

Title: Special Functions and Diff. Eqs. - Lecture 3

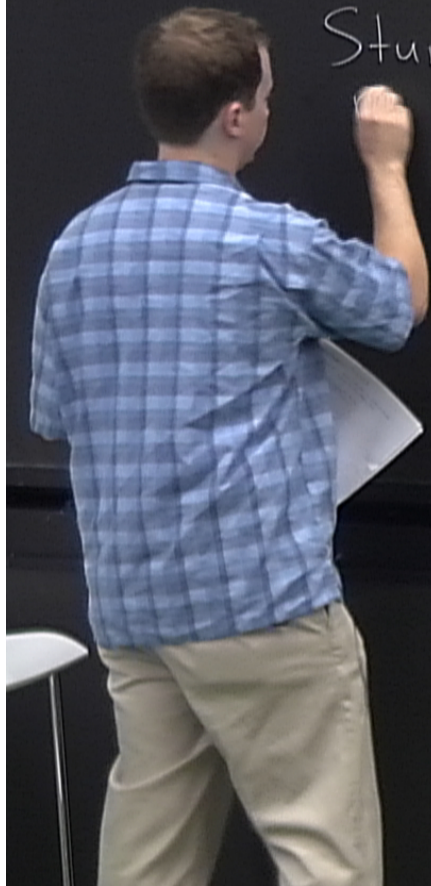
Date: Aug 23, 2012 09:00 AM

URL: <http://pirsa.org/12080022>

Abstract:

Orthogonal Functions

Sturm-Liouville Theory



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Sturm-Liouville Theory

Parseval's Theorem

Orthogonal Polynomials

Sturm-Liouville Theory

$$\mathcal{L}u_\lambda(x) + \lambda w(x)u_\lambda(x) = 0$$

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$$\mathcal{L} u_\lambda(x) + \lambda w(x) u_\lambda(x) = 0$$

$\mathcal{L}$ is a differential operator

Sturm-Liouville Theory

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$\mathcal{L}$ a differential operator

$w(x) > 0$ weight function

x in a finite interval $[a, b]$

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sols have nice properties if

Ω is self-adjoint or hermitian

$$\langle \Omega u_\lambda, u_{\lambda'} \rangle_W = \langle u_\lambda, \Omega u_{\lambda'} \rangle_W$$

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Ω is self adjoint iff

$$\Omega = \frac{d}{dx} \left[s(x) \frac{d}{dx} \right] + t(x)$$

and

$$\alpha_1 u_x(a) + \alpha_2 u'_x(a) = 0$$

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In this case

- the eigenvalues are real and distinct $-\infty < \lambda_0 < \lambda_1 < \lambda_2 < \dots < \infty$

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and

$$\alpha_1 u_\lambda(a) + \alpha_2 u'_\lambda(a) = 0$$

$$\alpha_1^2 + \alpha_2^2 > 0$$

$$\beta_1 u_\lambda(b) + \beta_2 u'_\lambda(b) = 0$$

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$(a, b]$ for $u_{\lambda}(x)$ λ

In this case:

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- the eigenvalues are real and distinct $-\infty < \lambda_0 < \lambda_1 < \lambda_2 < \dots < \infty$

- $u_{\lambda_n}(x)$ has exactly n zeros in (a, b)

- Zeros of $u_{\lambda_{n+1}}(x)$ are in between the zeros of $u_{\lambda_n}(x)$

$[a, b]$ for $u_\lambda(x)$ λ

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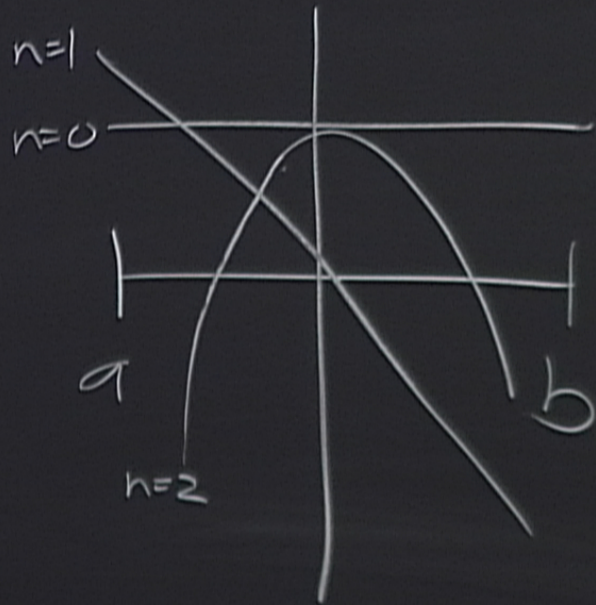
$$\alpha_1^2 + \alpha_2^2 > 0$$

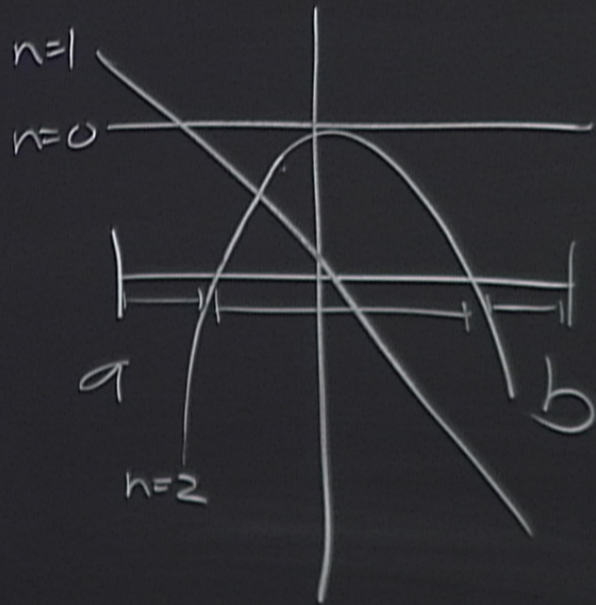
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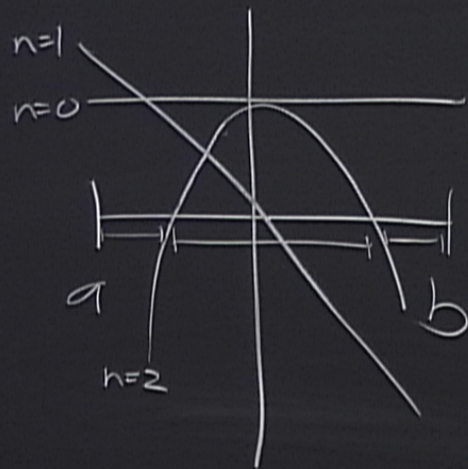
$$\beta_1^2 + \beta_2^2 > 0$$

- u_{λ_n} form an orthonormal basis of $L^2[a, b]_w$

$$\int_a^b u_{\lambda_n}(x) u_{\lambda_m}(x) w(x) dx = \delta_{nm}$$







Best approximation
Let u_n be orthonormal

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Let u_n be orthonormal. Best approximation to f with $u_0, \dots, u_N$

$$f = \sum_{n=0}^N a_n u_n \quad \text{with} \quad a_n = \langle u_n, f \rangle$$

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$$\Delta \equiv \left\| f - \sum_{n=0}^N a_n u_n \right\|^2 = \left\langle f - \sum_{n=0}^N a_n u_n, f - \sum_{n=0}^N a_n u_n \right\rangle$$
$$= \left\| \right\|^2$$

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$$\|f\|^2 - \sum_{n=0}^N a_n \langle f, u_n \rangle - \sum_{m=0}^N a_m^* \langle f, u_m \rangle + \sum_{n,m=0}^N |a_n|^2$$

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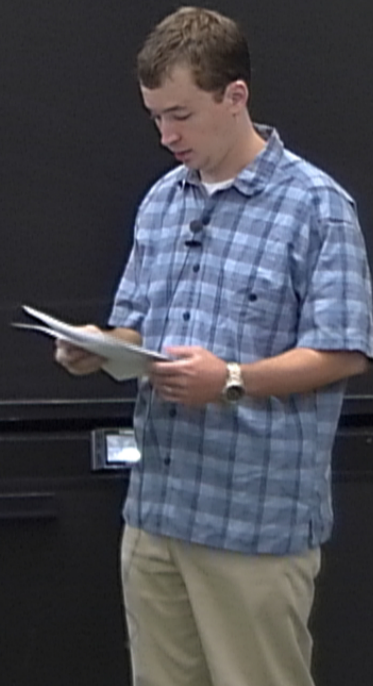
$$= \|f\|^2 - \sum_{n=0}^N a_n \langle f, u_n \rangle - \sum_{m=0}^N a_m^* \langle f, u_m \rangle + \underbrace{\sum_{n,m=0}^N a_m^* a_n \langle u_m, u_n \rangle}_{\sum_{n=0}^N |a_n|^2}$$

$$= \|f\|^2 + \sum_{n=0}^N |a_n - \langle u_n, f \rangle|^2 - \sum_{n=0}^N |\langle u_n, f \rangle|^2$$

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$$= \|f\|^2 + \underbrace{\sum_{n=0}^N |a_n - \langle u_n, f \rangle|^2}_{\geq 0} - \sum_{n=0}^N |\langle u_n, f \rangle|^2$$



Parseval's Theorem

Assume function space is complete

$$f = \sum_{n=0}^{\infty} a_n u_n$$

converges

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$$\begin{aligned}\Delta_N &= \|f\|^2 - \sum_{n=0}^N | \langle u_n, f \rangle |^2 \\ &= \|f\|^2 - \sum_{n=0}^N |a_n|^2\end{aligned}$$

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$$\Delta_N \xrightarrow{N \rightarrow \infty} 0 \Rightarrow \|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2$$

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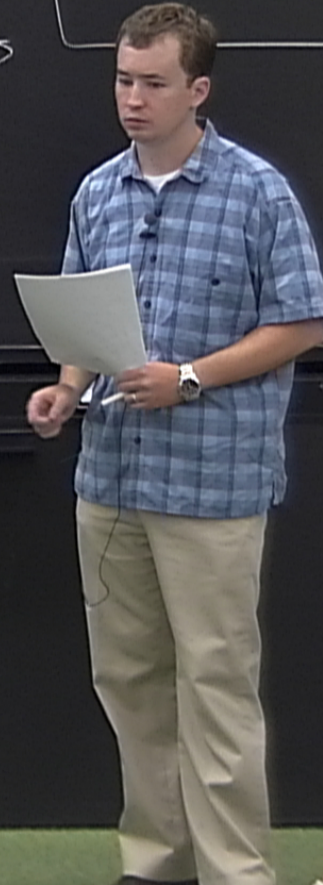
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If $\Delta_N \rightarrow 0$ as $N \rightarrow \infty$
+ then
 $f = \sum_{n=0}^{\infty} a_n u_n$
converges

$$= \|f\|^2 + \underbrace{\sum_{n=0}^N |a_n - \langle u_n, f \rangle|^2}_{\geq 0} - \sum_{n=0}^N |\langle u_n, f \rangle|^2$$

$$\Delta_N \geq 0 \Rightarrow \|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2$$

Example: Consider a wave with amplitude $f(x)$ and period L

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x/L}$$

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Example: Consider a wave with amplitude $f(x)$ and period L

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$$\text{intensity} = \text{energy/length} \propto \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |a_n|^2$$

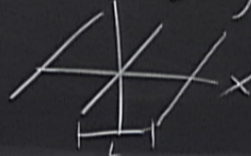
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Suppose $f(x) = x$



Sturm-Liouville Theory

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$\mathcal{L}$ is a differential operator
 $w(x) > 0$ weight function
 solve in a finite interval $[a, b]$ for $u(x)$

Sols have nice properties if

$\mathcal{L}$ is self-adjoint or hermitian

$$\langle \mathcal{L}u, v \rangle_w = \langle u, \mathcal{L}v \rangle_w$$

$$\langle f, g \rangle_w = \int_a^b f(x)g(x)w(x)dx$$

$$U = \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} x^2 dx$$

$$= \frac{L^2}{12}$$

$$a_n = 0 \quad \text{symmetry}$$

$$a_n = \frac{2}{L} \int_0^L x \sin(n\pi x/L) dx$$

Parseval's Theorem

Assume function space is complete

$$f = \sum_{n=0}^{\infty} a_n u_n$$

converges in norm

$$\Delta_n = \|f\|^2 - \sum_{n=0}^N |a_n|^2$$

$$= \|f\|^2 - \sum_{n=0}^N |a_n|^2$$

$$\Delta_n \geq 0 \Rightarrow \|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2$$

Example: Consider a wave with amplitude

$$f(x) = \sum_{n=0}^{\infty} a_n e^{2\pi i n x/L}$$

intensity density/length $\propto \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} |f(x)|^2 dx$

Suppose $f(x) = x$



solve in a finite interval $[a, b]$ for $u(x)$

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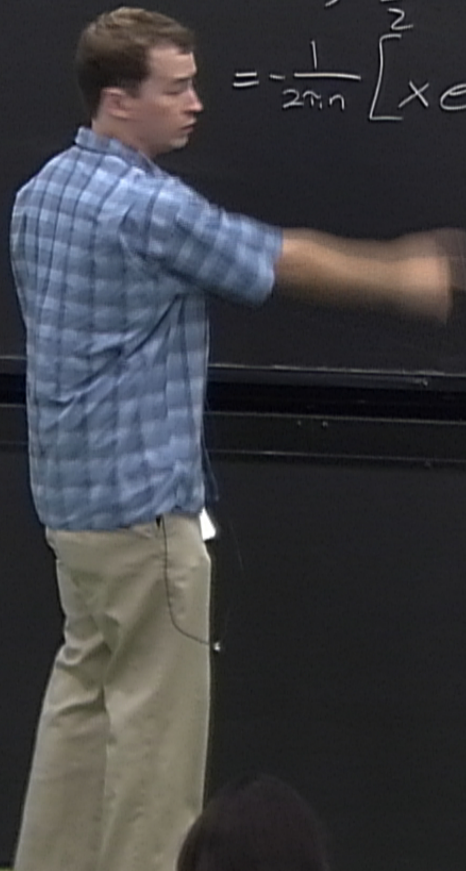
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$$a_n = \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} x e^{-2\pi i n x / L} dx$$

$$a_n = -\frac{1}{2\pi i n} \int_{-\frac{L}{2}}^{\frac{L}{2}} x \frac{d}{dx} (e^{-2\pi i n x / L}) dx$$

$$= -\frac{1}{2\pi i n} \left[x e^{-2\pi i n x / L} \Big|_{-\frac{L}{2}}^{\frac{L}{2}} - \int_{-\frac{L}{2}}^{\frac{L}{2}} e^{-2\pi i n x / L} dx \right]$$



solve in a finite interval $[a, b]$ for $u(x)$

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$$= \frac{L^2}{12}$$

by symmetry

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$$= -\frac{1}{2\pi i n} \left[\frac{L}{2} (e^{\pi i n} + e^{-\pi i n}) \right]$$

$$= \frac{L}{2\pi i n} (-1)^{n+1}$$

Parseval

$$\frac{L^2}{2} = \sum_{n=-\infty}^{\infty} \left| \frac{L}{2\pi i n} (-1)^{n+1} \right|^2$$

=

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$$\frac{L^2}{12} = \sum_{n=-\infty}^{\infty} \left| \frac{L}{2\pi i n} (-1)^{n+1} \right|^2$$

$$= \frac{L^2}{4\pi^2} 2 \sum_{n=0}^{\infty} \frac{1}{n^2}$$

$$\frac{L^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2)$$

Parseval

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Orthogonal Polynomials

Polynomials form a complete basis of $L^2[a,b]$

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Polynomials form a complete basis of $L^2[a,b]_w$

Orthogonalize using Gram-Schmidt:

$$P_0(x) = \frac{1}{\|1\|_w}$$

$$P_{n+1}(x) = \frac{xP_n - \sum_{i=0}^n P_i \langle P_i, xP_n \rangle_w}{\alpha_{n+1}}$$

$$\text{with } \alpha_{n+1} = \|xP_n - \sum_{i=0}^n P_i \langle P_i, xP_n \rangle_w\|_w$$

$$\Delta_N \geq 0 \Rightarrow \|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2$$

$$\begin{aligned} \langle P_i, x P_n \rangle_w &= \int_a^b w(x) P_i(x) P_n(x) dx \\ &= \int_a^b w[x P_i] P_n dx \\ &= 0 \text{ for } i < n-1 \end{aligned}$$

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$$\langle P_i, x P_n \rangle_w = \int_a^b w(x) P_i(x) P_n(x) dx$$

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$$= 0 \text{ for } i < n-1$$

$$P_n = a_n x P_n + b_n P_n + c_n P_{n-1}$$

$$\Delta_N \geq 0 \Rightarrow \|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2$$

$$\begin{aligned} \langle P_i, x P_n \rangle_w &= \int_a^b w(x) P_i(x) P_n(x) dx \\ &= \int_a^b w(x) [x P_i] P_n dx \\ &= 0 \text{ for } i < n-1 \end{aligned}$$

$$P_{n+1} = a_n x P_n + b_n P_n + c_n P_{n-1}$$

Explicit Construction

want P_n so that $\int_a^b P_n(x) w(x) x^p dx = 0$ for $p=0, \dots, n-1$

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Look at simple case $n=1, p=0$

$$\int_a^b P_1(x) w(x) dx = 0$$

Explicit Construction

want P_n so that $\int_a^b P_n(x) w(x) x^p dx = 0$ for $p=0, \dots, n-1$

Look at simple case $n=1, p=0$

$$\int_a^b P_1(x) w(x) dx = 0$$

$$w(x) P_1(x) \propto \frac{d}{dx}(Y_1) \quad \text{and require } Y_1(a) = 0 = Y_1(b)$$

$$P_{n+1} = a_n x P_n + b_n P_n + c_n P_{n-1}$$

Explicit Construction

want P_n so that $\int_a^b P_n(x) w(x) x^p dx = 0$ for $p=0, \dots, n-1$

Look at simple case $n=1, p=0$

$$\int_a^b P_1(x) w(x) dx = 0$$

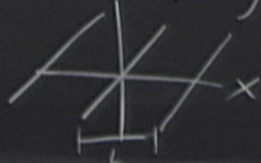
$w(x) P_1(x) \propto \frac{d}{dx}(Y_1)$ and require $Y_1(a)=0=Y_1(b)$ choose $Y_1 \propto (x-a)(x-b)$

der a wave with amplitude $f(x)$ and period L

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x/L}$$

$$\text{intensity} = \text{energy/length} \propto \frac{1}{L} \int_{-L/2}^{L/2} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |a_n|^2$$

Suppose $f(x) = x$



$$n=2 \quad p=1$$

$$\int_a^b w(x) P_2(x) x dx = 0$$

$$w(x) P_2(x) \propto \frac{d^2}{dx^2} (Y_2(x))$$

$$n=2 \quad p=1$$

$$\int_a^b w(x) P_2(x) x dx = 0$$

$$w(x) \propto \frac{d^2}{dx^2} (Y_2(x))$$

$$Y_2(x) \propto (x-a)^2 (x-b)^2$$

$$\text{In general } w(x) P_n \propto \frac{d^n}{dx^n} (Y_n) \text{ and } Y_n \propto (x-a)^n (x-b)^n$$

$$n=2 \quad p=1$$

$$\int_a^b w(x) P_2(x) x^p dx = 0$$

$$w(x) P_2(x) \propto \frac{d^2}{dx^2} (Y_2(x)) \quad Y_2(x) \propto (x-a)^2 (x-b)^2$$

In general if $w(x) P_n(x) \propto \frac{d^n}{dx^n} (Y_n)$ and $Y_n \propto (x-a)^n (x-b)^n$
 then $\int_a^b \frac{d^n}{dx^n} (Y_n) x^p dx = 0$ for $p < n$

What if a or $b = \pm\infty$?

$a = -\infty$ $b = \text{finite}$ $Y_n = (x-b)^n f_n(x)$ where $f_n(x) \rightarrow 0$ faster than $(x-b)^{-n}$ at $-\infty$

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$$w(x) P_n(x) = \frac{d^n}{dx^n} (Y_n)$$

$$Y_n \sim s^n(x) f_n(x)$$

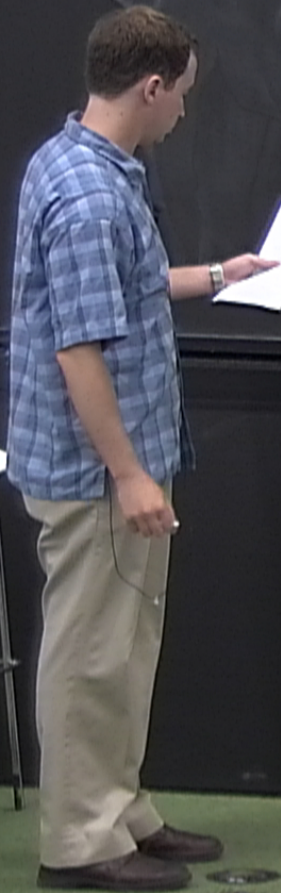
$$\text{with } s(x) = \begin{cases} (x-a)(x-b) & \text{if both finite} \\ (x-a) \text{ or } (x-b) & \text{if only } a \text{ or } b \text{ finite} \\ \text{constant} & \text{otherwise} \end{cases}$$

Generalized Rodrigues' Formula

$$P_n(x) = \frac{C_n}{w(x)} \frac{d^n}{dx^n} (s^n)$$

Generalized Rodrigues' Formula:

$$P_n(x) = \frac{C_n}{w(x)} \frac{d^n}{dx^n} (w S^n)$$



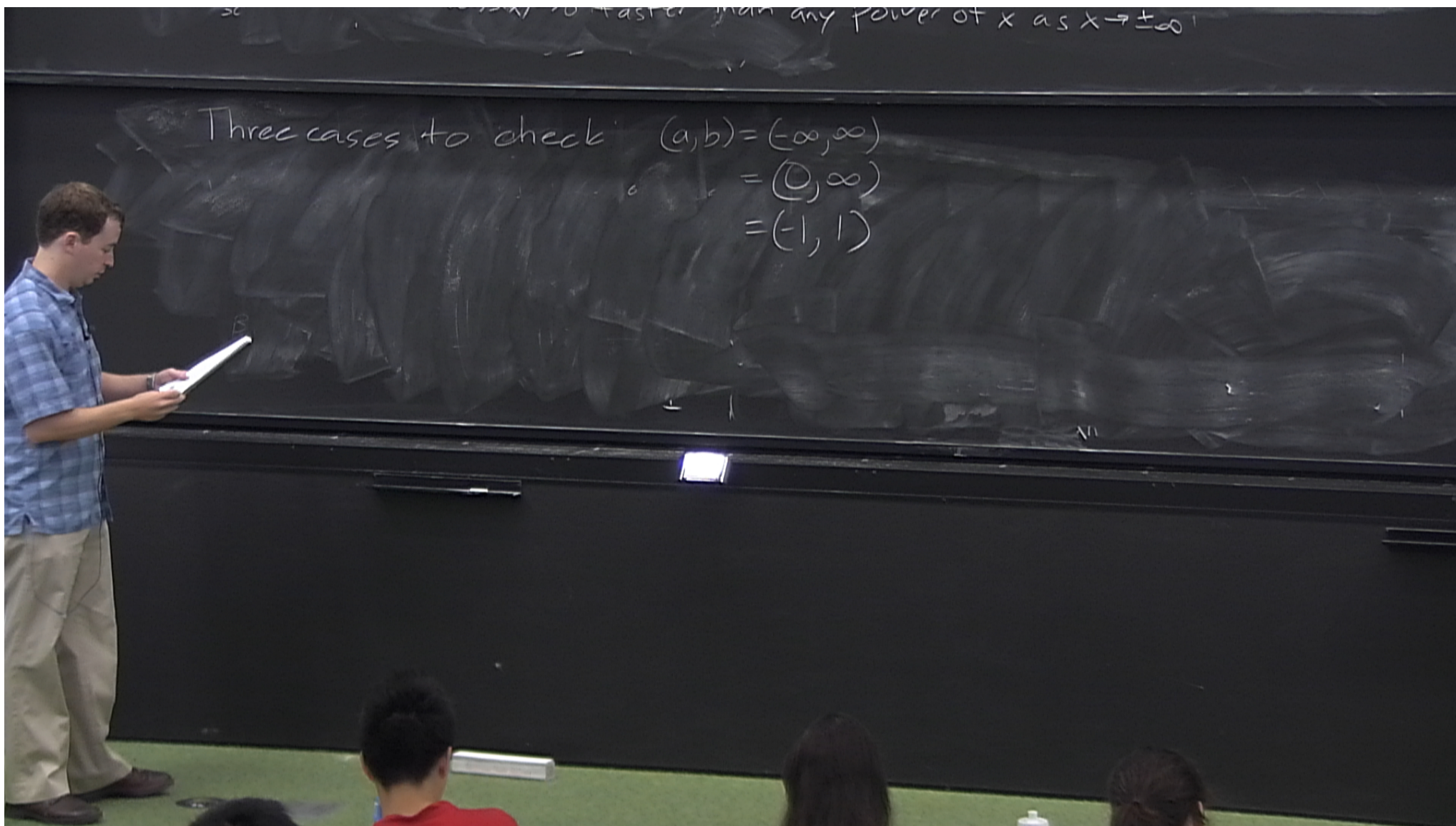
Generalized Rodrigues' Formula:

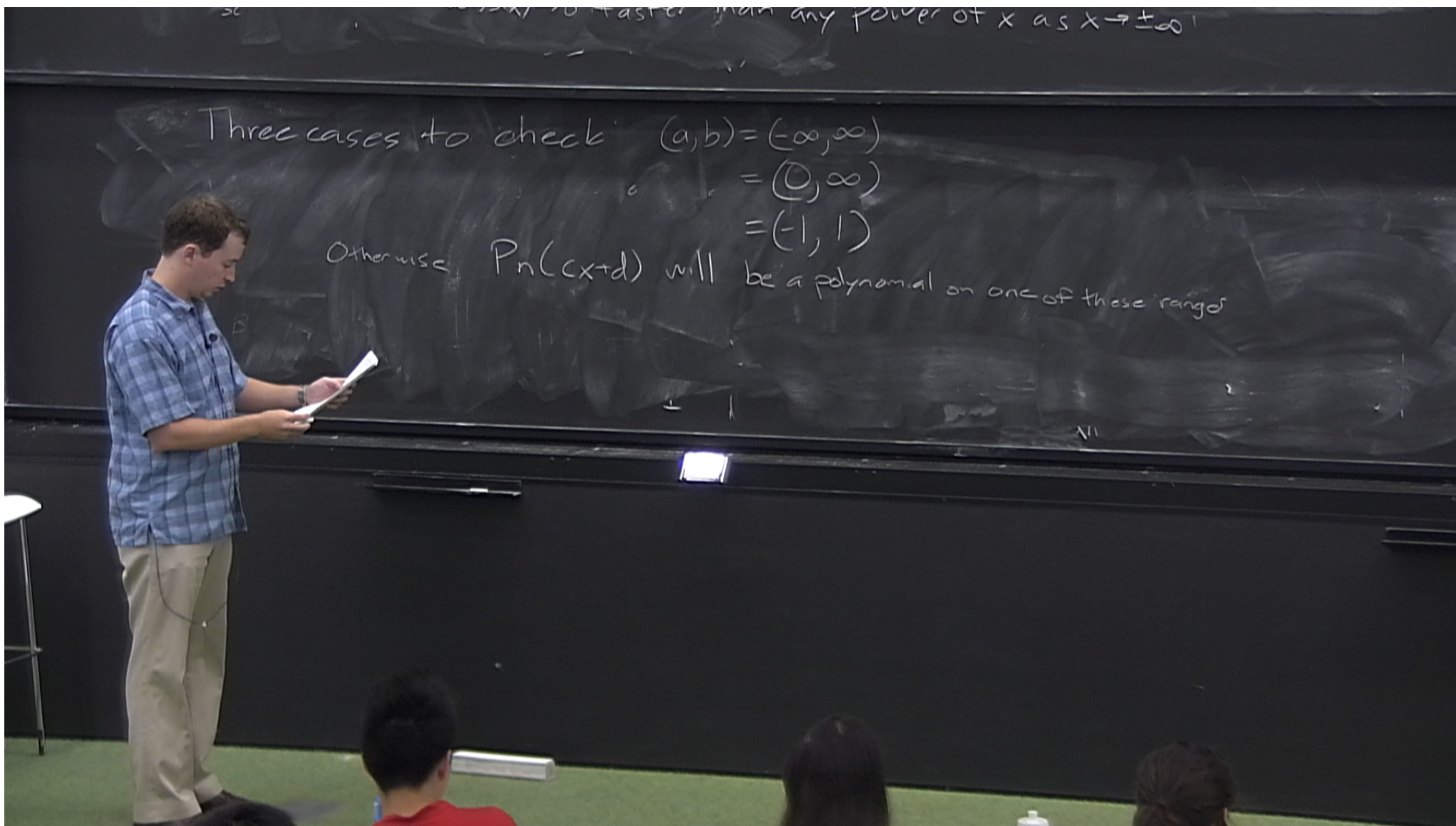
$$P_n(x) = \frac{C_n}{w(x)} \frac{d^n}{dx^n} (w s^n)$$

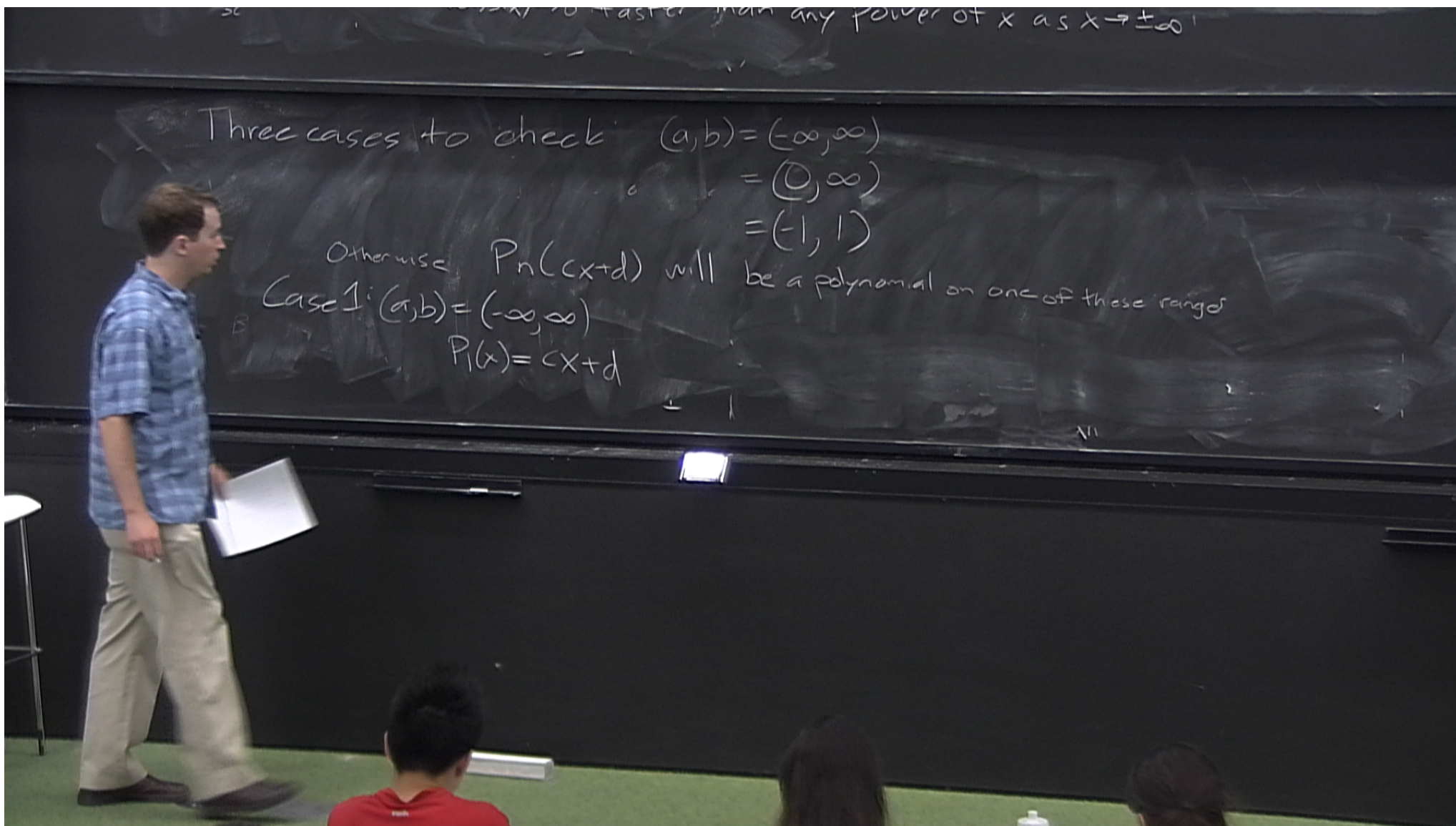
w is finite and differentiable

$w(a)s(a)=0=w(b)s(b)$ if a, b finite

$w(x)s(x) \rightarrow 0$ faster than any power of x as $x \rightarrow \pm\infty$







so $|x|$ is faster than any power of x as $x \rightarrow \pm\infty$

Three cases to check $(a,b) = (-\infty, \infty)$
 $= (0, \infty)$
 $= (-1, 1)$

Otherwise $P_n(cx+d)$ will be a polynomial on one of these ranges

Case 1: $(a,b) = (-\infty, \infty)$

$$P_1(x) = cx+d$$

linear transformation $\rightarrow P_1(x) = \pm 2x$

$w(x)P_1(x) \propto \frac{d}{dx}(1_1)$ and require $1_1(a) = 0 = 1_1(b)$ (choose $1_1(x) = \sqrt{x}$)

$$\frac{1}{w} \frac{dw}{dx} = \pm 2x$$

$w(x)P_1(x) \propto \frac{d}{dx}(1_1)$ and require $1_1(a) = 0 = 1_1(b)$ (choose $1_1(x) = \psi(x)$)

$$\frac{1}{w} \frac{dw}{dx} = \pm 2x$$
$$w(x) = \exp[\pm x^2]$$

$w(x)P_1(x) \propto \frac{d}{dx}(1)$ and require $y_1(a)=0=y_1(b)$ (choose $11 \propto (x-a)(x-b)$)

$$\frac{1}{w} \frac{dw}{dx} = \pm 2x$$

$$w(x) = \exp[\pm x^2]$$

$$P_n(x) = c_n \exp(x^2) \frac{d^n}{dx^n} (\exp(-x^2))$$

$$c_n = (-1)^n$$

Hermite polynomials

Three cases to check $(a,b) = (-\infty, \infty)$

$$= (0, \infty)$$

$$= (-1, 1)$$

Otherwise $P_n(cx+d)$ will be a polynomial on one of these ranges

Case 1: $(a,b) = (-\infty, \infty)$

$$P_1(x) = cx+d$$

$$P_1(ex+f) = cx+d$$

$$\text{linear transformation} \rightarrow P_1(x) = \pm 2x$$

Three cases to check $(a,b) = (-\infty, \infty)$

$$= (0, \infty)$$

$$(1, 1)$$

$$P_0 = \text{const} = \frac{1}{W} \text{ for}$$

Otherwise $P_n(cx+d)$ will

polynomial on one of these ranges

Case 1: $(a,b) = (-\infty, \infty)$

$$P_1(x) = cx+d$$

$$+f) = cx+d$$

$$\text{transformation} \rightarrow P_1(x) = \pm 2x$$