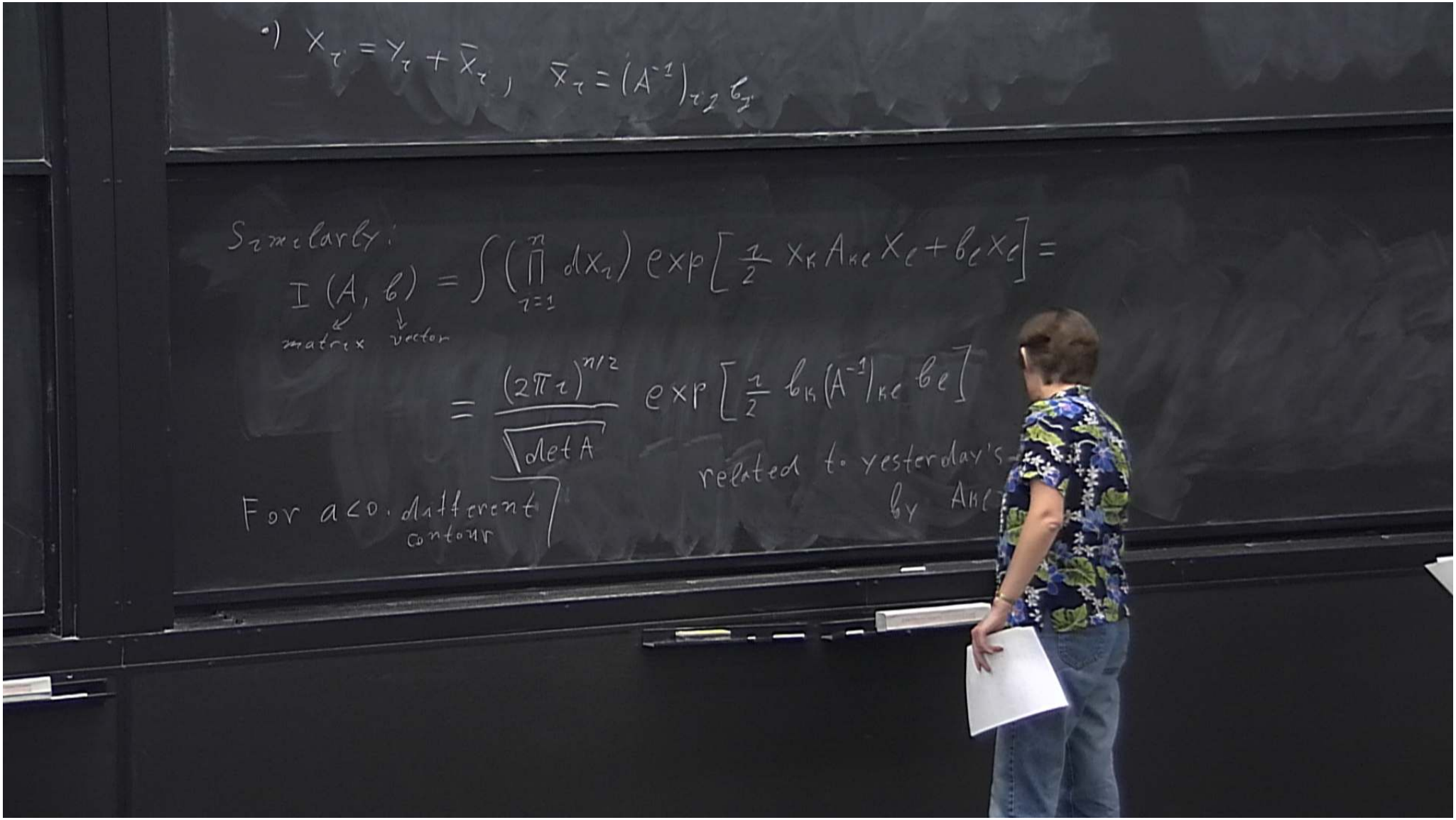

$= \lim_{R\to\infty} C_2$
On C_3: $z = r e^{i\frac{\pi}{4}}$
$\lim_{R\to\infty} \int_{C_3} e^{ia\frac{z^2}{2}} dz = -\int_{-\infty}^{\infty} dr\, e^{i\frac{\pi}{4}} e^{i\frac{a}{2}r^2} e^{i\frac{\pi}{2}} = -\sqrt{i}\int_{-\infty}^{\infty} dr\, e^{-a\frac{r^2}{2}} = -\sqrt{\frac{2\pi i}{a}}$
$\sqrt{i}$
$a \to \frac{a}{i}$
$\Rightarrow \boxed{I = \int_{-\infty}^{\infty} e^{ia\frac{x^2}{2}} dx = \sqrt{\frac{2\pi i}{a}}}$

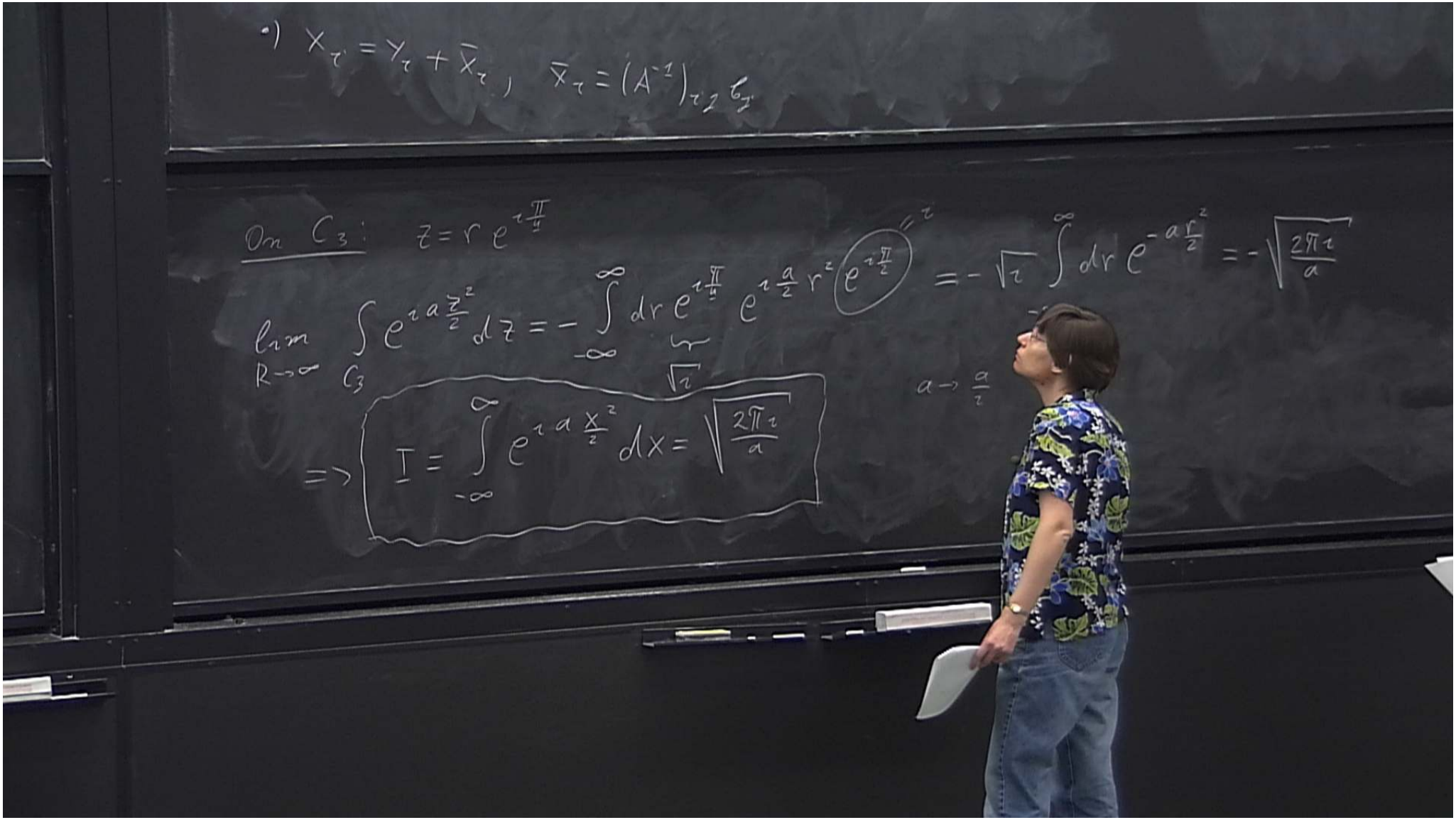

$= \lim_{R\to\infty} C_2$
$\theta = 2\theta = \frac{\pi}{2}$
Similarly:
$I(A, b) = \int \left(\prod_{i=1}^{n} dx_i\right) \exp\left[\frac{1}{2} x_k A_{ke} x_e + b_e x_e\right] =$
matrix
vector
$= \frac{(2\pi i)^{n/2}}{\sqrt{\det A}} \exp\left[\frac{1}{2} b_k (A^{-1})_{ke} b_e\right]$
related to yesterday's result
by $A_{ke} \to \frac{A_{ke}}{2}$

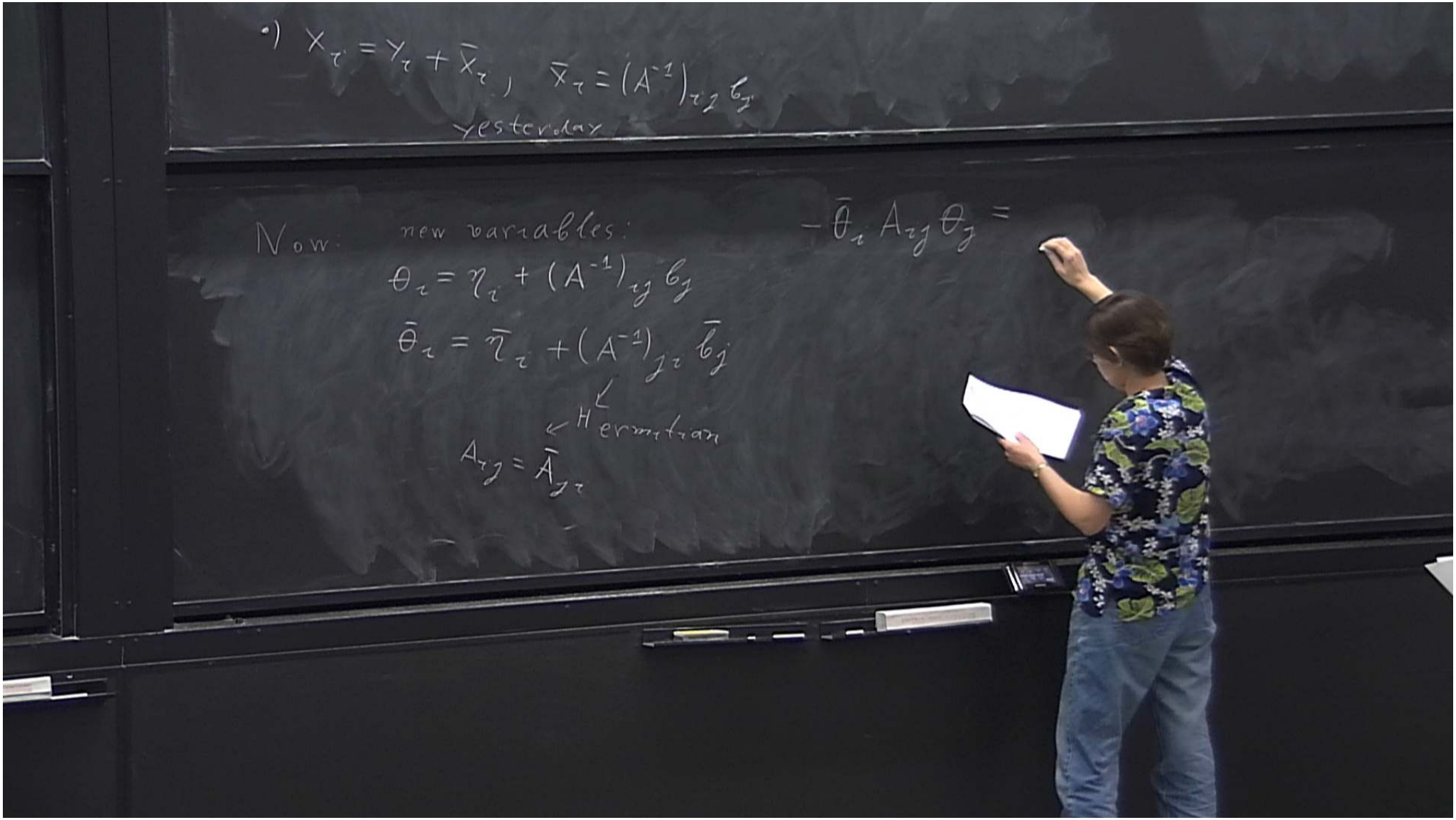

Wick's theorem
all possible pairings
On C_1.
$\lim_{R\to\infty} \int_{C_1} e^{iaz^2/2}\,dz = I$
what we want to compute
$\oint_C e^{iaz^2/2}\,dz = 0 = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}$
no pole inside C
$C = C_1 + C_2 + C_3 + C_4$

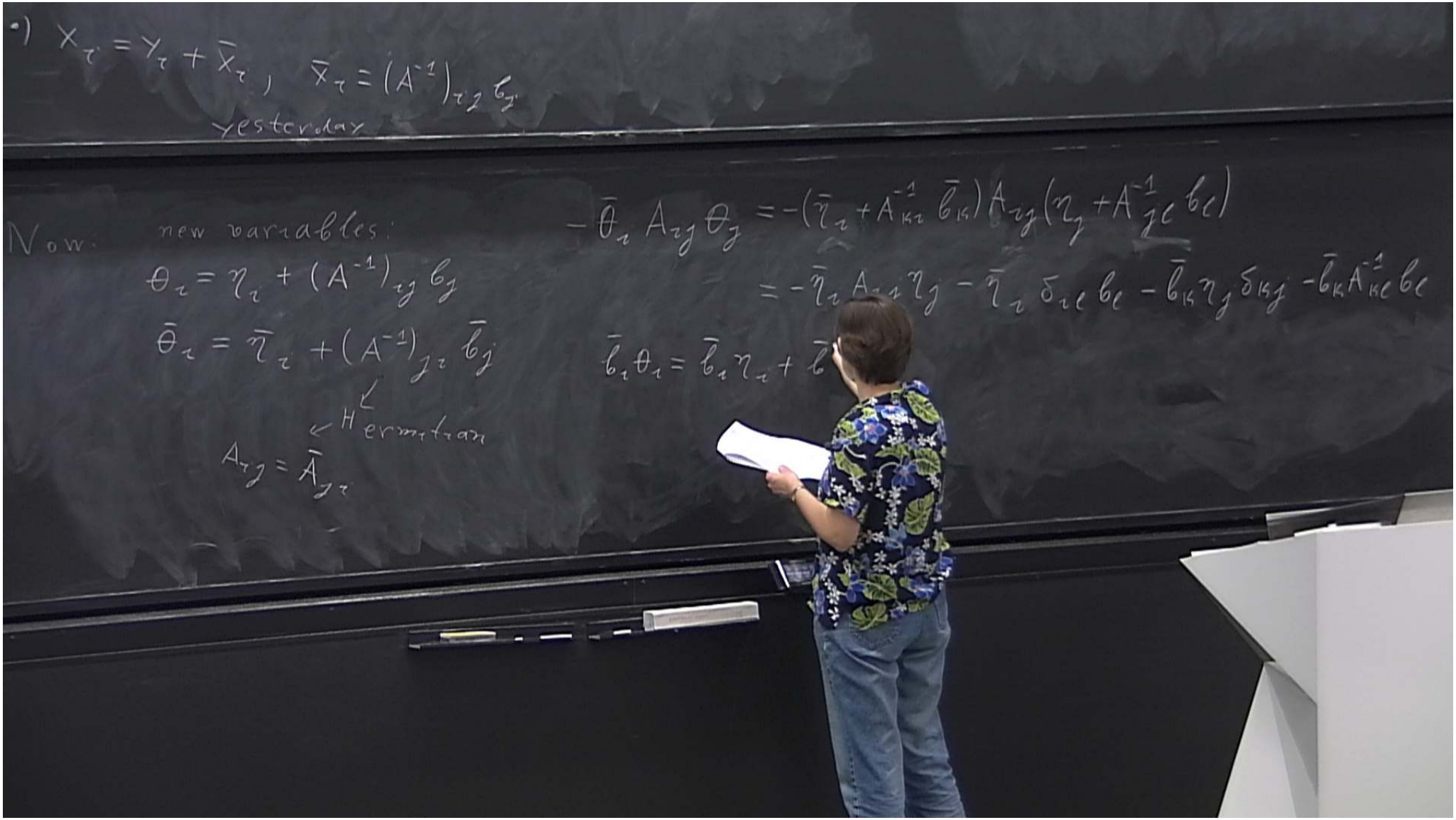

$\exp\left[-\sum_{j=1}^{n} \lambda_j \bar{\theta}_j \theta_j\right] =$
x, y
$z = x + iy$
$\bar{z} = x - iy$
$(x,y) \hookleftarrow (z,\bar{z})$
$\theta = \theta_1 + i\theta_2$
$\bar{\theta} = \theta_1 - i\theta_2$

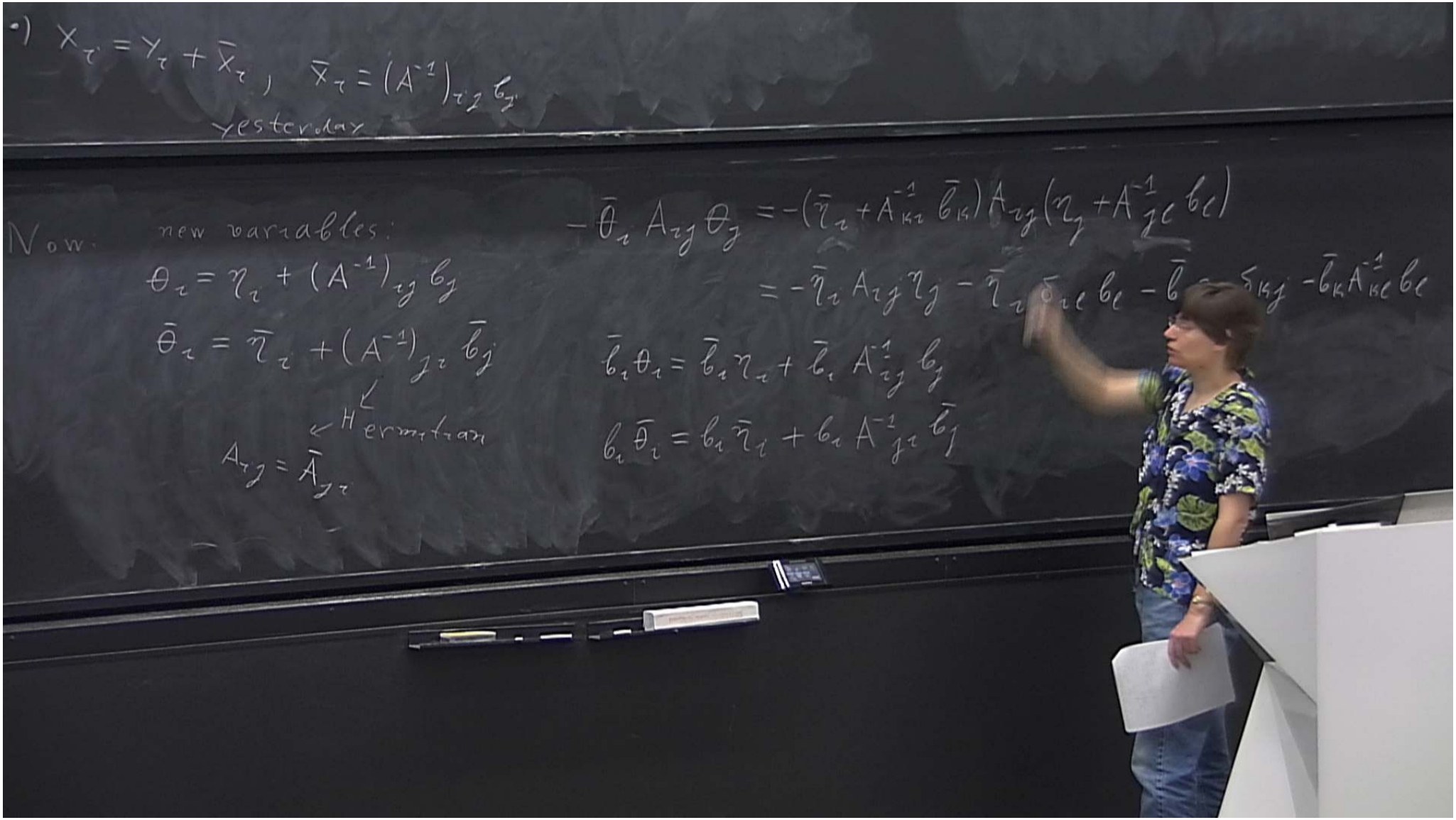

III) Gaussian integrals with Grassmann variables
On C2
Similarly
independent variables!
For a co. different contour
related

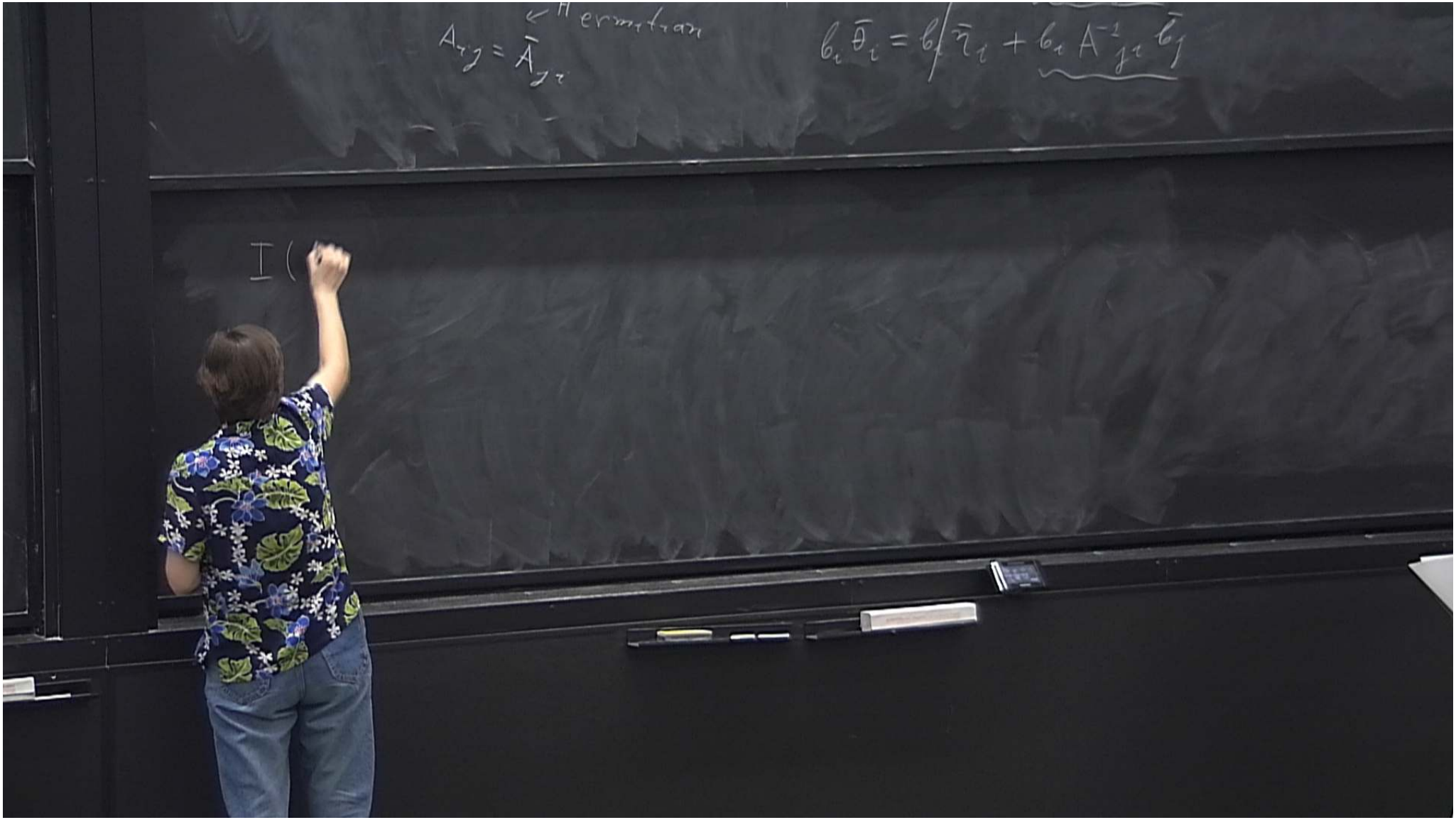

•) $x_i = y_i + \bar{x}_i$, $\bar{x}_i = (A^{-1})_{ij} b_j$
Similarly:
$I(A, b) = \int \left(\prod_{i=1}^{n} dx_i\right) \exp\left[\frac{i}{2} x_k A_{kl} x_l + b_l x_l\right] =$
matrix
vector
$= \frac{(2\pi i)^{n/2}}{\sqrt{\det A}} \exp\left[\frac{i}{2} b_k (A^{-1})_{kl} b_l\right]$
For a<0. different contour
related to yesterday's
by A_{kl}

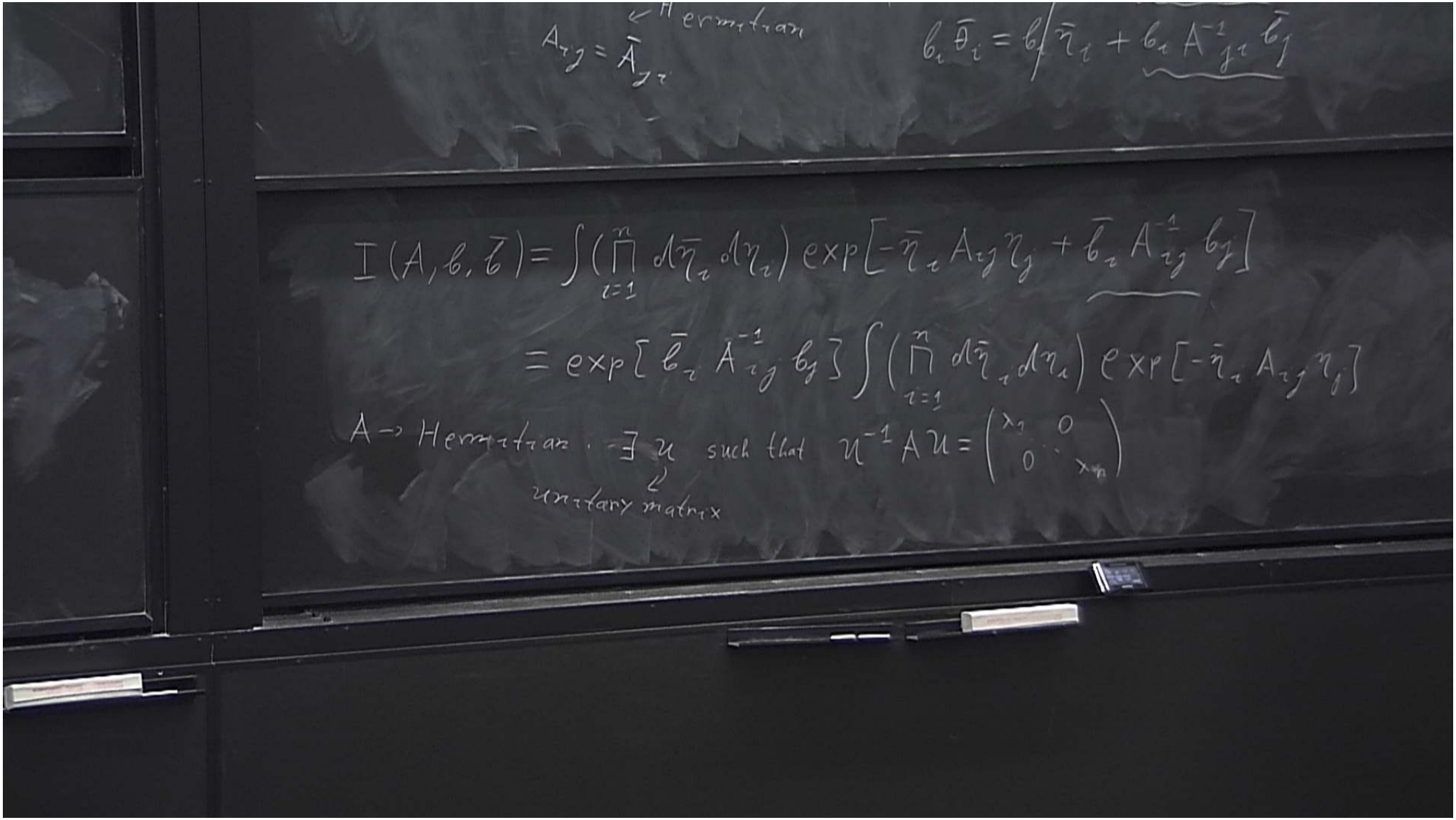

• $x_i = y_i + \bar{x}_i$, $\bar{x}_i = (A^{-1})_{ij} b_j$
On C_3: $z = re^{i\frac{\pi}{4}}$
$\lim_{R\to\infty} \int_{C_3} e^{ia\frac{z^2}{2}} dz = -\int_{-\infty}^{\infty} dr\, e^{i\frac{\pi}{4}} e^{i\frac{a}{2}r^2} e^{i\frac{\pi}{2}} = -\sqrt{i}\int_{-\infty}^{\infty} dr\, e^{-a\frac{r^2}{2}} = -\sqrt{\frac{2\pi i}{a}}$
$\sqrt{i}$
$a \to \frac{a}{i}$
$\Rightarrow \boxed{I = \int_{-\infty}^{\infty} e^{ia\frac{x^2}{2}} dx = \sqrt{\frac{2\pi i}{a}}}$

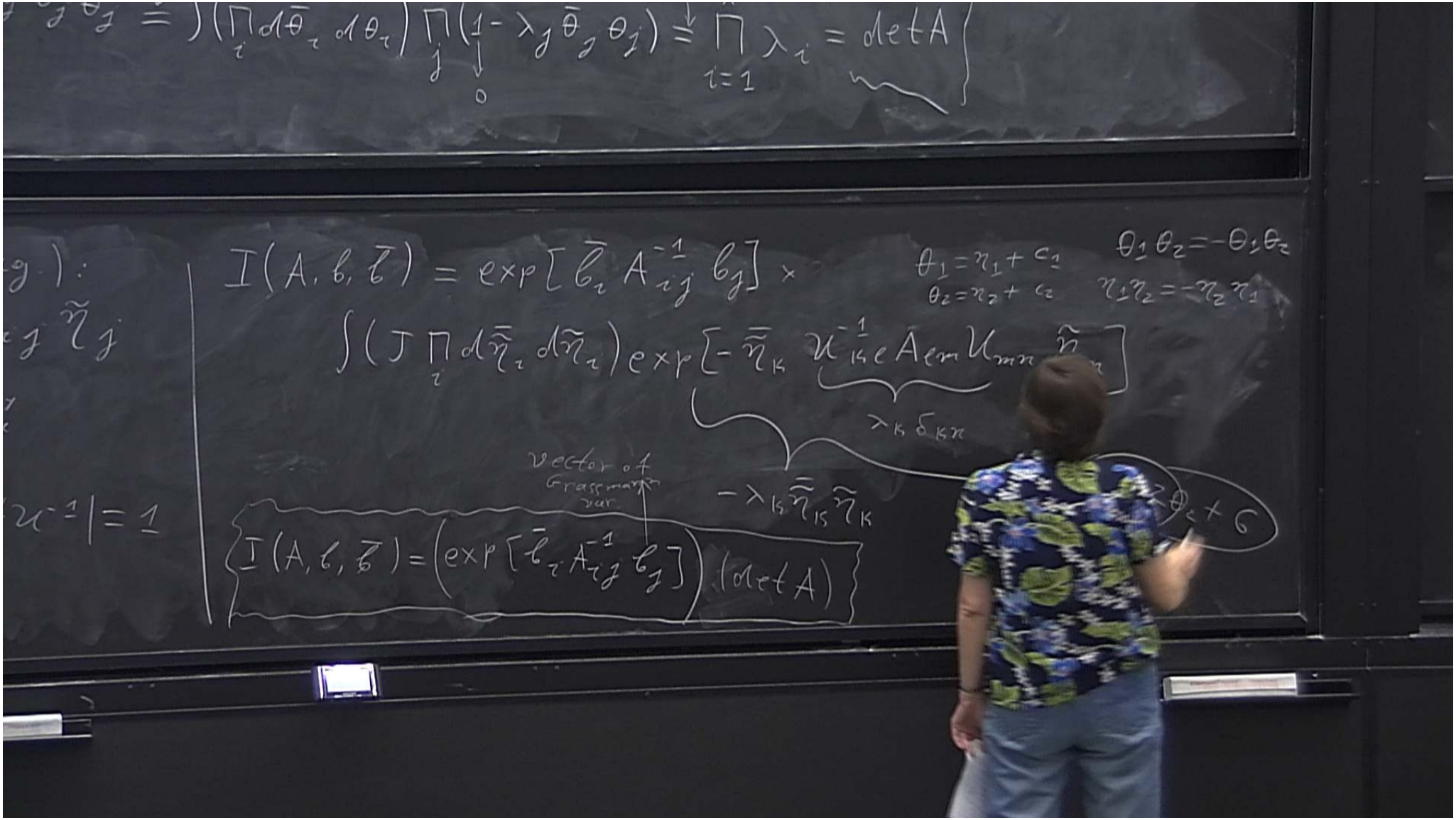

•) $x_i = y_i + \bar{x}_i$, $\bar{x}_i = (A^{-1})_{ij} b_j$
yesterday
Now: new variables:
$\theta_i = \eta_i + (A^{-1})_{ij} b_j$
$\bar{\theta}_i = \bar{\eta}_i + (A^{-1})_{ji} \bar{b}_j$
Hermitian
$A_{ij} = \bar{A}_{ji}$
$-\bar{\theta}_i A_{ij} \theta_j =$


•) $X_i = Y_i + \bar{X}_i$, $\bar{X}_i = (A^{-1})_{ij} b_j$
yesterday
Now: new variables:
$\theta_i = \eta_i + (A^{-1})_{ij} b_j$
$\bar{\theta}_i = \bar{\eta}_i + (A^{-1})_{ji} \bar{b}_j$
Hermitian
$A_{ij} = \bar{A}_{ji}$
$-\bar{\theta}_i A_{ij} \theta_j = -(\bar{\eta}_i + A^{-1}_{ki} \bar{b}_k) A_{ij} (\eta_j + A^{-1}_{jc} b_c)$
$= -\bar{\eta}_i A_{ij} \eta_j - \bar{\eta}_i \delta_{ic} b_c - \bar{b}_k \eta_j \delta_{kj} - \bar{b}_k A^{-1}_{kc} b_c$
$\bar{b}_i \theta_i = \bar{b}_i \eta_i + \bar{b}$


•) $x_i = y_i + \bar{x}_i$, $\bar{x}_i = (A^{-1})_{ij} b_j$
yesterday
Now: new variables:
$\theta_i = \eta_i + (A^{-1})_{ij} b_j$
$\bar{\theta}_i = \bar{\eta}_i + (A^{-1})_{ji} \bar{b}_j$
← Hermitian
$A_{ij} = \bar{A}_{ji}$
$-\bar{\theta}_i A_{ij} \theta_j = -(\bar{\eta}_i + A^{-1}_{ki} \bar{b}_k) A_{ij} (\eta_j + A^{-1}_{jl} b_l)$
$= -\bar{\eta}_i A_{ij} \eta_j - \bar{\eta}_i \delta_{il} b_l - \bar{b}_k \delta_{kj} - \bar{b}_k A^{-1}_{kl} b_l$
$\bar{b}_i \theta_i = \bar{b}_i \eta_i + \bar{b}_i A^{-1}_{ij} b_j$
$b_i \bar{\theta}_i = b_i \bar{\eta}_i + b_i A^{-1}_{ji} \bar{b}_j$


Hermitian
$A_{ij} = \bar{A}_{ji}$
$b_i \bar{\theta}_i = b_i \bar{x}_i + b_i A^{-1}_{ji} b_j$
$I($


Hermitian
$A_{ij} = \bar{A}_{ji}$
$b_i \bar{\theta}_i = b_i \bar{\eta}_i + b_i A^{-1}_{ji} \bar{b}_j$
$I(A, b, \bar{b}) = \int \left(\prod_{i=1}^{n} d\bar{\eta}_i d\eta_i \right) \exp\left[-\bar{\eta}_i A_{ij} \eta_j + \bar{b}_i A^{-1}_{ij} b_j \right]$
$= \exp\left[\bar{b}_i A^{-1}_{ij} b_j \right] \int \left(\prod_{i=1}^{n} d\bar{\eta}_i d\eta_i \right) \exp\left[-\bar{\eta}_i A_{ij} \eta_j \right]$
$A \to$ Hermitian $\Rightarrow \exists\, U$ such that $U^{-1} A U = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_n \end{pmatrix}$
unitary matrix


$\prod_i d\bar{\theta}_i\, d\theta_i \prod_j (1 - \lambda_j \bar{\theta}_j \theta_j) = \prod_{i=1}^{} \lambda_i = \det A$
$I(A, b, \bar{b}) = \exp[\bar{b}_i A^{-1}_{ij} b_j] \times$
$\theta_1 = \eta_1 + c_1$
$\theta_2 = \eta_2 + c_2$
$\theta_1 \theta_2 = -\theta_1 \theta_2$
$\eta_1 \eta_2 = -\eta_2 \eta_1$
$\int (J \prod_i d\bar{\eta}_i\, d\eta_i) \exp[-\bar{\eta}_k$
$\lambda_k \delta_{kn}$
vector of Grassmann var
$-\lambda_k \bar{\eta}_k \eta_k$
$I(A, b, \bar{b}) = (\exp[\bar{b}_i A^{-1}_{ij} b_j])(\det A)$
$\theta_2 + c$