

Title: Special Functions and Diff. Eqs. - Lecture 2

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Abstract:

Solution Methods

- Reduction of Order
- Variation of Parameters
- Power Series Method
- WKB Approximation

$$\frac{d^2 y}{dx^2} + a_1(x) \frac{dy}{dx} + a_0(x) y = 0$$

Solution Methods

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$$\frac{d^2 y}{dx^2} + a_1(x) \frac{dy}{dx} + a_0(x) y(x) = \begin{cases} 0 & \text{homogeneous} \\ f(x) & \end{cases}$$

Power Series Method
- WKB Approximation

Reduction of Order



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Reduction of Order

Suppose we know $u(x)$ solves our ODE

Power Series Method
- WKB Approximation

Reduction of Order

Suppose we know $u(x)$ solves our ODE

$$y_2 = u(x)v(x)$$

$$y_2' = u'v + uv' \\ y_2'' = u''v + 2u'v' + uv''$$

Power Series Method
- WKB Approximation

Reduction of Order

Suppose we know $u(x)$ solves our ODE

$$y_2(x) = u(x) v(x)$$

$$y_2'(x) = u'v + uv'$$

$$y_2''(x) = u''v + 2u'v' + uv''$$

$$v(x)(u'')$$

Power Series Method
- WKB Approximation

Reduction of Order

Suppose we know $u(x)$ solves our ODE

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$$y_2''(x) = u''v + 2u'v' + uv''$$

$$v(x)(u'')$$

$u(x)$ solves our ^{homogeneous} ODE
 $v(x)$
 $+ uv'$
 $2u'v' + uv''$

$$v(x)(u'' + a_1 u' + a_0 u) + v'(2u' + a_1 u) + v''u = 0$$

$$y_2(x) = u v + u v$$

$$y_2''(x) = u''v + 2u'v' + uv''$$

$$\frac{dz}{z} + \frac{2du}{u} + a_1 dx = 0$$

$$\ln z + 2 \ln u + \int^x a_1(x') dx' = C$$

$$z = \frac{A}{u^2} \exp\left[-\int^x a_1(x') dx'\right]$$

$$v(x) = \int z(x) dx$$

=

$$= u''v + 2u'v' + uv''$$

$$z'(x) + z(x) \left(-\frac{1}{u} \right) = 0$$

$$a_1 dx = 0$$

$$v(x) = \int z(x) dx$$

$$= A \int^x \frac{dx'}{u^2} \exp \left[- \int^{x'} a_1(x'') dx'' \right] + B$$

General Solution

$$y(x) = A u(x) \int^x \frac{dx'}{u^2} \exp \left[- \int^{x'} a_1(x'') dx'' \right] + B u(x)$$

$$\int^x a_1(x') dx' = C$$

$$\exp \left[- \int^x a_1(x') dx' \right]$$

Example: $y'' + 2xy' - 2y = 0$
 $y = x$

$$y(x) = A_x \int^x \frac{dx'}{(x')^2} \exp\left[-\int^{x'} 2x'' dx''\right] + B_x$$

$$= A_x \int^x \frac{dx'}{(x')^2} \exp[-x'^2] + B_x$$

$$= A_x \int^x \frac{dx'}{(x')^2}$$

$$\begin{aligned} &= A \times \int^x \frac{dx'}{(x')^2} \left[1 - (x')^2 + \frac{1}{2}(x')^4 - \dots \right] + Bx \\ &= A \times \left[-\frac{1}{x} - x + \frac{x^3}{6} + \dots \right] + Bx \end{aligned}$$

$$= A \times \left[-\frac{1}{x} - x + \frac{x^3}{6} + \dots \right] + Bx$$

Variation of Parameters

technique for solving inhomogeneous eq given sol. to homogeneous eq

Assume y_1, y_2 solve homogeneous ODE

$$= A \times \left[-\frac{1}{x} - x + \frac{x^3}{6} + \dots \right] + Bx$$

Variation of Parameters

technique for solving inhomogeneous eq given sol. to homogeneous eq.

Assume y_1, y_2 solve homogeneous ODE

$$y(x) = v_1(x)y_1(x) + v_2(x)y_2(x)$$

$$= A \times \left[-\frac{1}{x} - x + \frac{x^3}{6} + \dots \right] + Bx$$

Variation of Parameters

technique for solving inhomogeneous eq given sol. to homogeneous eq.

Assume y_1, y_2 sol. homogeneous ODE

$$y(x) = v_1(x)y_1(x) + v_2(x)y_2(x)$$

$$y' = v_1 y_1' + v_2 y_2' + v_1' y_1 + v_2' y_2$$

$$= A x \left[-\frac{1}{x} - x + \frac{x^3}{6} + \dots \right] + B x$$

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Variation of Parameters

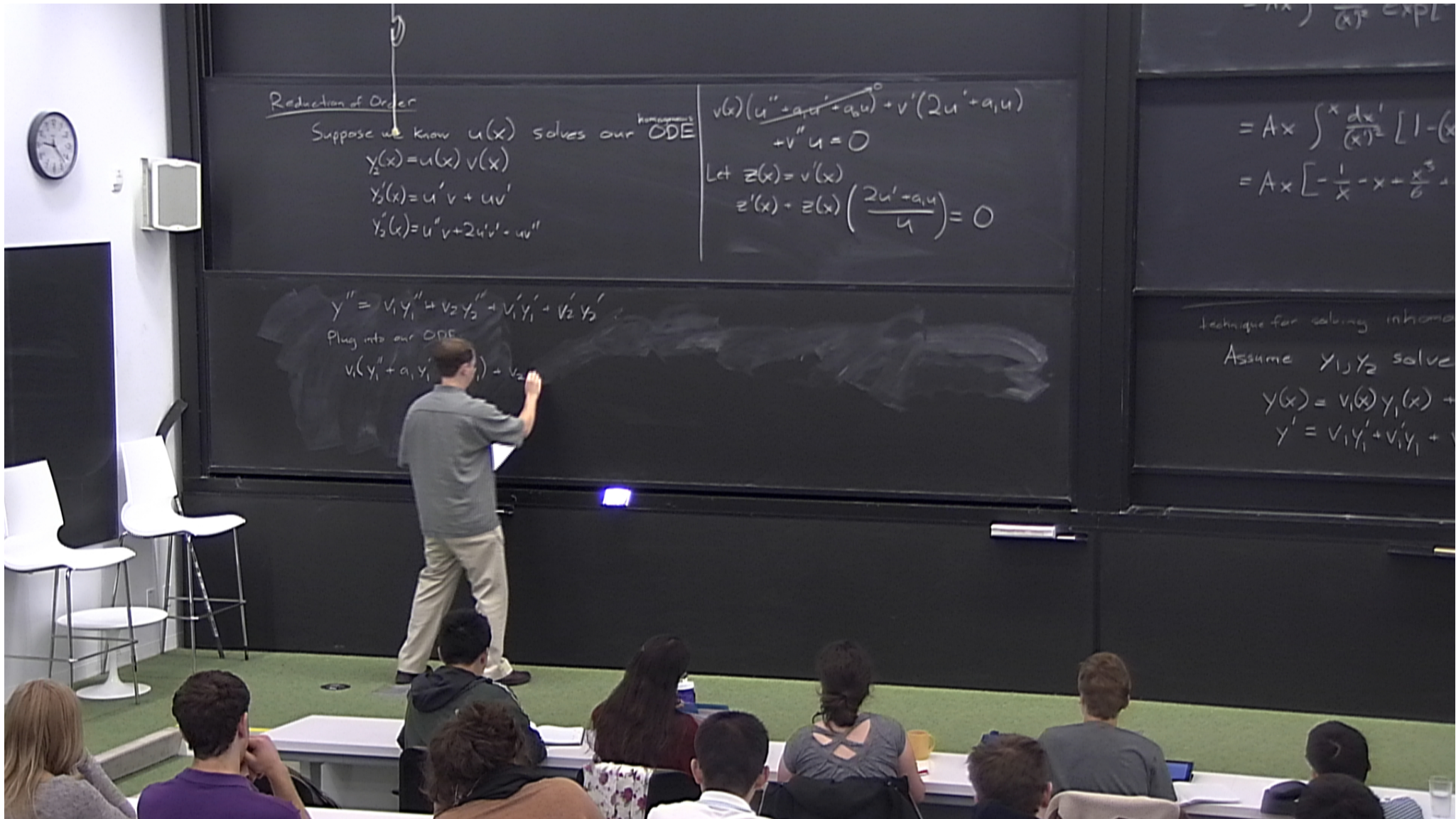
technique for solving inhomogeneous eq given sol. to homogeneous eq

Assume y_1, y_2 solve homogeneous ODE

$$y(x) = v_1(x) y_1(x) + v_2(x) y_2(x)$$

$$y' = v_1 y_1' + v_1' y_1 + v_2 y_2' + v_2' y_2$$

Assume $v_1' y_1 + v_2' y_2 = 0$ need to check later



Reduction of Order

Suppose we know $u(x)$ solves our ^{homogeneous} ODE

$$y_2(x) = u(x) v(x)$$

$$y_2'(x) = u'v + uv'$$

$$y_2''(x) = u''v + 2u'v' + uv''$$

$$v(x)(u'' + a_1u' + a_0u) + v'(2u' + a_1u) + v''u = 0$$

$$\text{Let } z(x) = v'(x)$$

$$z'(x) + z(x)\left(\frac{2u' + a_1u}{u}\right) = 0$$

$$y'' = v_1 y_1'' + v_2 y_2'' + v_1' y_1' + v_2' y_2'$$

Plug into our ODE

$$v_1(y_1'' + a_1 y_1' + a_0 y_1) + v_2(y_2'' + a_1 y_2' + a_0 y_2) = 0$$

$$= A \times \int^x \frac{dx'}{(x')^2} [1 - (x')^3]$$
$$= A \times \left[-\frac{1}{x'} - x' + \frac{x'^3}{3}\right]$$

Technique for solving inhomogeneous

Assume y_1, y_2 solve

$$y(x) = v(x) y_1(x) +$$

$$y' = v_1 y_1' + v_1' y_1 +$$

Reduction of Order

Suppose we know $u(x)$ solves our ODE

$$y_2(x) = u(x)v(x)$$

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Plug into our ODE

$$v_1(y_1'' + a_1y_1' + a_0y_1) + v_2(y_2'' + a_1y_2' + a_0y_2) + v_1'y_1' + v_2'y_2' = f$$

$$= Ax \int^x \frac{dx'}{(x')^2} [1 - (x')^2]$$

$$= Ax \left[-\frac{1}{x} - x + \frac{x^3}{6} \right]$$

Technique for solving inhomogeneous

Assume y_1, y_2 solve

$$y(x) = v(x)y_1(x) +$$

$$y' = v_1y_1' + v_1'y_1 +$$

Example: Forced Damped HM

$$m y'' + c y' + k y = F(t)$$

$$y(t) = A e^{\lambda_1 t} + B e^{\lambda_2 t}$$

where λ_1, λ_2 solve

$$m \lambda^2 + c \lambda + k = 0$$

and $\lambda_1 \neq \lambda_2$

$$W = \begin{vmatrix} e^{\lambda_1 t} & e^{\lambda_2 t} \\ \lambda_1 e^{\lambda_1 t} & \lambda_2 e^{\lambda_2 t} \end{vmatrix} = \lambda_2 e^{(\lambda_1 + \lambda_2)t} - \lambda_1 e^{(\lambda_2 + \lambda_1)t}$$

Power Series Method

Want to solve: $p_0 y'' + p_1 y' + p_2 y = 0$ near $x = x_0$

If $\frac{p_1}{p_0}$ and $\frac{p_2}{p_0}$ are analytic near x_0 , x_0 is an ordinary point otherwise it is a

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If x_0 is ordinary, then there is a solution $y = \sum_{n=0}^{\infty} a_n(x-x_0)^n$ that converges near x_0

geous eq given sol. to homogeneous eq.

solve homogeneous ODE

$$y_1(x) + v_2(x)y_2(x)$$

$$+ v_2 y_1' + v_2' y_2$$

Assume $v_1' y_1 + v_2' y_2 = 0$ need to check later

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Example: Airy equation $y'' - xy = 0$ around $x_0 = 0$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\sum_{n=1}^{\infty} (n+3)(n+2) a_{n+3} x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$n=-1$ term gives $a_2 = 0$
also have $(n+3)(n+2) a_{n+3} - a_n = 0$

$$y(x) = a_0 \left(1 + \frac{x^3}{6} + \frac{x^6}{180} \right)$$

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$n=-1$ term gives $a_2 = 0$
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$$y(x) = a_0 \left(1 + \frac{x^3}{6} + \frac{x^6}{180} + \dots \right) + a_1 \left(x + \frac{x^4}{12} + \frac{x^7}{504} + \dots \right)$$

If $x_1, x_2 \rightarrow \infty$ we need to use reduction of order to find second solution
 y_1 will still be a solution

$$y_2 = y_1 \ln(x-x_0) + (x-x_0) \sum_{n=0}^{\infty} b_n (x-x_0)^n$$

WKB Approximation

$$\frac{d^2 \psi}{dx^2} + \frac{2m}{\hbar^2} (E - V(x)) \psi(x) = 0$$

Define

$$p(x) = \sqrt{2m(E - V(x))}$$

$$\frac{d^2 \psi}{dx^2} + \frac{p^2}{\hbar^2} \psi = 0$$

If $V = \text{constant}$ $p(x) = \text{constant}$
 $\psi(x) = e^{\pm i p x / \hbar}$

we need to use reduction of order to find second solution
 y_1 will still be a solution

$$y_2 = y_1 \ln(x-x_0) + (x-x_0) \sum_{n=0}^{\infty} b_n (x-x_0)^n$$

approximation

$$\frac{\psi}{x^2} + \frac{2m}{\hbar^2} (E - V(x)) \psi(x) = 0$$

Define $p(x) = \sqrt{2m(E - V(x))}$

$$\frac{d^2\psi}{dx^2} + \frac{p^2}{\hbar^2} \psi = 0$$

If $V = \text{constant}$ $p(x) = \text{constant} \pm ipx/\hbar$

$$\psi(x) =$$

What if $V(x)$ is "slowly varying" with x
 $\psi(x) \approx e^{\pm iW(x)/\hbar}$

If $\frac{1}{p_0}$ and $\frac{p_2}{p_0}$ are analytic near x_0 , x_0 is an ordinary point of

If x_0 is ordinary, then there is a solution $y = \sum_{n=0}^{\infty} a_n (x-x_0)^n$ that converges

$$\psi'(x) = \frac{iW'}{\hbar} \exp(iW/\hbar)$$

$$\psi''(x) = -\frac{(W')^2}{\hbar^2} \exp(iW/\hbar) + \frac{iW''}{\hbar} \exp(iW/\hbar)$$

$$i\hbar W'' - (W')^2 + p^2 = 0$$

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$$i\hbar W'' - (W')^2 + p^2 = 0$$

$$W(x) = W_0(x) + \hbar W_1(x) + \hbar^2 W_2(x) + \dots$$

Plug in and satisfy

$$(W')^2 = p^2$$

$$i\hbar W_0'' = 2$$

If $\frac{1}{P_0}$ and $\frac{P_2}{P_0}$ are analytic near x_0 , x_0 is an ordinary point otherwise it is a singular point.

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$$i\hbar W'' - (W')^2 + P^2 = 0$$

$$W(x) = W_0(x) + \hbar W_1(x) + \hbar^2 W_2(x) + \dots$$

Plug in and satisfy at each order in $\hbar$.

$$(W_0')^2 = P^2$$

$$i\hbar W_0' = 2W_0' W_1'$$

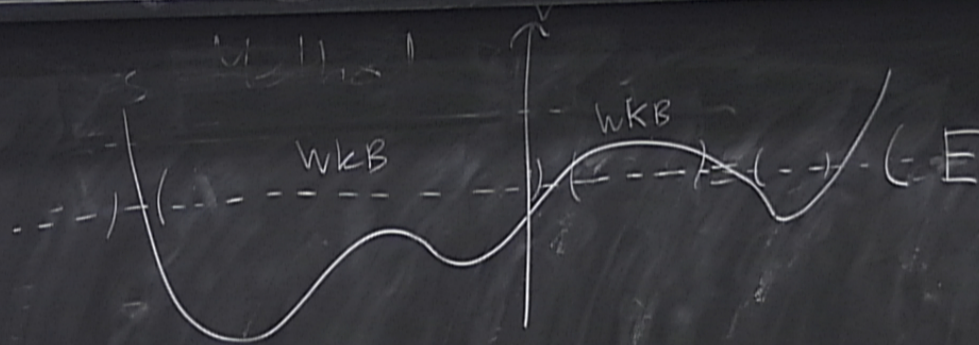
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Expansion breaks down if $P=0$

$$i\hbar W'' - (W')^2 + P = 0$$

$$W(x) = W_0(x) + \hbar W_1(x) + \hbar^2 W_2(x) + \dots$$

Expansion breaks down if $P=0$



match solutions in classically allowed and classically forbidden regions