

Title: Algebra - Lecture 3

Date: Aug 16, 2012 10:30 AM

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Abstract:

A symmetry transformation (in QM) is a change in our point of view that does not change the results of possible experiments.

Weinberg - The Quantum Theory of Fields

- Rotations in 3-d Euclidean Space $SO(3)$

in 3-d Euclidean Space
n-d

$SO(3)$
 $SO(n)$

$\det = +1$

- Rotations in 3-d Euclidean Space $SO(3)$ $\det = 1$
n-d $SO(n)$
- Rotations in Hilbert Space $SU(n)$
- Rotations in 4-d Minkowski Space $SO(3,1)$

$$|\psi\rangle \rightarrow U|\psi\rangle = |\psi'\rangle$$

$$\langle\psi|\psi\rangle = \langle\psi'|\psi'\rangle = \langle\psi|U^\dagger U|\psi\rangle$$

$$UU^\dagger = \mathbb{1} \quad U(n) \quad n \times n \text{ unitary matrices}$$

$$SU(n) \quad \det U = +1$$

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$$SU(n) \quad \det U = +1$$

$|\psi\rangle$

- unitary, linear ops

- anti-

unitary matrices

$\lambda = +$

$SU(2)$: The most general $SU(2)$ matrix

$$U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}, \quad a, b: \text{complex}$$

Wigner

The most general $SU(2)$ matrix

$$U = \begin{pmatrix} a & b \\ -b^* & a \end{pmatrix}$$

, a, b : complex parameters

$$|a|^2 + |b|^2 = 1$$

Wigner

$SU(2)$: The most general $SU(2)$ matrix

$$U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$$

, a, b : complex parameters

$$|a|^2 + |b|^2 = 1$$

$$\begin{aligned} a &= x + iy \\ b &= u + iv \end{aligned}$$

$$\left. \begin{aligned} a &= x + iy \\ b &= u + iv \end{aligned} \right\} \begin{aligned} x^2 + y^2 + u^2 + v^2 &= 1 \rightarrow 3\text{-d} \\ (x, y, u, v) \end{aligned}$$

$$|x\rangle = X |\psi\rangle$$

$\downarrow u$

$$|x'\rangle = X' |\psi'\rangle$$

$$U|x\rangle = X' U|\psi\rangle$$

$$|\chi\rangle = X |\psi\rangle$$

$$\downarrow u$$

$$|\chi'\rangle = X' |\psi'\rangle$$

$$u^\dagger u |\chi\rangle = u^\dagger X' u |\psi\rangle$$

$$|\chi\rangle = u^\dagger X' u |\psi\rangle$$

$$|\chi\rangle = X |\psi\rangle \quad (1)$$

$$\downarrow u$$

$$|\chi'\rangle = X' |\psi'\rangle$$

$$u^\dagger u |\chi\rangle = u^\dagger X' u |\psi\rangle$$

$$|\chi\rangle = u^\dagger X' u |\psi\rangle \quad (2)$$

$$X = u^\dagger X' u$$

$$\boxed{X' = u X u^\dagger}$$

↳ Action of $SU(2)$ on

$$X |\psi\rangle \quad (1)$$

$$X' |\psi'\rangle$$

$$= U^\dagger X' U |\psi\rangle$$

$$= U^\dagger X' U |\psi\rangle \quad (2)$$

$$X = U^\dagger X' U$$

$$\boxed{X' = U X U^\dagger}$$

↳ Action of $SU(2)$ on ops

Homomorphism between $SU(2)$ & $SO(3)$

Homomorphism between $SU(2)$ & $SO(3)$

- Start from a 3-vector $\vec{x}$ on which $SO(3)$ acts
- Map $\vec{x}$ into 2×2 complex matrix

Homomorphism between $SU(2)$ & $SO(3)$

- Start from a 3-vector $\vec{x}$ on which $SO(3)$ acts
- Turn $\vec{x}$ into 2×2 complex matrix

$$\underset{\substack{\uparrow \\ \text{MATRIX}}}{X} = x^i \sigma_i \quad \hookrightarrow \text{Pauli Matrices}$$

isomorphism between $SU(2)$ & $SO(3)$

map from a 3-vector $\vec{x}$ on which $SO(3)$ acts

into 2×2 complex matrix

$$\vec{x} = x^i \sigma_i$$

σ_i $\rightarrow$ Pauli Matrices

$$y = y^i \sigma_i$$
$$x \cdot y = (x^i \sigma_i) \cdot (y^j \sigma_j)$$

MATRIX
PRODUCT

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

SU(2) & SO(3)

$\vec{x}$ on which SO(3) acts

matrix $\left| \begin{array}{l} y = y^i \sigma_i \\ x \cdot y = (x^i \sigma_i) \cdot (y^j \sigma_j) = y^j \sigma_i \sigma_j \end{array} \right.$

MATRIX PRODUCT $\left| \begin{array}{l} \text{Tr}(xy) = \text{Tr}(\sigma_i \sigma_j) \end{array} \right.$

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

SL(2, C)

complex, invertible 2x2 matrices

MATRIX

σ_i
Pauli Matrices

MATRIX
PRODUCT

$$\text{Tr}(xy) = x$$

$$\sigma_i \sigma_j = \delta_{ij} \mathbb{1} + i \epsilon_{ijk} \sigma^k$$

$$\sigma_1 = \sigma^1$$

$$\epsilon_{123}$$

MATRIX

Pauli Matrices

MATRIX
PRODUCT

$\text{Tr}(xy) =$

$$\sigma_i \sigma_j = \delta_{ij} \mathbb{1} + i \epsilon_{ijk} \sigma^k$$

$$\sigma_3 = \sigma^3$$

$$\epsilon_{123} = +1$$

$$\epsilon_{132} = -1$$

$$\epsilon_{312} = +1$$

MATRIX

Pauli Matrices

MATRIX
PRODUCT

$\text{Tr}(xy) =$

$$\sigma_i \sigma_j = \delta_{ij} \mathbb{1} + i \epsilon_{ijk} \sigma^k$$

$$\sigma_3 = \sigma^3$$

$$\epsilon_{123} = +1$$

$$\epsilon_{132} = -1$$

$$\epsilon_{312} = +1$$

$$\begin{aligned} \text{Tr}(\sigma_i \sigma_j) &= \delta_{ij} \text{Tr} \mathbb{1} + i \epsilon_{ijk} \underbrace{\text{Tr} \sigma^k}_0 \\ &= 2 \delta_{ij} \end{aligned}$$

$$\text{Tr}(x \cdot y) = 2 \delta_{ij} x^i y^j = 2 \vec{x} \cdot \vec{y}$$

MATRIX
PRODUCT

$$\text{Tr}(xy) = x^i y^j \text{Tr}(\sigma_i \sigma_j)$$

$$\sigma_3 = \sigma^3$$

$$i \epsilon_{ijk} \underbrace{\text{Tr} \sigma^k}_0$$

SCALAR
PRODUCT

$$y^j = 2 \vec{x} \cdot \vec{y}$$

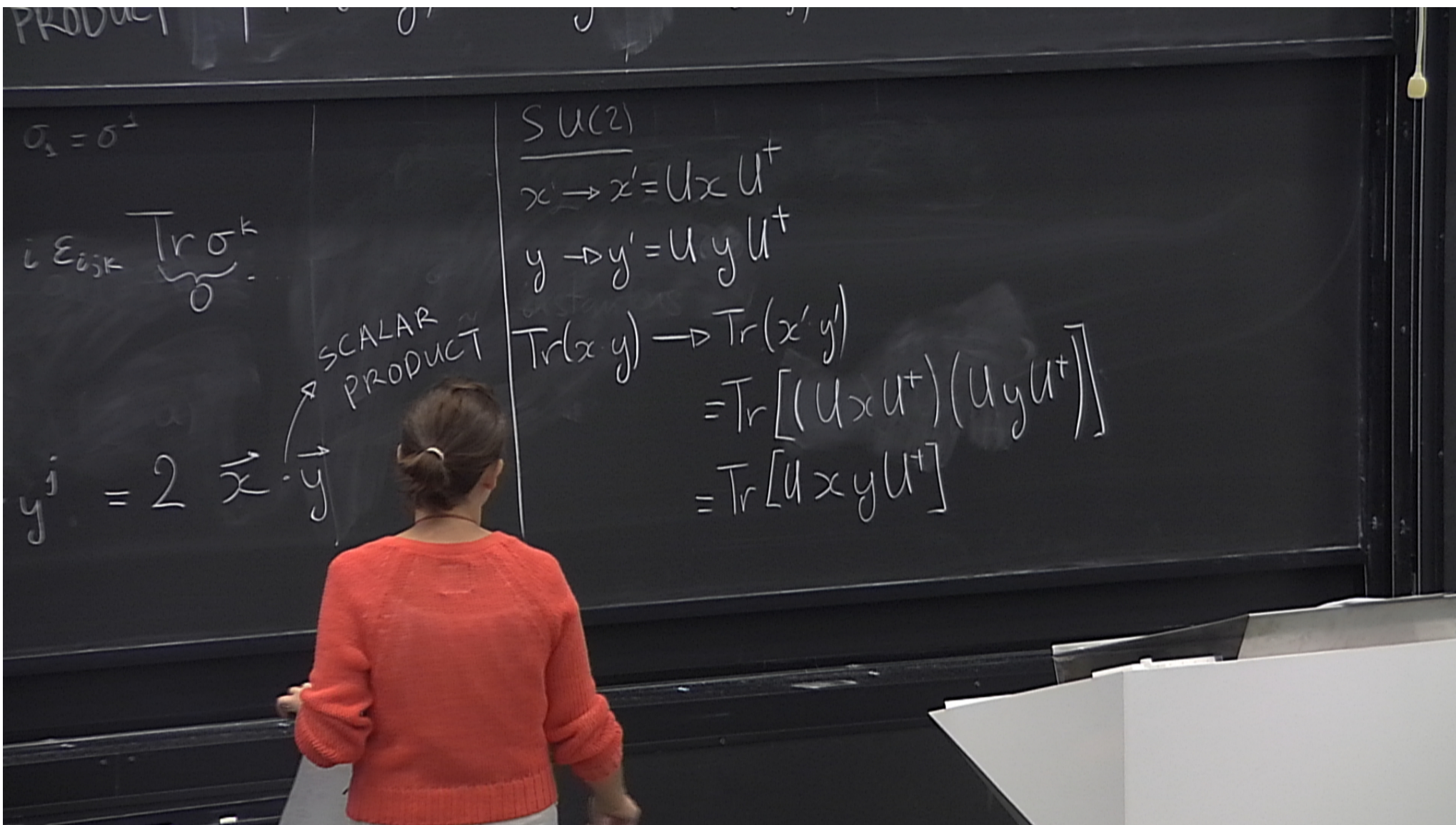
SU(2)

$$x \rightarrow x' = Ux$$

$$y \rightarrow y' = Uy$$

$$\begin{aligned} \text{Tr}(xy) &\rightarrow \text{Tr}(x'y') \\ &= \text{Tr}[(Ux)(Uy)] \end{aligned}$$

=



$$\underline{SU(2)}$$

$$x \rightarrow x'$$

$$y \rightarrow y'$$

$$\text{Tr}(x \cdot y) = \text{Tr}(x' \cdot y')$$

SU(2)

$$x \rightarrow x'$$

$$y \rightarrow y'$$

$$\text{Tr}(x \cdot y) = \text{Tr}(x' \cdot y') \Rightarrow \vec{x}' \cdot \vec{y}' = \vec{x} \cdot \vec{y} \Rightarrow$$

SU(2)

$$x \rightarrow x'$$

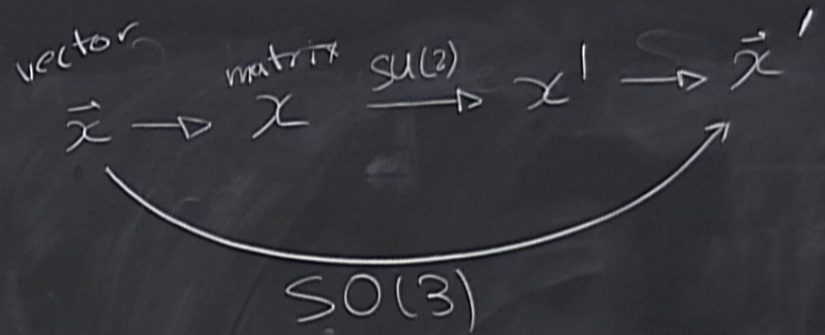
$$y \rightarrow y'$$

$$\text{Tr}(x \cdot y) = \text{Tr}(x' \cdot y') \Rightarrow \vec{x}' \cdot \vec{y}' = \vec{x} \cdot \vec{y} \Rightarrow \text{SO}(3)$$

W

SU(2)

$$\begin{aligned}x &\rightarrow x' \\ y &\rightarrow y'\end{aligned}$$



$$\text{Tr}(x \cdot y) = \text{Tr}(x' \cdot y') \Rightarrow \vec{x}' \cdot \vec{y}' = \vec{x} \cdot \vec{y} \Rightarrow SO(3)$$

Wigner

$\vec{x}'$
 $\uparrow$
Homomorphism, but not isomorphism
 $U \rightarrow -U$: same $SO(3)$ transformation

$\vec{x} \rightarrow \vec{y} \Rightarrow SO(3)$ transformation

Wigner

Homomorphism, but not isomorphism

$U \rightarrow -U$: same $SO(3)$ transformation

$SU(2)$ double cover of $SO(3)$ $SO(3) \simeq SU(2)/\mathbb{Z}_2$

$\Rightarrow SO(3)$ transformation

$$\psi |U^\dagger U| \psi \rangle$$

$n \times n$ unitary m

$$\det U = +1$$

$$\bar{x} \bar{y} = x^i \delta_{ij} y^j$$

$$\psi |U^\dagger U| \psi \rangle$$

$n \times n$ unitary matrix

$$\det U = +1$$

$$\bar{x} \bar{y} = x^i \delta_{ij} y^j$$

$$x^i \rightarrow R^i_j x^j$$

$$\vec{x} \cdot \vec{y} = x^i \delta_{ij} y^j$$

$$x^i \rightarrow R^i_k x^k$$

$$R^i_k \delta_{ij} R^j_l = \delta_{kl}$$

preserves
metric

$$\vec{x}' \cdot \vec{y}' \stackrel{!}{=} \vec{x} \cdot \vec{y}$$

$$x'^i \delta_{ij} y'^j = \boxed{R^i_k x^k \delta_{ij} R^j_l y^l} \stackrel{!}{=} x^k \delta_{kl} y^l$$

ary matrices

$$\vec{x} \cdot \vec{y} = x^i \delta_{ij} y^j$$

$$x^i \rightarrow R^i_k x^k$$

$$R^i_k \delta_{ij} R^j_l = \delta_{kl}$$

preserves
metric / scalar
product

$$\vec{x}' \cdot \vec{y}' \stackrel{!}{=} \vec{x} \cdot \vec{y}$$

$$x'^i \delta_{ij} y'^j = \boxed{R^i_k x^k \delta_{ij} R^j_l y^l} \stackrel{!}{=} x^k \delta_{kl} y^l$$

$$\vec{x} \cdot \vec{y} = x^i \delta_{ij} y^j$$

$$x^i \rightarrow R^i{}_j x^j$$

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$$\vec{x}' \cdot \vec{y}' \stackrel{!}{=} \vec{x} \cdot \vec{y}$$

$$x'^i \delta_{ij} y'^j = \boxed{R^i{}_k x^k \delta_{ij} R^j{}_l y^l} \stackrel{!}{=} x^k \delta_{kl} y^l$$

Minkowski Space-time 4dimensional

event (t, x, y, z)

space-time interval between two events (t, x, y, z) $(t + \Delta t, x + \Delta x, y + \Delta y, z + \Delta z)$

ski Space-time 4dimensional

(t, x, y, z)

interval between two events (t, x, y, z) & $(t + \Delta t, x + \Delta x, y + \Delta y$

Minkowski Space-time 4dimensional

event (t, x, y, z)

space-time interval between two events (t, x, y, z) & $(t + \Delta t, x + \Delta x, y + \Delta y, z + \Delta z)$

$$(\Delta s)^2 = - (c \Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

Minkowski Space-time 4dimensional

event (t, x, y, z)

space-time interval between two events (t, x, y, z) & $(t + \Delta t, x + \Delta x, y + \Delta y, z + \Delta z)$

$$(\Delta s)^2 = - (c \Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

SR: $(\Delta s)^2 = \text{invariant}$

MATRIX

→ Pauli Matrices

PRODUCT

Transformations that leave $(\Delta s)^2$ invariant

MATRIX

→ Pauli Matrices

PRODUCT

Transformations that leave $(\Delta s)^2$ invariant

- translations $x^\mu \rightarrow x'^\mu = x^\mu + a^\mu \rightarrow \text{constant}$

- "rotations"

MATRIX

→ Pauli Matrices

PRODUCT

Transformations that leave $(\Delta s)^2$ invariant

- translations $x^\mu \rightarrow x'^\mu = x^\mu + a^\mu \rightarrow \text{constant}$

- "rotations" $x^\mu \rightarrow x'^\mu = \Lambda^\mu_\nu x^\nu$

MATRIX

→ Pauli Matrices

PRODUCT

Transformations that leave $(\Delta s)^2$ invariant

- translations $x^\mu \rightarrow x'^\mu = x^\mu + a^\mu \rightarrow \text{constant}$

- "rotations" $x^\mu \rightarrow x'^\mu = \Lambda^\mu_\nu x^\nu$

$$\boxed{\Lambda^\mu_\kappa \eta_{\mu\nu} \Lambda^\nu_\lambda = \eta_{\kappa\lambda}}$$

MATRIX

→ Pauli Matrices

PRODUCT

Transformations that leave $(\Delta s)^2$ invariant

- translations $x^\mu \rightarrow x'^\mu = x^\mu + a^\mu \rightarrow \text{constant}$

- "rotations" $x^\mu \rightarrow x'^\mu = \Lambda^\mu_\nu x^\nu$

$$\boxed{\Lambda^\mu_\kappa \eta_{\mu\nu} \Lambda^\nu_\lambda = \eta_{\kappa\lambda}}$$

$$\Lambda^T \eta \Lambda = \eta$$

MATRIX

→ Pauli Matrices

PRODUCT

Transformations that leave $(\Delta s)^2$ invariant

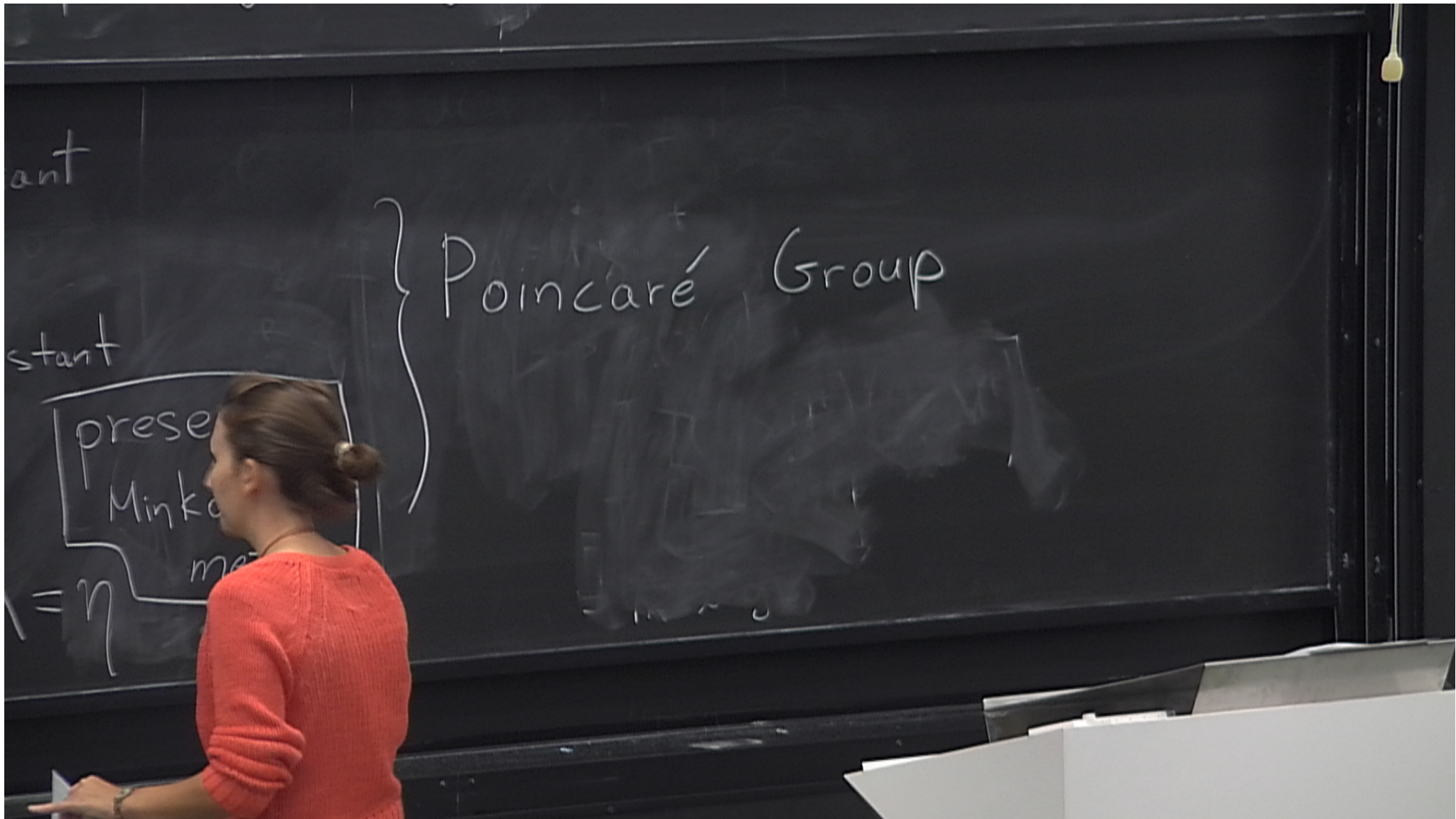
- translations $x^\mu \rightarrow x'^\mu = x^\mu + a^\mu \rightarrow \text{constant}$

- Lorentz transf. $x^\mu \rightarrow x'^\mu = \Lambda^\mu_\nu x^\nu$

preserve
Minkowski
metric

$$\Lambda^\mu_\kappa \eta_{\mu\nu} \Lambda^\nu_\lambda = \eta_{\kappa\lambda}$$

$$\Lambda^T \eta \Lambda = \eta$$



$\rho(3)$ acts

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$SL(2, \mathbb{C})$

complex, invertible 2×2 matrices

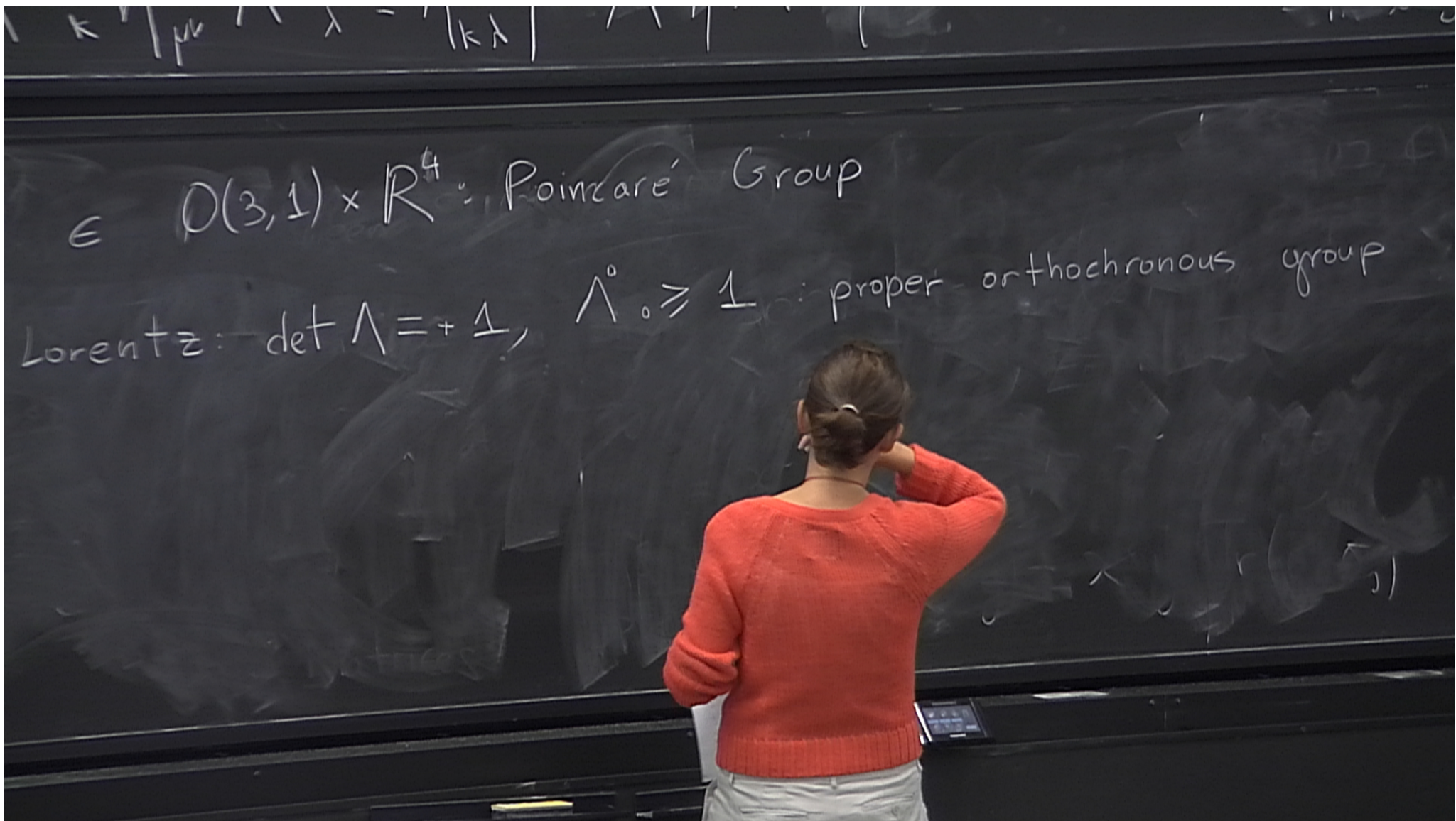
$$(\sigma_i) \cdot (y^j \sigma_j) = x^i y^j \sigma_i \sigma_j$$

$$| \text{Tr}(xy) = x^i y^j \text{Tr}(\sigma_i \sigma_j)$$

$T(\Lambda, a) \in O(3,1) \times \mathbb{R}^3$: Poincaré Group

$T(\Lambda, a) \in O(3,1) \times \mathbb{R}^4$: Poincaré Group

subgroup of Lorentz: $\det \Lambda = +1$, $\Lambda^0_0 \geq 1$



$\in O(3,1) \times \mathbb{R}^4$: Poincaré Group

Lorentz: $\det \Lambda = +1$, $\Lambda^0_0 \geq 1$: proper orthochronous group

$\in O(3,1) \times \mathbb{R}^4$: Poincaré Group

Lorentz: $\det \Lambda = +1$, $\Lambda^0_0 \geq 1$: proper orthochronous group

$SO^+(3,1)$

$SO^+(3,1) \times P$

T
time

inversion

a) $\in O(3,1) \times \mathbb{R}^4$: Poincaré Group

of Lorentz: $\det \Lambda = +1$, $\Lambda^0_0 \geq 1$: proper orthochronous group
 $SO^+(3,1)$

$= SO^+(3,1) \times \underset{\text{inversion}}{P} \times \underset{\text{inversion}}{T}$
space time