

Title: Algebra - Lecture 4

Date: Aug 20, 2012 10:30 AM

URL: <http://pirsa.org/12080019>

Abstract:

Poincaré Group = Lorentz + Translations
10 param 6 parameters 4

$$x^\mu \rightarrow \Lambda^\mu{}_\nu x^\nu + c^\mu$$

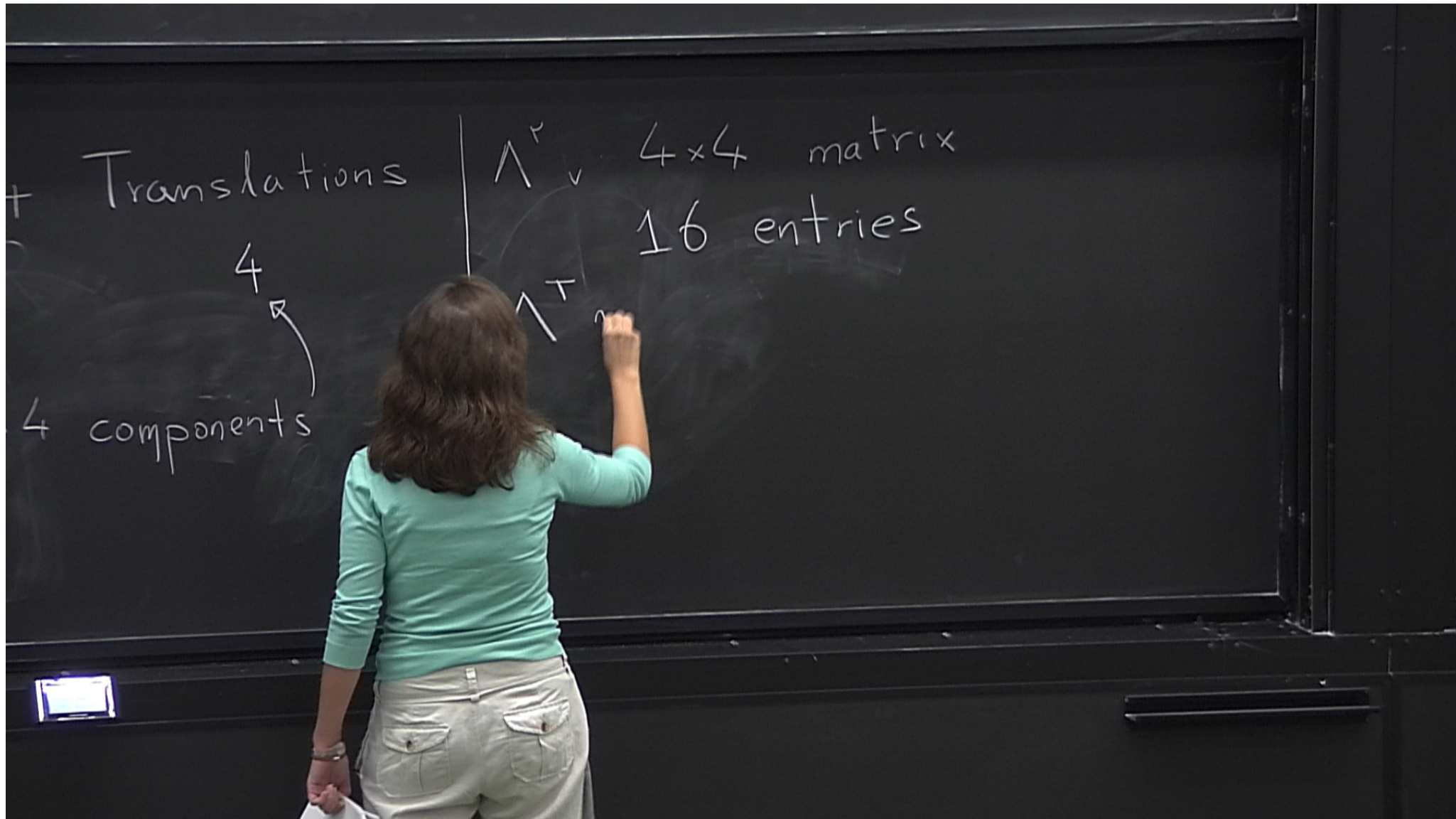
Poincaré Group = Lorentz + Translations

10 param

6 parameters

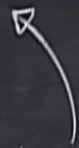
4

$$x^\mu \rightarrow \Lambda^\mu_\nu x^\nu + \alpha^\mu \rightarrow 4 \text{ components}$$



+ Translations

4



4 components

Λ^μ_ν 4x4 matrix

16 entries

$$\underbrace{\Lambda^\top \eta \Lambda}_{4 \times 4} = \eta$$

→ 4x4 matrix

z + Translations

4 components

Λ^μ_ν 4x4 matrix

16 entries

$$\underbrace{\Lambda^\mu_\nu \eta_{\mu\nu}}_{4 \times 4} = \eta_{\mu\nu}$$

4x4 matrix

z + Translations

4
→ 4 components

Λ^μ_ν 4x4 matrix

16 entries

$$\Lambda^\mu_\nu \eta_{\mu\nu} \Lambda^\alpha_\beta = \eta_{\alpha\beta}$$

4x4

→ 4x4 matrix

10 independent eqns

→ 6 parameters

Minkowski 4-d space-time

Tangent space at a point P : T_P

set of all vectors defined at P .

basis $\{\hat{e}_0, \hat{e}_1, \hat{e}_2, \hat{e}_3\} = \hat{e}_\mu$

$$V \in T_P: V = V^\mu \hat{e}_\mu$$

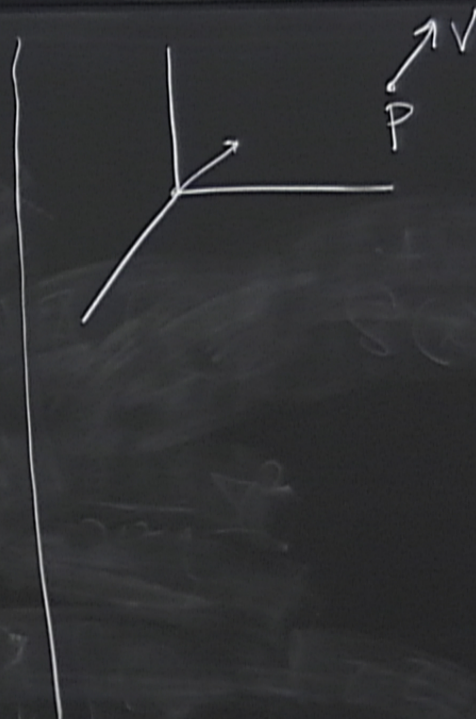
4-d space-time

space at a point P T_P

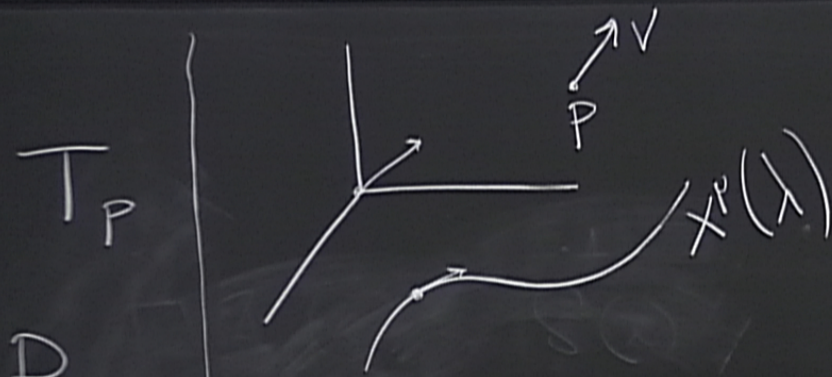
all vectors defined at P .

$$\{\hat{e}_0, \hat{e}_1, \hat{e}_2, \hat{e}_3\} = \hat{e}_\mu$$

$$V = V^\mu \hat{e}_\mu$$



→ 0 parameters

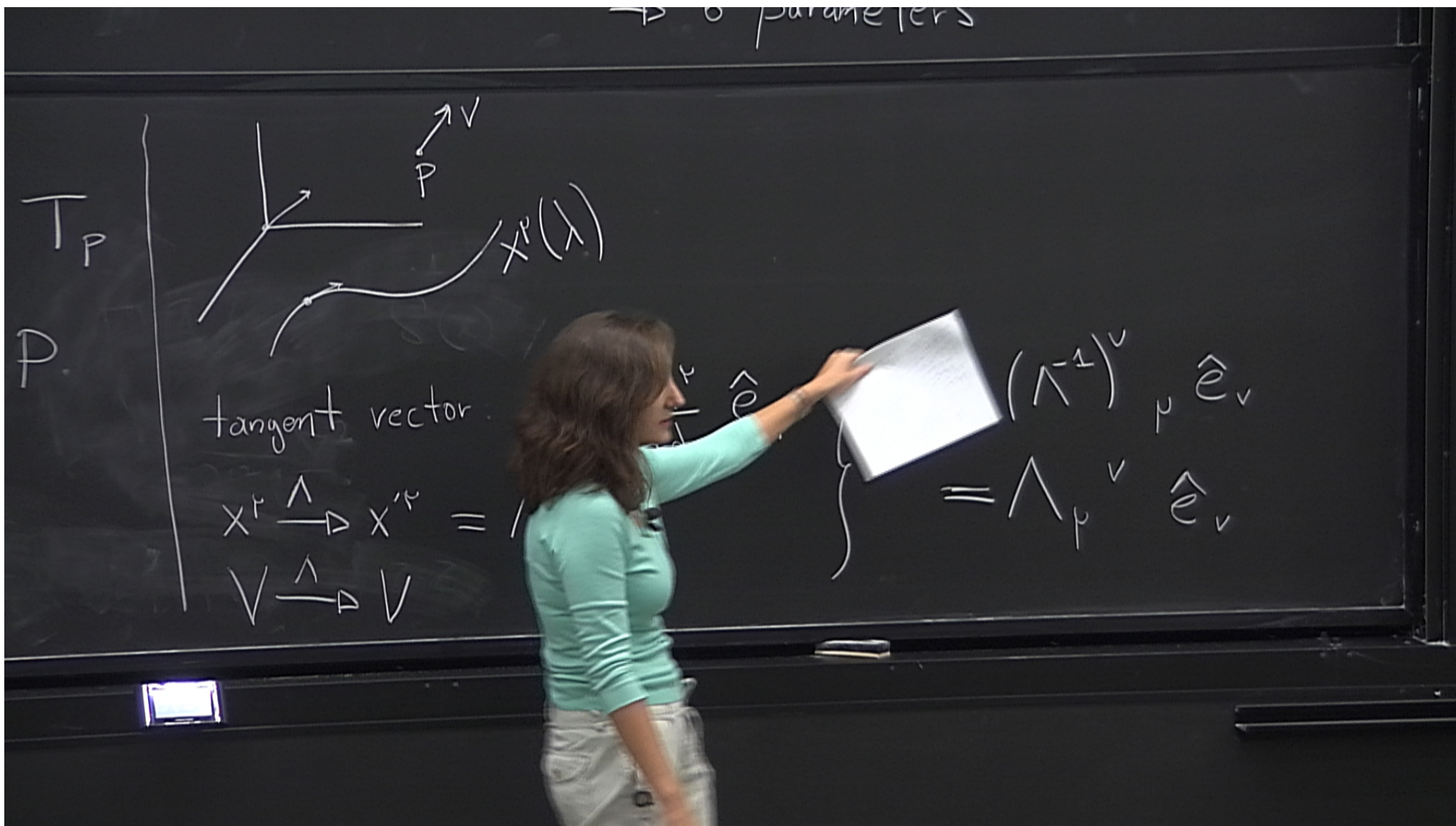


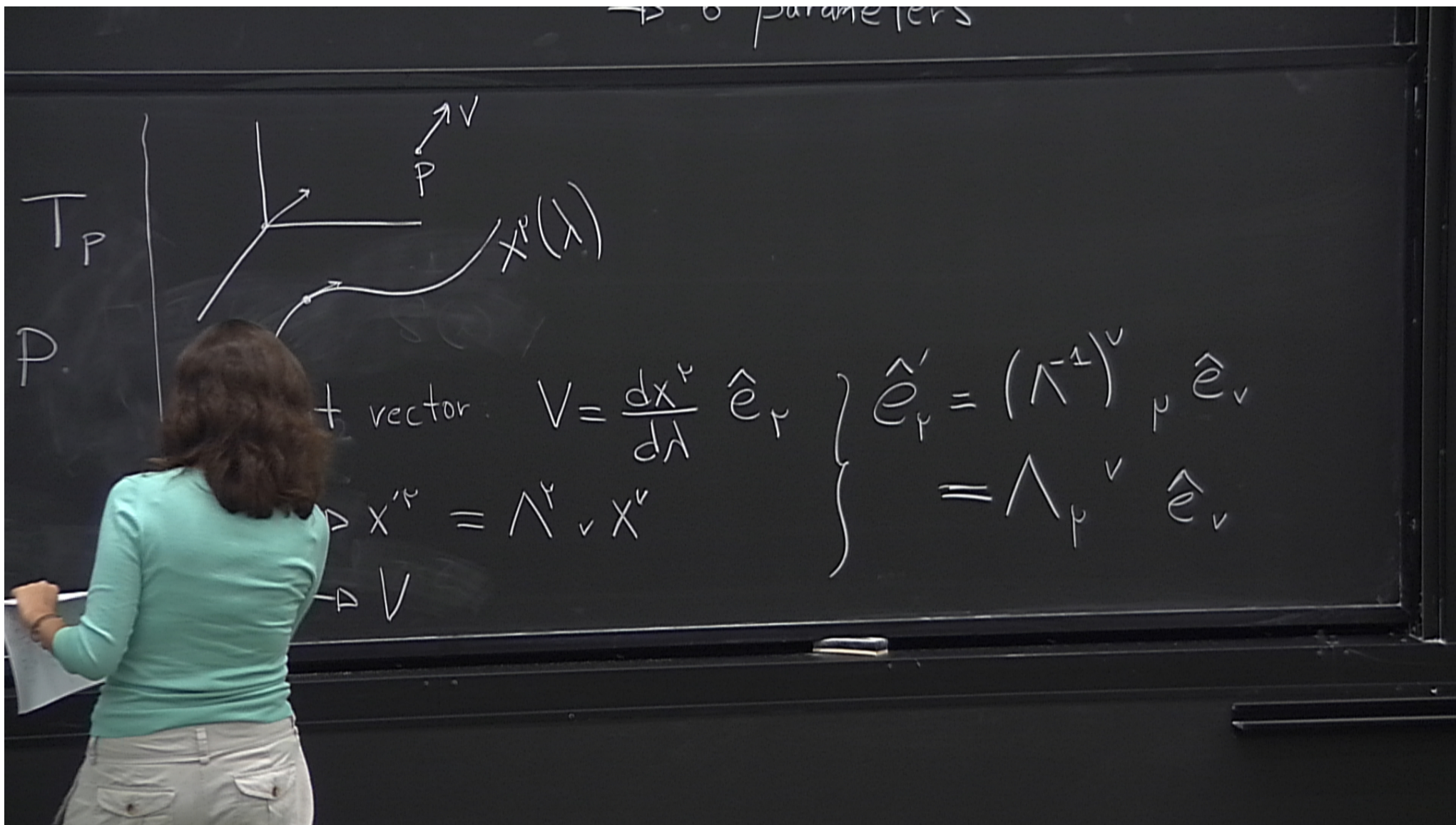
tangent vector $V = \frac{dx^r}{d\lambda} \hat{e}_r$

$$x^r \xrightarrow{\Lambda} x'^r = \Lambda^r_{\nu} x^{\nu}$$

$$V \xrightarrow{\Lambda} V$$

$$\left. \begin{array}{l} V = \frac{dx^r}{d\lambda} \hat{e}_r \\ x^r \xrightarrow{\Lambda} x'^r = \Lambda^r_{\nu} x^{\nu} \\ V \xrightarrow{\Lambda} V \end{array} \right\} \hat{e}'_r = (\Lambda^{-1})^{\nu}_r \hat{e}_{\nu}$$





Cotangent Space T_p^*
dual basis $\{\hat{\theta}^\mu\}$

$$\hat{\theta}^\mu(\hat{e}_\nu) = \delta^\mu_\nu \quad (T_p)$$

Tangent Space T_p^*
 dual basis $\{\hat{\theta}^\mu\}$ $\hat{\theta}^\mu(\hat{e}_\nu) = \delta^\mu_\nu$ $(T_p \xrightarrow{\hat{\theta}^\mu} \mathbb{R})$

$$V \in T_p: V = V^\mu \hat{e}_\mu$$

$$V \xrightarrow{\hat{\omega}} \mathbb{R}$$

Cotangent Space T_p^*

dual basis $\{\hat{\theta}^\mu\}$

$$\hat{\theta}^\mu(\hat{e}_\nu) = \delta_\nu^\mu$$

$$(T_p \xrightarrow{\hat{\theta}^\mu} \mathbb{R})$$

$$\omega = \omega_\mu \hat{\theta}^\mu \in T_p^*$$

$$\omega(V) = \omega_\mu \hat{\theta}^\mu(V^\nu \hat{e}_\nu) = \omega_\mu V^\nu \hat{\theta}^\mu(\hat{e}_\nu) = \omega_\mu V^\nu \delta_\nu^\mu = \omega_\mu V^\mu$$

$$\delta_\nu^r \quad (T_p \xrightarrow{\hat{\theta}^r} \mathbb{R})$$

$$\Theta^r(\hat{e}_\nu) = \omega_\mu V^\mu \delta_\nu^r = \omega_\mu V^\mu$$

$$\hat{\theta}'^r = \Lambda^\mu_\nu \hat{\theta}^\nu$$

$$\omega'_r = (\Lambda^{-1})^\mu_r \omega_\mu$$

Example of co-vector.
 $\varphi(x)$ scalar function

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$$d\varphi = \frac{\partial \varphi}{\partial x^\mu} \hat{\theta}^\mu \quad (\hat{\theta}^\mu = dx^\mu)$$

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$$x^\mu \xrightarrow{\Lambda} \Lambda^\mu{}_\nu x^\nu$$

$$\frac{\partial \varphi}{\partial x^\mu} \rightarrow \frac{\partial \varphi}{\partial x^\nu}$$

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$$\frac{\partial \varphi}{\partial x^\mu} \xrightarrow{\Lambda} \frac{\partial \varphi}{\partial x'^\mu} = \frac{\partial \varphi}{\partial x^\nu} \frac{\partial x^\nu}{\partial x'^\mu}$$

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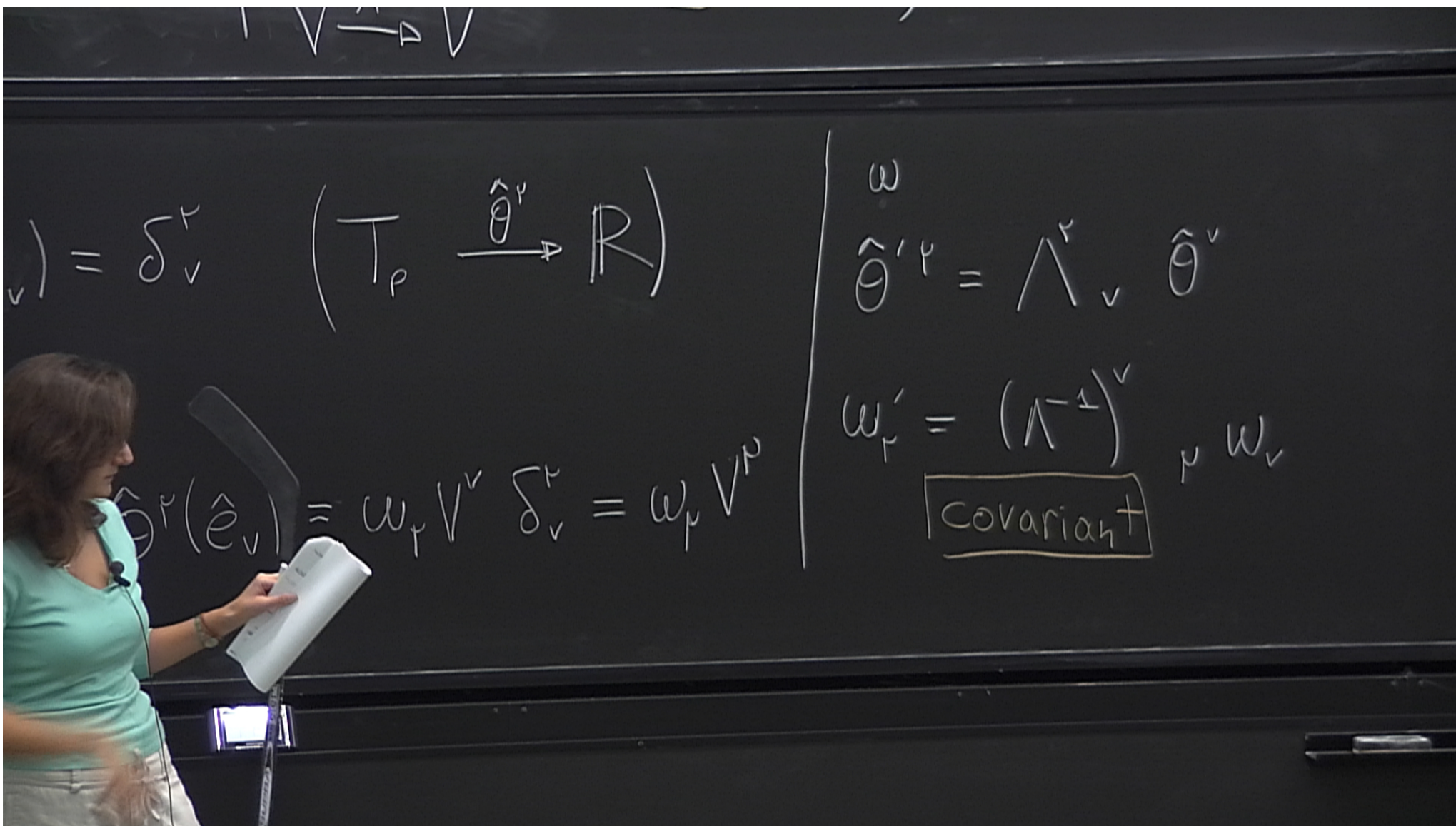
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Tensors of rank (k, l)

$$\text{map: } \underbrace{T_p^* \otimes \dots \otimes T_p^*}_k \otimes \underbrace{T_p \otimes \dots \otimes T_p}_l \rightarrow \mathbb{R}$$

Tensor of rank (k, l)

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basis:

Tensor of rank (k, l)

$$\text{map: } \underbrace{T_p^* \otimes \dots \otimes T_p^*}_k \otimes \underbrace{T_p \otimes \dots \otimes T_p}_l \rightarrow \mathbb{R}$$

$$\text{basis: } \hat{e}_{\mu_1} \otimes \dots \otimes \hat{e}_{\mu_k} \otimes \hat{\theta}^{\nu_1} \otimes \dots \otimes \hat{\theta}^{\nu_l}$$

T

$$\frac{\partial}{\partial x^i} \rightarrow \partial x^i \quad \partial x^i \partial x^j \rightarrow \partial x^i \partial x^j$$

Tensor of rank (k, l)

map: $\underbrace{T_p^* \otimes \dots \otimes T_p^*}_k \otimes \underbrace{T_p \otimes \dots \otimes T_p}_l \rightarrow \mathbb{R}$

basis: $\hat{e}_{\mu_1} \otimes \dots \otimes \hat{e}_{\mu_k} \otimes \hat{\theta}^{\nu_1} \otimes \dots \otimes \hat{\theta}^{\nu_l}$

$$\begin{matrix} T & \mu_1 \dots \mu_k \\ & \nu_1 \dots \nu_l \end{matrix}$$

$$\begin{array}{c}
 \text{rank } (k, l) : \underbrace{T_p \otimes \dots \otimes T_p}_k \otimes \underbrace{T_p^* \otimes \dots \otimes T_p^*}_l \mid T^{\alpha} \text{ } \delta \epsilon \\
 \\
 \underbrace{\otimes \dots T_p^*}_k \otimes \underbrace{T_p \otimes \dots \otimes T_p}_l \rightarrow R \\
 \\
 \left| \begin{array}{c} T^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_l} = \Lambda^{\mu_1}_{\rho_1} \dots \Lambda^{\mu_k}_{\rho_k} T^{\rho_1 \dots \rho_k}_{\sigma_1 \dots \sigma_l} \\
 \hat{e}_{\mu_k} \otimes \hat{\theta}^{\nu_1} \otimes \dots \otimes \hat{\theta}^{\nu_l} \end{array} \right. \begin{array}{c} (\Lambda^{-1})^{\sigma_1}_{\nu_1} \dots (\Lambda^{-1})^{\sigma_l}_{\nu_l} \end{array}
 \end{array}$$

$$\begin{array}{c}
 \text{--- } \bigotimes T_p \bigotimes \underbrace{T_p^* \bigotimes \dots \bigotimes T_p^*}_l \text{ ---} \\
 \hline
 \bigotimes T_p \longrightarrow \mathbb{R} \\
 \hline
 \begin{array}{c}
 T_{\substack{\mu_1 \dots \mu_k \\ \nu_1 \dots \nu_l}} = \bigwedge_{\rho_1 \dots \rho_k}^{\mu_1 \dots \mu_k} \bigwedge_{\sigma_1 \dots \sigma_l}^{\nu_1 \dots \nu_l} T_{\substack{\rho_1 \dots \rho_k \\ \sigma_1 \dots \sigma_l}} \\
 \qquad \qquad \qquad \underbrace{\qquad \qquad \qquad}_{\substack{(\bigwedge^{-1})^{\sigma_1}_{\nu_1} \dots (\bigwedge^{-1})^{\sigma_l}_{\nu_l}}}
 \end{array}
 \end{array}$$

$v_1 \dots v_l$

$\eta_{\mu\nu}$: rank $(0, 2)$

$$\eta(V, U) = \eta_{\mu\nu} V^\mu U^\nu = V_\mu U^\mu$$

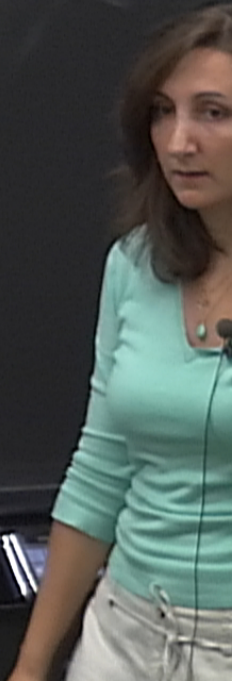
$$V, U \in T_p$$

$V_1 \dots V_l$

$\eta_{\mu\nu} : \text{rank } (0, 2)$

$$\eta(V, U) = \eta_{\mu\nu} V^\mu U^\nu = V_\mu U^\mu$$

$V, U \in T_p$



$$V \in l_p: V = V^r \hat{e}_r$$

Isomorphism between A , B

$$V \in l_p: V = V^r \hat{e}_r$$

Isomorphism between A, B

$$A_i \in A$$

$$\varphi: A_i \longrightarrow \varphi(A_i) \in B$$

$$V \in l_p: V = V^r \hat{e}_r$$

$$V \xrightarrow{\Lambda} V$$

Isomorphism between A, B

$$A \in A$$

$$\varphi: A \longrightarrow \varphi(A) \in B$$

one-to-one

$$A, A' \in A \longrightarrow \varphi(A), \varphi(A') \in B$$

$$V \xrightarrow{\Delta} V$$

$$\varphi(A) * \varphi(A') = \varphi(A * A')$$

$$\varphi(A) * \varphi(k) = \varphi(kA)$$

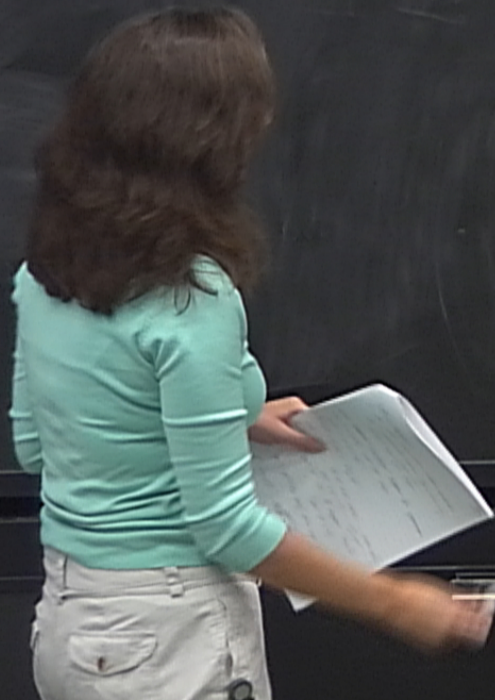
$\downarrow$
 number

$$k * A = kA$$

e-to-one

Clifford Algebra

V : real vector space & orthonormal basis $\{e_i, i=1, \dots, n\}$



Clifford Algebra

V : real vector space & "orthonormal" basis $\{e_i, i=1, \dots, n\}$
 $(e_i, e_j) = \begin{cases} \pm 1 & i=j \\ 0 & i \neq j \end{cases}$

Clifford Algebra

V : real vector space & "orthonormal" basis $\{e_i, i=1, \dots, n\}$
 $(e_i, e_j) = \begin{cases} \pm 1 & i=j \\ 0 & i \neq j \end{cases} = g_{ij}$

associative, generated by $\{1, e_i\}$

$$\begin{array}{l|l} 1 * e_i = e_i & e_i e_j + e_j e_i = 2 g_{ij} \\ e_i * 1 = e_i & \end{array}$$

Clifford Algebra

V : real vector space & "orthonormal" basis $\{e_i, i=1, \dots\}$
 $(e_i, e_j) = \begin{cases} \pm 1 & i=j \\ 0 & i \neq j \end{cases} = g_{ij}$

→ associative, generated by $\{1, e_i\}$
 $1 * e_i = e_i$
 $e_i * 1 = e_i$ | $e_i * e_j + e_j * e_i = 2 g_{ij}$

$$\begin{array}{l}
 \text{basis } \{e_i, i=1, \dots, n\} \\
 \begin{array}{l}
 i=j \\
 i \neq j
 \end{array} \left. \vphantom{\begin{array}{l} i=j \\ i \neq j \end{array}} \right\} = g_{ij} \\
 , e_i \}
 \end{array}
 \quad \Bigg| \quad
 \begin{array}{l}
 \text{basis of algebra:} \\
 \{1, e_i, e_i * e_j, e_i * e_j * e_k, \dots, \\
 e_{i_1} * e_{i_2} * \dots * e_{i_n}\}
 \end{array}$$

$$\underline{\text{Ex}}: n=1, g_{11}=-1$$

$$1 * e_1 = e_1 = e_1 * 1$$

$$e_1 * e_1 = -1$$

$$\{1, e_1\}$$

$$\dim(\mathbb{C}) = 2$$

$V_1 \dots V_\ell$

$n=4$

$\rightarrow$ spin $\frac{1}{2}$ - particle

$$\gamma^\mu \frac{\partial}{\partial x^\mu} |\psi\rangle = |\psi\rangle$$

$V_1 \dots V_\ell$

$n=4$

$\rightarrow$ spin $\frac{1}{2}$ -particle

Combine Dirac

$$\gamma^\mu \frac{\partial}{\partial x^\mu} |\psi\rangle = |\psi\rangle \quad \underline{\text{Dirac}}$$

$\downarrow$
operators ^{μ} , $\mu=0,1,2,3$

-particle

rac

Combine Dirac + Klein-Gordon eqns

$$\eta^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} |\psi\rangle = m^2 |\psi\rangle$$

-particle

rac

Combine Dirac + Klein-Gordon eqns

$$\eta^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} |\psi\rangle = m^2 |\psi\rangle$$

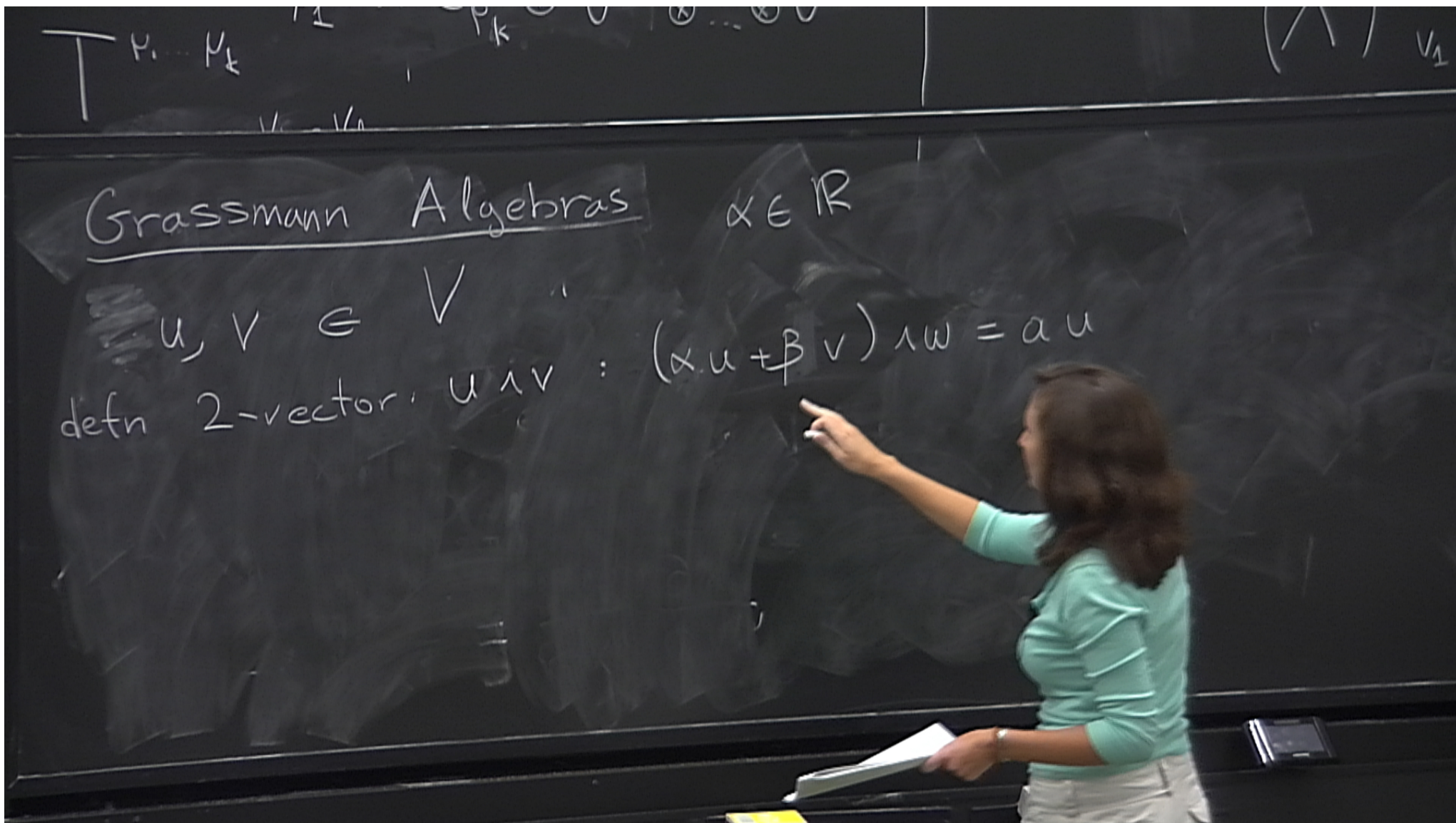
$$\boxed{\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 \eta^{\mu\nu}}$$

$$\begin{array}{l|l}
 \text{basis } \{e_i, i=1, \dots, n\} & \text{basis of algebra:} \\
 \begin{array}{l}
 i=j \\
 i \neq j
 \end{array} \left. \vphantom{\begin{array}{l} i=j \\ i \neq j \end{array}} \right\} = g_{ij} & \left\{ 1 + n + \binom{n}{2} + \binom{n}{3} \right. \\
 \{e_i\} & \left. \{1, e_i, e_i * e_j, e_i * e_j * e_k, \dots, \right. \\
 & \left. e_{i_1} * e_{i_2} * \dots * e_{i_n}\} \right. \\
 & + \binom{n}{n} \\
 & 1 = 2^n
 \end{array}$$

$$g_{ij} = 1$$

$$e_i * e_i + e_i * e_i = 2g_{ii} = \pm 2$$

$$e_i * e_i = \pm 1$$



Grassmann Algebras $\alpha \in \mathbb{R}$

$$u, v \in V$$

defn 2-vector: $u \wedge v : (\alpha u + \beta v) \wedge w = \alpha u \wedge w + \beta v \wedge w$

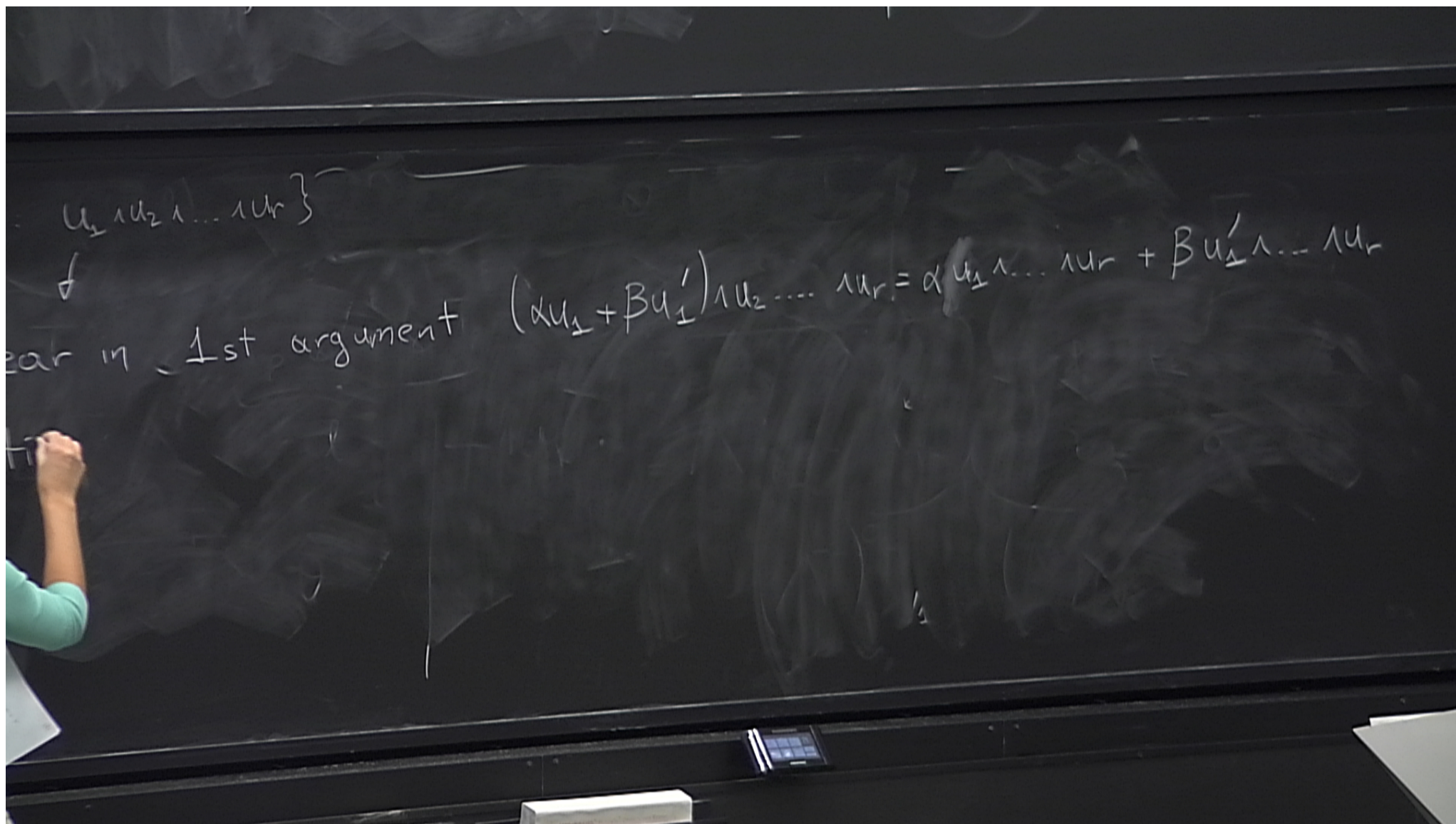
$$u \wedge v = -v \wedge u$$

$$\wedge^2(V) = \{2\text{-vectors}\}$$

vectors: $u_1, u_2, \dots, u_r$

↓

- linear in 1st argument $(\alpha u_1 + \beta u_1') u_2 \dots u_r = \alpha u_1 u_2 \dots u_r + \beta u_1' u_2 \dots u_r$



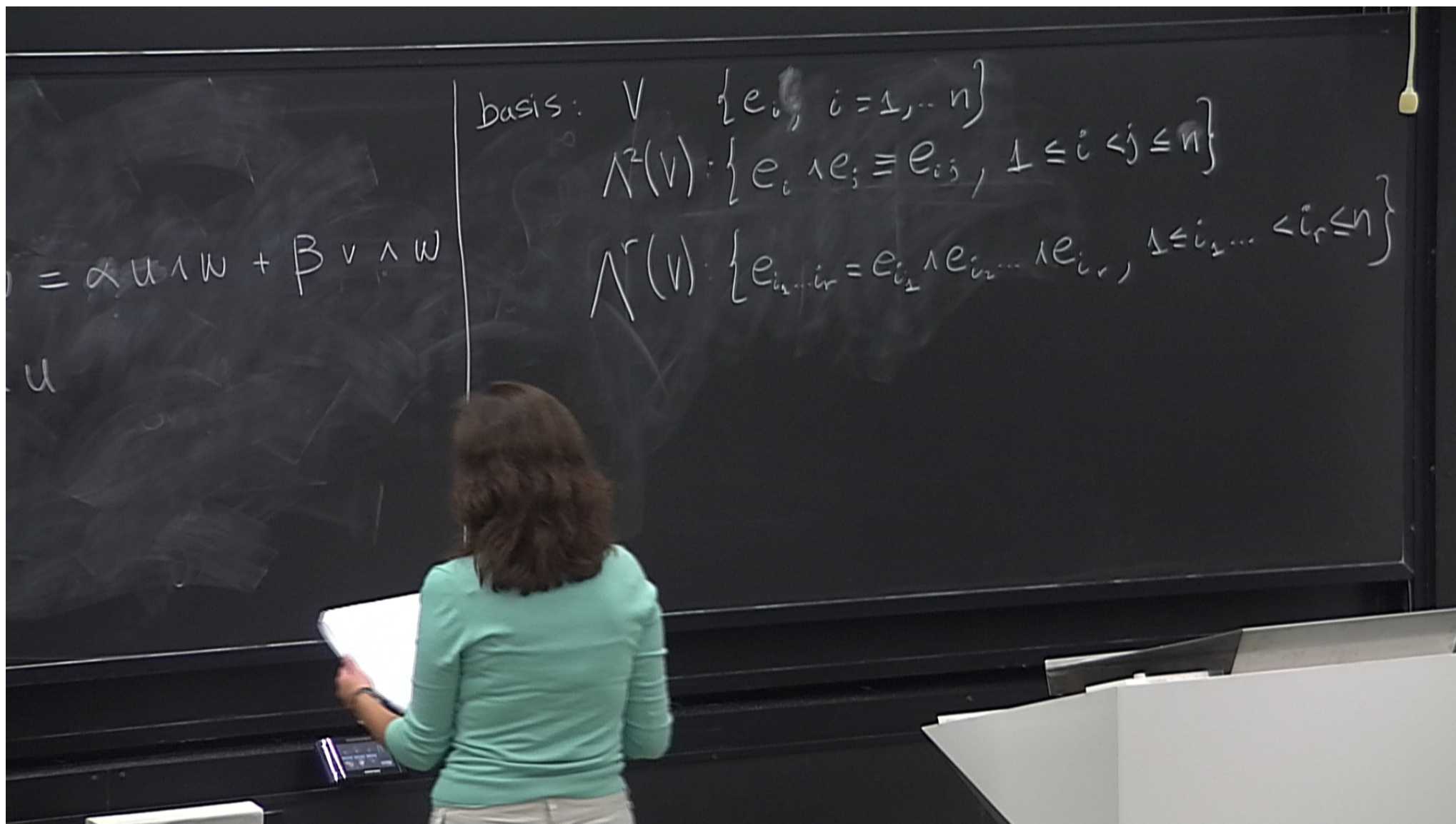
r-vectors: $u_1 \wedge u_2 \wedge \dots \wedge u_r$

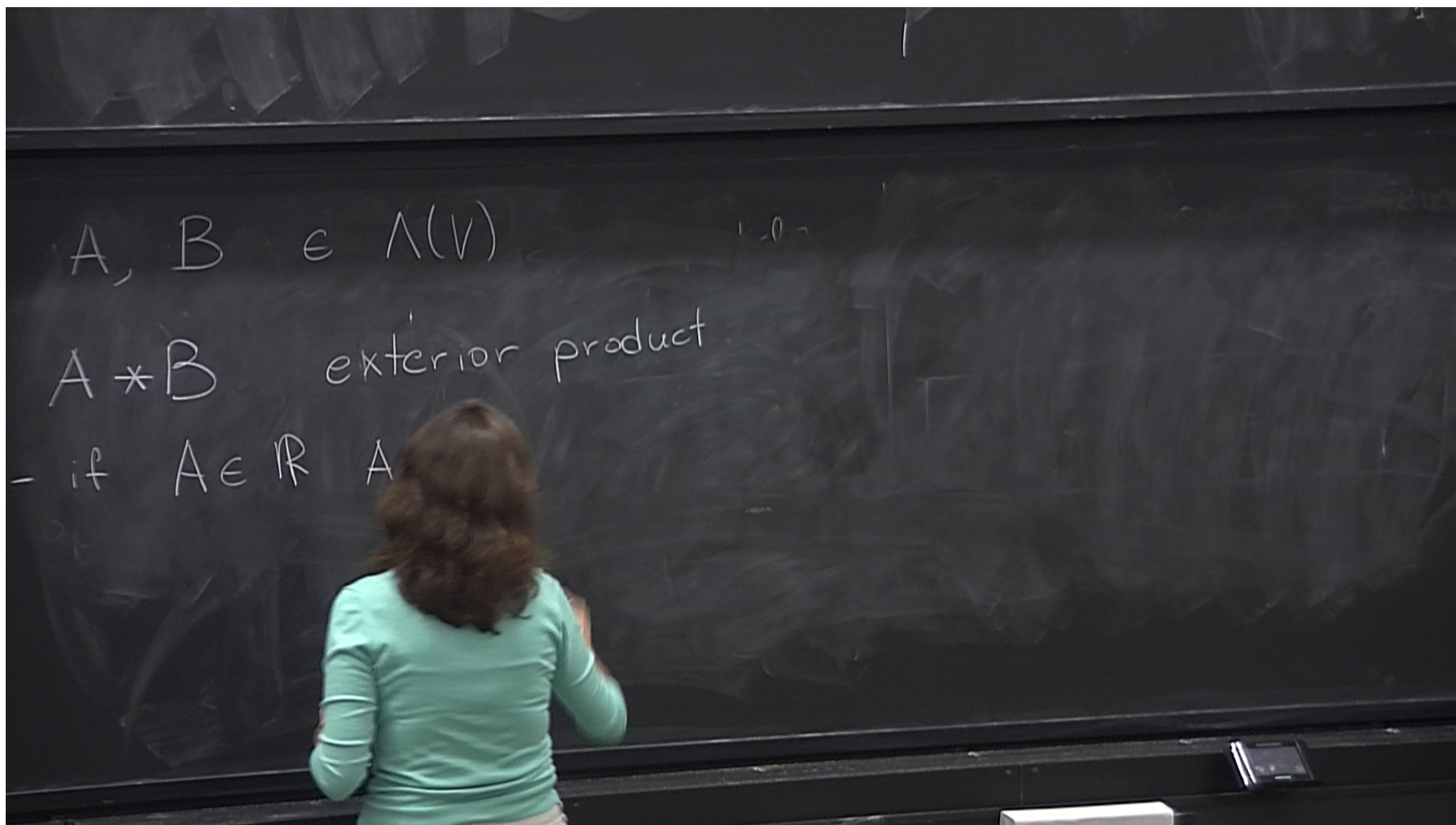
- linear in 1st argument $(\alpha u_1 + \beta u_1') \wedge u_2 \wedge \dots \wedge u_r = \alpha u_1 \wedge \dots \wedge u_r + \beta u_1' \wedge \dots \wedge u_r$
- anti-symmetric in any pair: $u_1 \wedge \dots \wedge u_{i-1} \wedge u_i \wedge u_i \wedge \dots \wedge u_r = 0$

$$u_1, u_2, \dots, u_r \}$$

or in 1st argument $(\alpha u_1 + \beta u_1') u_2 \dots u_r = \alpha u_1 u_2 \dots u_r + \beta u_1' u_2 \dots u_r$

-symmetric in any pair : $u_1 \wedge \dots \wedge u_{i-1} \wedge u_i \wedge u_{i+1} \wedge \dots \wedge u_r = -u_1 \wedge \dots \wedge u_i \wedge u_{i-1} \wedge \dots \wedge u_r$





$$A, B \in \Lambda(V)$$

$A * B$ exterior product

- if $A \in \mathbb{R}$ A

$$A, B \in \Lambda(V)$$

$A * B$ exterior product

- if $A \in \mathbb{R}$ $A * B = B * A = AB$

$$A, B \in \Lambda(V)$$

- linear in both

$A * B$ exterior product

- if $A \in \mathbb{R}$ $A * B = B * A = AB$

- $A \in \Lambda^r(V), B \in \Lambda^s(V)$ $A * B \in \Lambda^{r+s}(V)$

$$\begin{array}{ccc} \parallel & \parallel & \parallel \\ u_1 \wedge u_2 \dots \wedge u_r & v_1 \wedge v_2 \dots \wedge v_s & u_1 \wedge u_2 \dots \wedge u_r \wedge v_1 \dots v_s \end{array}$$

$\cdot$ product
 $= B * A = AB$
 $(V) \quad A * B \in \wedge^{r+s}(V)$
 $\parallel$
 $u_1 \wedge u_2 \dots \wedge u_r \wedge v_1 \dots v_s$

$-$ linear in both arguments
 $\{u_1, \dots, u_m\}$ linearly dependent $\Leftrightarrow$
 $u_1 \wedge u_2 \wedge \dots \wedge u_m = 0$

