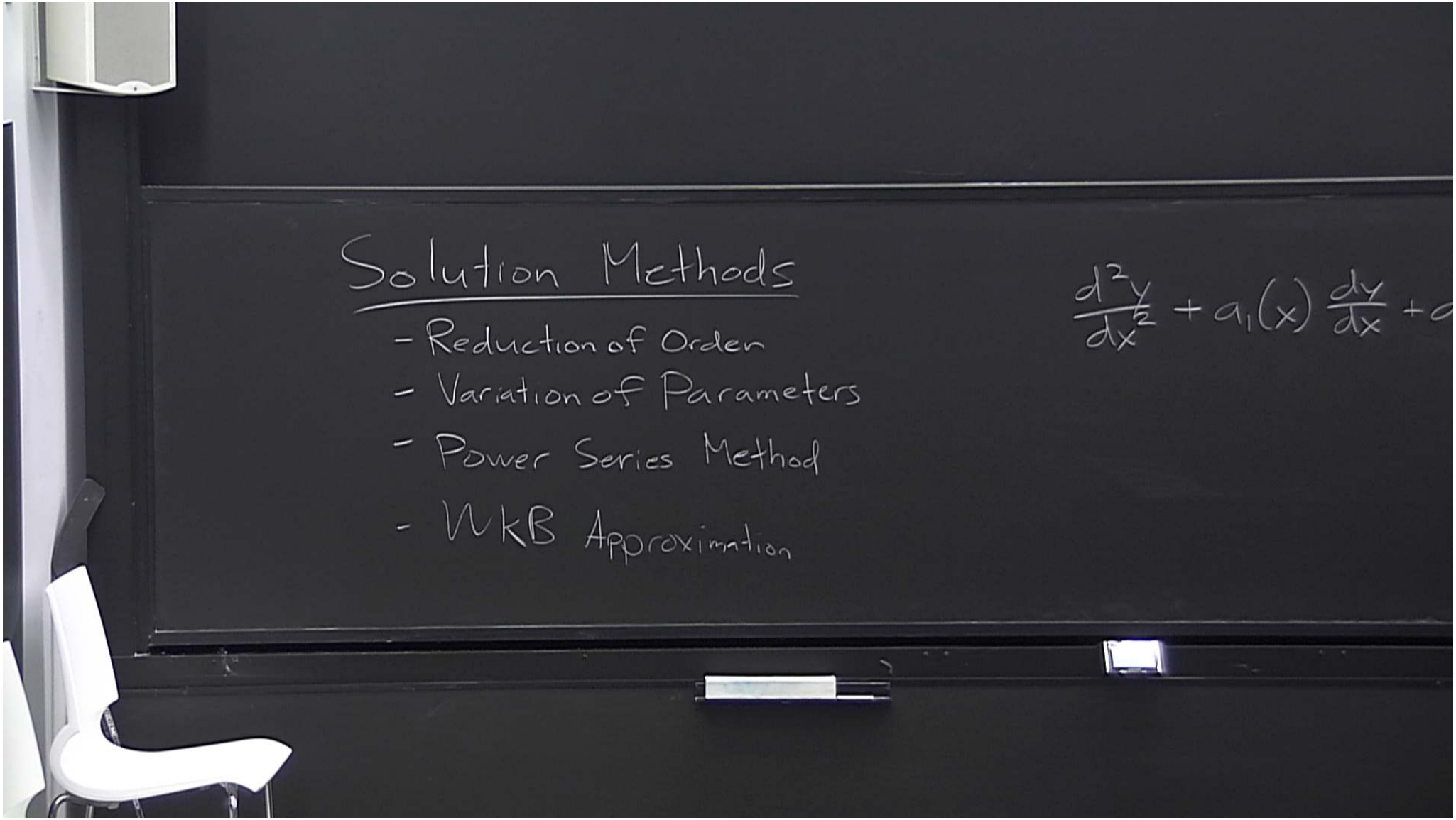
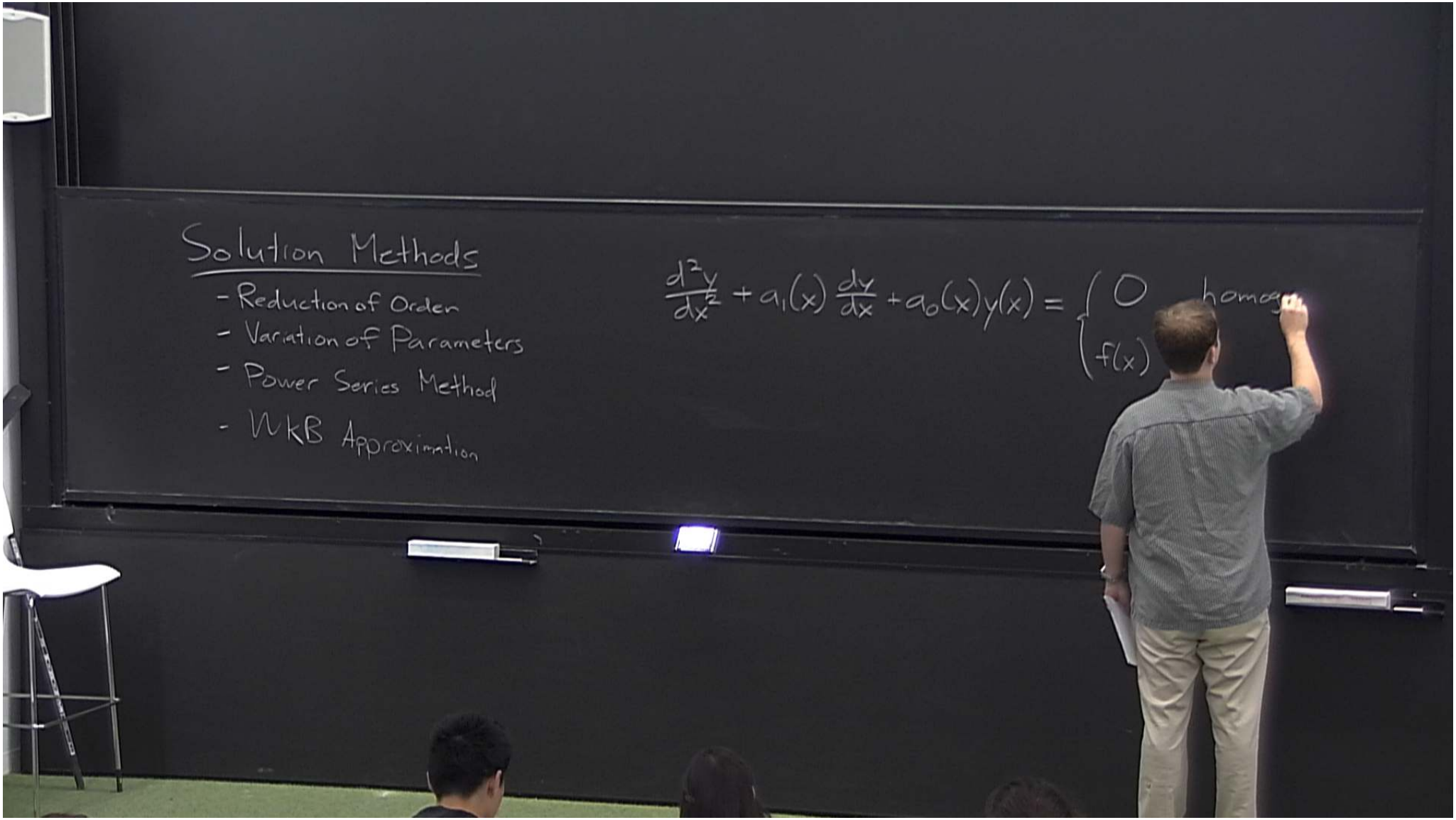
Title: Special Functions and Diff. Eqs. - Lecture 2

Date: Aug 22, 2012 09:00 AM

URL: http://pirsa.org/12080021

Abstract:


Solution Methods
- Reduction of Orden
- Variation of Parameters
- Power Series Method
- WkB Approximation
$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} +$


Solution Methods
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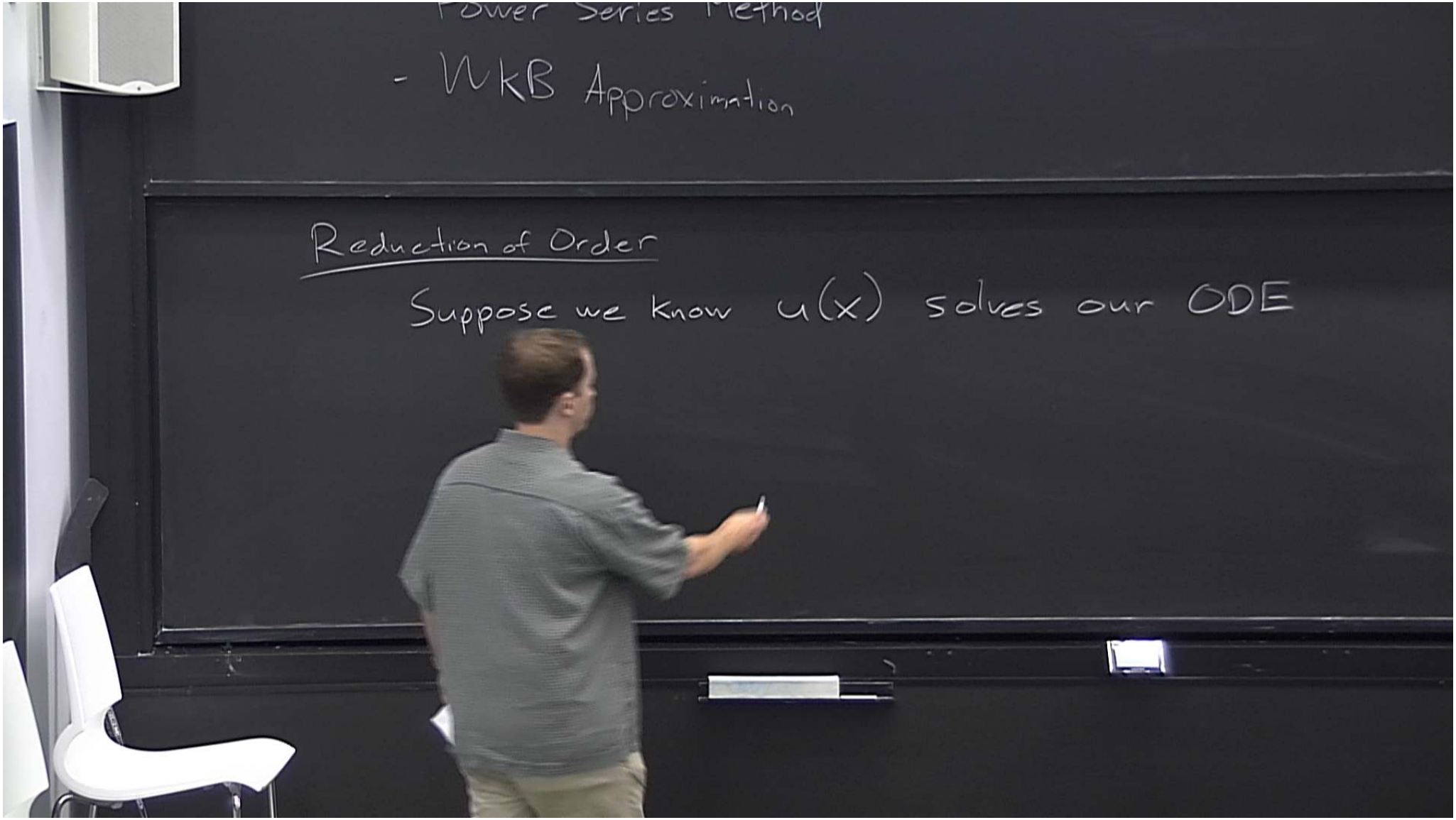
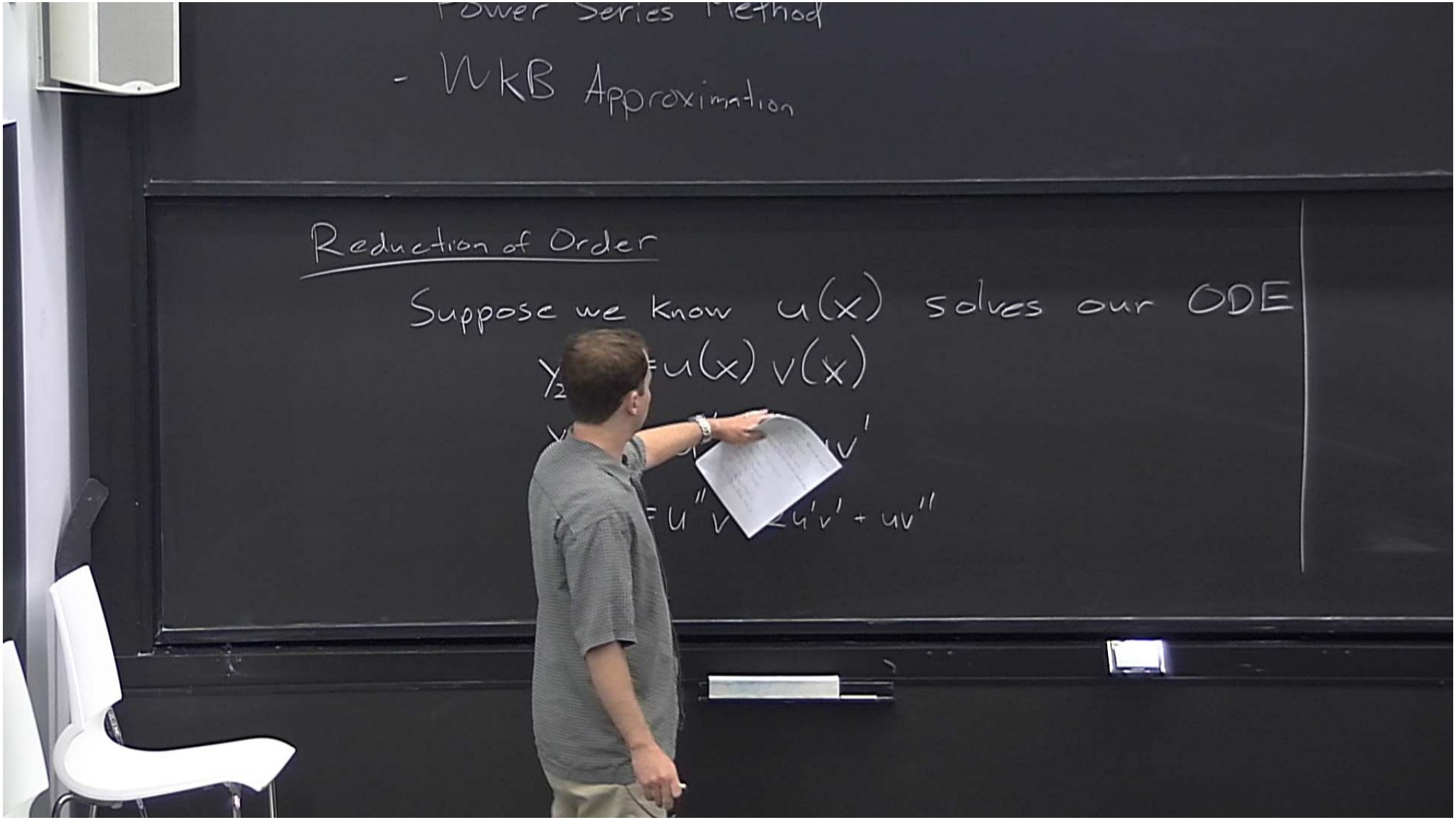
- WkB Approximation
$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y(x) = \begin{cases} 0 & \text{homog} \\ f(x) & \end{cases}$

Power Series Method
- WkB Approximation
Reduction of Order


Power Series Method
- WkB Approximation
Reduction of Order
Suppose we know u(x) solves our ODE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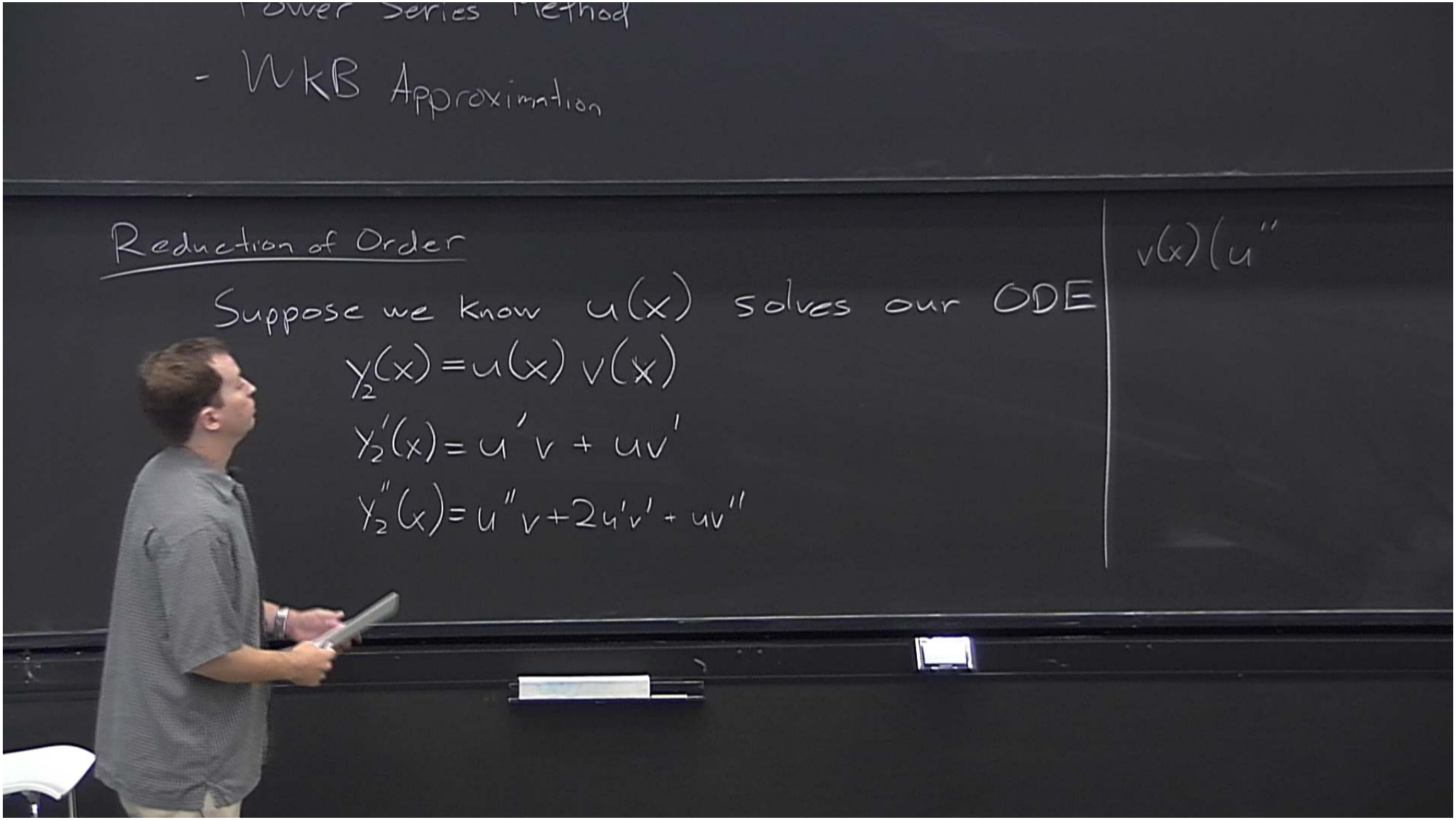
Suppose we know u(x) solves our ODE
$y_2 = u(x)\,v(x)$


Power Series Method
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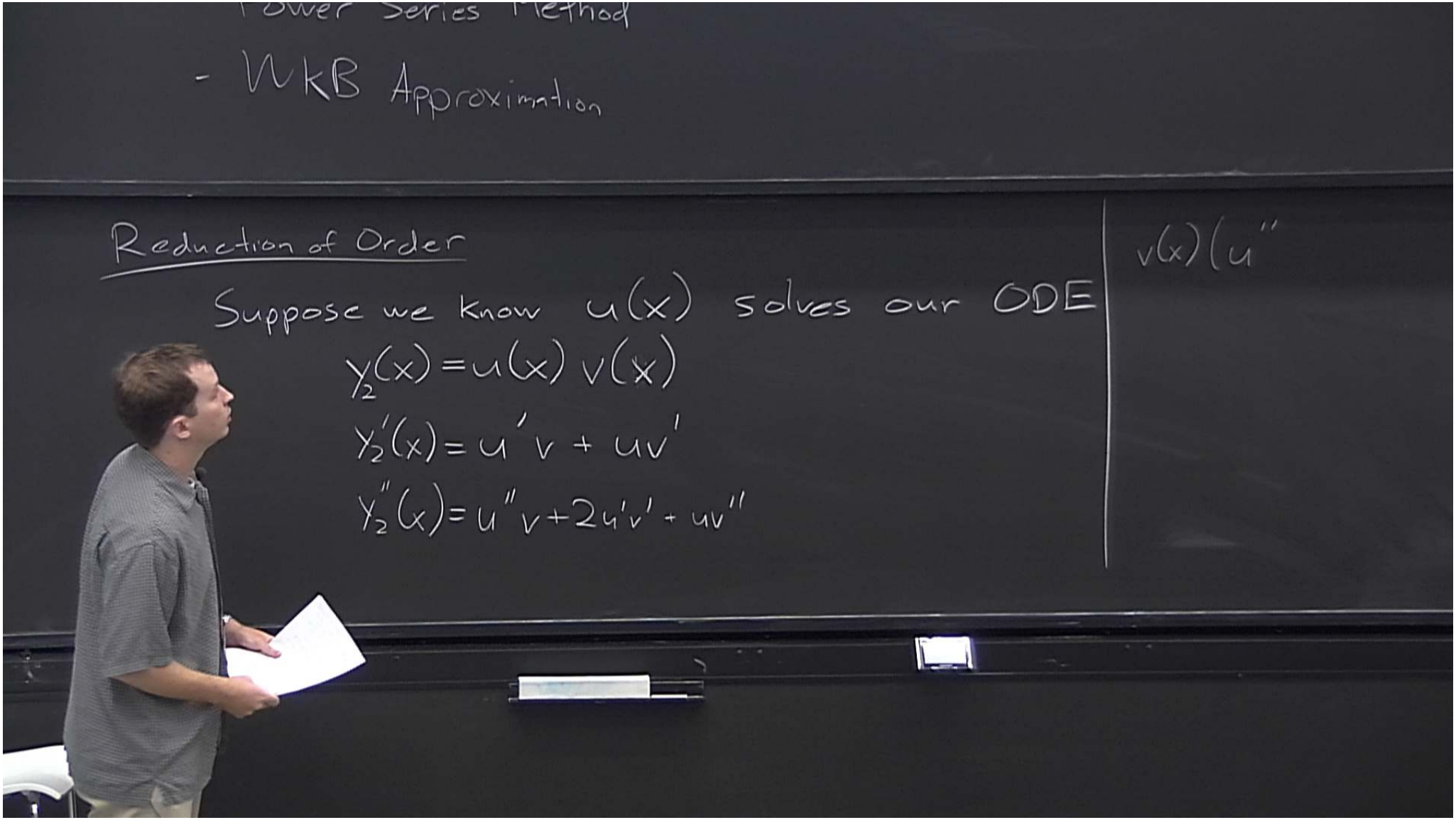
Suppose we know u(x) solves our ODE
$y_2(x) = u(x)\,v(x)$
$y_2'(x) = u'v + uv'$
$y_2''(x) = u''v + 2u'v' + uv''$
v(x)(u"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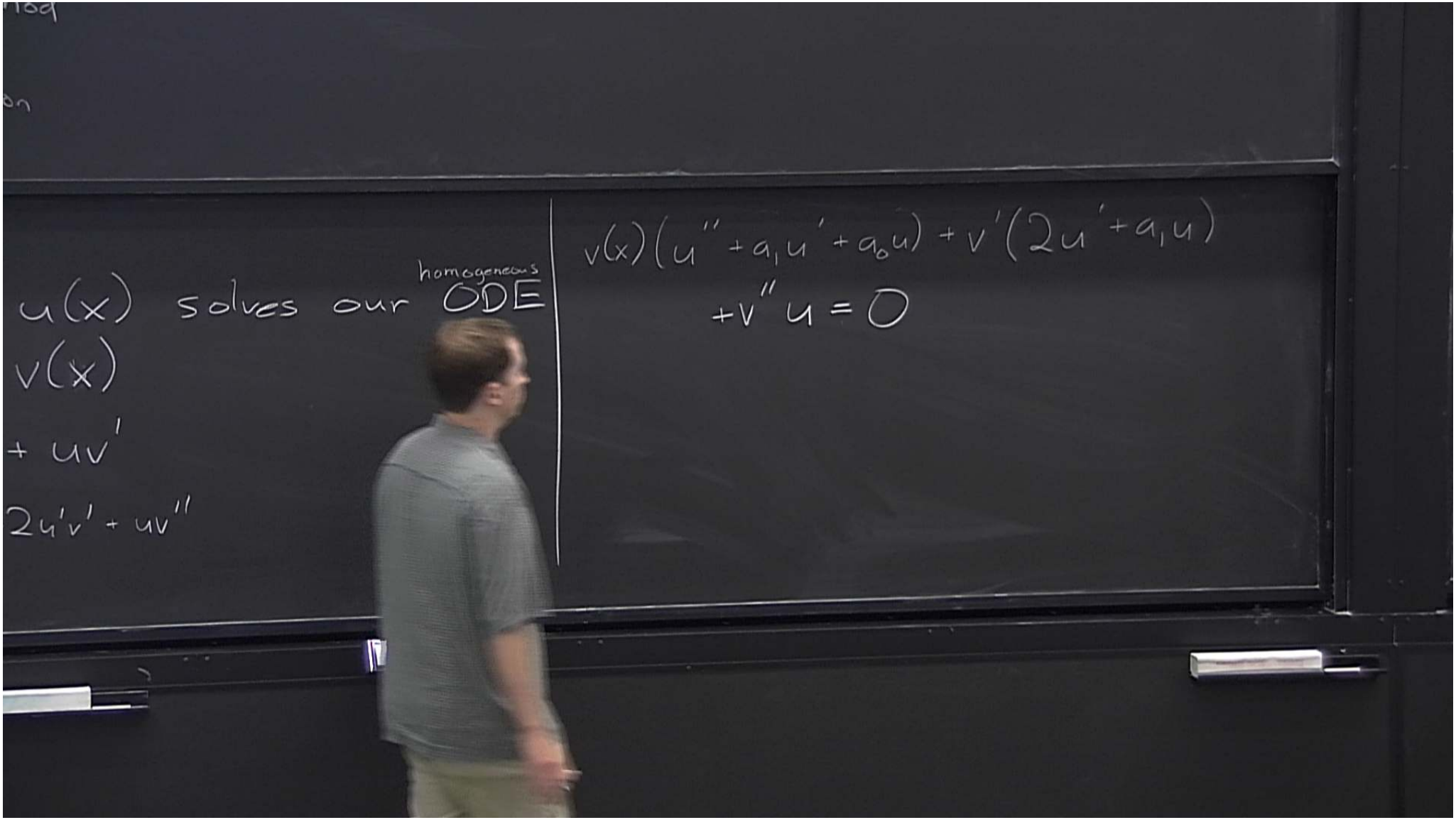
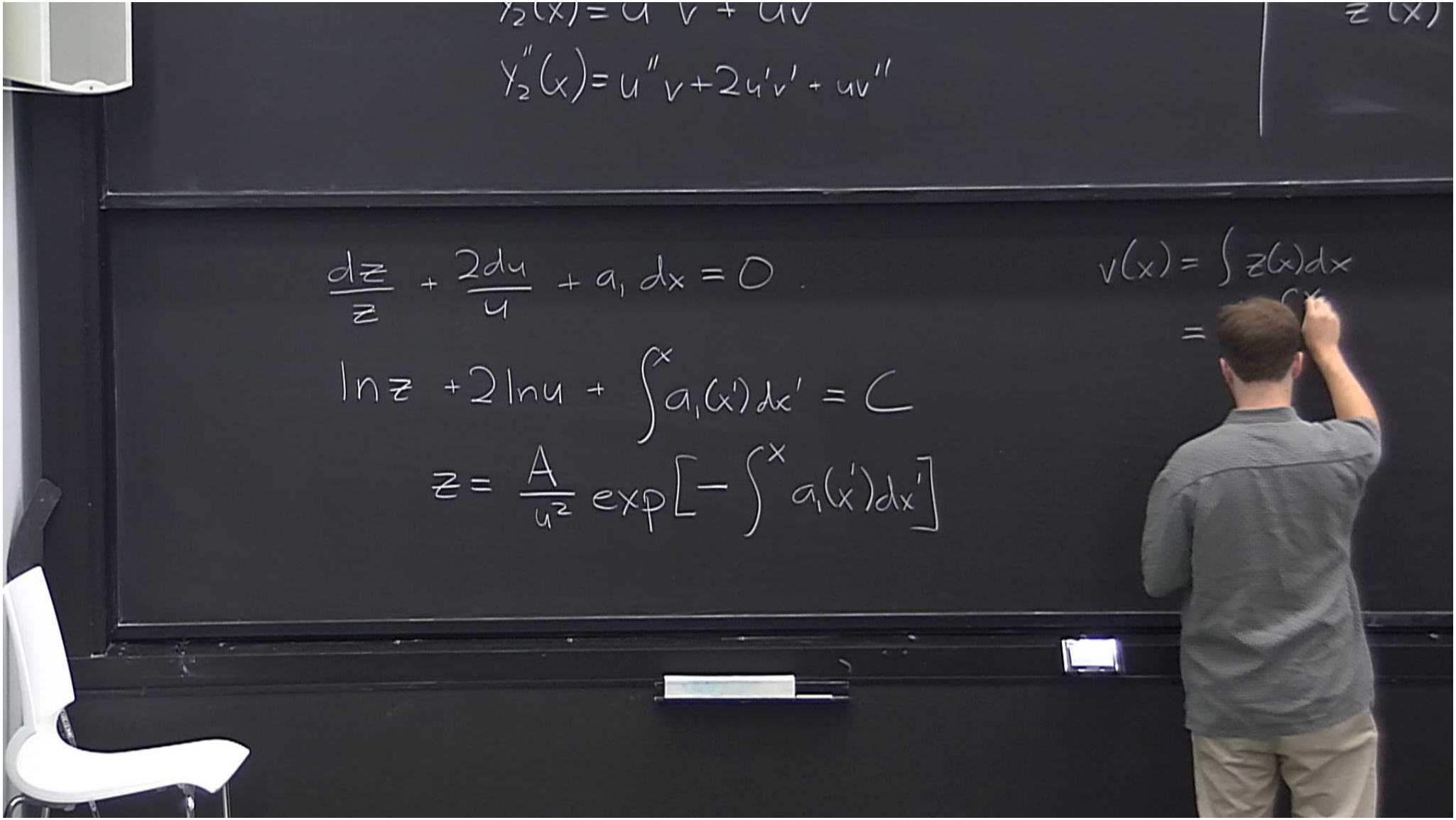

$u(x)$ solves our homogeneous ODE
$v(x)$
$+ uv'$
$2u'v' + uv''$
$v(x)(u'' + a_1u' + a_0u) + v'(2u' + a_1u)$
$+v''u = 0$


$y_2(x) = u'v + uv'$
$y_2''(x) = u''v + 2u'v' + uv''$
$z(x)$
$\frac{dz}{z} + \frac{2du}{u} + a_1\,dx = 0$
$v(x) = \int z(x)\,dx$
$\ln z + 2\ln u + \int^x a_1(x')\,dx' = C$
$z = \frac{A}{u^2}\exp\left[-\int^x a_1(x')\,dx'\right]$

$= u'' v + 2u'v' + uv''$

$z'(x) + z(x)\left(\frac{\quad}{u}\right) = 0$

$a_1 dx = 0$

$\int a_1(x')dx' = C$

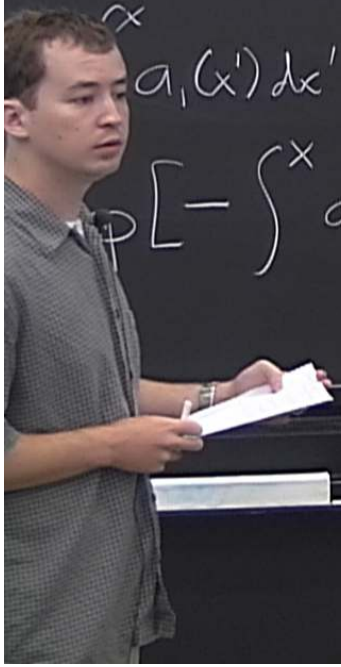
$\exp\left[-\int^x a_1(x')dx'\right]$

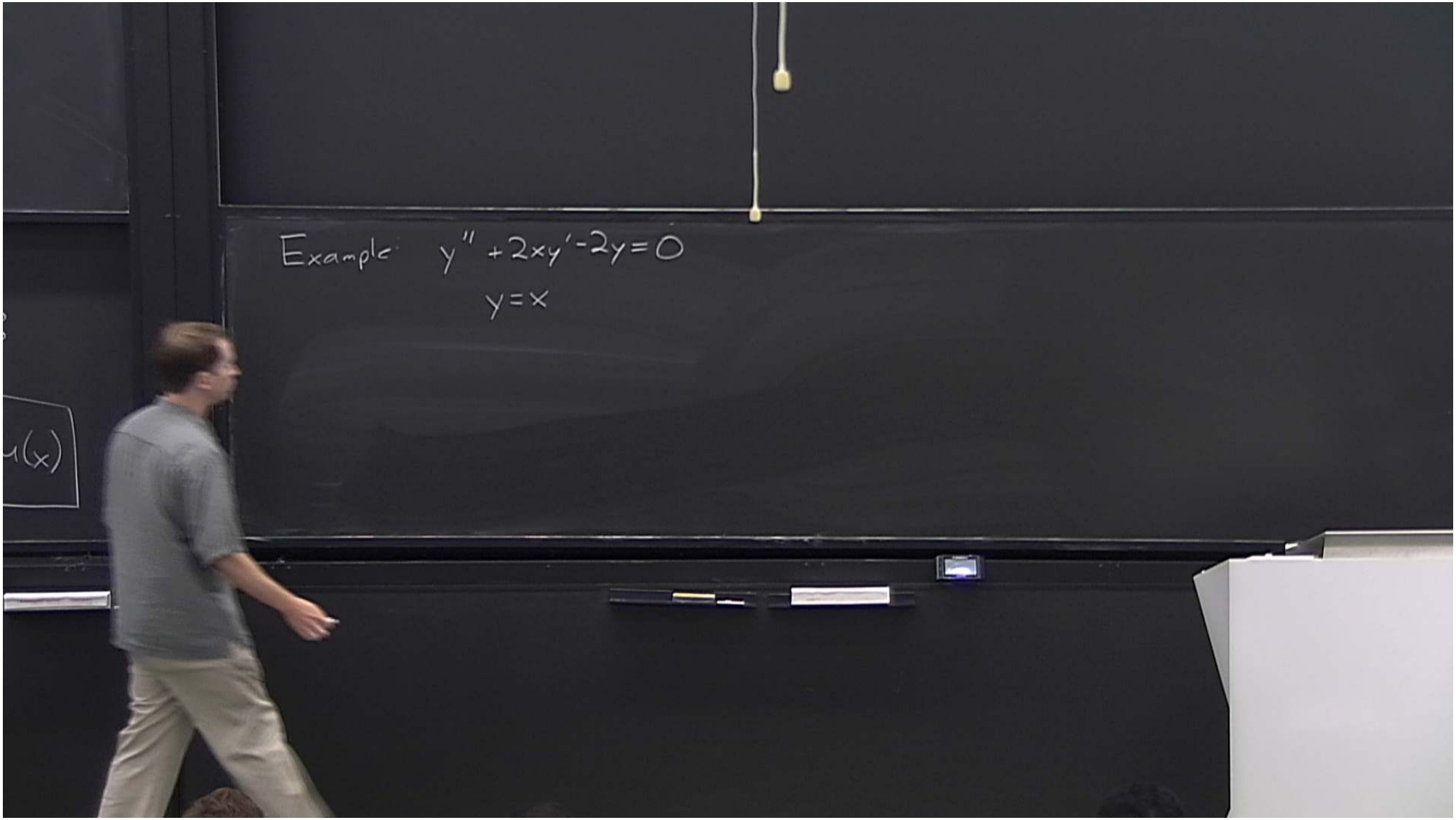
$v(x) = \int z(x)dx$

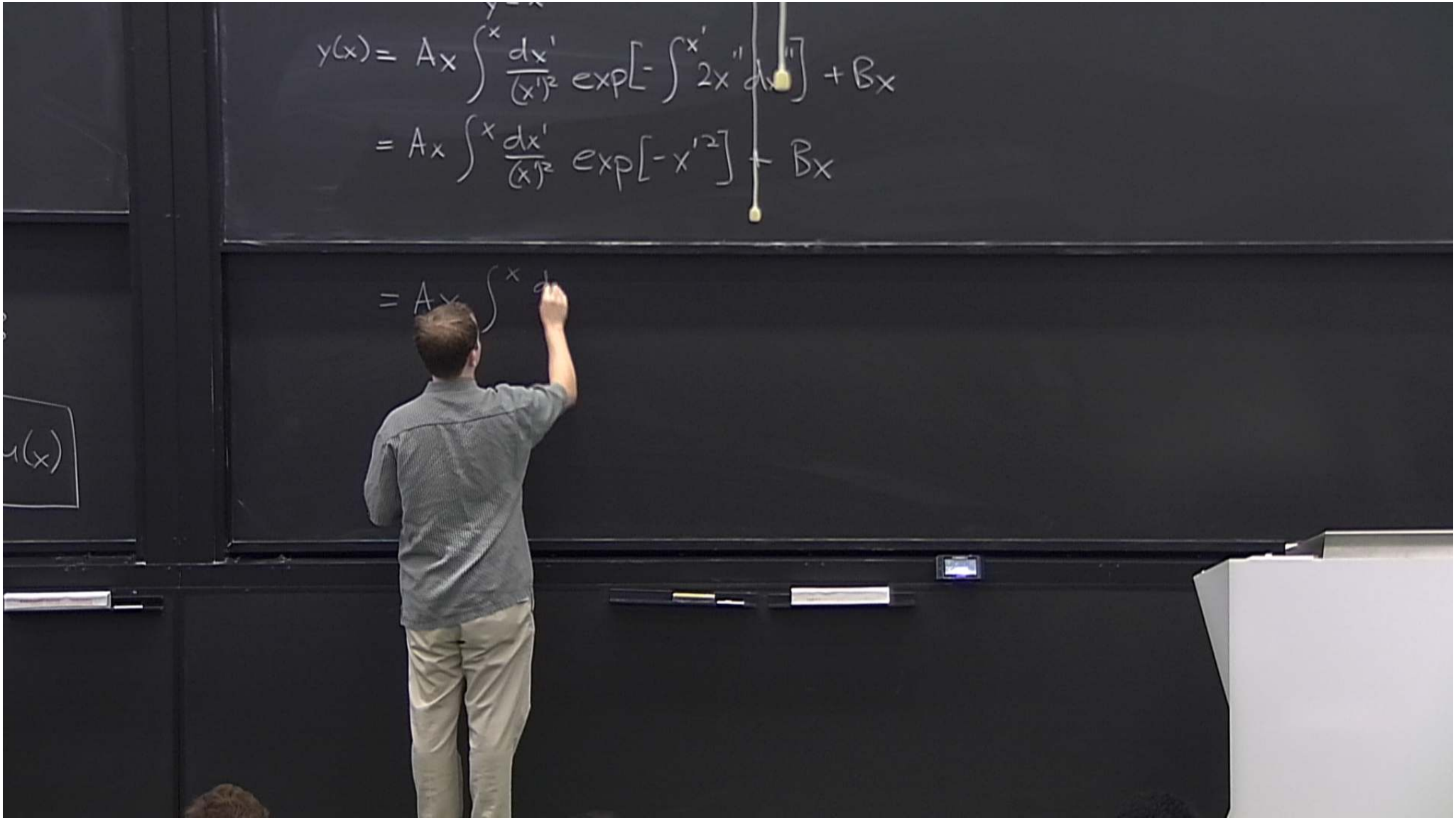
$= A\int^x \frac{dx'}{u^2} \exp\left[-\int^{x'} a_1(x'')dx''\right] + B$

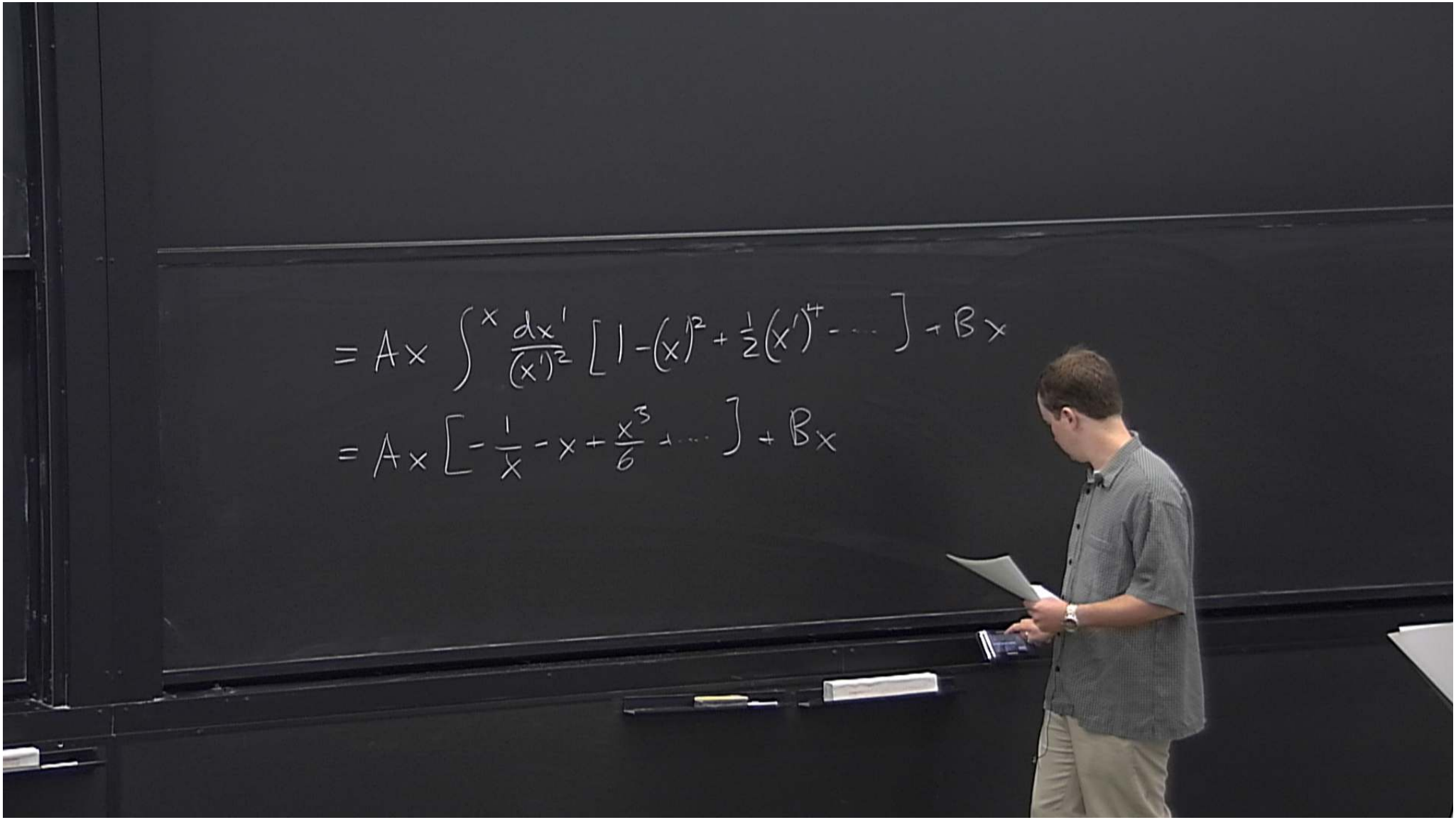
General Solution

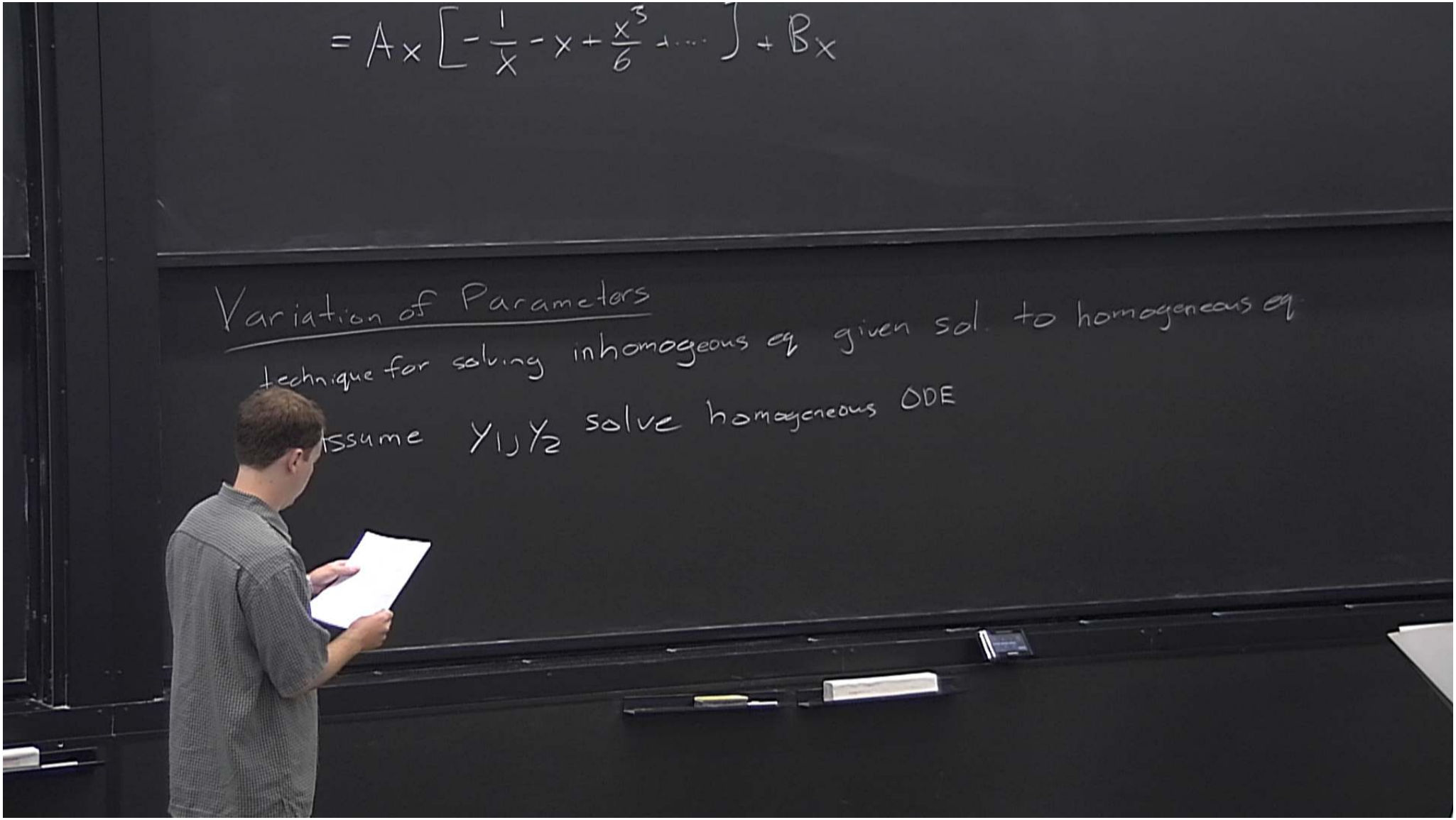
$$y(x) = A\,u(x)\int^x \frac{dx'}{u^2}\exp\left[-\int^{x'} a_1(x'')dx''\right] + B\,u(x)$$




Example $y'' + 2xy' - 2y = 0$
$y = x$


$y(x) = A_x \int^x \frac{dx'}{(x')^2} \exp\left[-\int^{x'} 2x'' dx''\right] + B_x$
$= A_x \int^x \frac{dx'}{(x')^2} \exp[-x'^2] + B_x$
$= A_x \int^x$


$= A x \int^{x} \frac{dx'}{(x')^2} \left[1 - (x')^2 + \frac{1}{2}(x')^4 - \cdots \right] + Bx$
$= A x \left[-\frac{1}{x} - x + \frac{x^3}{6} + \cdots \right] + Bx$


$= Ax\left[-\frac{1}{x} - x + \frac{x^3}{6} + \cdots\right] + Bx$
Variation of Parameters
technique for solving inhomogeneous eq given sol. to homogeneous eq
Assume y_1, y_2 solve homogeneous ODE

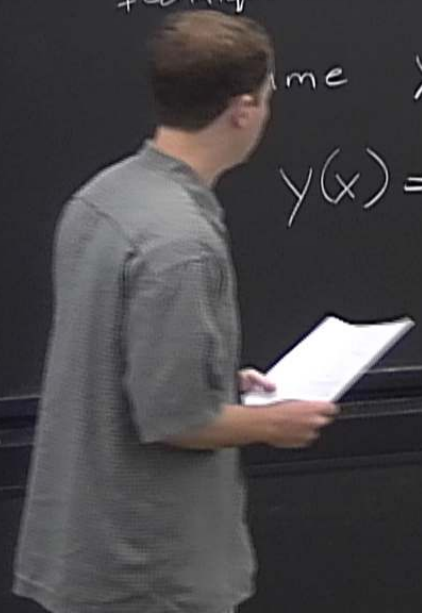
$$= Ax\left[-\frac{1}{x} - x + \frac{x^3}{6} + \cdots\right] + Bx$$

Variation of Parameters

technique for solving inhomogeneous eq given sol. to homogeneous eq

me y_1, y_2 solve homogeneous ODE

$$y(x) = v_1(x)y_1(x) + v_2(x)y_2(x)$$



$$= Ax\left[-\frac{1}{x} - x + \frac{x^3}{6} + \cdots\right] + Bx$$

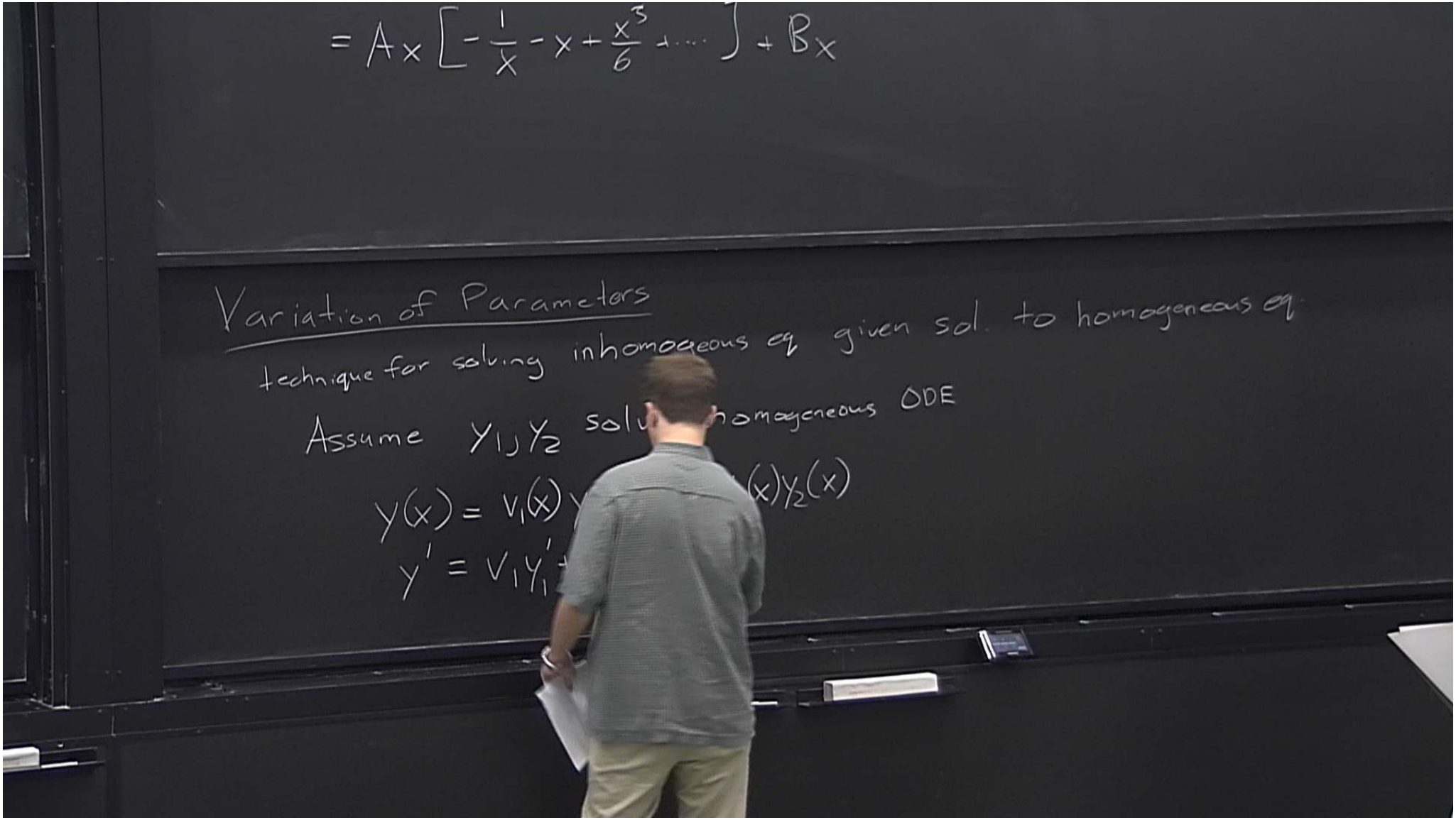
Variation of Parameters

technique for solving inhomogeneous eq given sol. to homogeneous eq

Assume y_1, y_2 solu[illegible]homogeneous ODE

$$y(x) = v_1(x)[illegible](x)y_2(x)$$

$$y' = v_1 y_1' +$$



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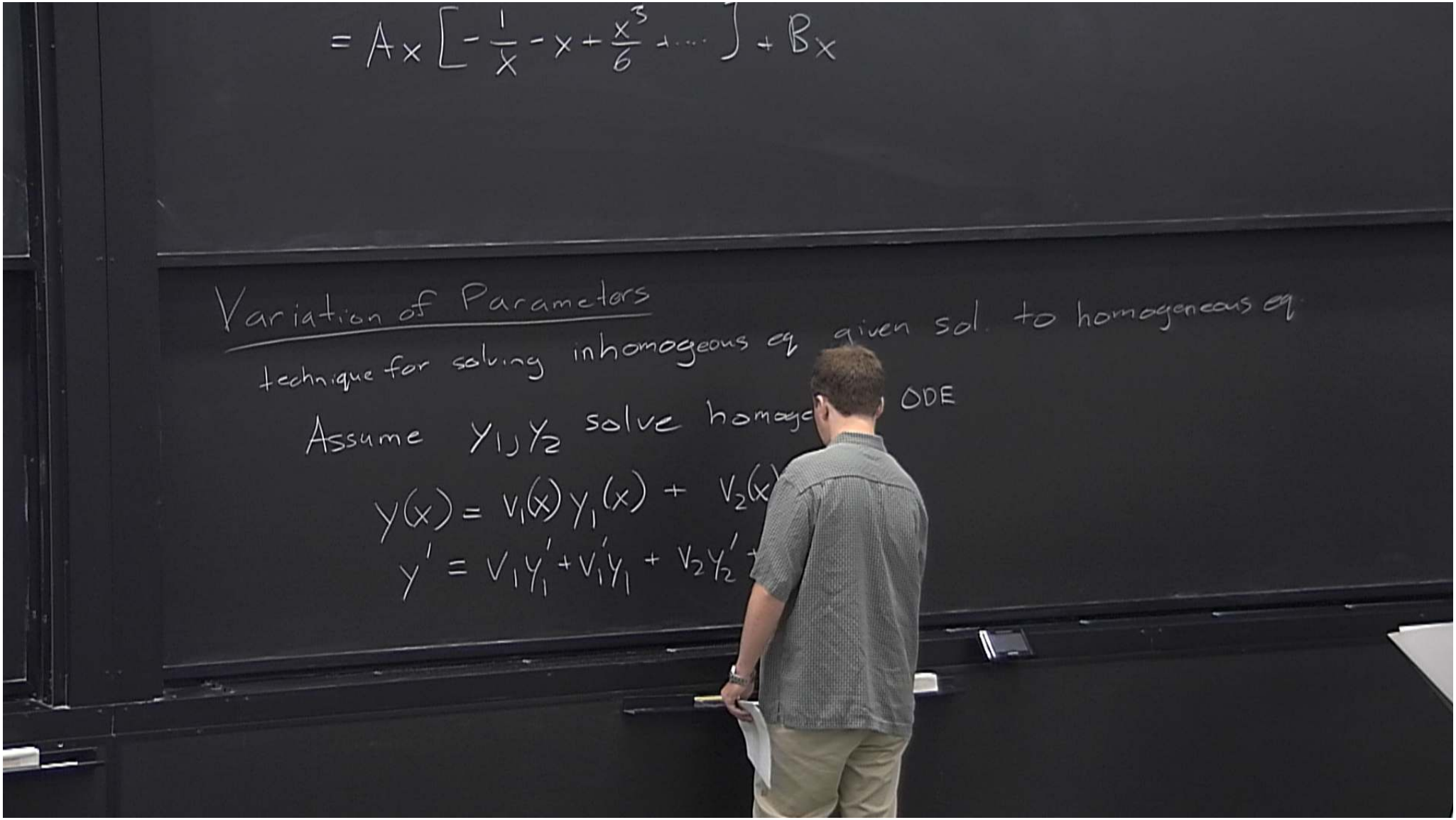
Variation of Parameters

technique for solving inhomogeous eq given sol. to homogeneous eq

Assume y_1, y_2 solve homoge ODE

$$y(x) = v_1(x)y_1(x) + v_2(x)$$

$$y' = v_1y_1' + v_1'y_1 + v_2y_2' +$$



$$= Ax\left[-\frac{1}{x} - x + \frac{x^3}{6} - \cdots\right] + Bx$$

Variation of Parameters

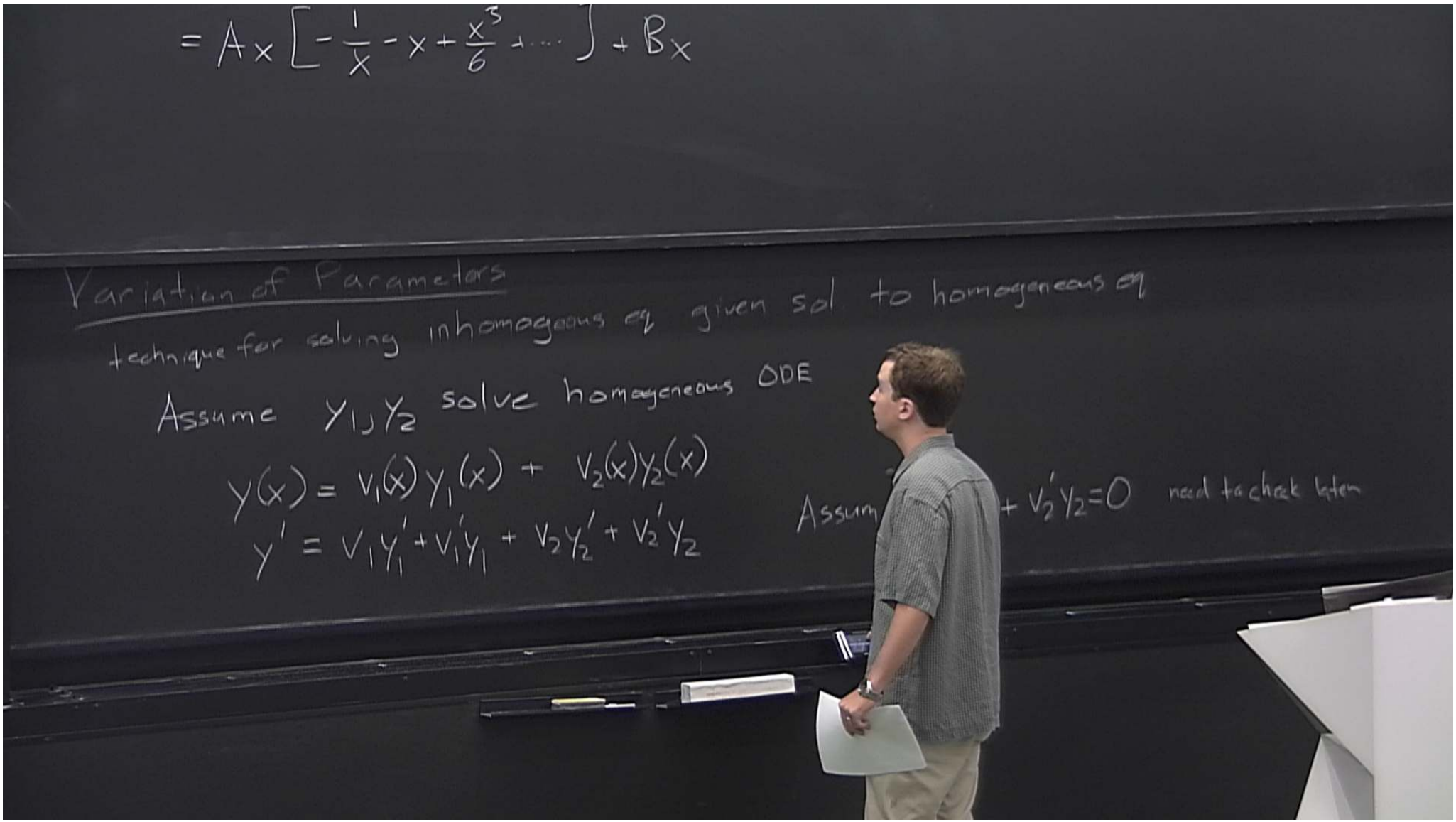
technique for solving inhomogeneous eq given sol to homogeneous eq

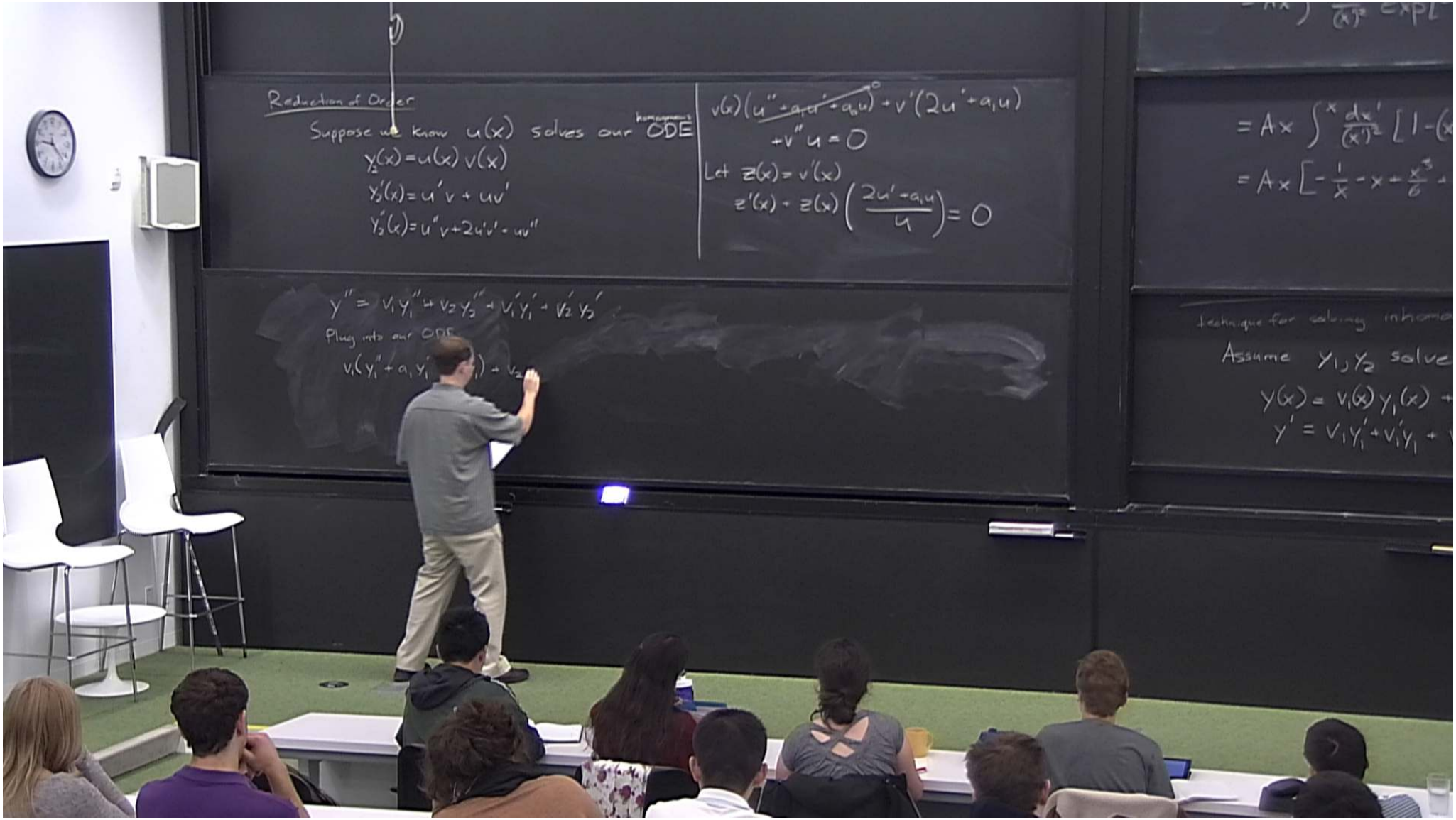
Assume y_1, y_2 solve homogeneous ODE

$$y(x) = v_1(x)y_1(x) + v_2(x)y_2(x)$$

$$y' = v_1y_1' + v_1'y_1 + v_2y_2' + v_2'y_2$$

Assum $+ v_2'y_2 = 0$ need to check later




Reduction of Order
Suppose we know $u(x)$ solves our homogeneous ODE
$y_2(x) = u(x)\,v(x)$
$y_2'(x) = u'v + uv'$
$y_2''(x) = u''v + 2u'v' + uv''$
$v(u'' + a_1u' + a_0u) + v'(2u' + a_1u) + v''u = 0$
Let $z(x) = v'(x)$
$z'(x) + z(x)\left(\frac{2u' + a_1u}{u}\right) = 0$
$y'' = v_1y_1'' + v_2y_2'' + v_1'y_1' + v_2'y_2'$
Plug into our ODE
technique for solving
Assume y_1, y_2 solve
$y(x) = v_1(x)y_1(x) +$
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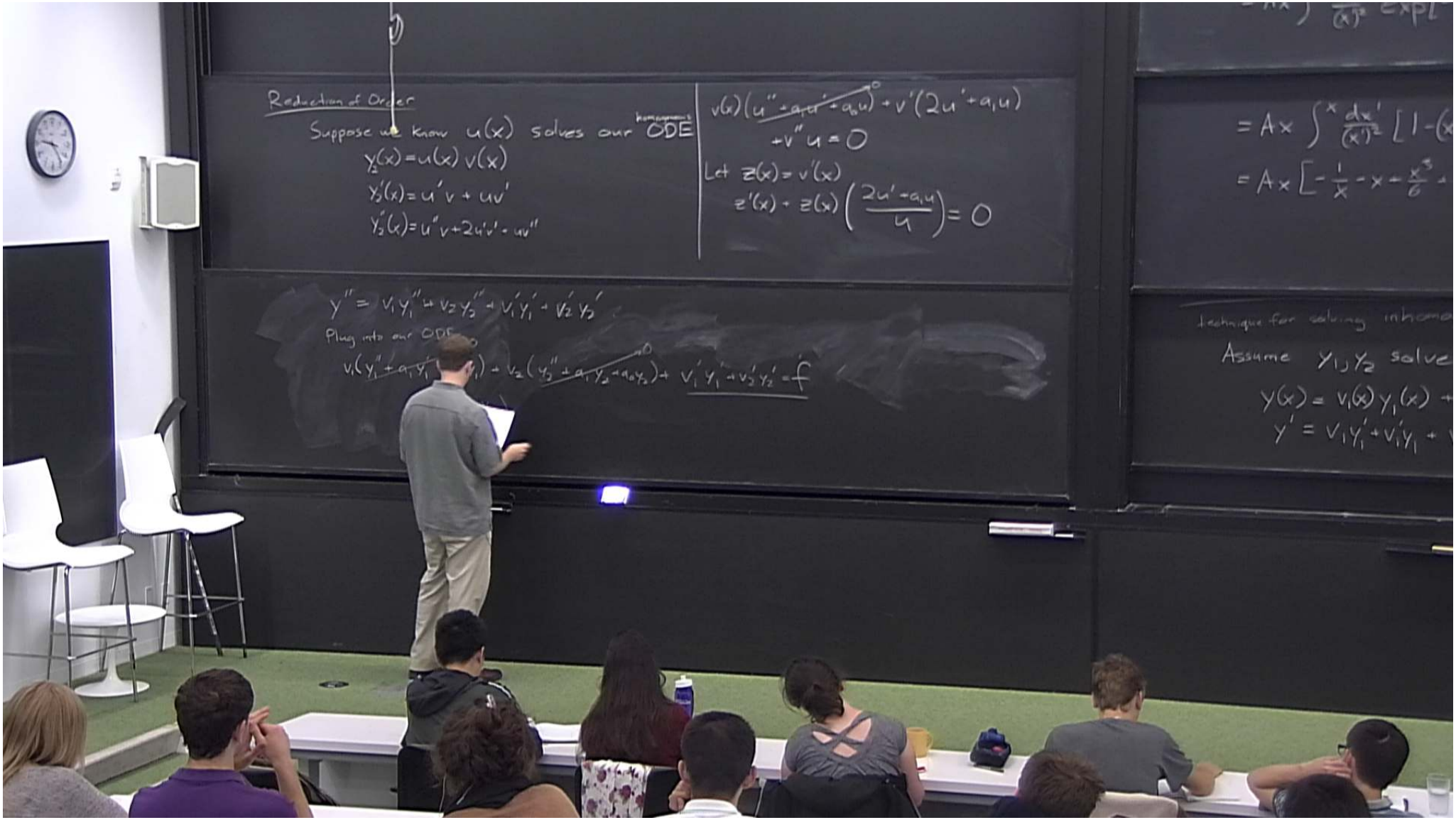
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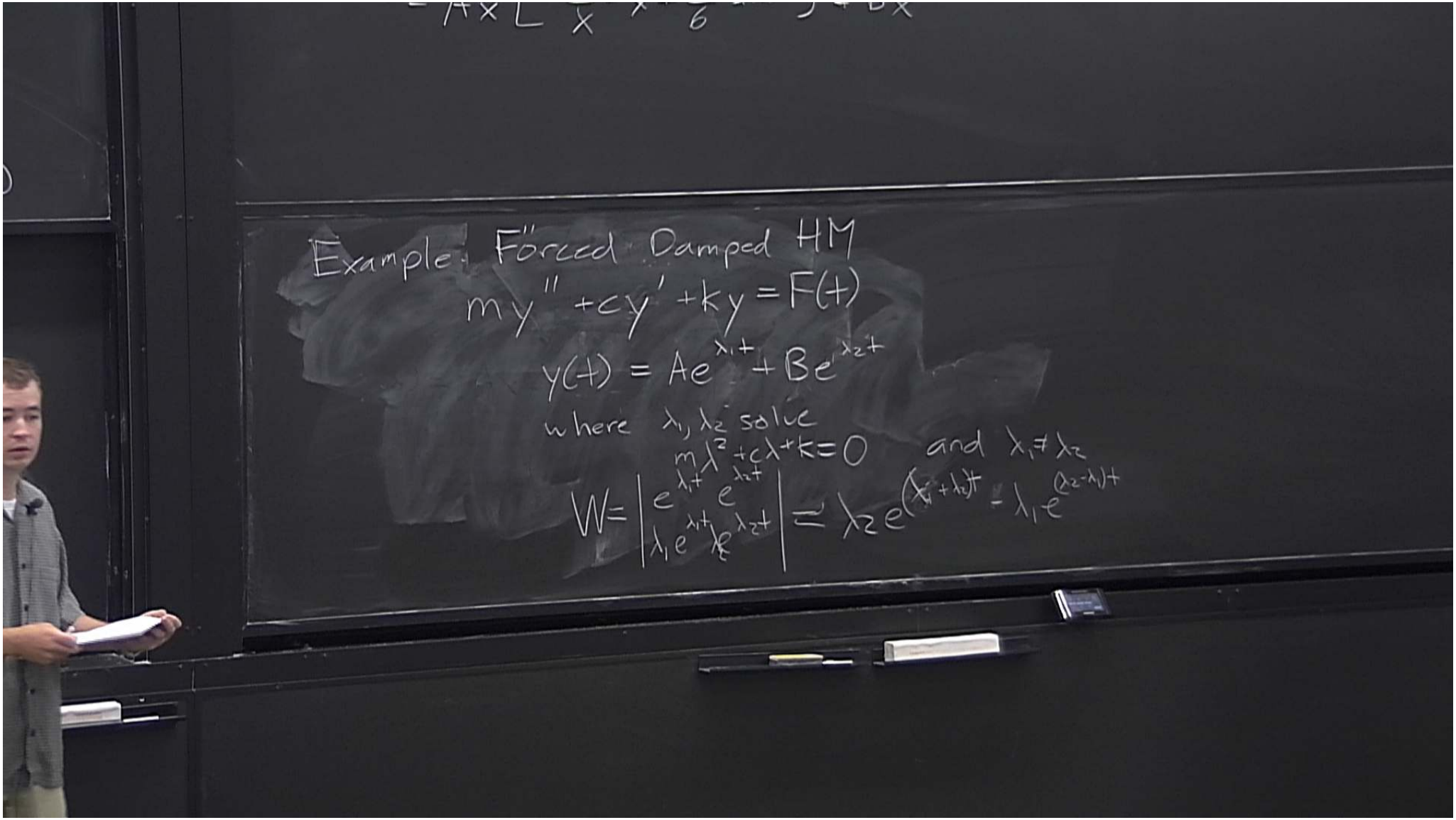
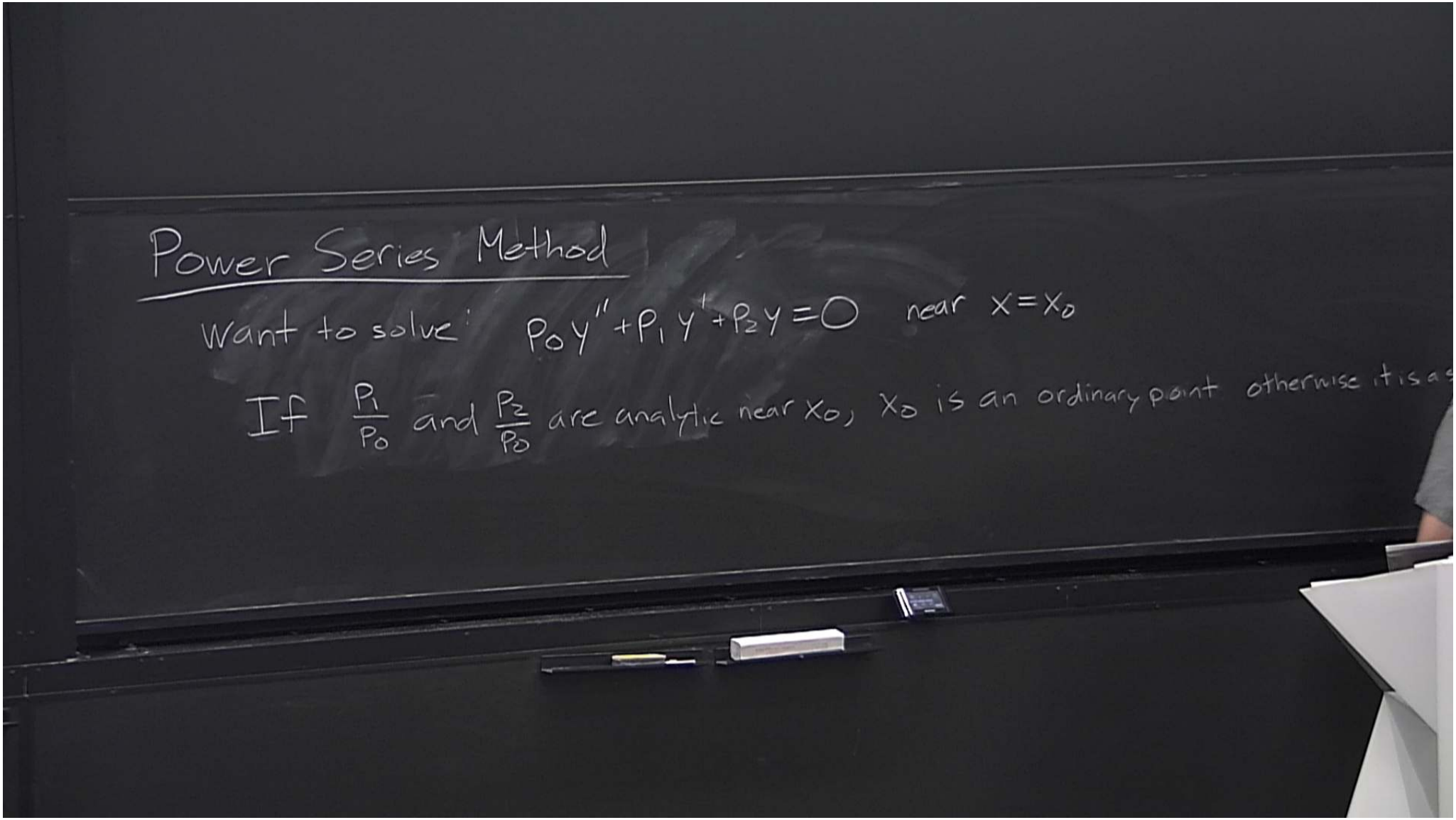
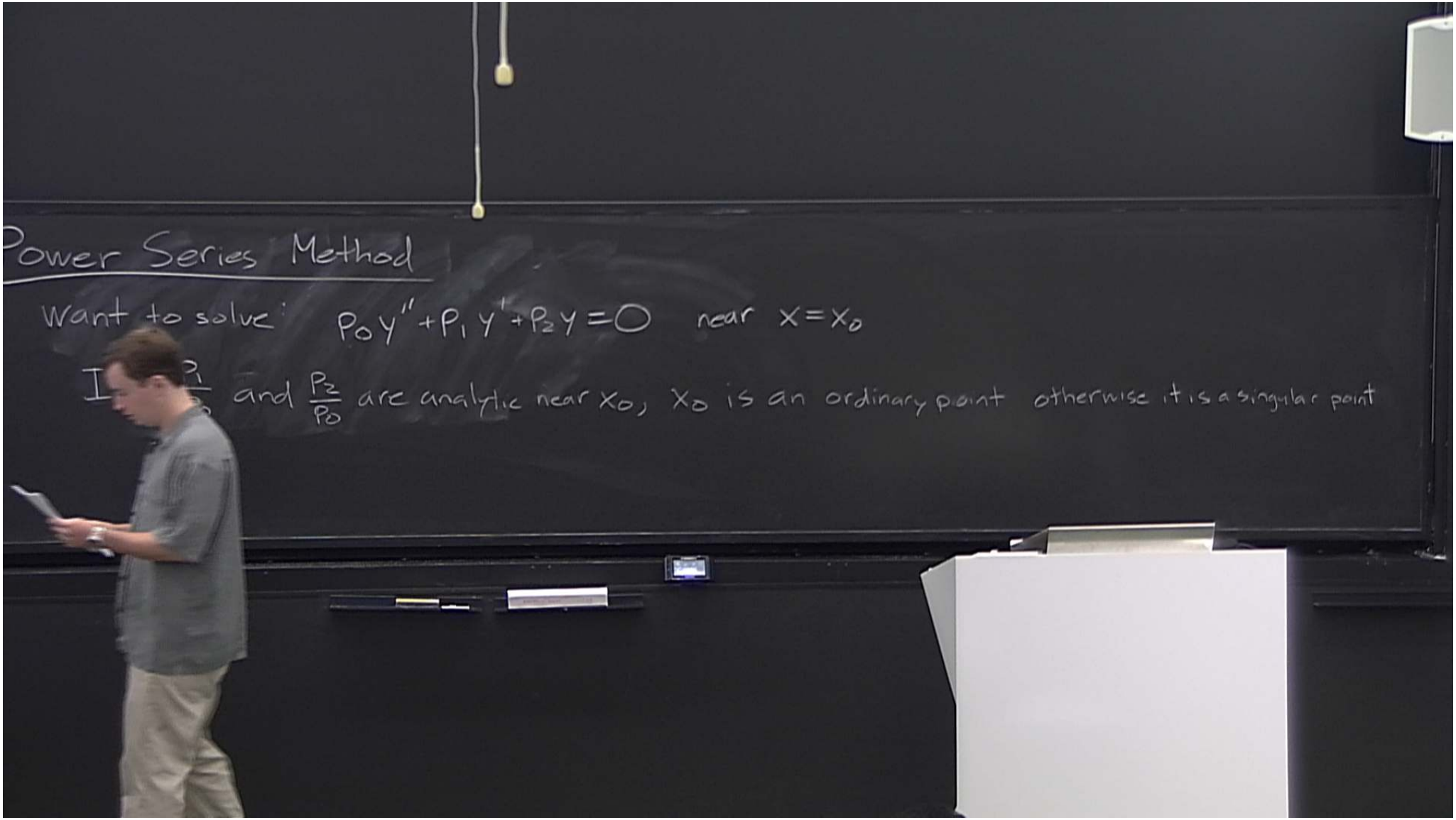
$v(u'' + a_1u' + a_0u) + v'(2u' + a_1u) + v''u = 0$
Let $z(x) = v'(x)$
$z'(x) + z(x)\left(\frac{2u' + a_1u}{u}\right) = 0$
$y'' = v_1y_1'' + v_2y_2'' + v_1'y_1' + v_2'y_2'$
Plug into our ODE
$v_1(y_1'' + a_1y_1' + a_0y_1) + v_2(y_2'' + a_1y_2' + a_0y_2) + v_1'y_1' + v_2'y_2' = f$
Technique for solving inhom
Assume y_1, y_2 solve
$y(x) = v_1(x)y_1(x) +$
$y' = v_1y_1' + v_1'y_1 +$


Example: Forced Damped HM
$my'' + cy' + ky = F(t)$
$y(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$
where λ_1, λ_2 solve
$m\lambda^2 + c\lambda + k = 0$ and $\lambda_1 \neq \lambda_2$
$W = \begin{vmatrix} e^{\lambda_1 t} & e^{\lambda_2 t} \\ \lambda_1 e^{\lambda_1 t} & \lambda_2 e^{\lambda_2 t} \end{vmatrix} = \lambda_2 e^{(\lambda_1+\lambda_2)t} - \lambda_1 e^{(\lambda_2-\lambda_1)t}$


Power Series Method
Want to solve: $P_0 y'' + P_1 y' + P_2 y = 0$ near $x = x_0$
If $\frac{P_1}{P_0}$ and $\frac{P_2}{P_0}$ are analytic near x_0, x_0 is an ordinary point otherwise it is a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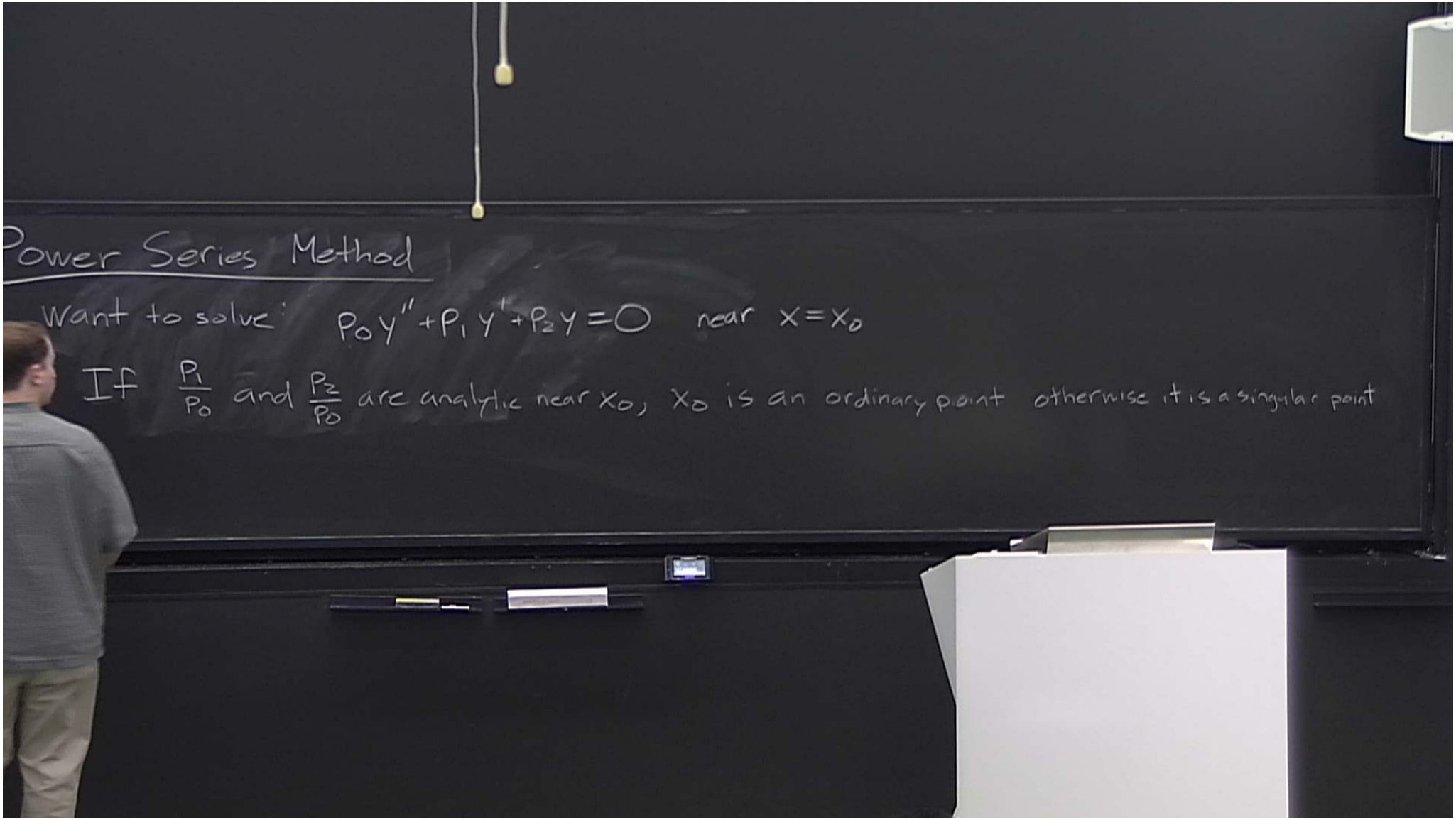
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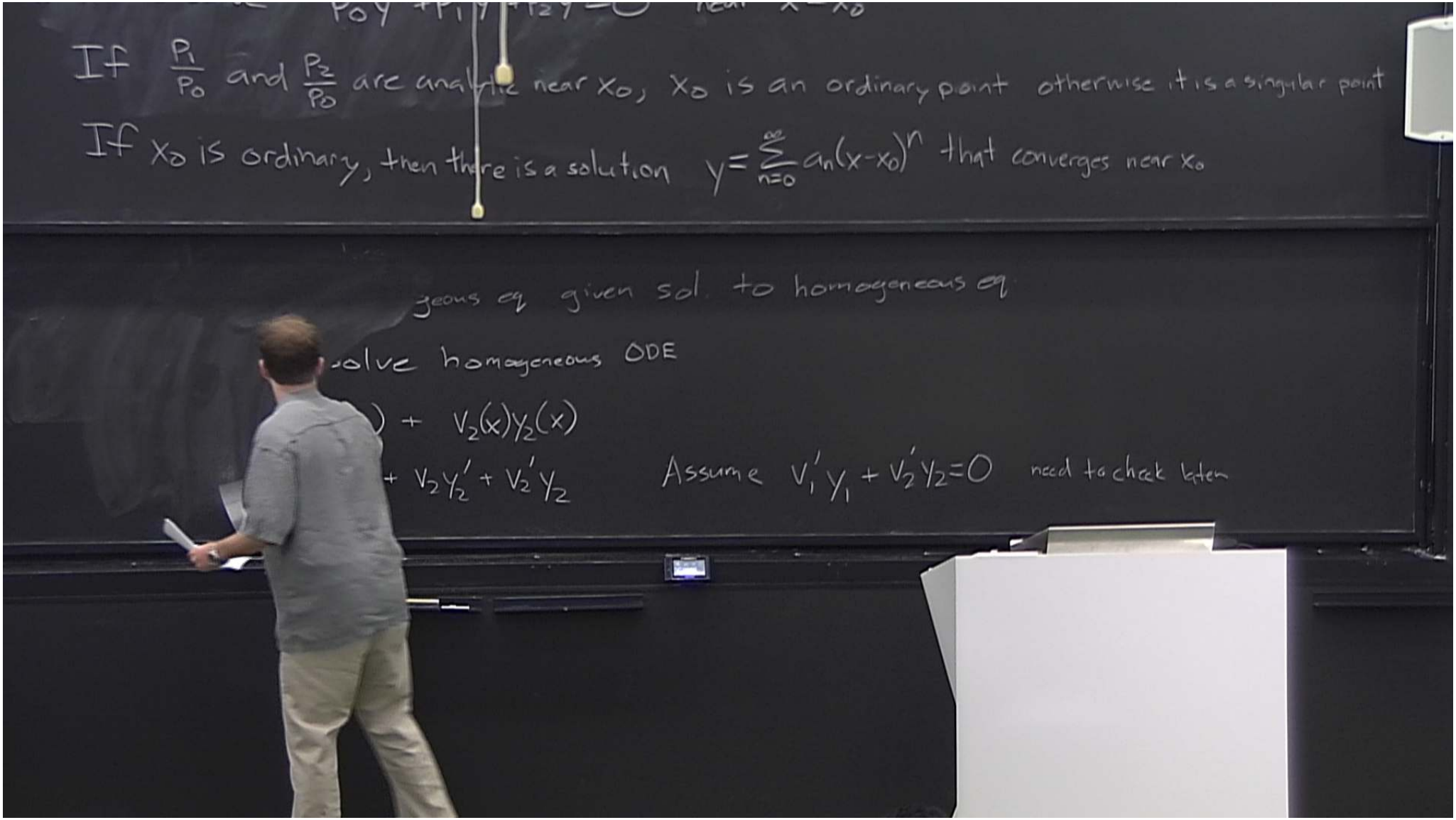
If $\frac{P_1}{P_0}$ and $\frac{P_2}{P_0}$ are analytic near x_0, x_0 is an ordinary point otherwise it is a singular point
If x_0 is ordinary, then there is a solution $y=\sum_{n=0}^{\infty} a_n(x-x_0)^n$ that converges near x_0
geous eq given sol. to homogeneous eq
olve homogeneous ODE
$+ V_2(x)y_2(x)$
$+ V_2y_2' + V_2'y_2$
Assume $V_1'y_1 + V_2'y_2=0$ need to check later

If $\frac{P_1}{P_0}$ and $\frac{P_2}{P_0}$ are analytic near x_0, x_0 is an ordinary point otherwise it is a

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Example: Airy equation $y'' - xy = 0$ around $x_0 = 0$

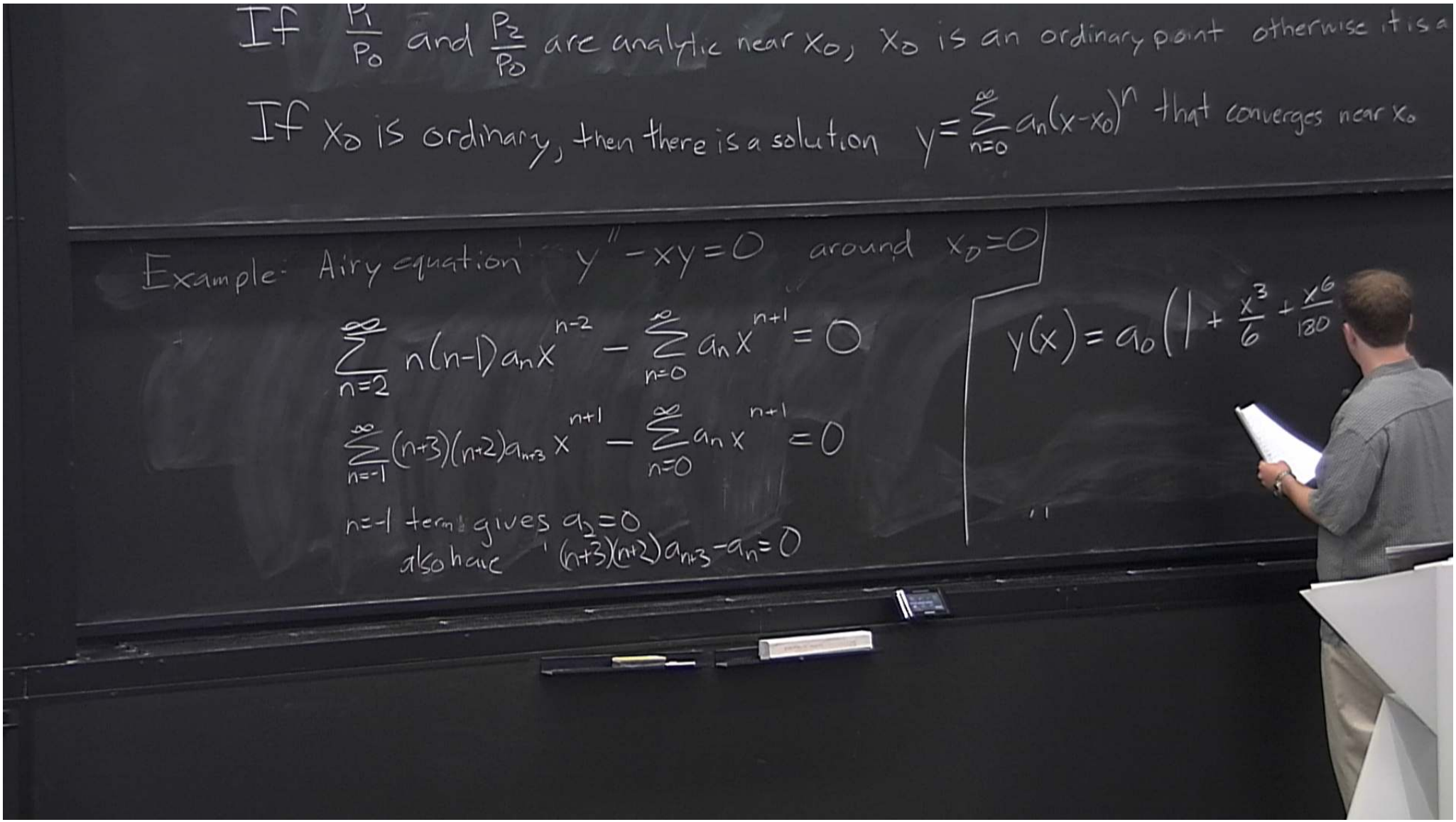
$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\sum_{n=-1}^{\infty} (n+3)(n+2) a_{n+3} x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$n=-1$ term gives $a_2 = 0$

also have $(n+3)(n+2) a_{n+3} - a_n = 0$

$$y(x) = a_0 \left(1 + \frac{x^3}{6} + \frac{x^6}{180} \right.$$



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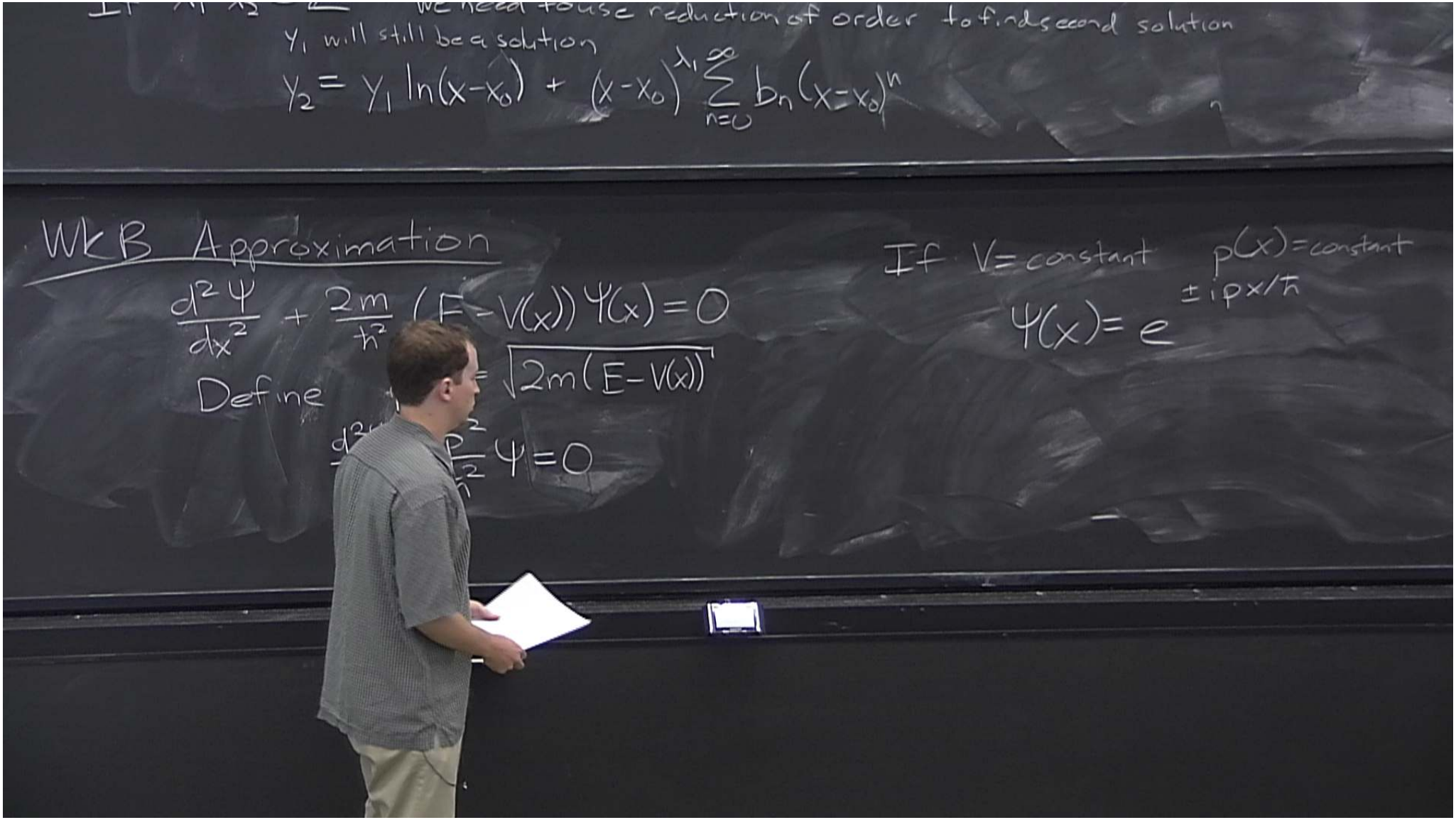
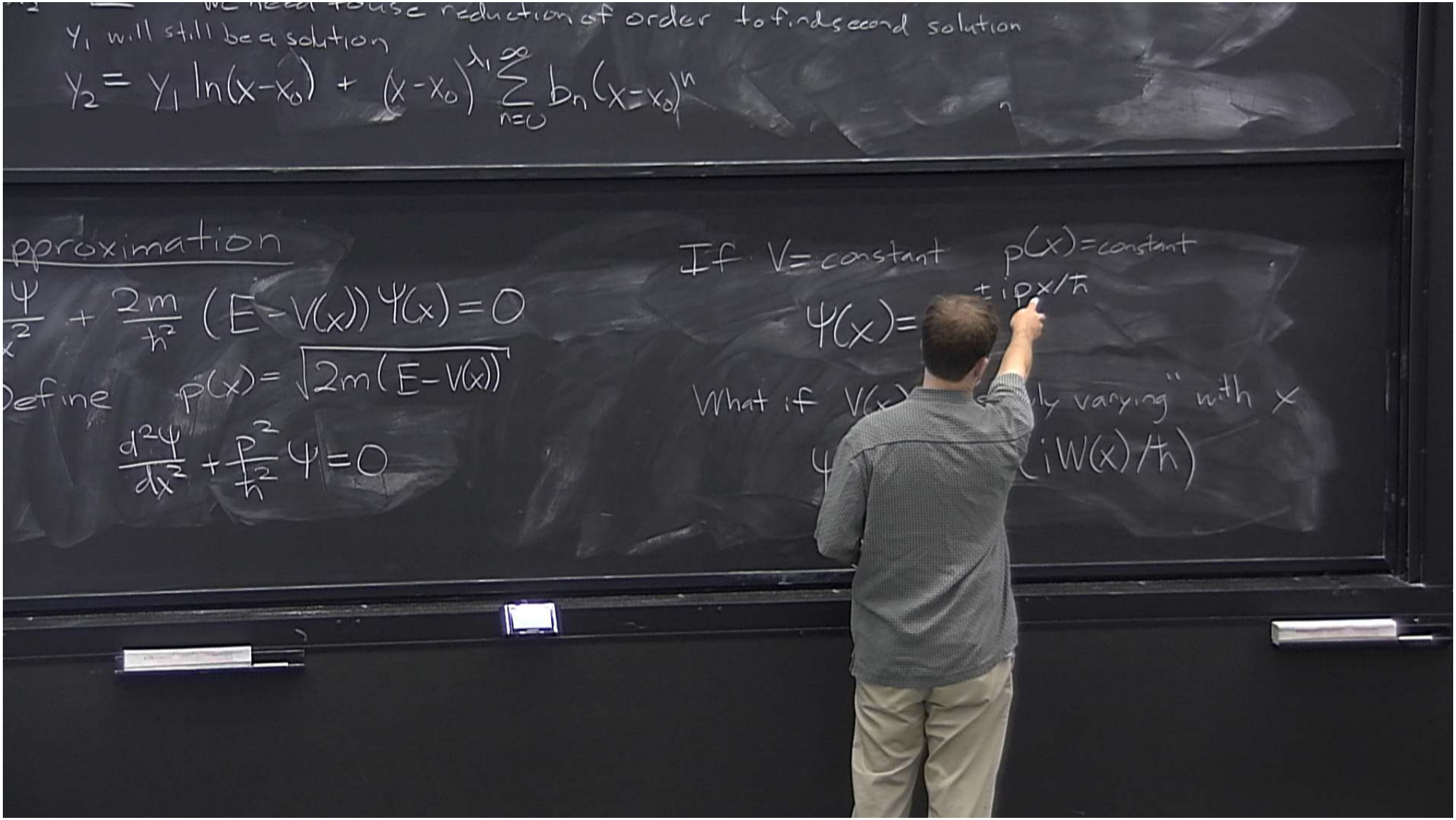
$n=-1$ term gives $a_2 = 0$

also have $(n+3)(n+2) a_{n+3} - a_n = 0$

$$y(x) = a_0 \left(1 + \frac{x^3}{6} + \frac{x^6}{180} + \cdots\right) + a_1 \left(x + \frac{x^4}{12} + \frac{x^7}{504} + \cdots\right)$$


we need to use reduction of order to find second solution
y_1 will still be a solution
$y_2 = y_1 \ln(x-x_0) + (x-x_0)^{\lambda_1} \sum_{n=0}^{\infty} b_n (x-x_0)^n$
WKB Approximation
$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E-V(x))\psi(x) = 0$
Define
$= \sqrt{2m(E-V(x))}$
$\frac{p^2}{\hbar^2}\psi = 0$
If V = constant p(x) = constant
$\psi(x) = e^{\pm ipx/\hbar}$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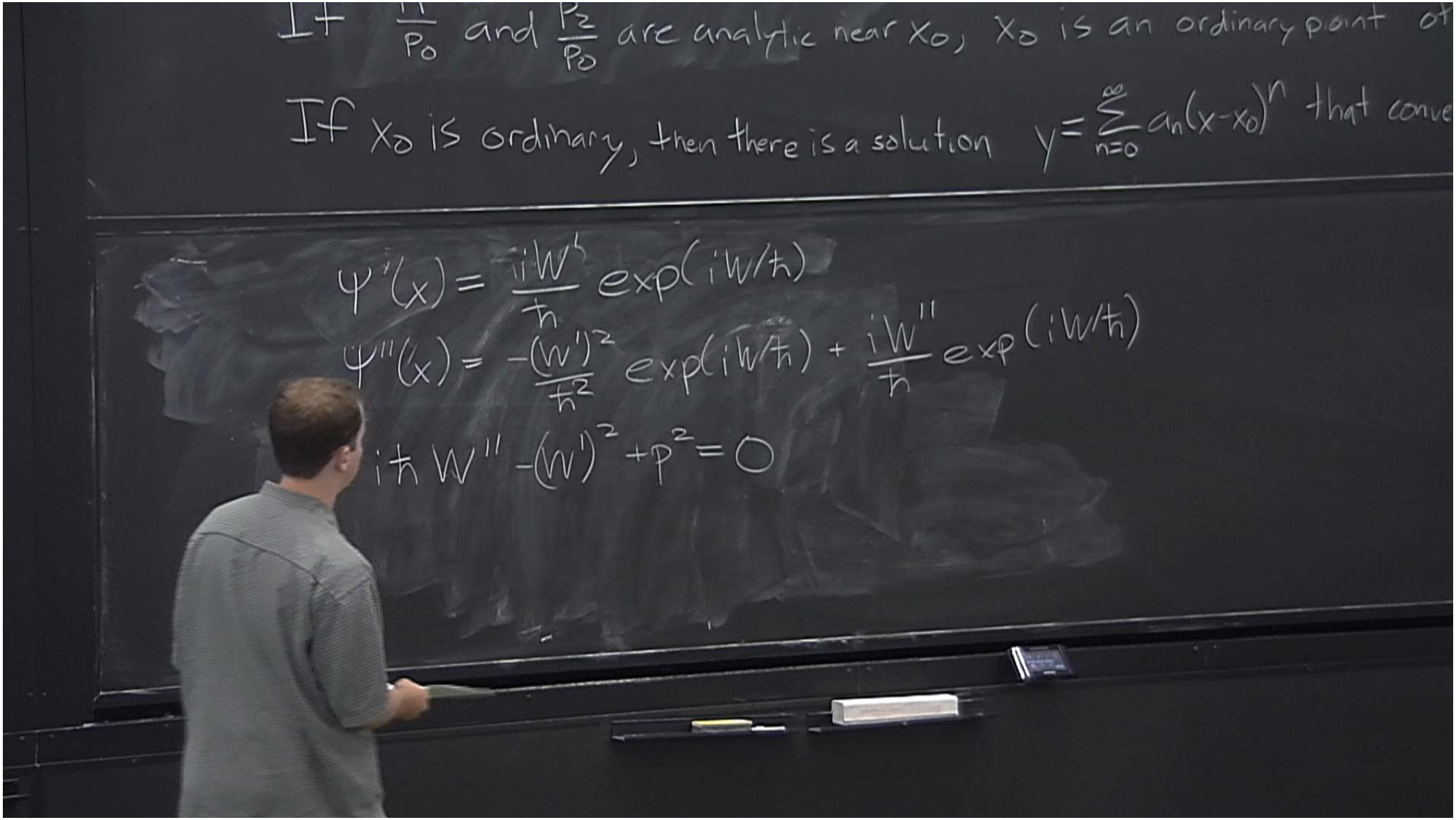
$y_2 = y_1 \ln(x-x_0) + (x-x_0)^{\lambda_1} \sum_{n=0}^{\infty} b_n (x-x_0)^n$
pproximation
$\frac{\psi}{x^2} + \frac{2m}{\hbar^2}(E-V(x))\psi(x) = 0$
Define $p(x) = \sqrt{2m(E-V(x))}$
$\frac{d^2\psi}{dx^2} + \frac{p^2}{\hbar^2}\psi = 0$
If V = constant $p(x)$ = constant
$\psi(x) = \pm ipx/\hbar$
What if $V(x)$ ly varying" with x
ψ (iW(x)/ħ)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If x_0 is ordinary, then there is a solution $y = \sum_{n=0}^{\infty} a_n (x-x_0)^n$ that conve
$\psi'(x) = \frac{iW'}{\hbar} \exp(iW/\hbar)$
$\psi''(x) = -\frac{(W')^2}{\hbar^2} \exp(iW/\hbar) + \frac{iW''}{\hbar} \exp(iW/\hbar)$
$i\hbar W'' - (W')^2 + p^2 = 0$

If $\frac{P_1}{P_0}$ and $\frac{P_2}{P_0}$ are analytic near x_0, x_0 is an ordinary point of

If x_0 is ordinary, then there is a solution $y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ that conve

$$\psi'(x) = \frac{iW'}{\hbar} \exp(iW/\hbar)$$

$$\psi''(x) = -\frac{(W')^2}{\hbar^2} \exp(iW/\hbar) + \frac{iW''}{\hbar} \exp(iW/\hbar)$$

$$i\hbar W'' - (W')^2 + p^2 = 0$$

If $\frac{P_1}{P_0}$ and $\frac{P_2}{P_0}$ are analytic near x_0, x_0 is an ordinary point of

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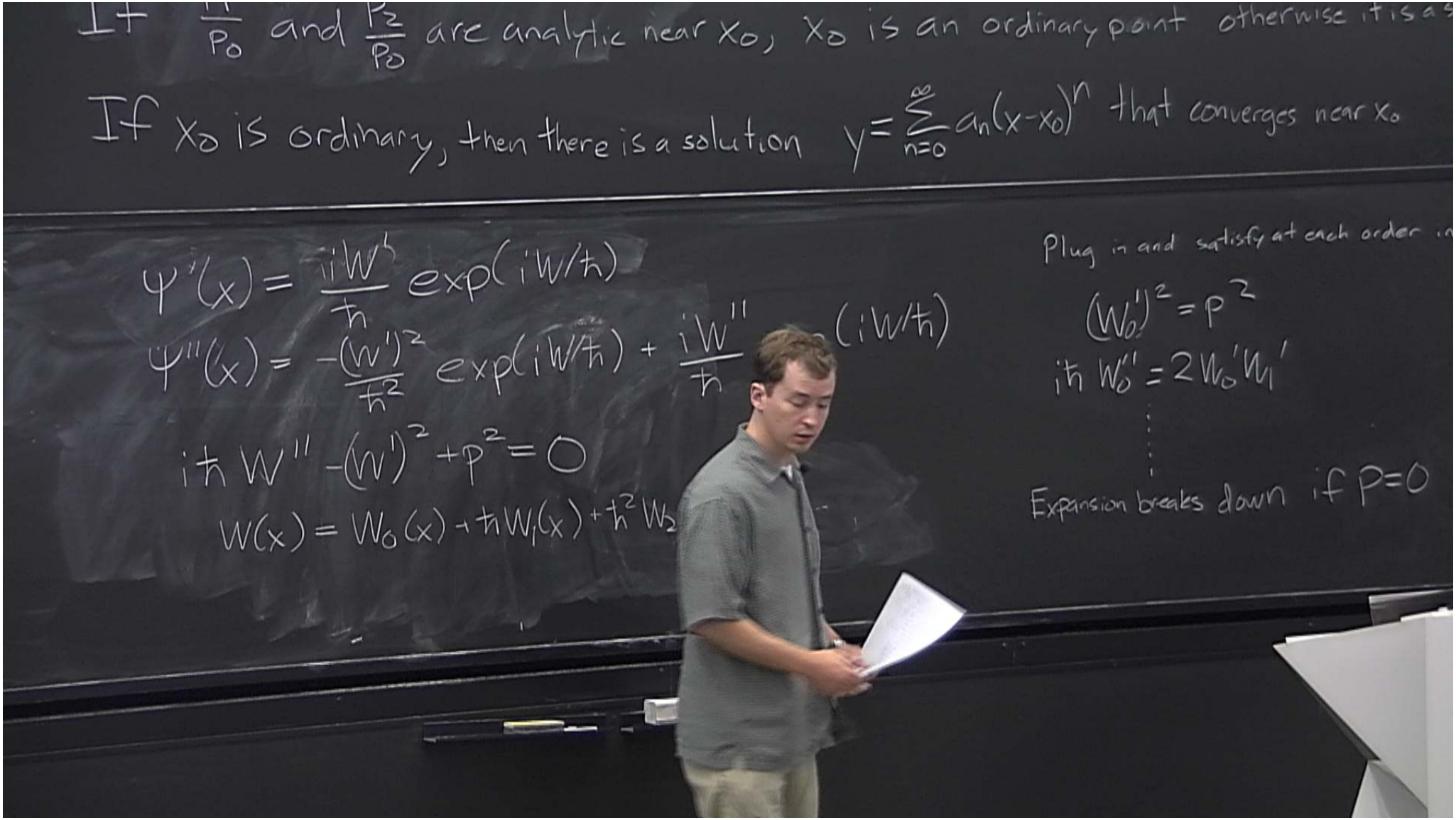
$$W(x) = W_0(x) + \hbar W_1(x) + \hbar^2 W_2(x) + \cdots$$

Plug in and satisf

$$(W')^2 = p^2$$

$$i\hbar W_0'' = 2$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If x_0 is ordinary, then there is a solution $y = \sum_{n=0}^{\infty} a_n (x-x_0)^n$ that converges near x_0
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$W(x) = W_0(x) + \hbar W_1(x) + \hbar^2 W_2$
Plug in and satisfy at each order in
$(W_0')^2 = p^2$
$i\hbar W_0'' = 2W_0'W_1'$
Expansion breaks down if $p=0$

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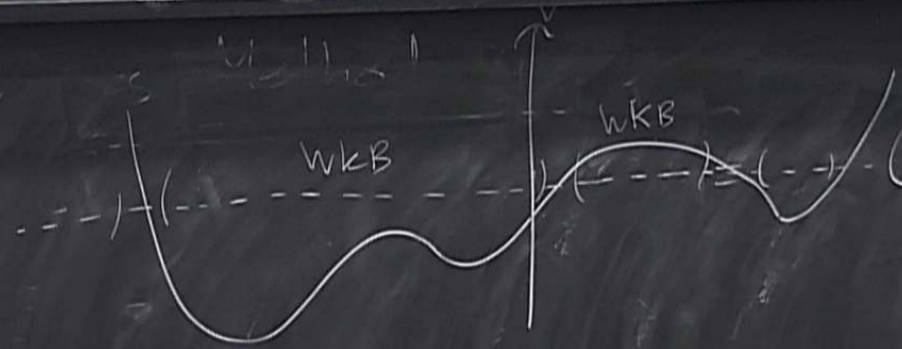

match solutions in classically allowed and classically forbidden region