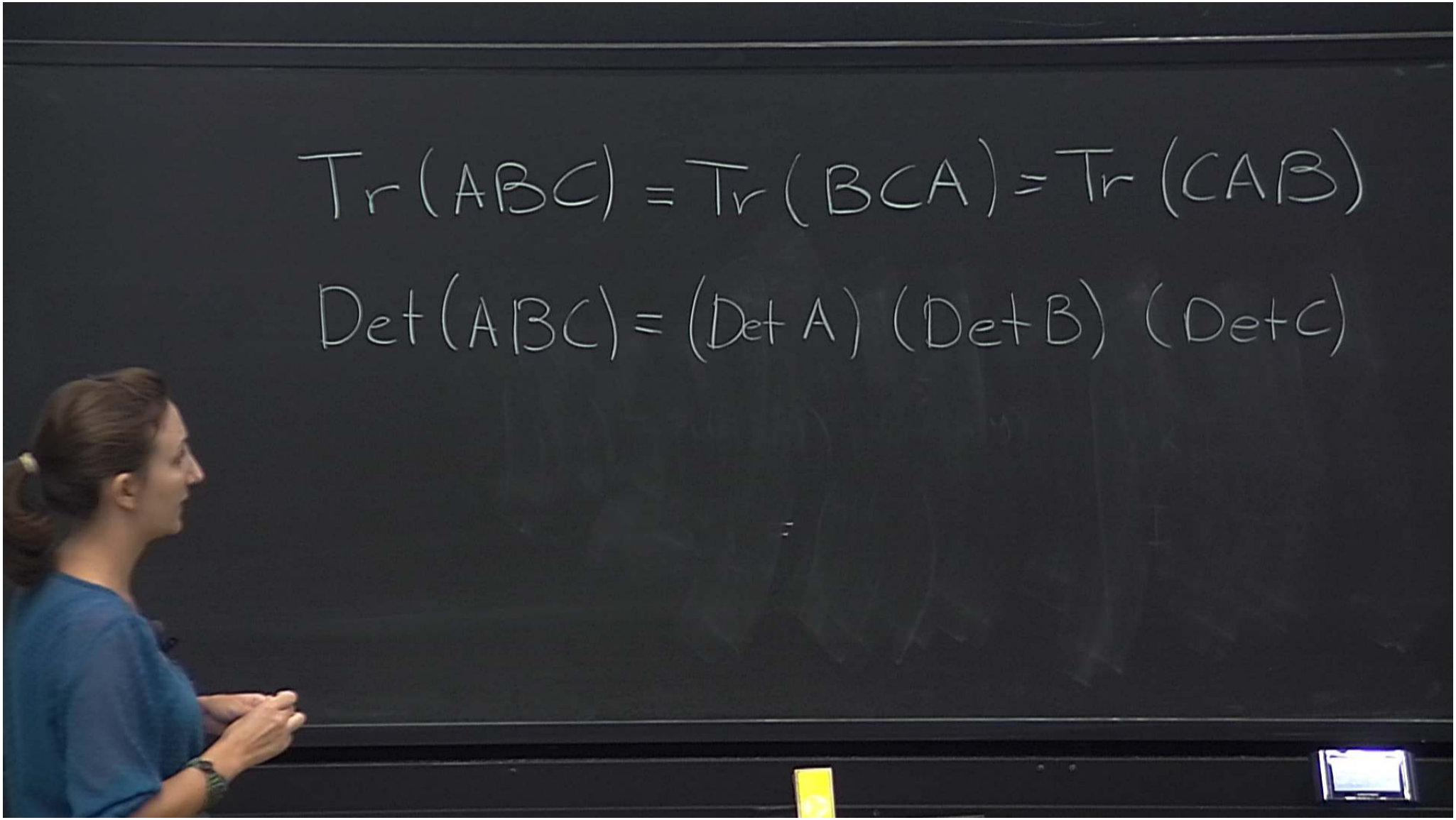
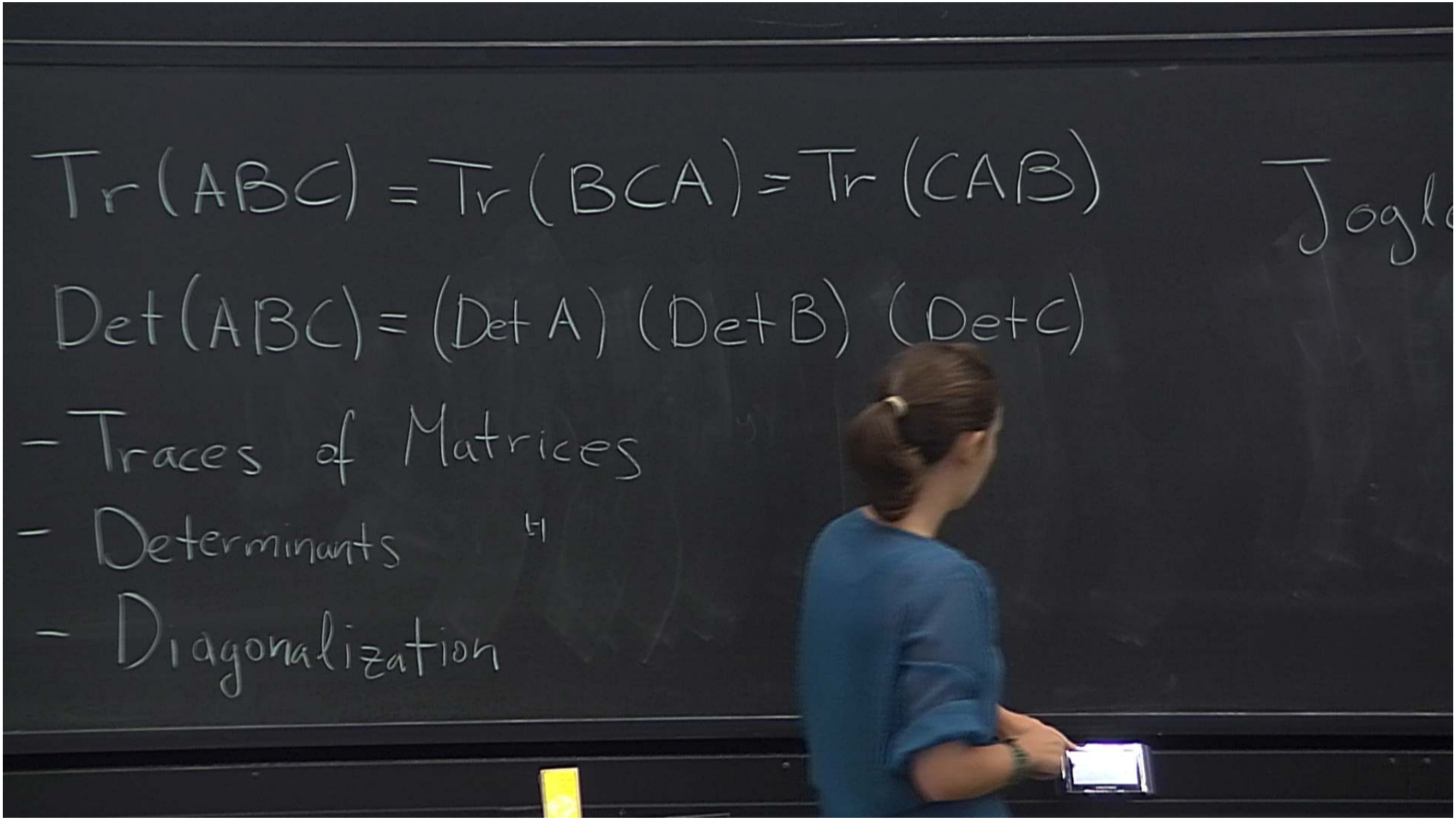
Title: Algebra - Lecture 2

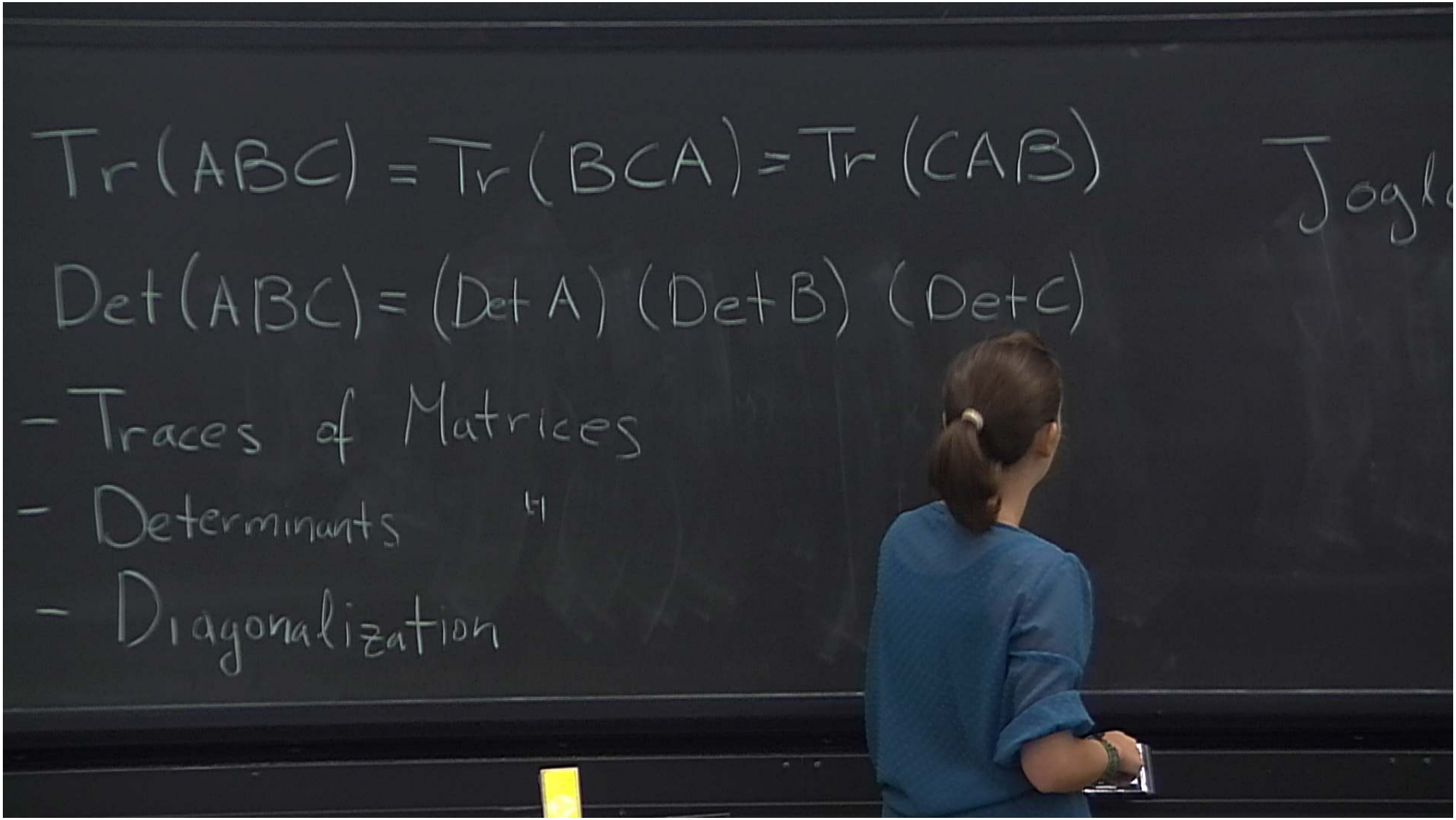
Date: Aug 15, 2012 10:30 AM

URL: http://pirsa.org/12080016

Abstract:


Tr(ABC) = Tr(BCA) = Tr(CAB)
Det(ABC) = (Det A)(Det B)(Det C)


Tr(ABC) = Tr(BCA) = Tr(CAB)
Det(ABC) = (Det A)(Det B)(Det C)
- Traces of Matrices
- Determinants
- Diagonalization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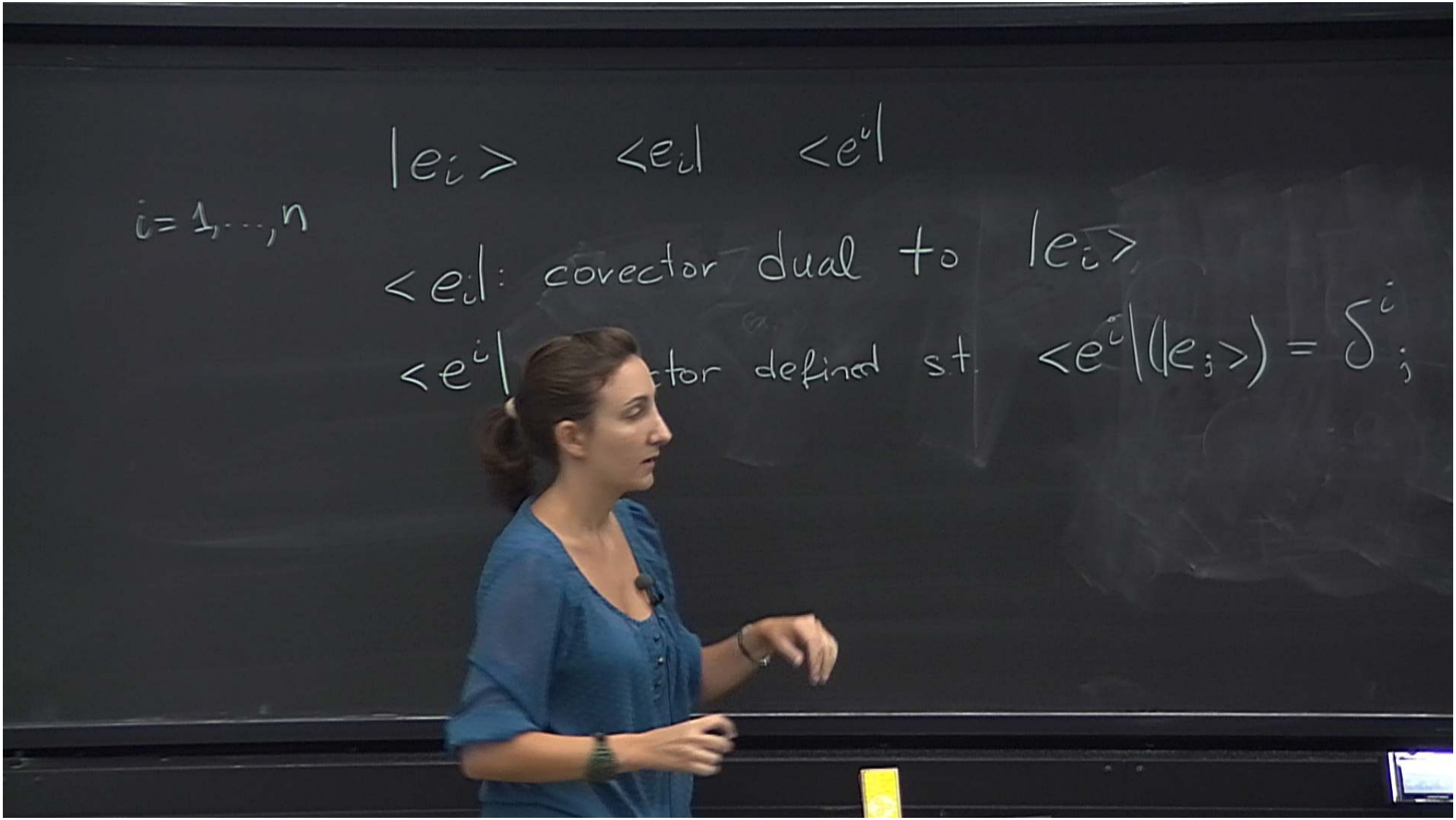
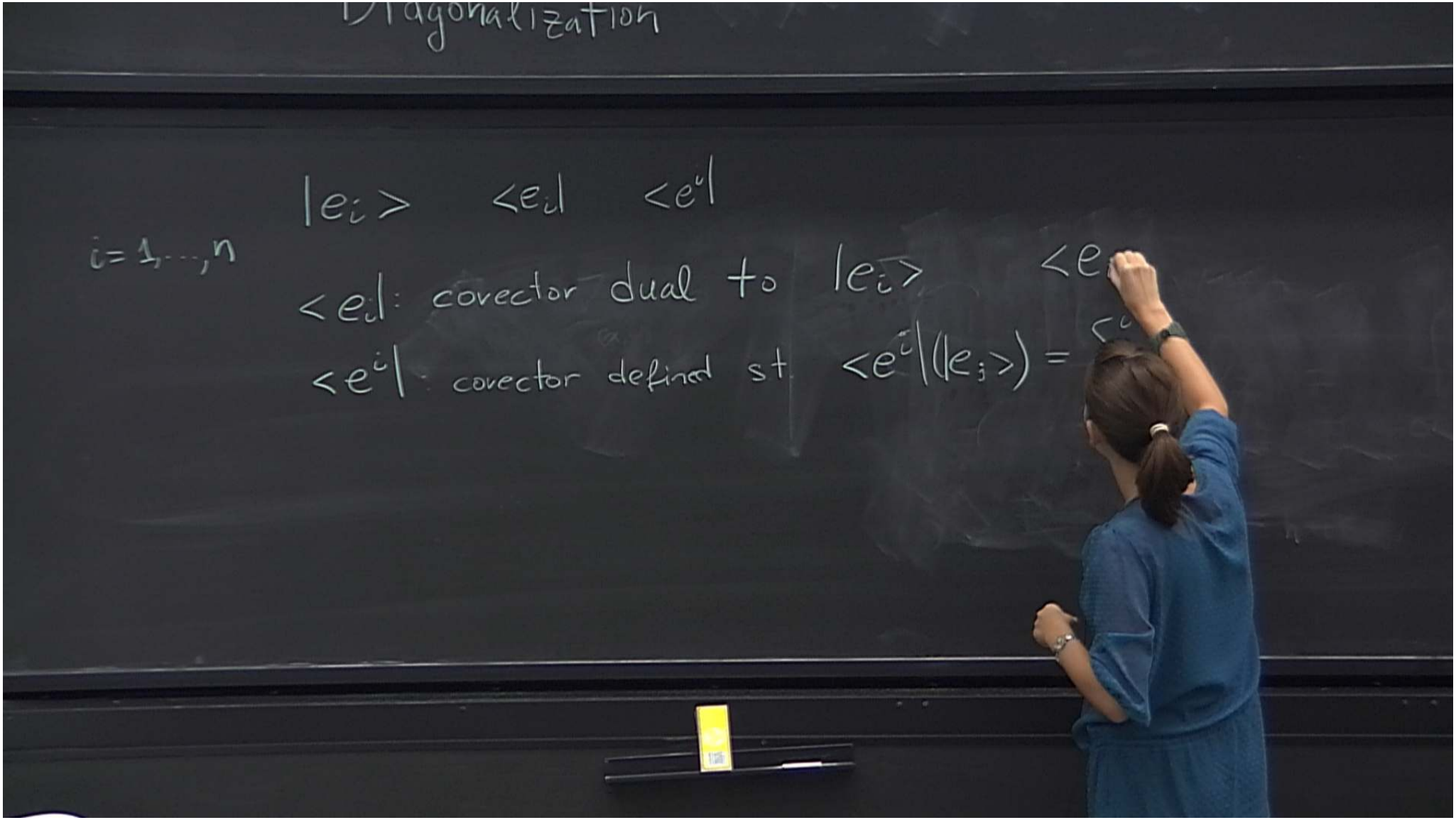

$|e_i\rangle$ $\langle e_i|$ $\langle e^i|$
$i = 1, \ldots, n$
$\langle e_i|$: covector dual to $|e_i\rangle$
$\langle e^i|$ ctor defined s.t. $\langle e^i|(|e_j\rangle) = \delta^i_j$


Diagonalization
$|e_i\rangle$ $\langle e_i|$ $\langle e^i|$
$i = 1, \dots, n$
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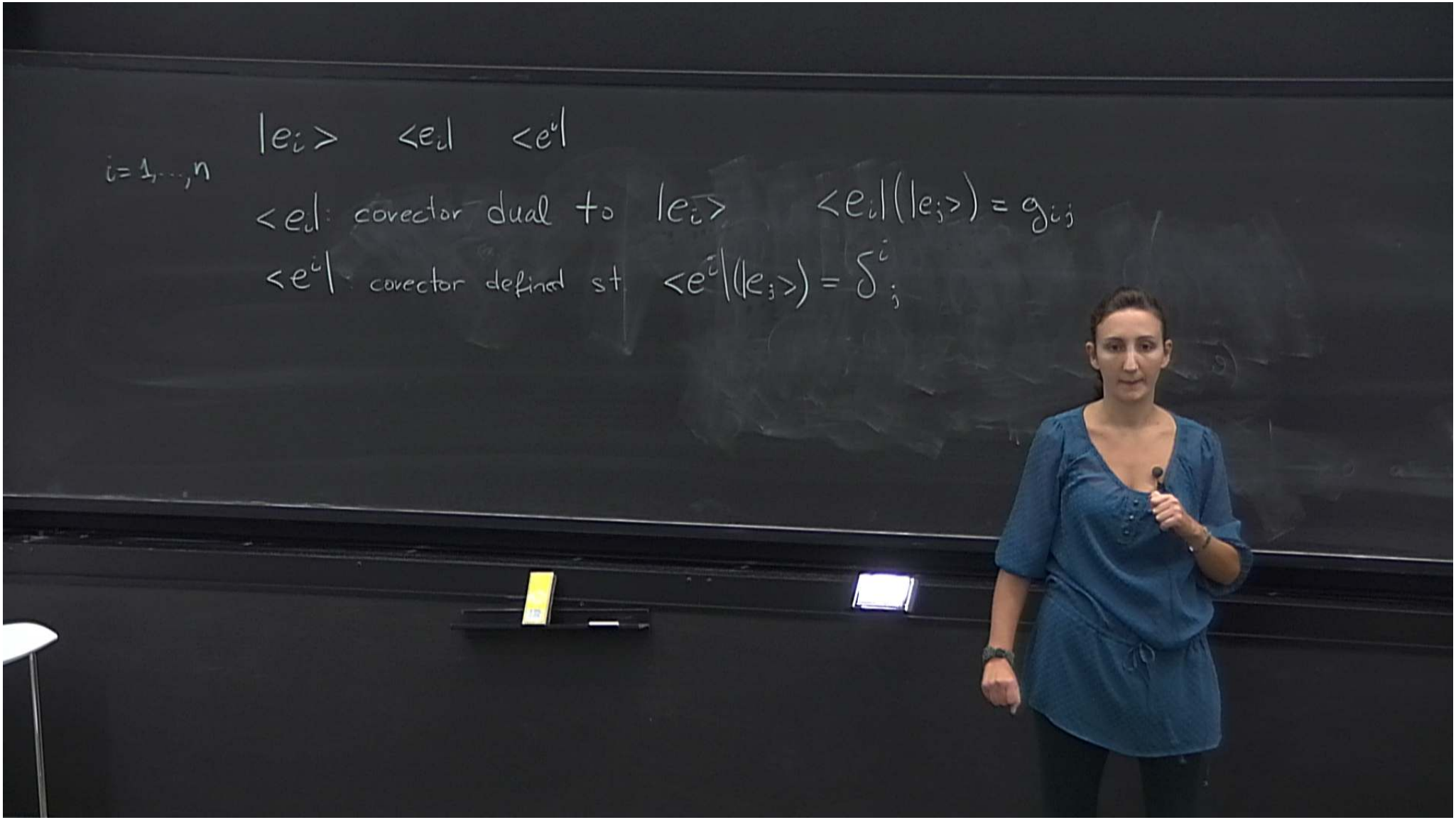
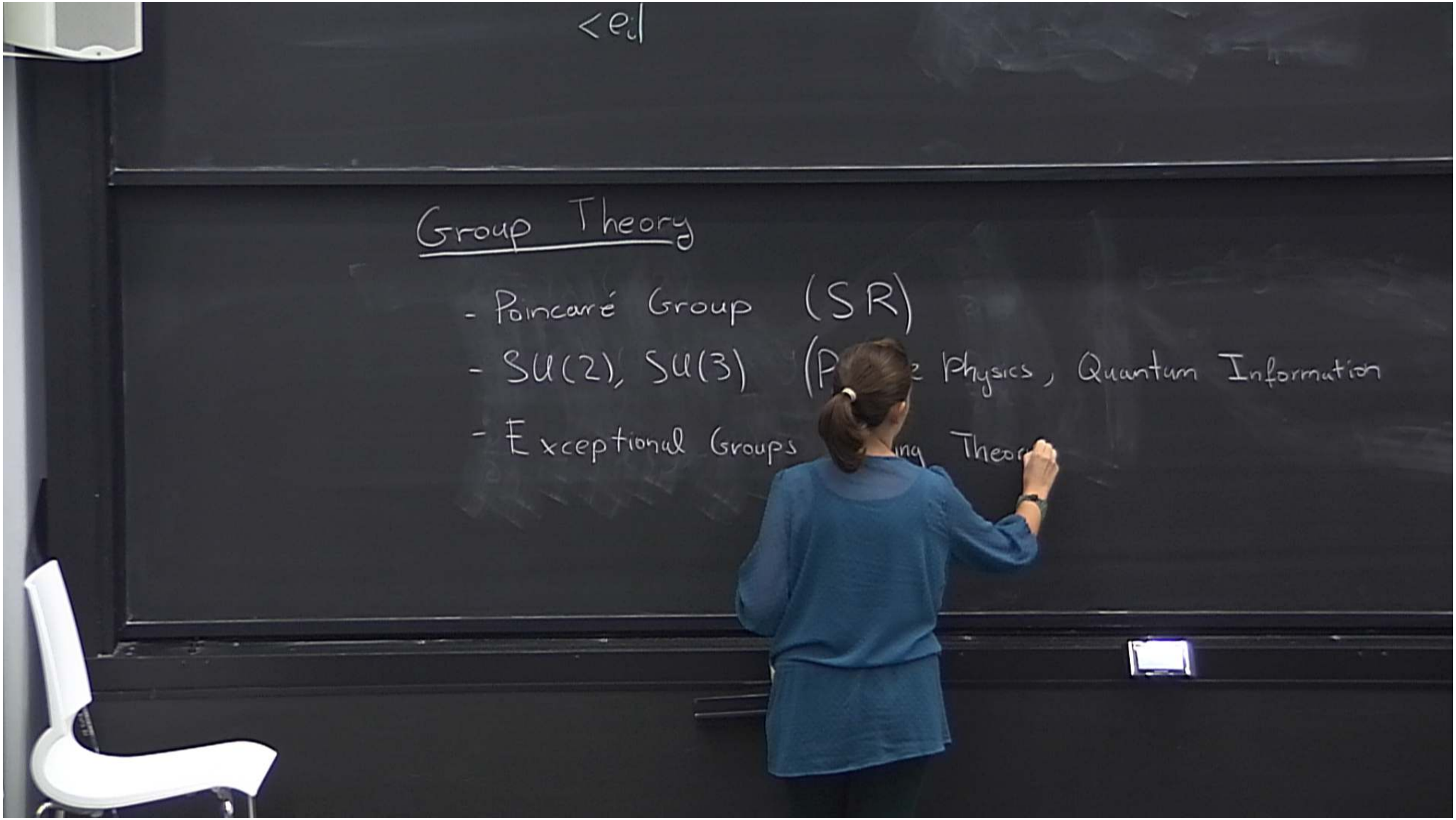
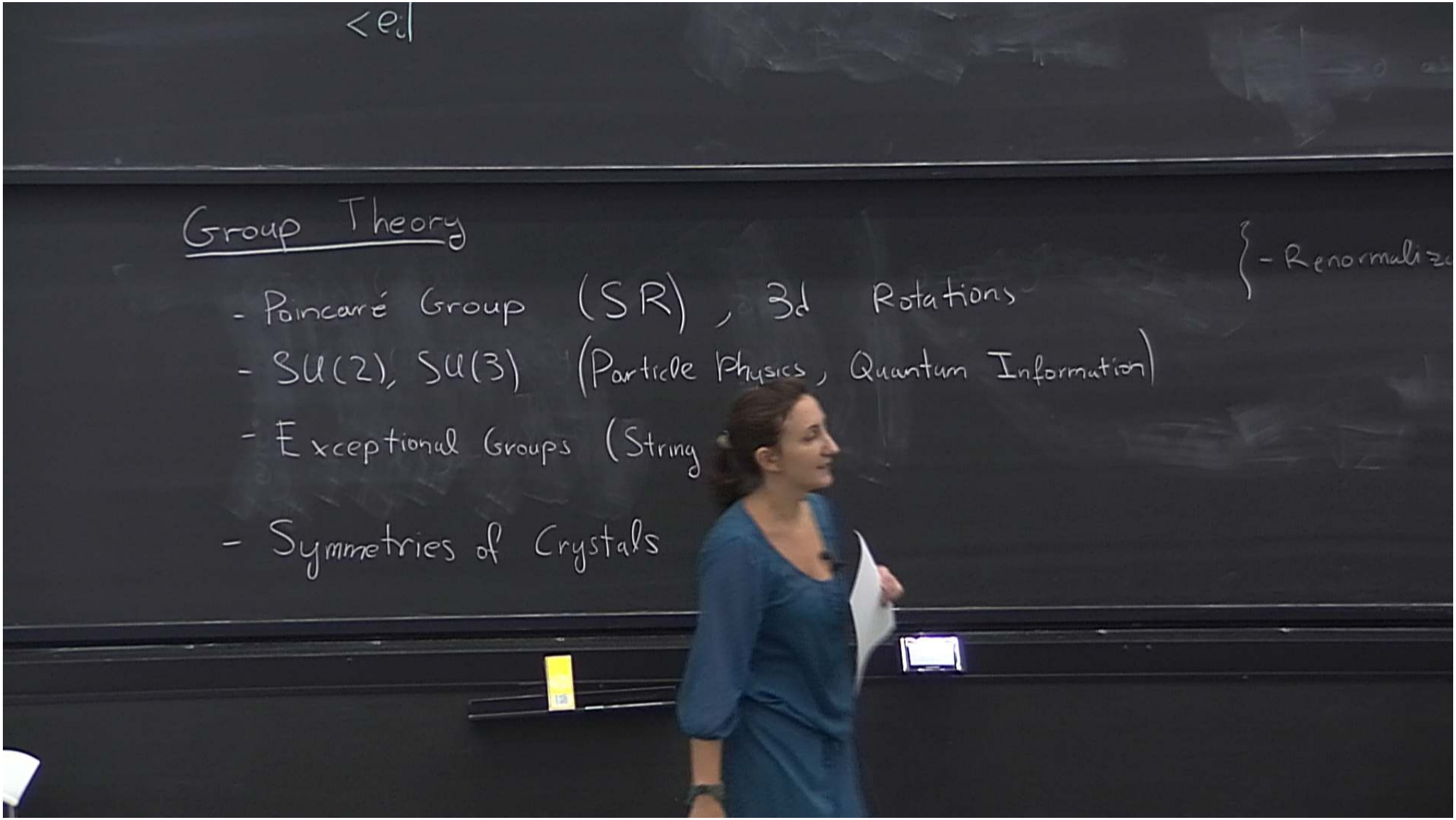
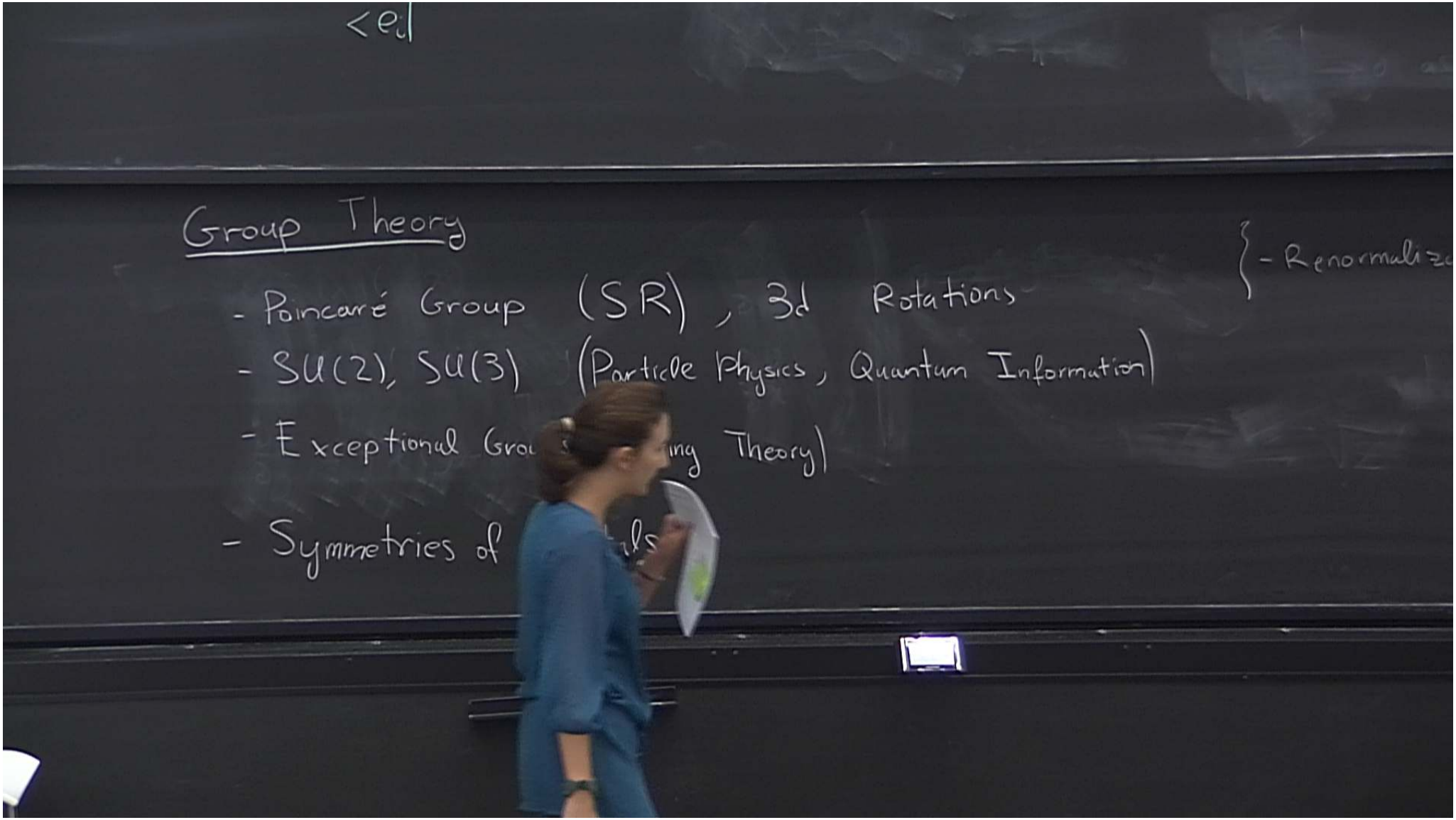

$i = 1, \cdots, n$
$|e_i\rangle$ $\langle e_i|$ $\langle e^i|$
$\langle e_i|$: covector dual to $|e_i\rangle$ $\langle e_i|(|e_j\rangle) = g_{ij}$
$\langle e^i|$: covector defined st $\langle e^i|(|e_j\rangle) = \delta^i_j$


Group Theory
- Poincaré Group (SR)
- SU(2), SU(3)
Physics, Quantum Information
- Exceptional Groups


$\langle e_i|$
Group Theory
- Poincaré Group (SR), 3d Rotations
- SU(2), SU(3) (Particle Physics, Quantum Information)
- Exceptional Groups (String
- Symmetries of Crystals
- Renormali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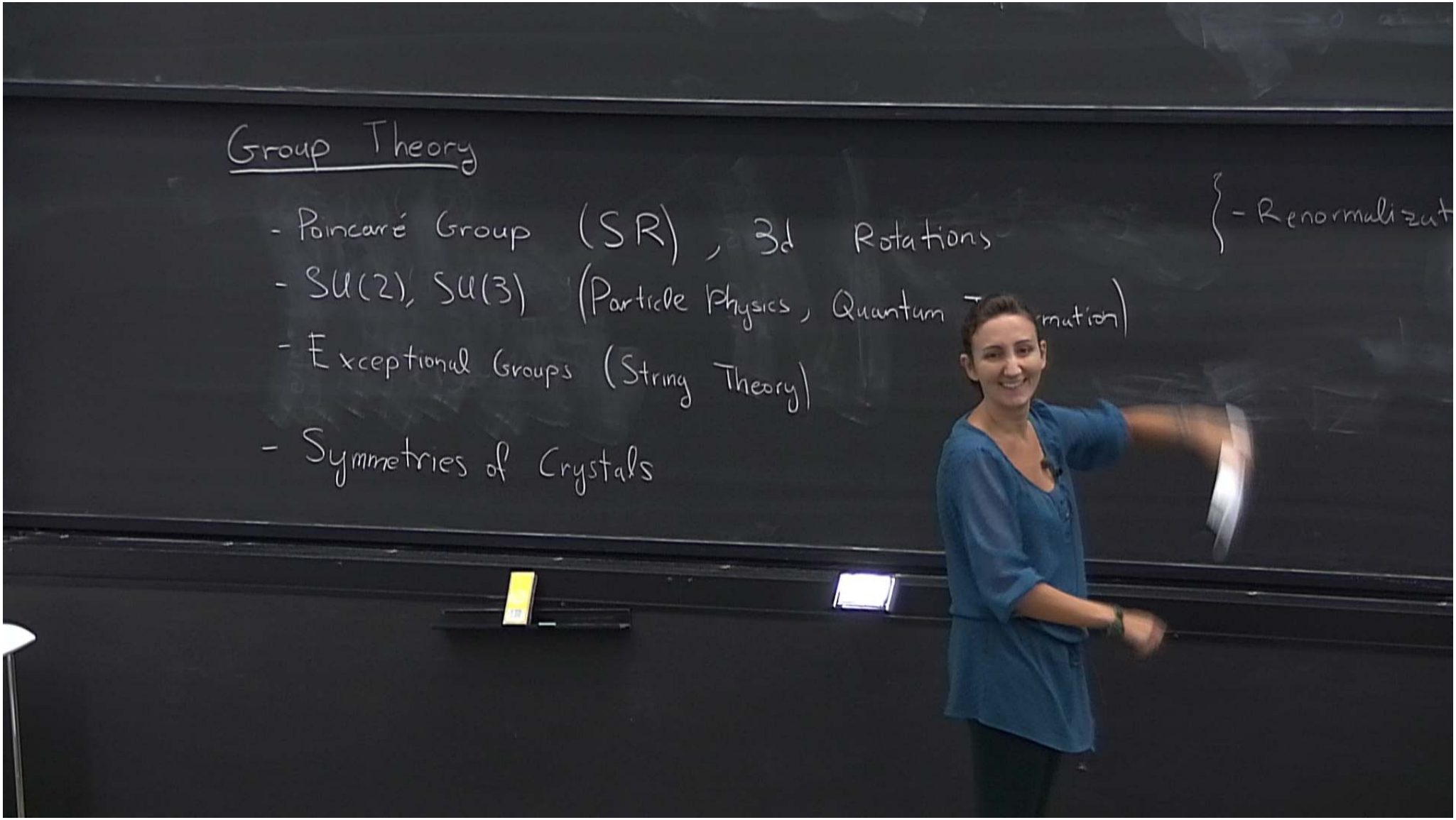
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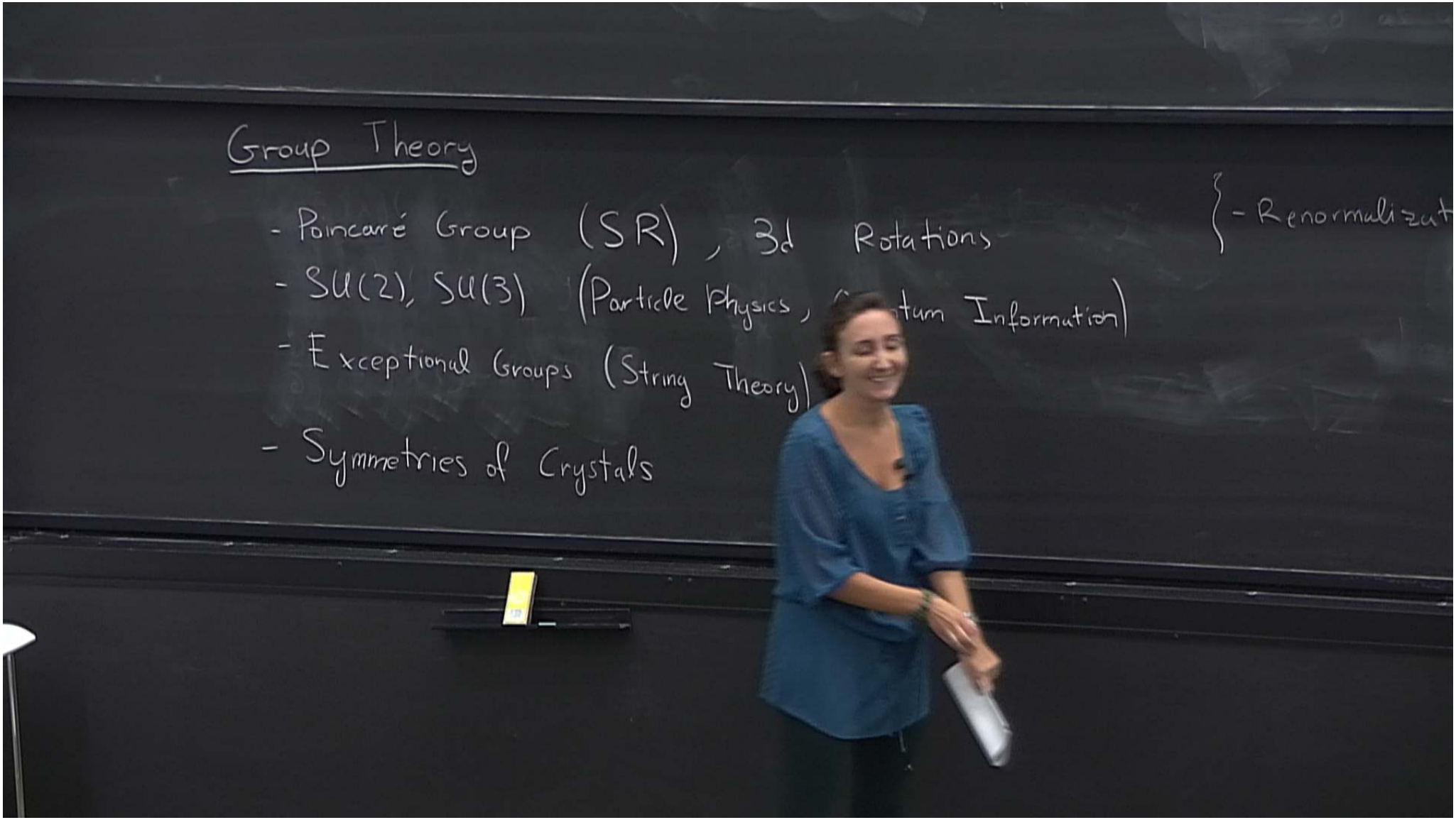
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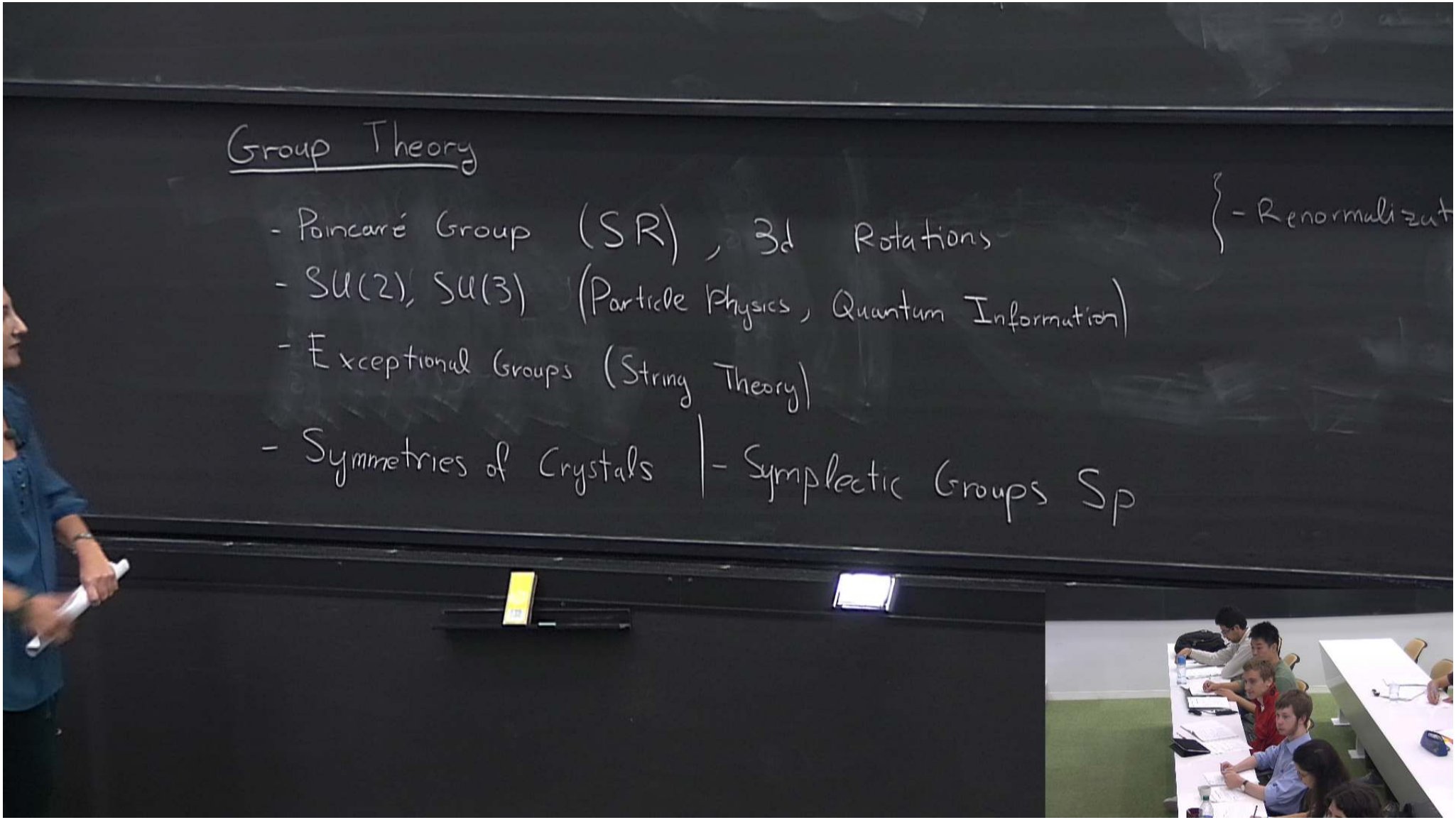
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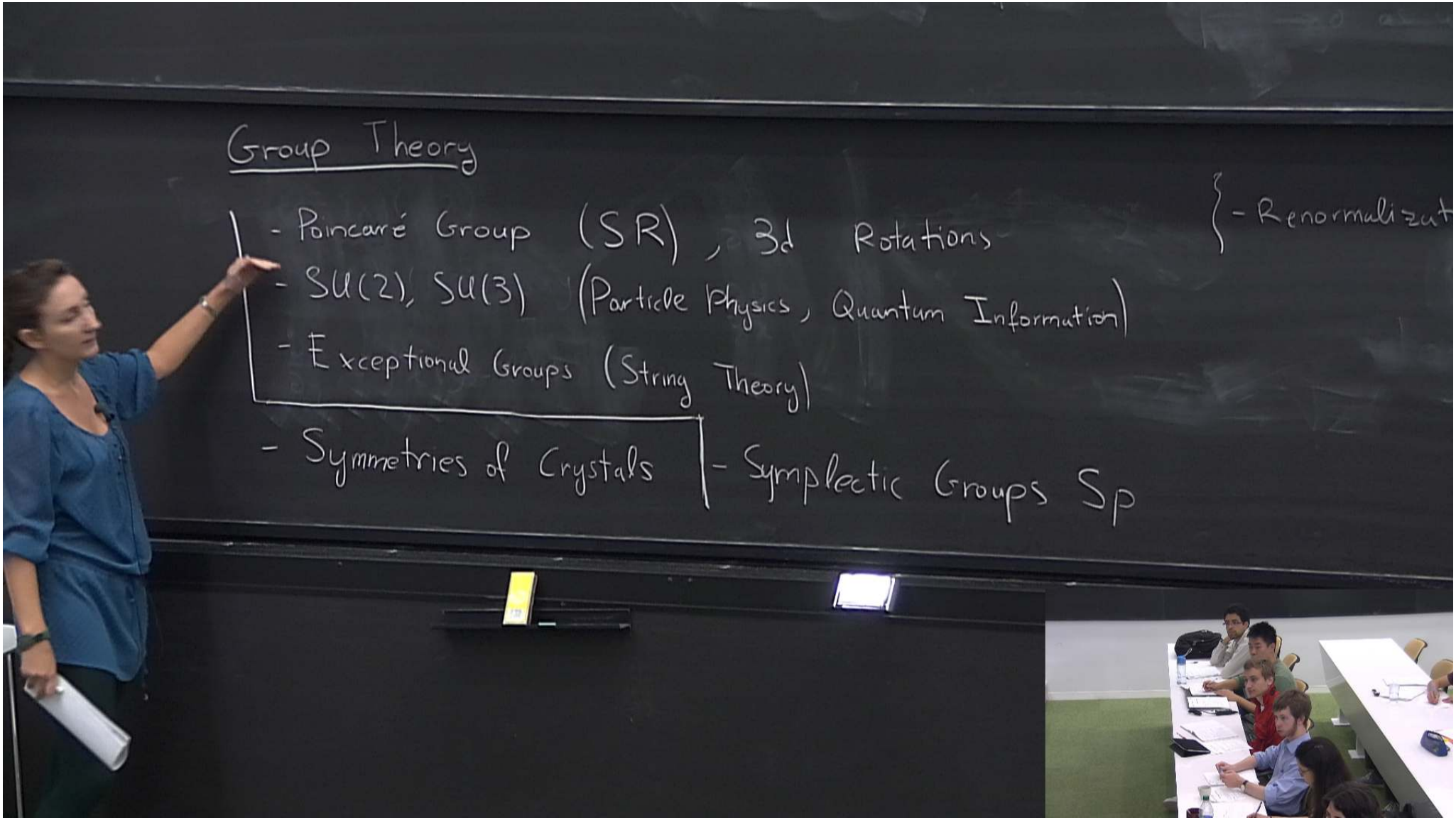
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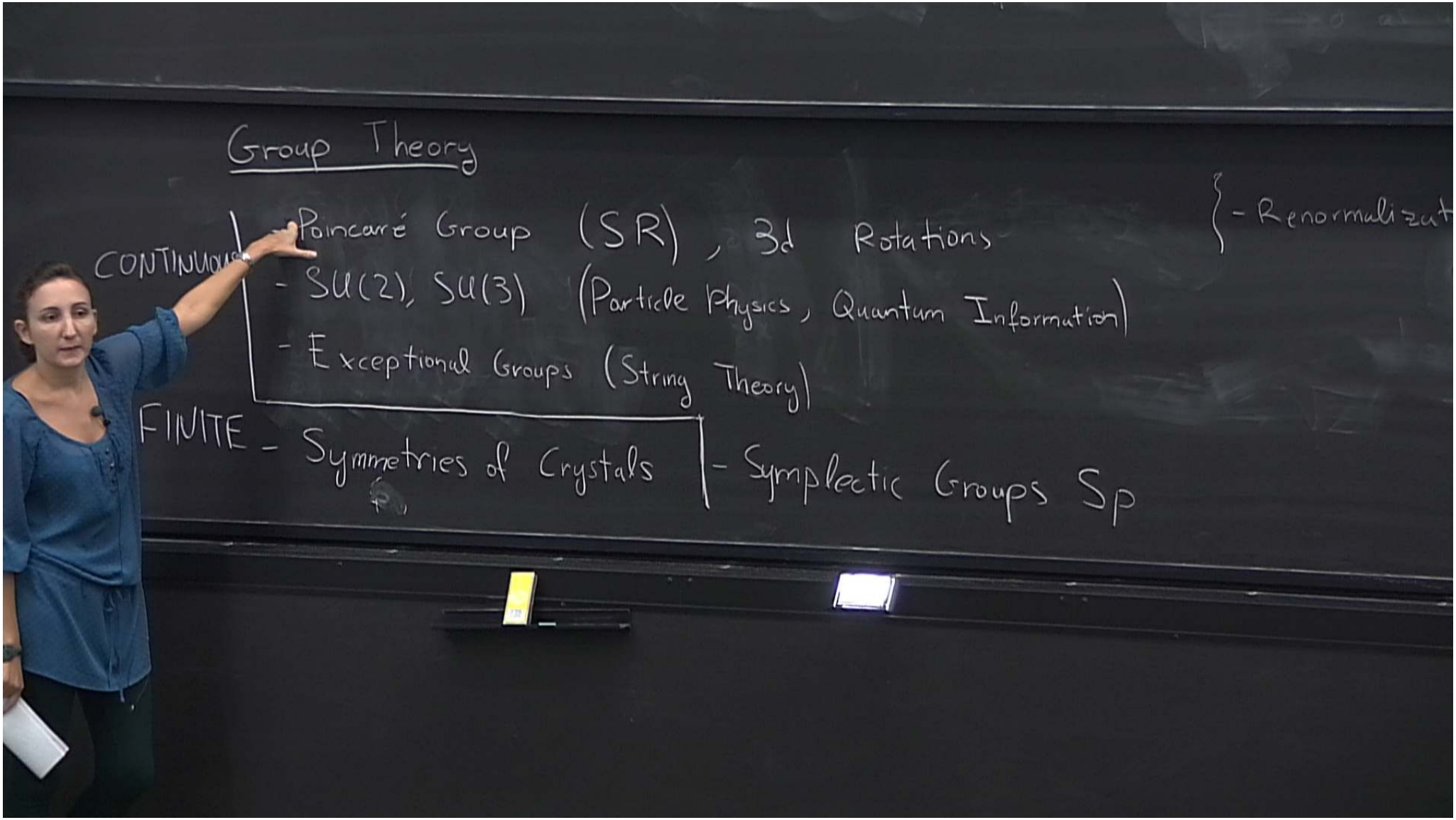
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- Poincaré Group (SR), 3d Rotations
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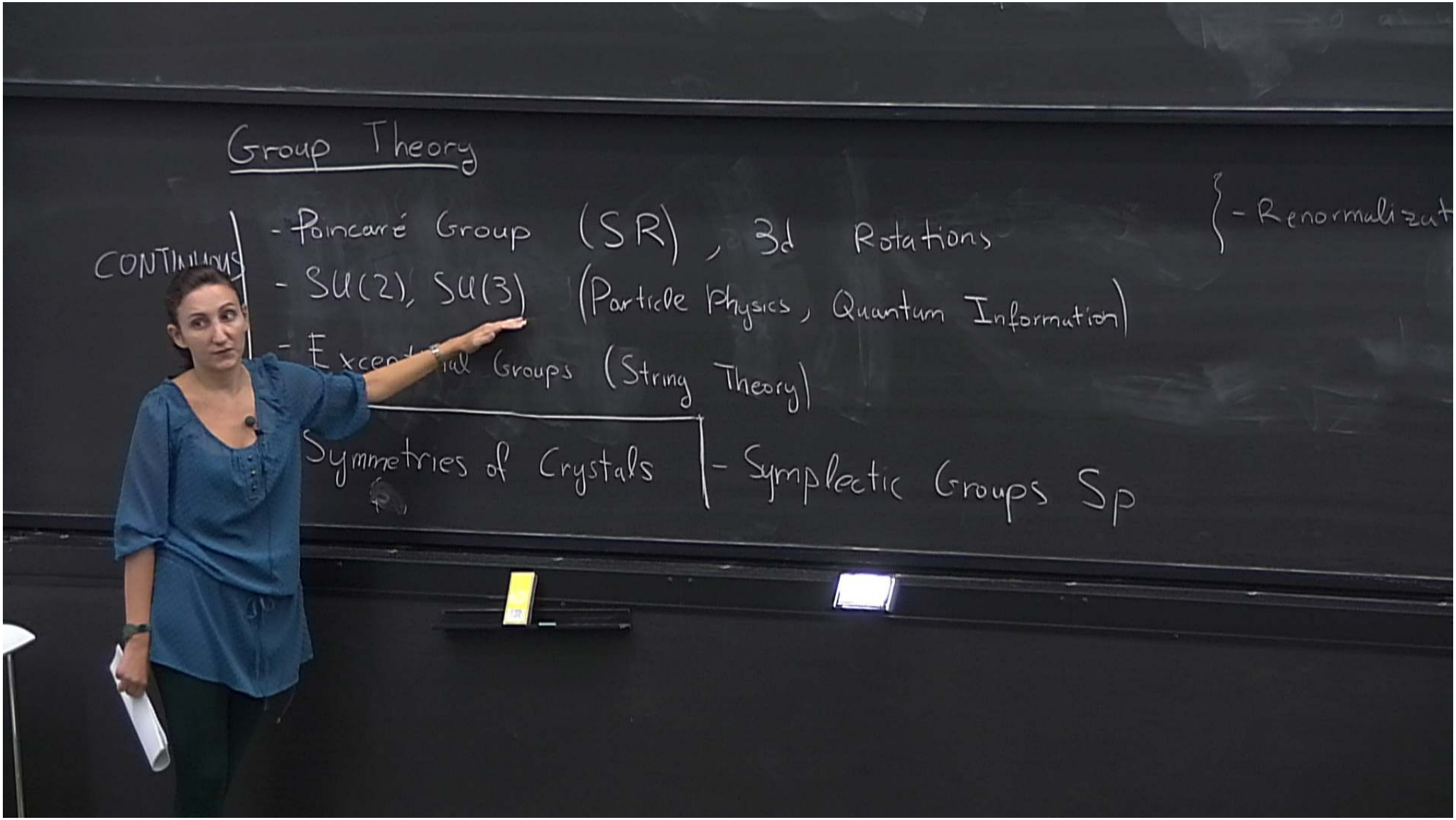
FINITE - Symmetries of Crystals
- Symplectic Groups Sp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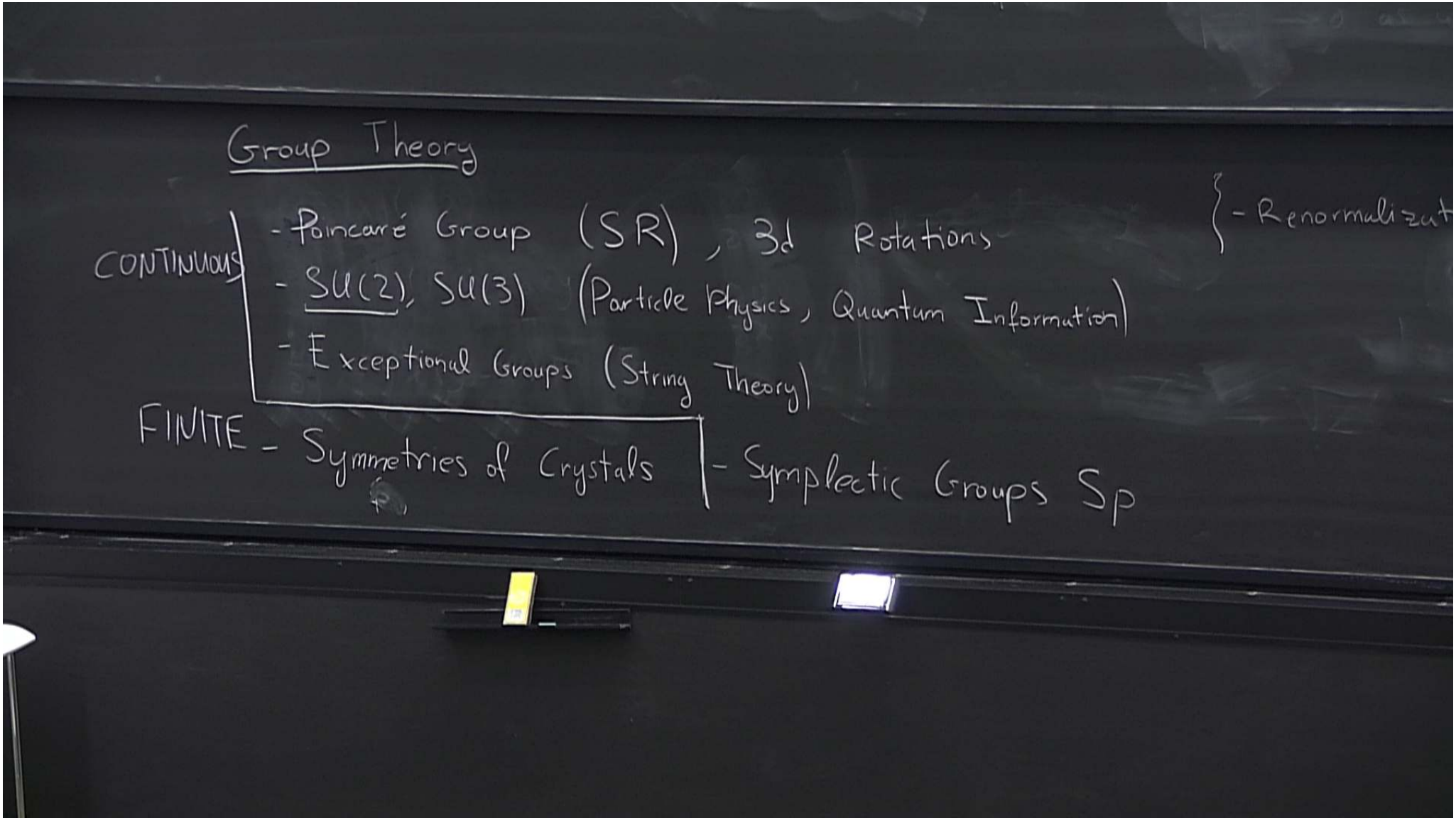
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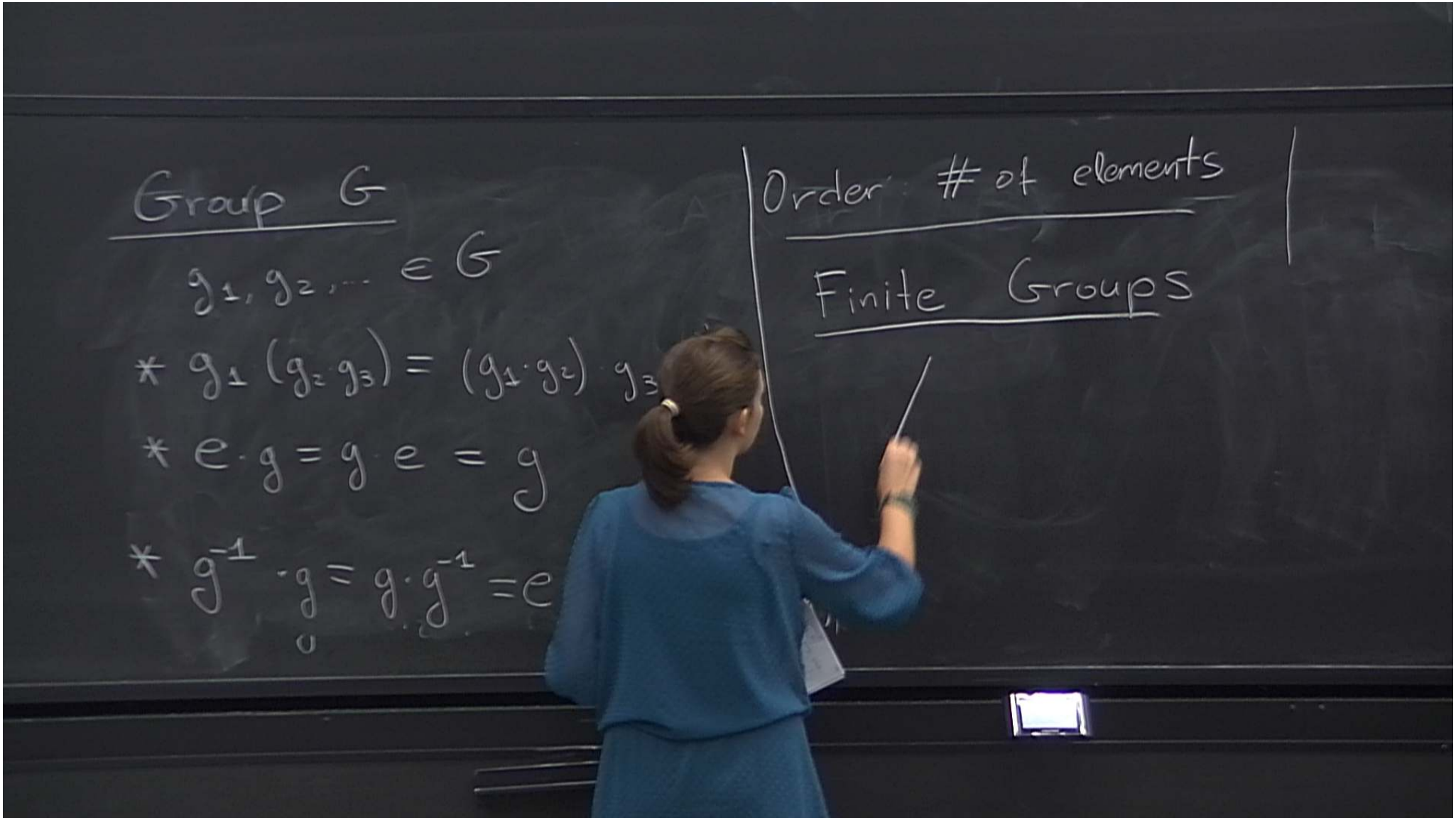
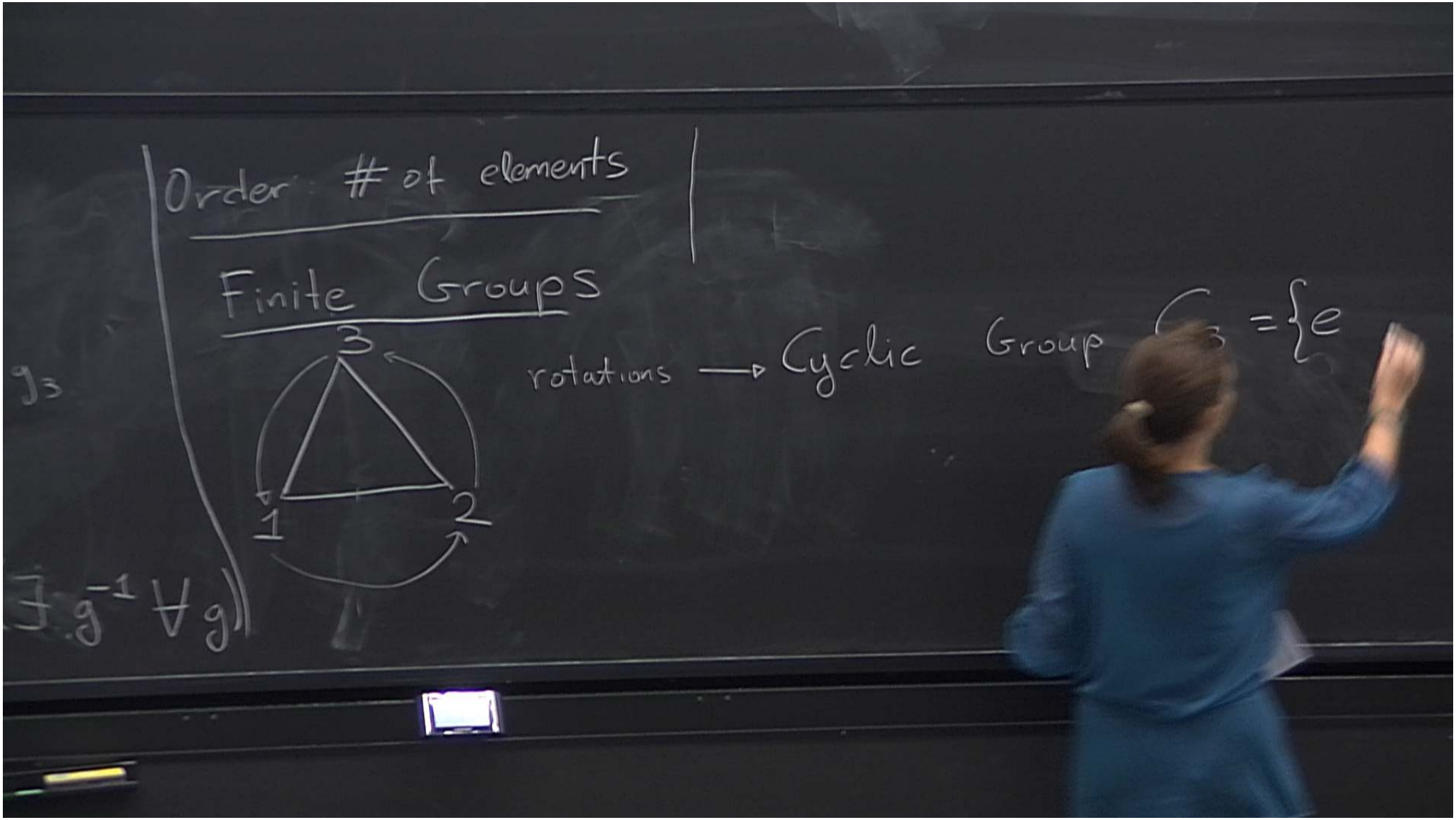
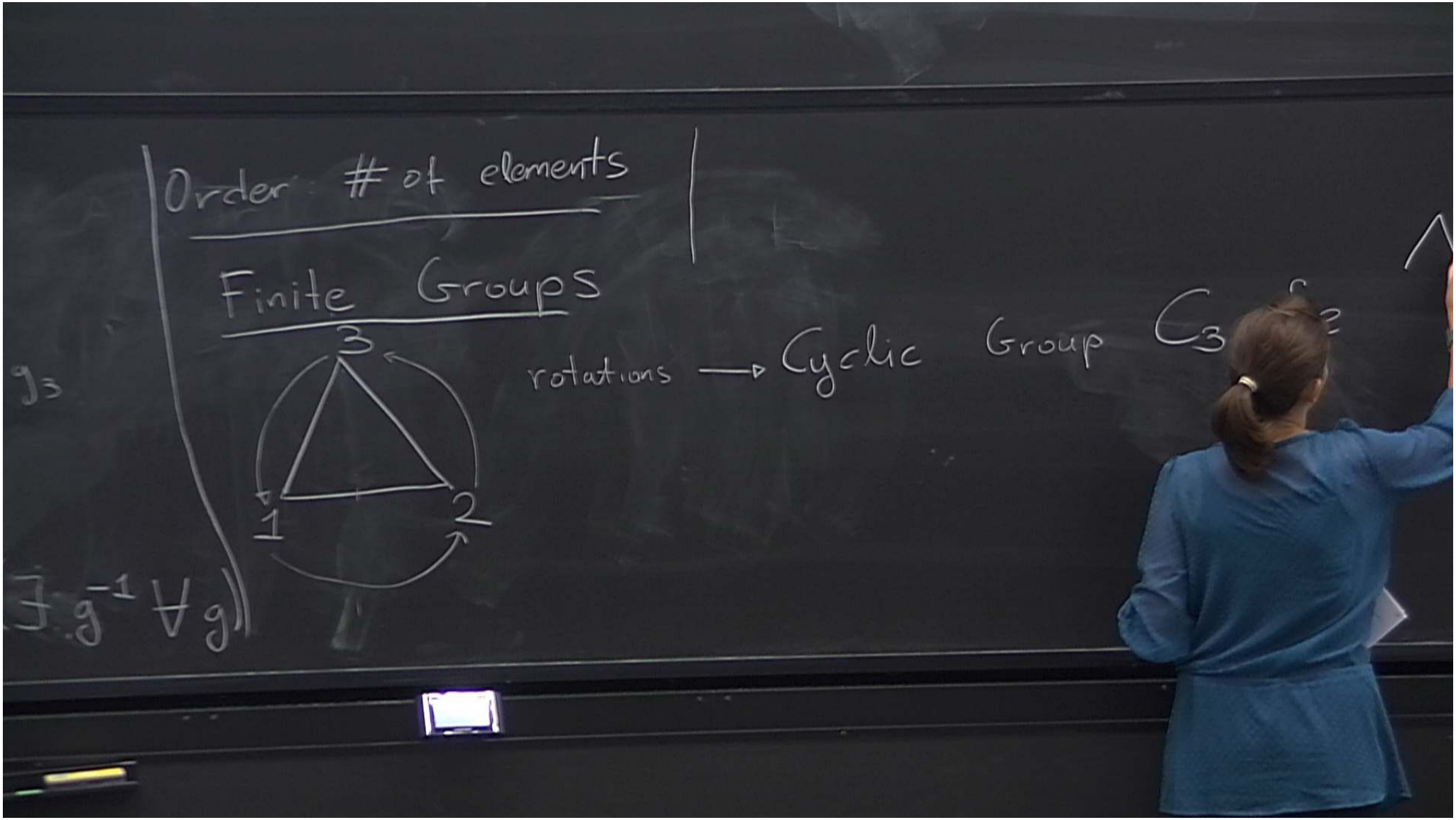
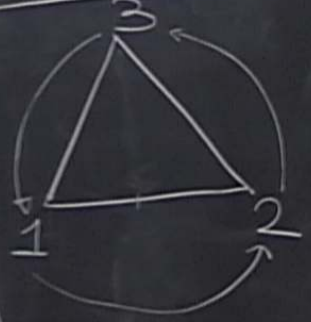

Group G
$g_1, g_2, \ldots \in G$
$* \; g_1 (g_2 g_3) = (g_1 \cdot g_2) \; g_3$
$* \; e \cdot g = g \; e = g$
$* \; g^{-1} \cdot g = g \cdot g^{-1} = e$
Order: # of elements
Finite Groups


Order: # of elements
Finite Groups
3
1
2
rotations → Cyclic Group
=

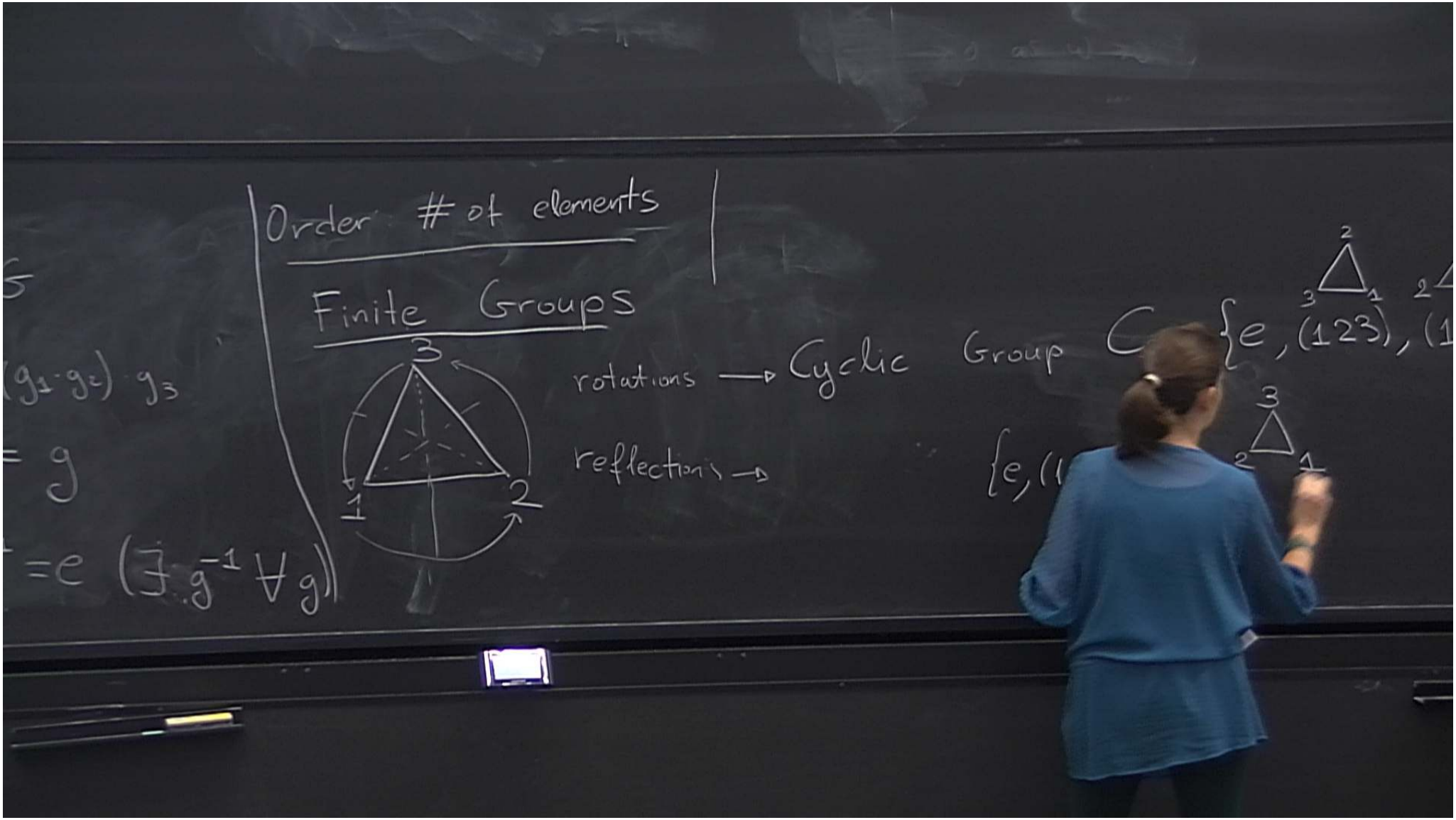

Order # of elements
Finite Groups
3
1
2
rotations → Cyclic Group C_3
$\exists g^{-1} \forall g$

Order # of elements

Finite Groups

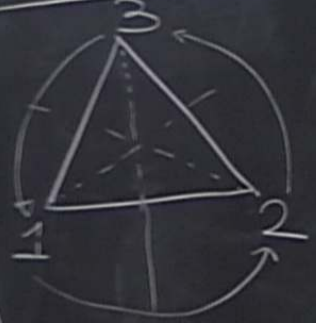


rotations → Cyclic Group $C_3 = \{e, (123), (132)\}$


Order # of elements
Finite Groups
3
1
2
rotations → Cyclic Group C
reflections →
{e, (123),
2
3
1
3
2
1

der # of elements

Finite Groups



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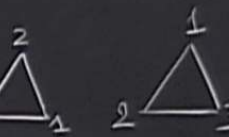
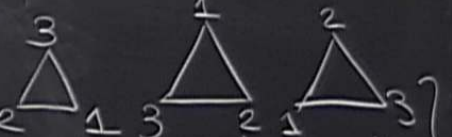
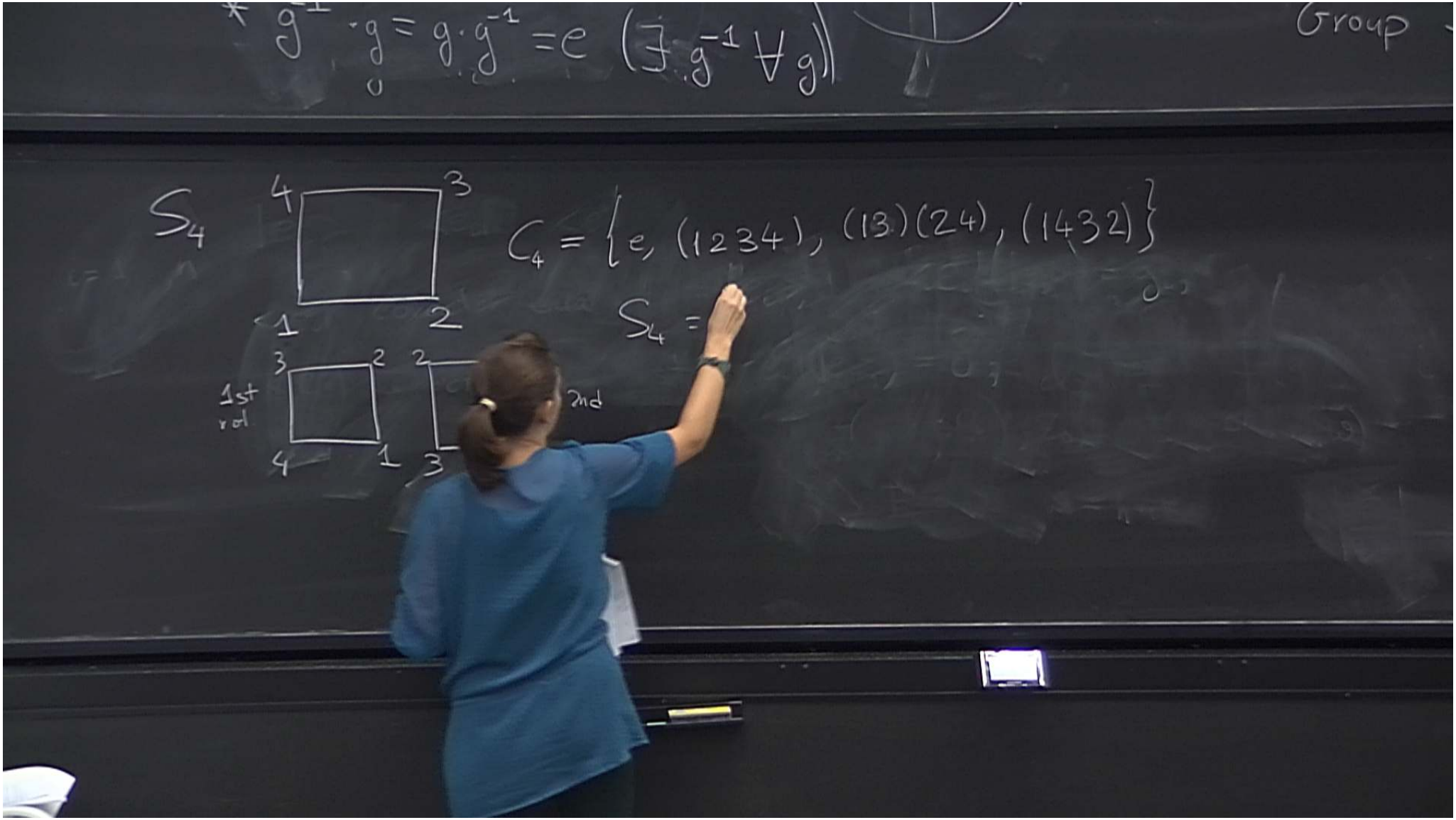
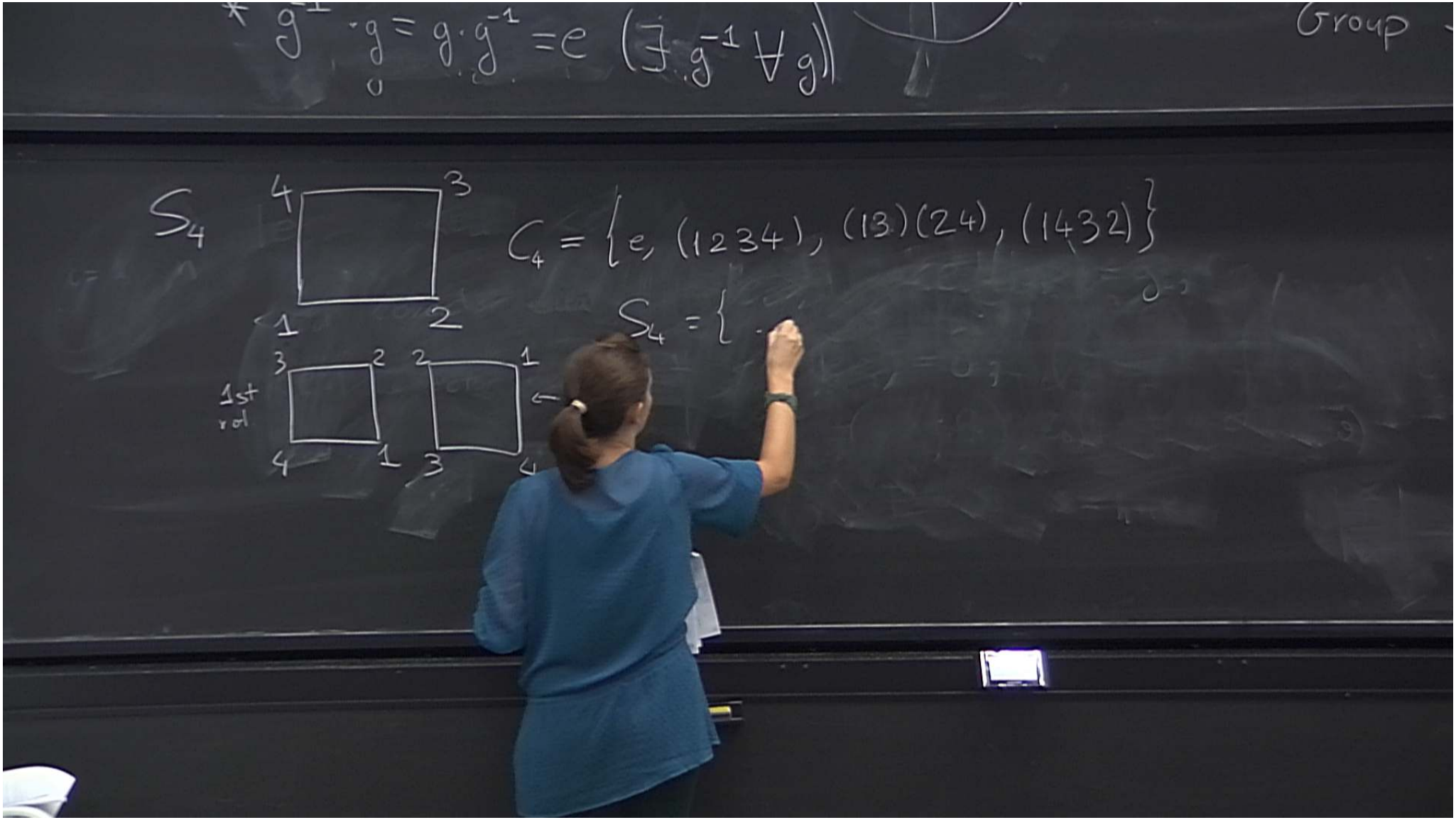


reflections → Permutation Group S_3 $\{e, (123), (132), (12), (13), (23)\}$




$g^{-1} \cdot g = g \cdot g^{-1} = e \quad (\exists\, g^{-1} \; \forall\, g)$
Group
S_4
4 3 1 2
$C_4 = \{e, (1234), (13)(24), (1432)\}$
$S_4 =$
1st rot
2nd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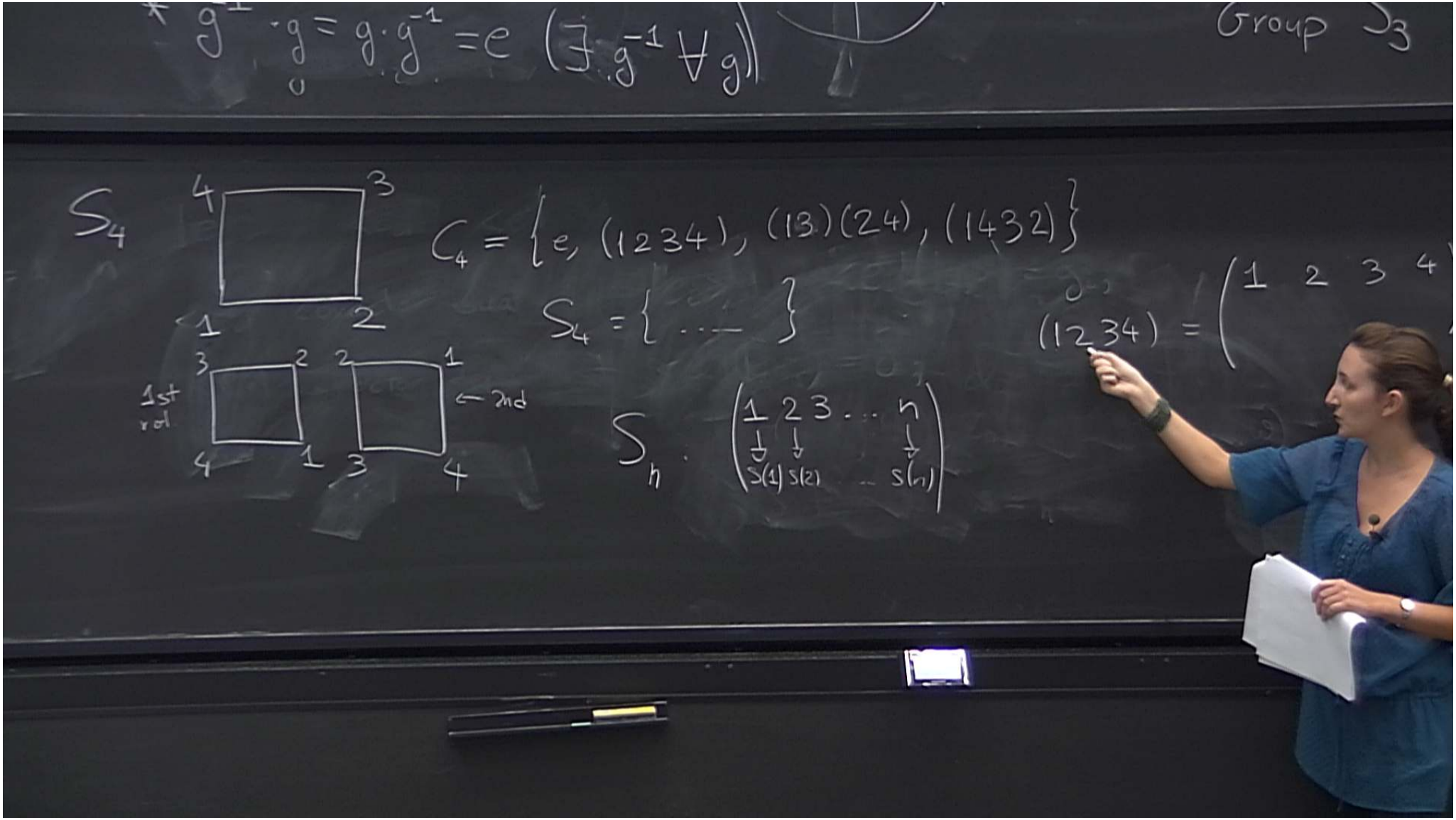
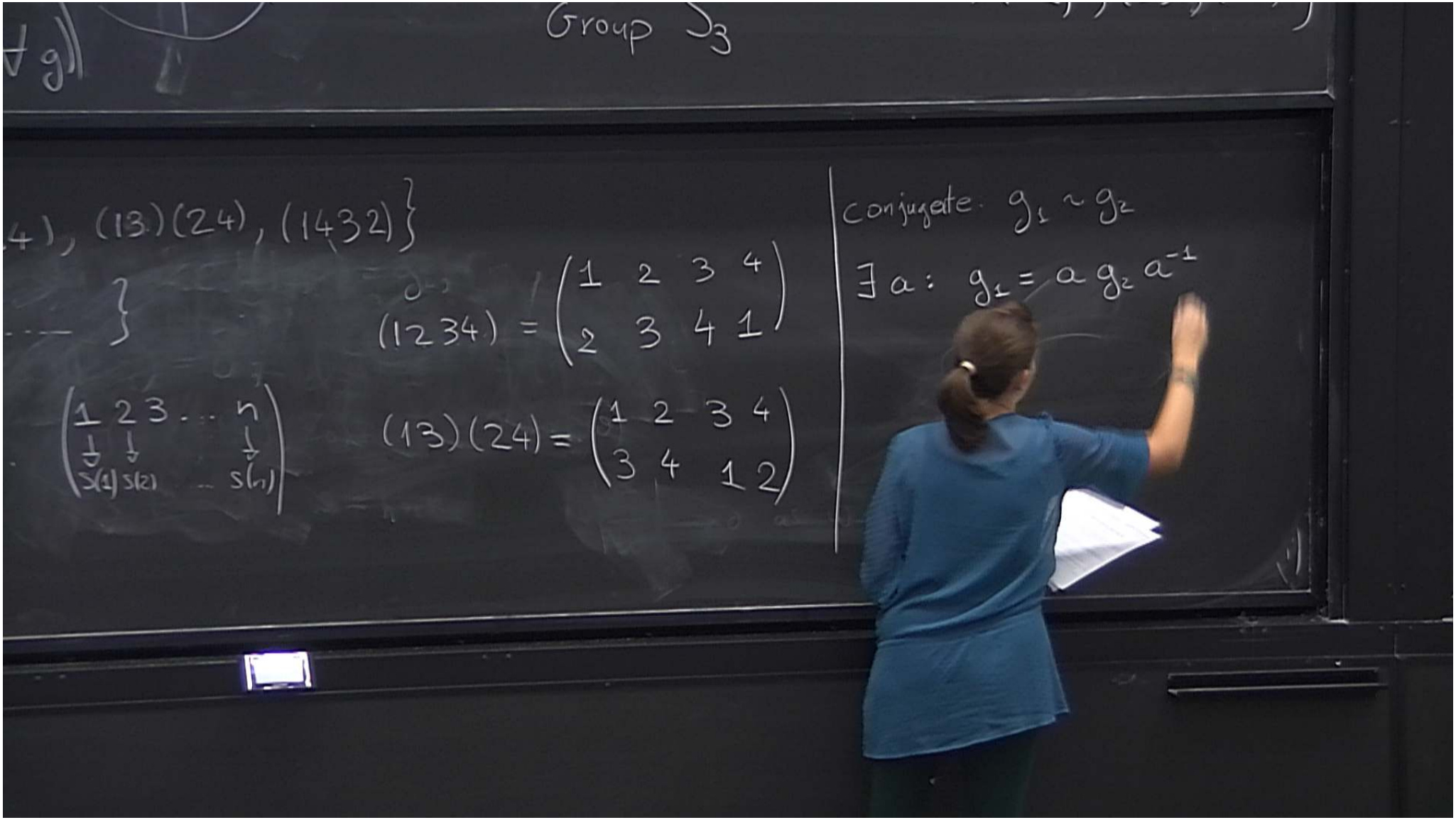
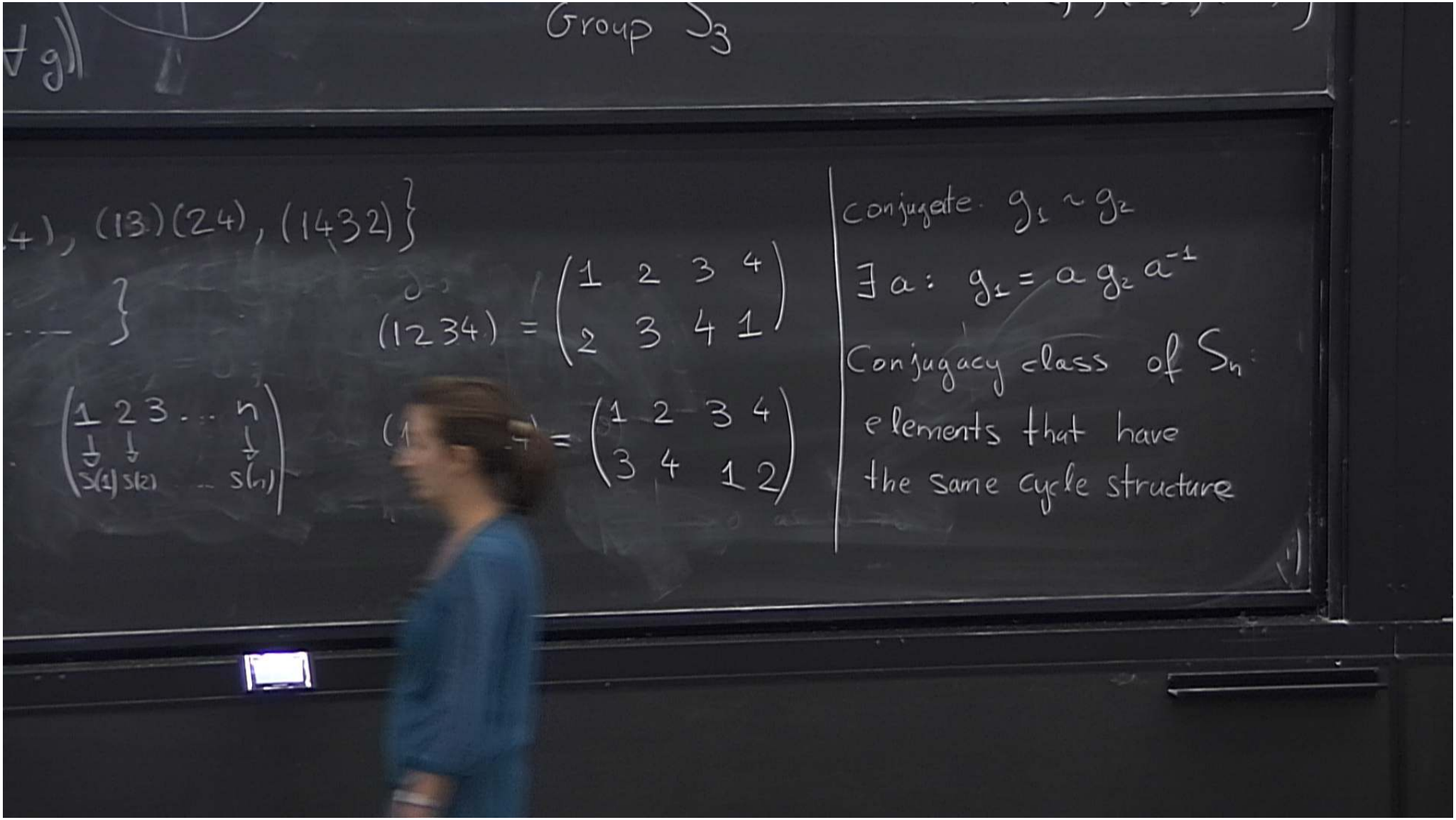
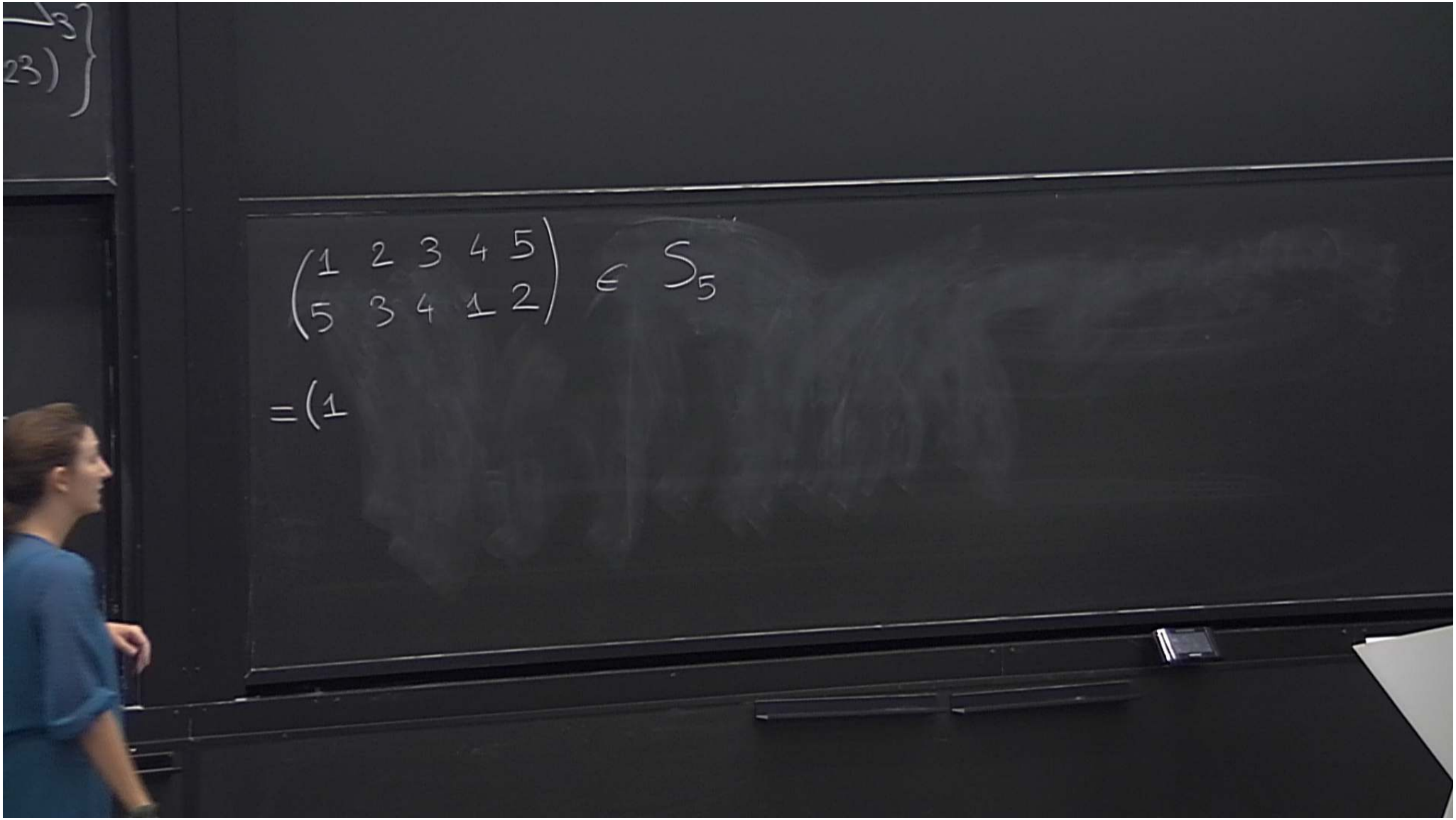
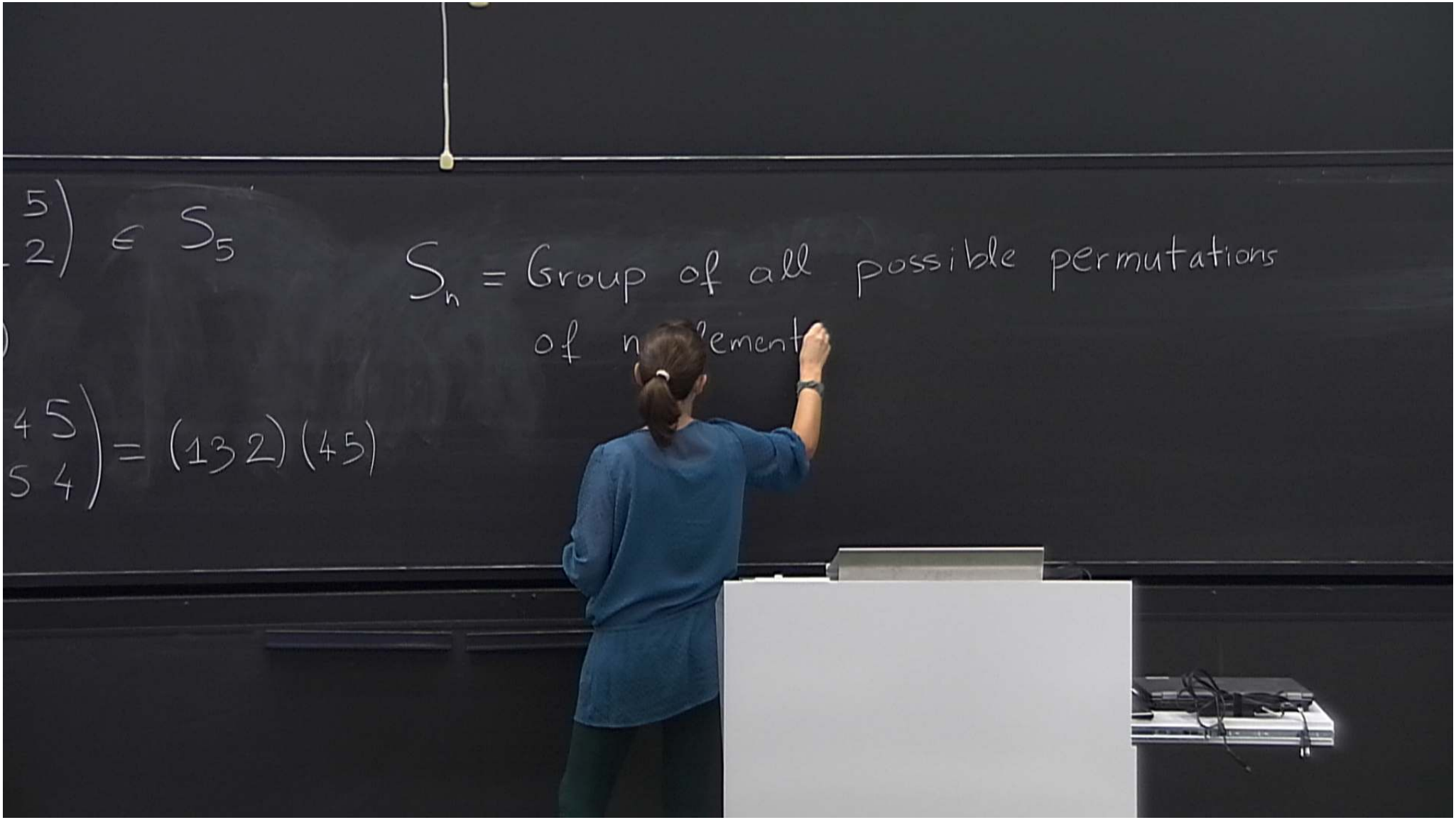
1st rot
3 2 4 1
2 1 3 4


$g^{-1} \cdot g = g \cdot g^{-1} = e \quad (\exists\, g^{-1}\ \forall\, g)$
Group S_3
S_4
4 3 1 2
$C_4 = \{e, (1234), (13)(24), (1432)\}$
$S_4 = \{ \ldots \}$
$(1234) = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}$
3 2 4 1
1st rot
2 1 3 4
← 2nd
$S_n \quad \begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ \downarrow & \downarrow & & & \downarrow \\ s(1) & s(2) & & \ldots & s(n) \end{pmatrix}$


Group S_3
$(4), (13)(24), (1432)\}$
$(1234) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$
$\begin{pmatrix} 1 & 2 & 3 & \dots & n \\ \downarrow & \downarrow & & & \downarrow \\ s(1) & s(2) & & \dots & s(n) \end{pmatrix}$
$(13)(24) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$
conjugate $g_1 \sim g_2$
$\exists a: g_1 = a g_2 a^{-1}$


Group S_3
$(13)(24)$, (1432)
$(1234) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$
$\begin{pmatrix} 1 & 2 & 3 & \dots & n \\ \downarrow & \downarrow & & & \downarrow \\ s(1) & s(2) & \dots & & s(n) \end{pmatrix}$
$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$
conjugate $g_1 \sim g_2$
$\exists a: g_1 = a g_2 a^{-1}$
Conjugacy class of S_n:
elements that have
the same cycle structure

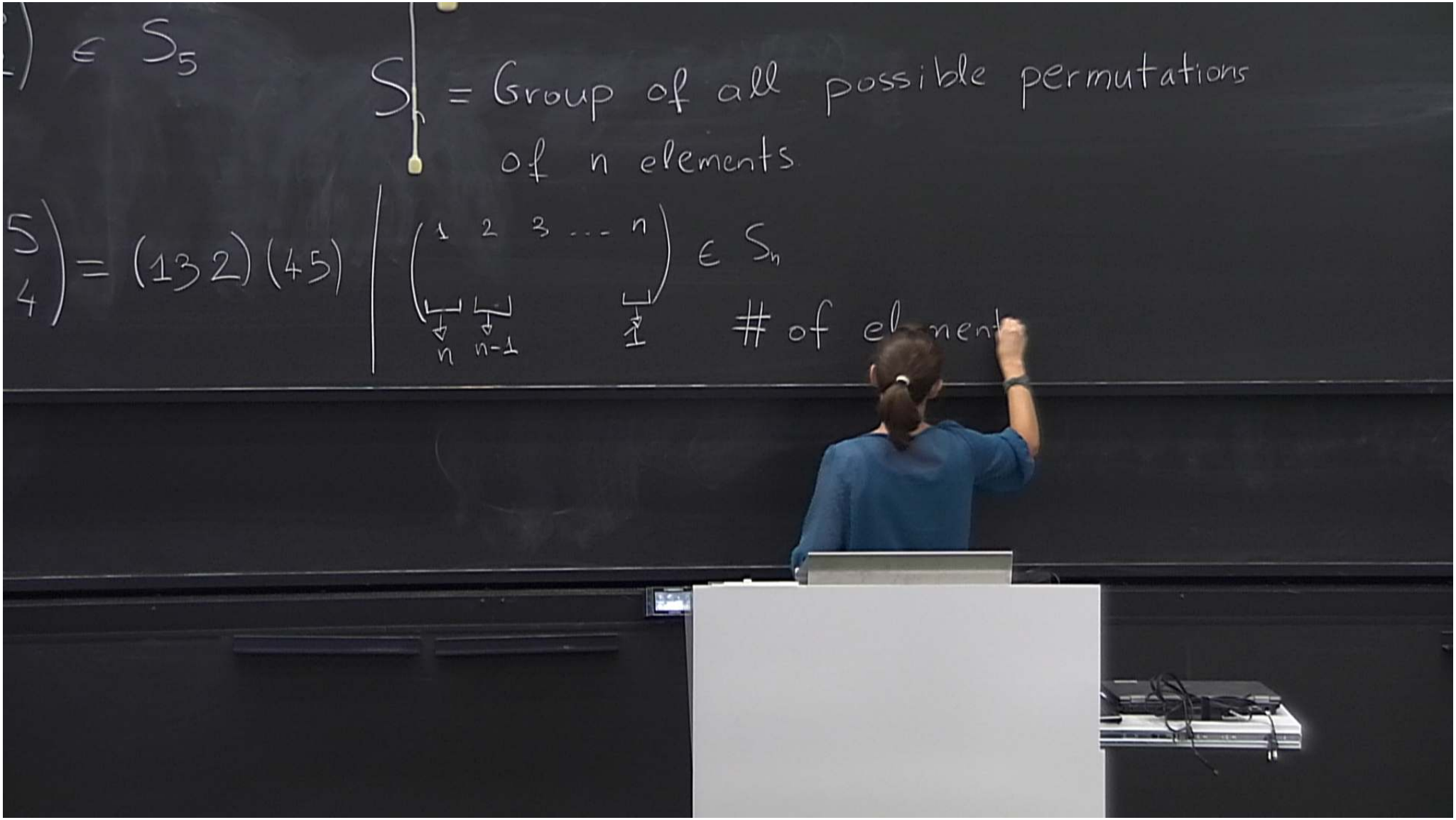

23)
$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 4 & 1 & 2 \end{pmatrix} \in S_5$
$= (1$

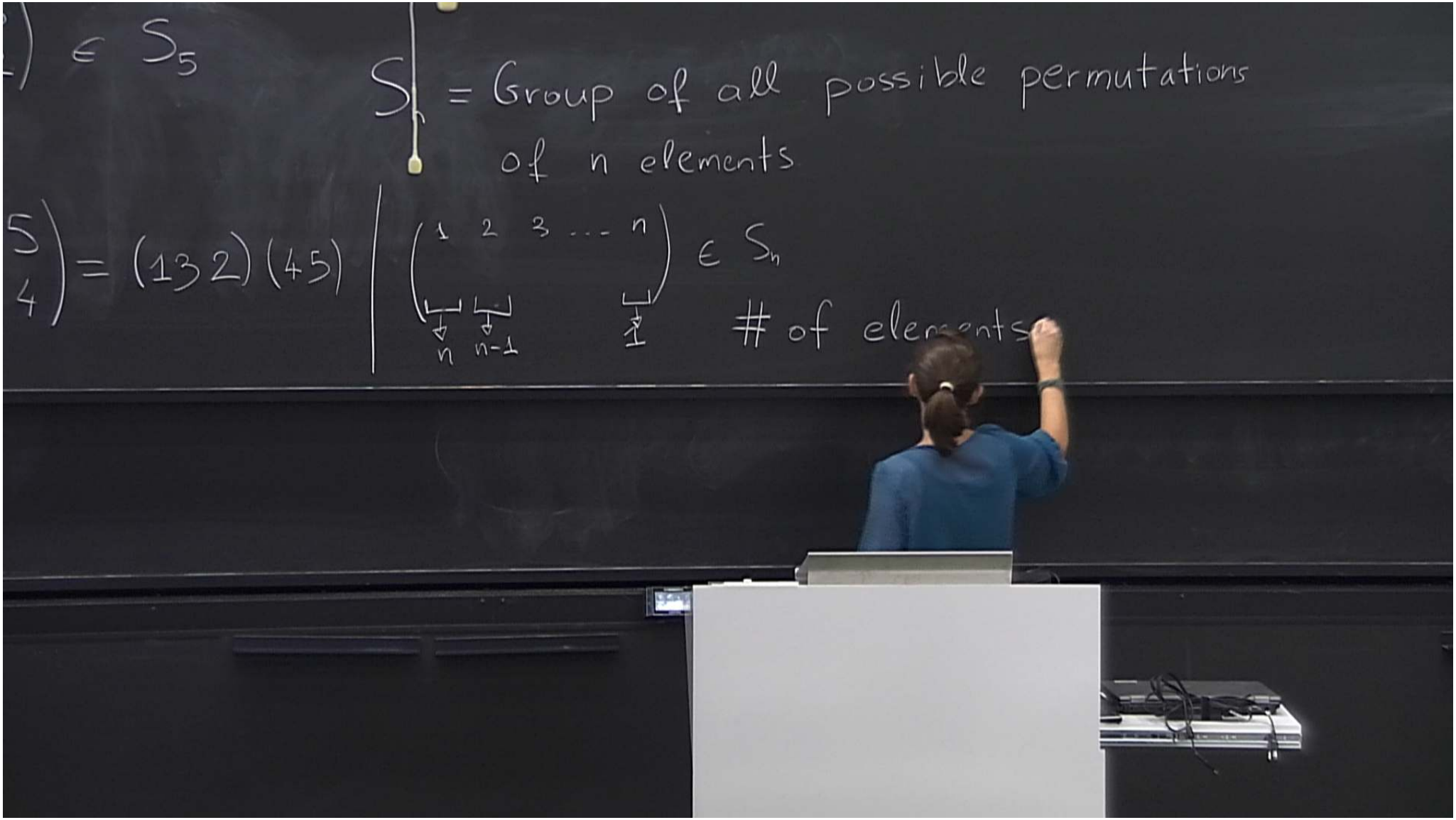

ϵS_5
S_n = Group of all possible permutations
of n
$= (132)(45)$

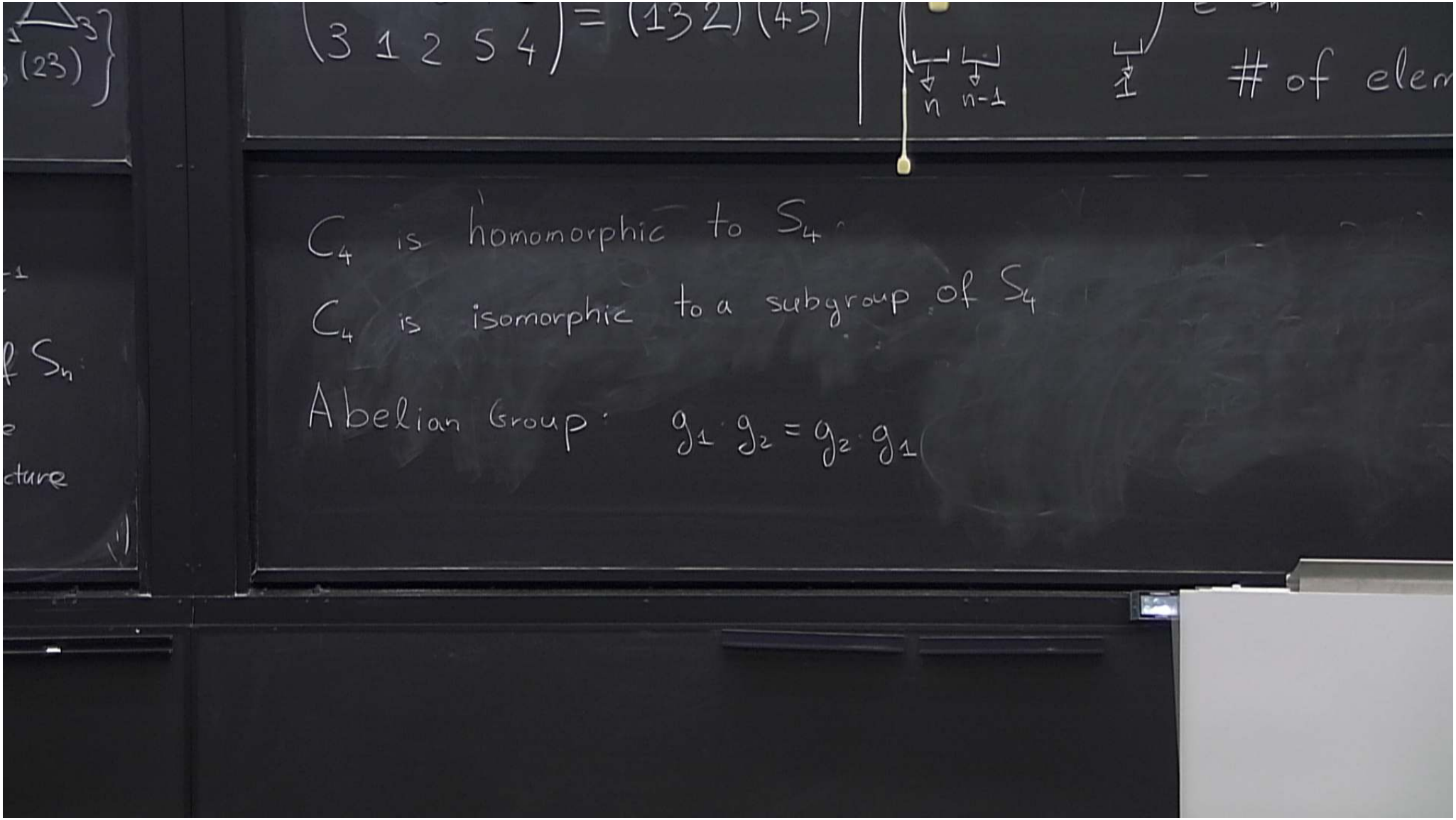
$\begin{pmatrix} 5 \\ 2 \end{pmatrix} \in S_5$

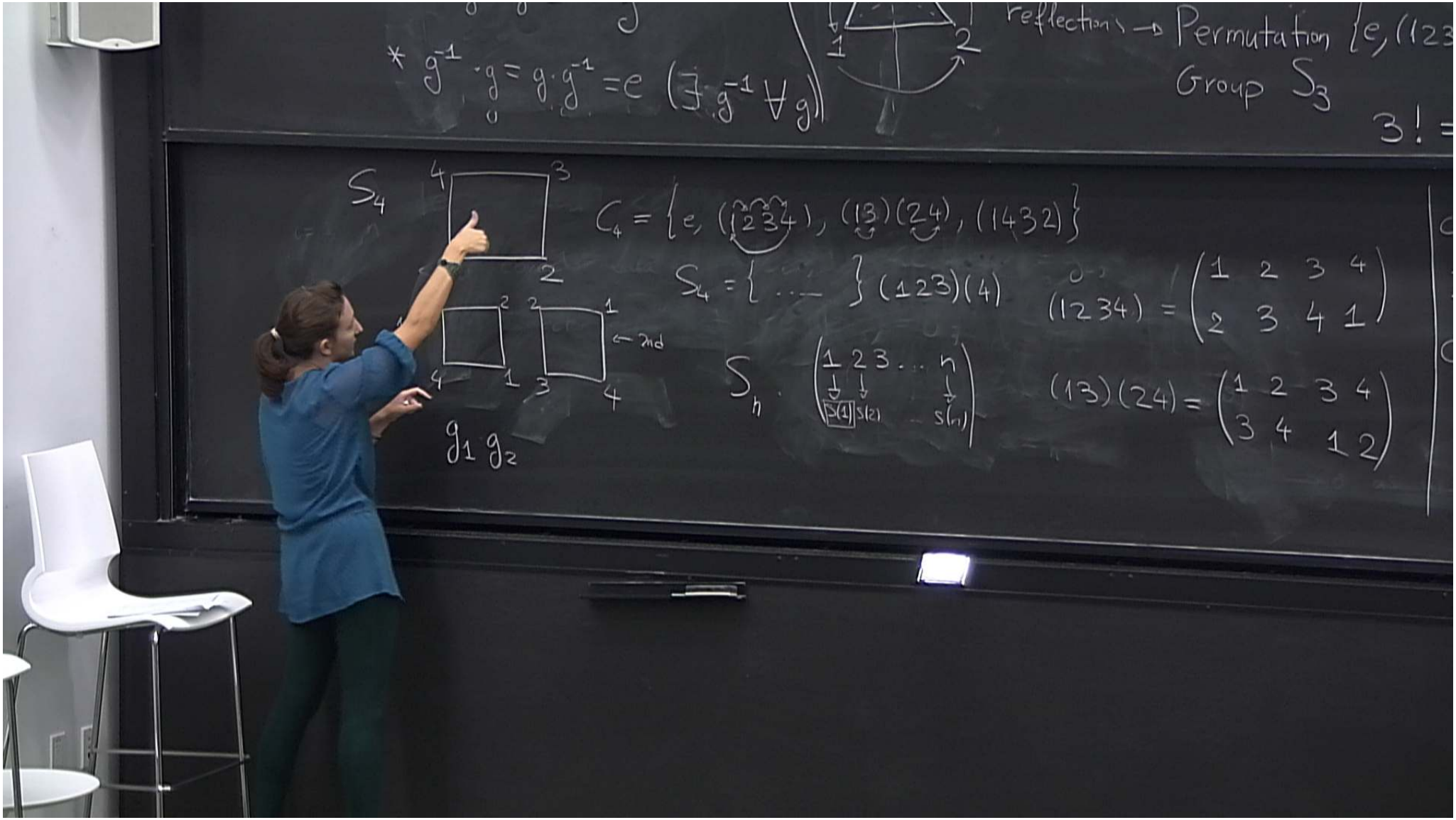
S_n = Group of all possible permutations of n elements

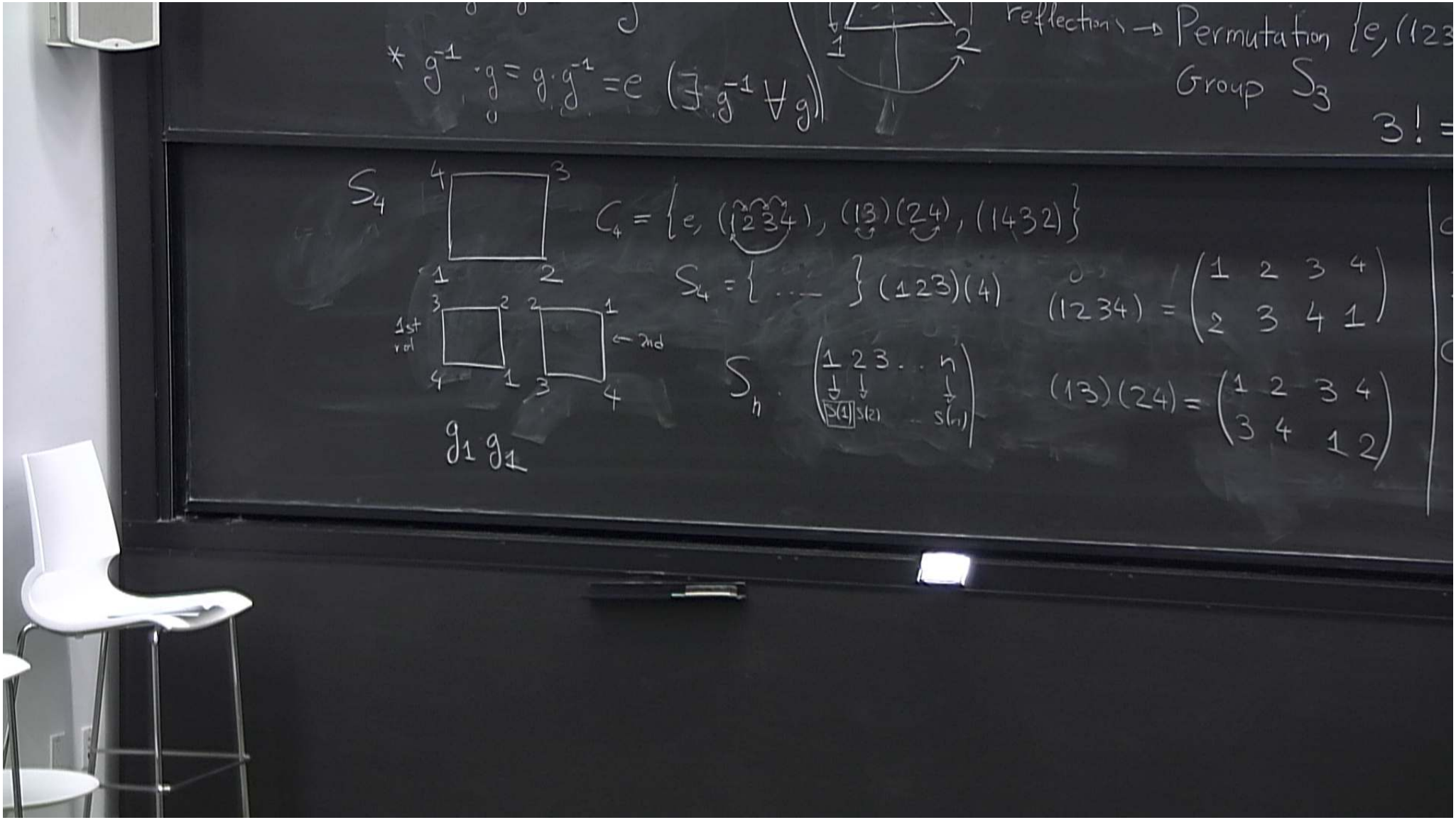
$\begin{pmatrix} 4 & 5 \\ 5 & 4 \end{pmatrix} = (132)(45)$

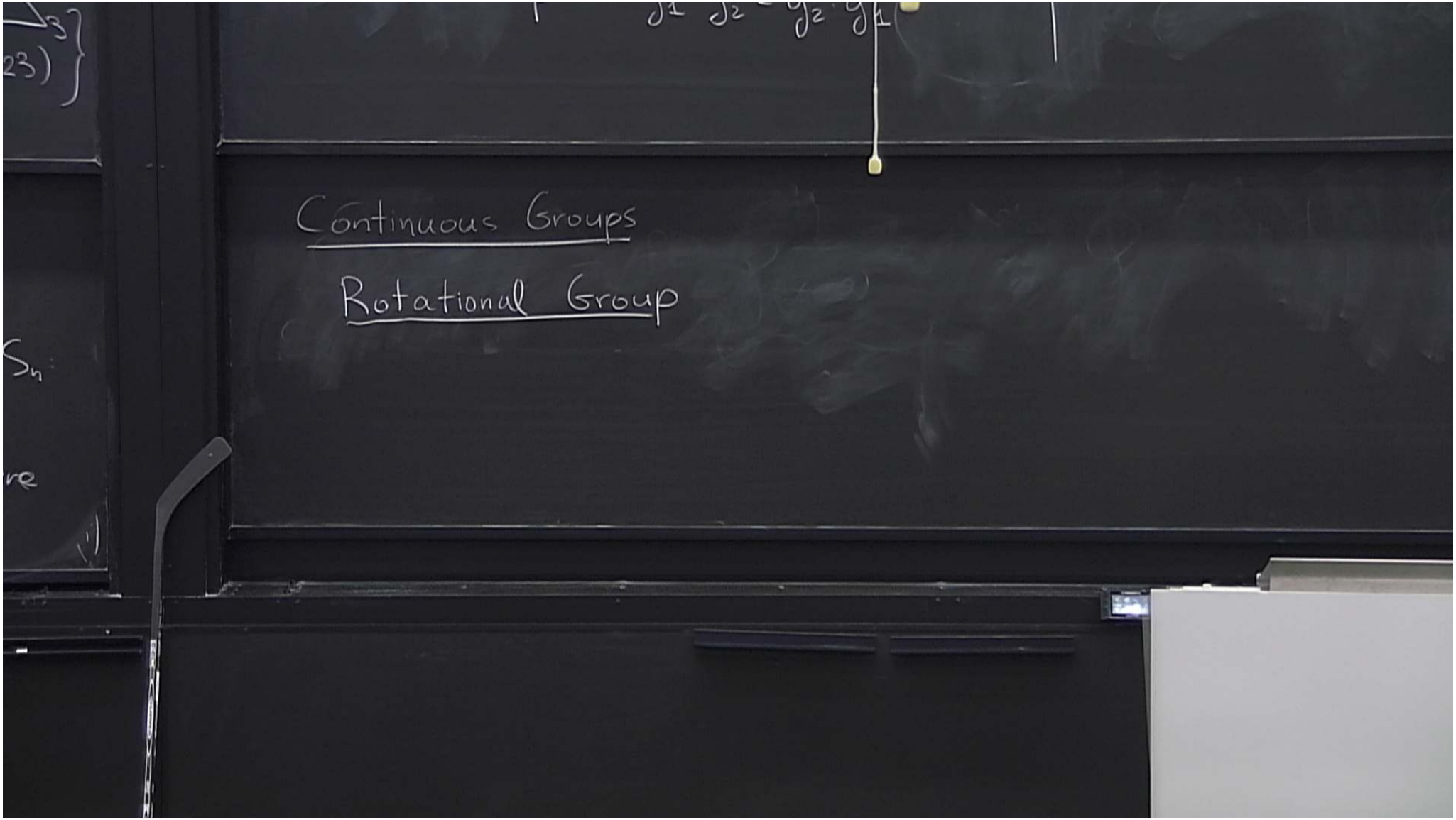

$\in S_5$
S_n = Group of all possible permutations of n elements
$= (132)(45)$
$\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \downarrow & \downarrow & & & \downarrow \\ n & n-1 & & & 1 \end{pmatrix} \in S_n$
# of el ment

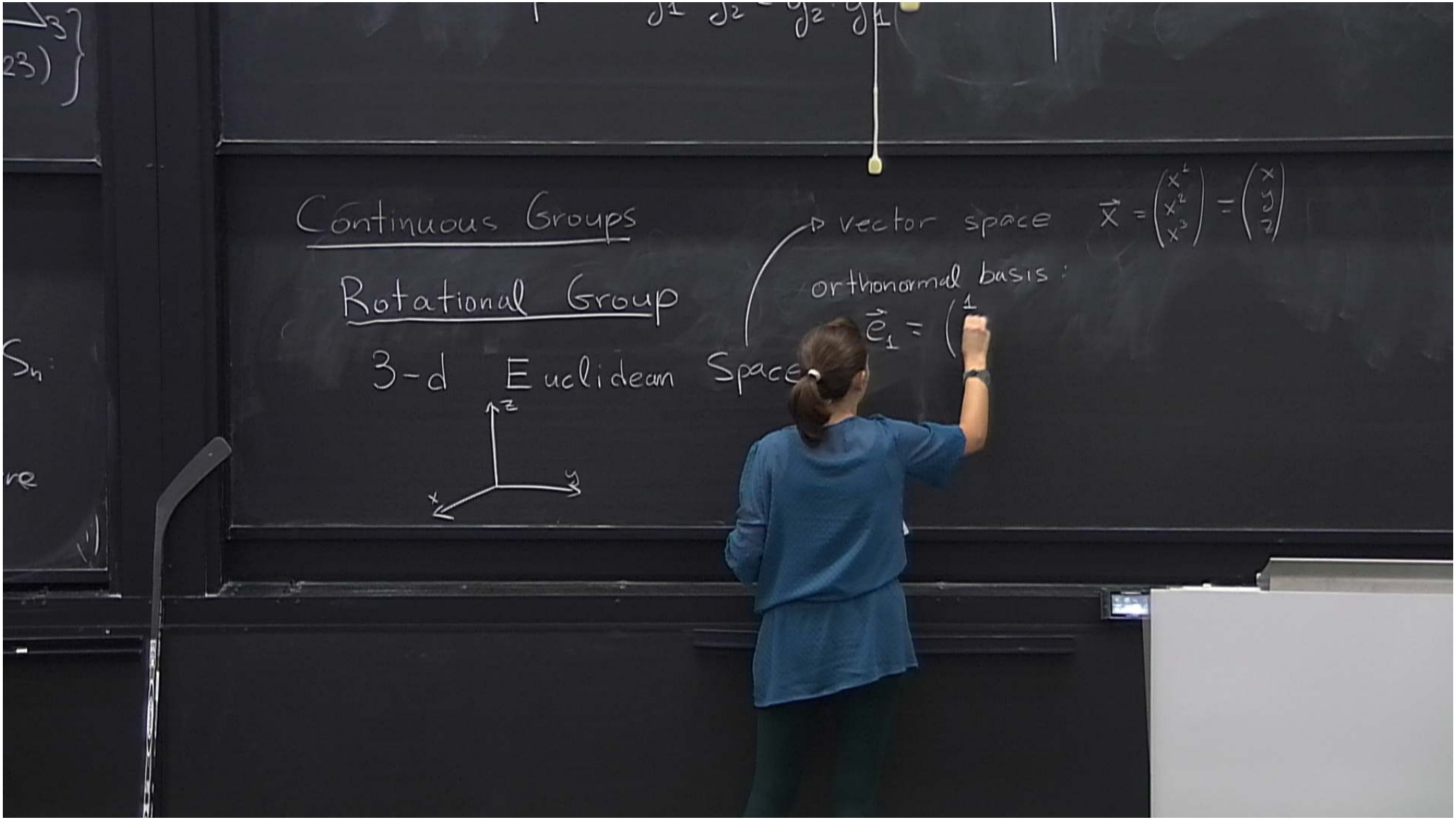

$\in S_5$
S_n = Group of all possible permutations of n elements
$= (132)(45)$
$\begin{pmatrix} 1 & 2 & 3 & \dots & n \\ & & & & \end{pmatrix} \in S_n$
n
n-1
1
# of elements

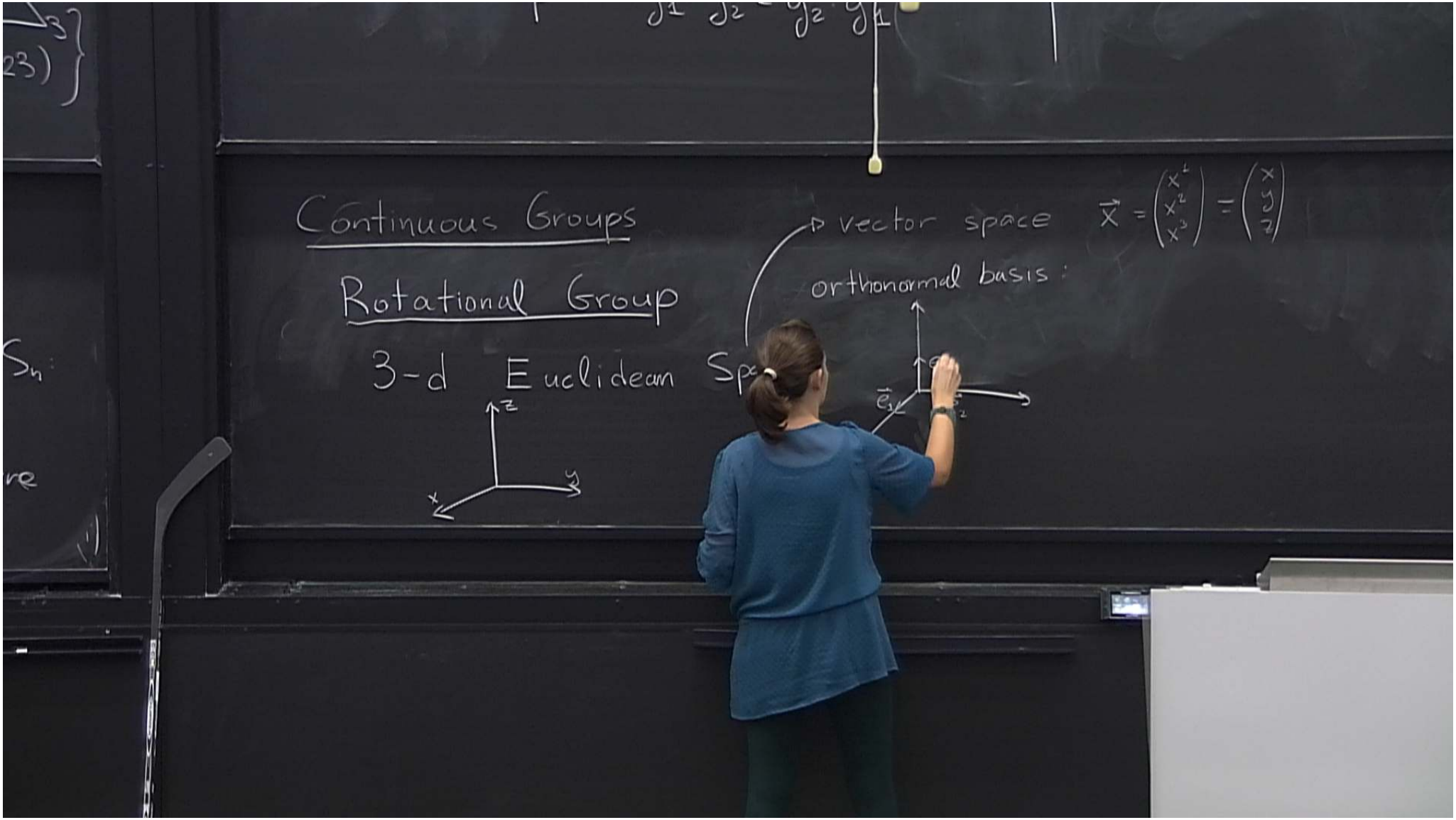

(23)
$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix} = (132)(45)$
n
n-1
1
# of elem
C_4 is homomorphic to S_4.
C_4 is isomorphic to a subgroup of S_4
Abelian Group: $g_1 \cdot g_2 = g_2 \cdot g_1$


* $g^{-1} \cdot g = g \cdot g^{-1} = e$ $(\exists\, g^{-1}\ \forall g)$
reflections → Permutation {e, (123
Group S_3
3! =
S_4
$C_4 = \{e, (1234), (13)(24), (1432)\}$
$S_4 = \{\dots\}$ (123)(4)
$(1234) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$
$(13)(24) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$
S_n
$\begin{pmatrix} 1 & 2 & 3 & \dots & n \\ \downarrow & \downarrow & & & \downarrow \\ s(1) & s(2) & & & s(n) \end{pmatrix}$
← 2nd
g_1 g_2


reflections → Permutation {e, (123
Group S_3
3! =
S_4
$C_4 = \{e, (1234), (13)(24), (1432)\}$
$S_4 = \{\dots\}$ (123)(4)
(1234) = $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$
(13)(24) = $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$
1st rot
2nd
$g_1\, g_1$


Continuous Groups
Rotational Group


Continuous Groups
Rotational Group
3-d Euclidean Space
vector space
orthonormal basis:
z
y
x


Continuous Groups
Rotational Group
3-d Euclidean Sp
vector space
orthonormal basis:
z
x
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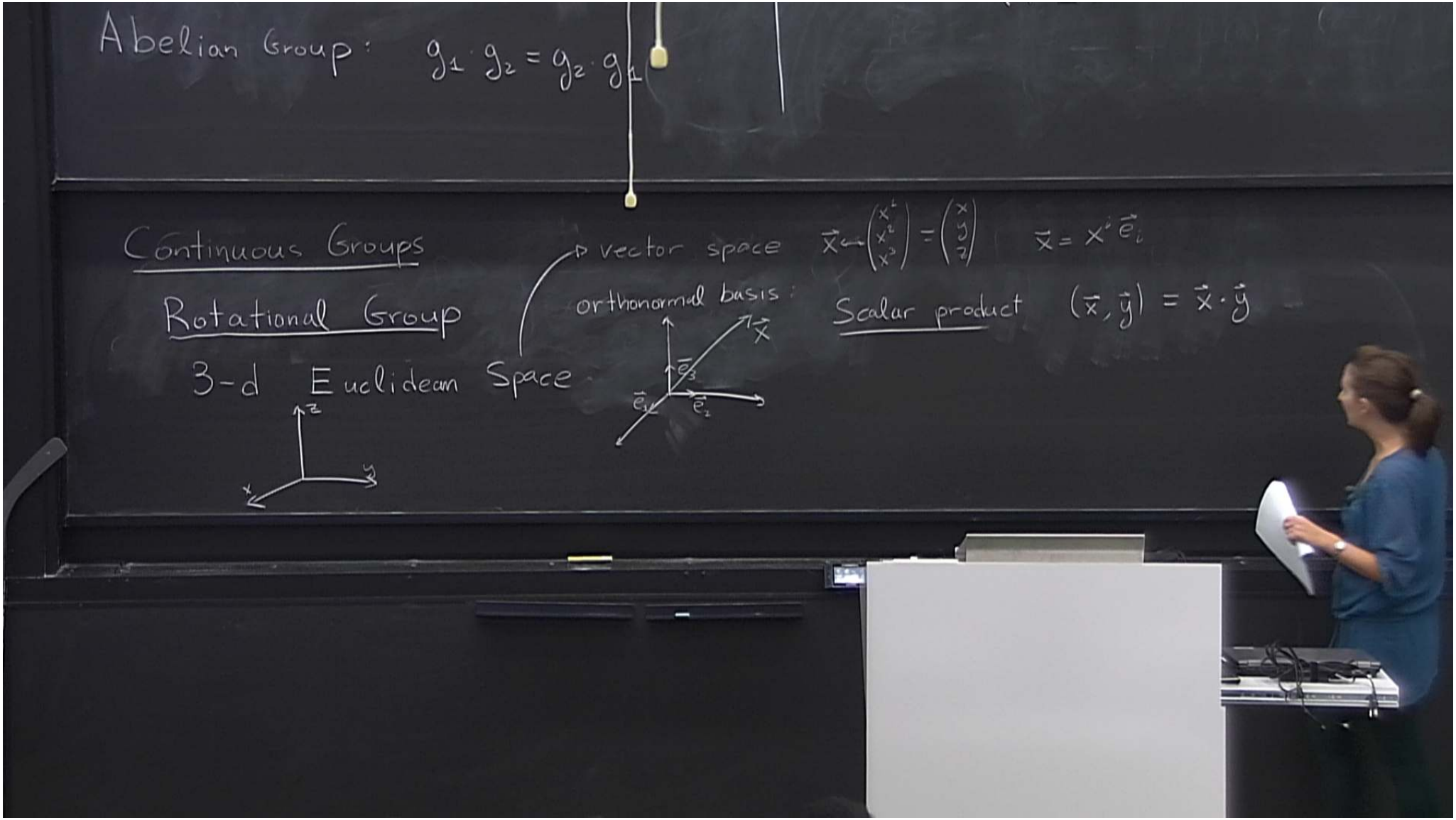
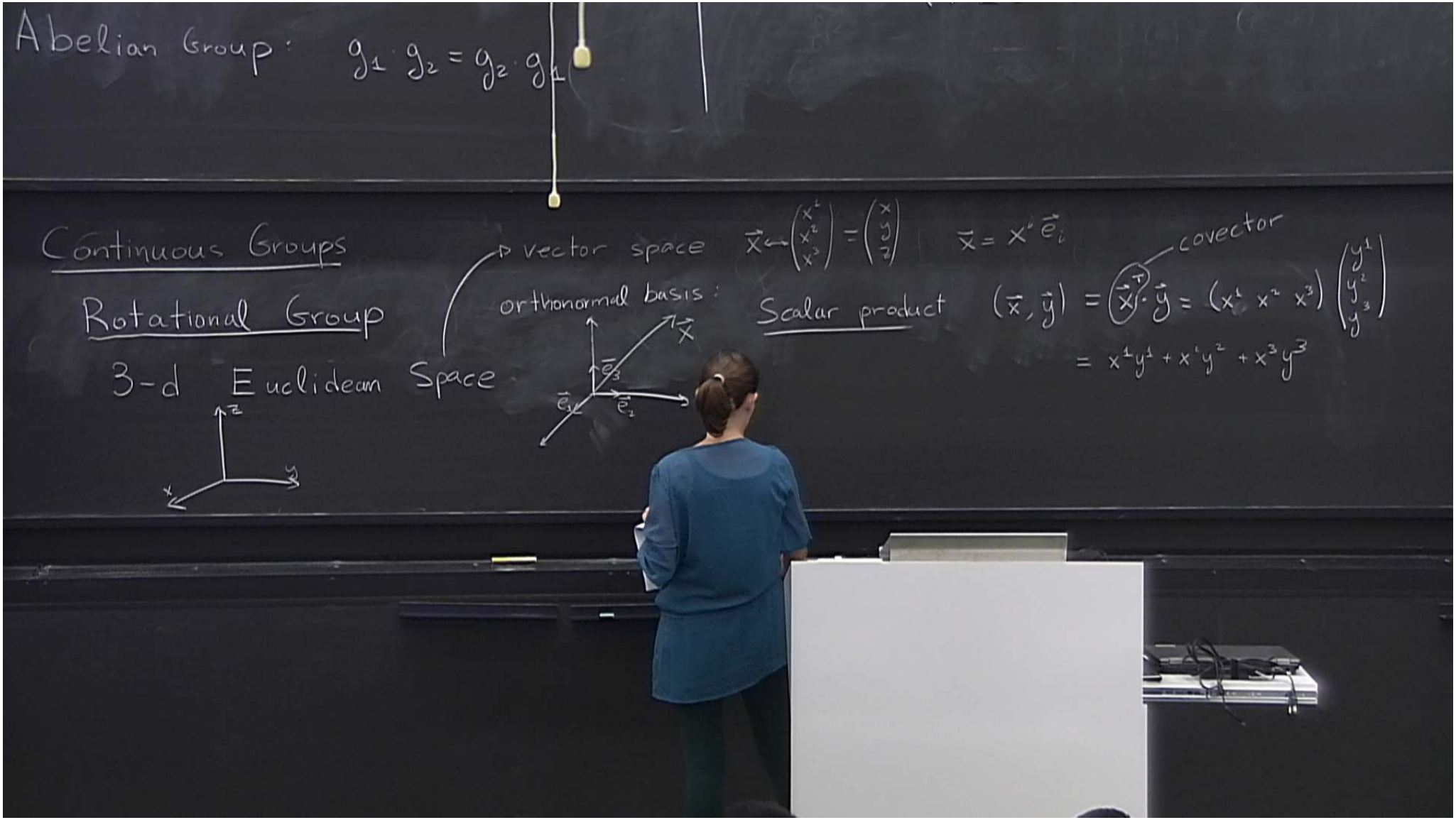

Abelian Group: $g_1 \cdot g_2 = g_2 \cdot g_1$
Continuous Groups
Rotational Group
3-d Euclidean Space
vector space
$\vec{x} \leftrightarrow \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$
$\vec{x} = x^i \vec{e}_i$
orthonormal basis:
$\vec{x}$
$\vec{e}_3$
$\vec{e}_1$
$\vec{e}_2$
Scalar product $(\vec{x}, \vec{y}) = \vec{x} \cdot \vec{y}$
z
y
x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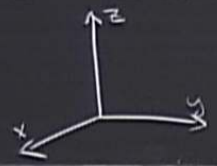
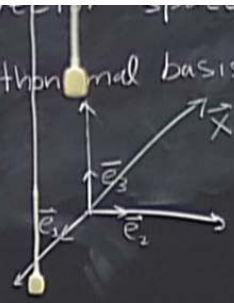
$\vec{x} = x^i \vec{e}_i$
orthonormal basis:
Scalar product
covector
$(\vec{x}, \vec{y}) = \vec{x}^T \cdot \vec{y} = (x^1 \; x^2 \; x^3) \begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix}$
$= x^1 y^1 + x^2 y^2 + x^3 y^3$

Rotational Group

3-d Euclidean Space



$x_i = x^i$



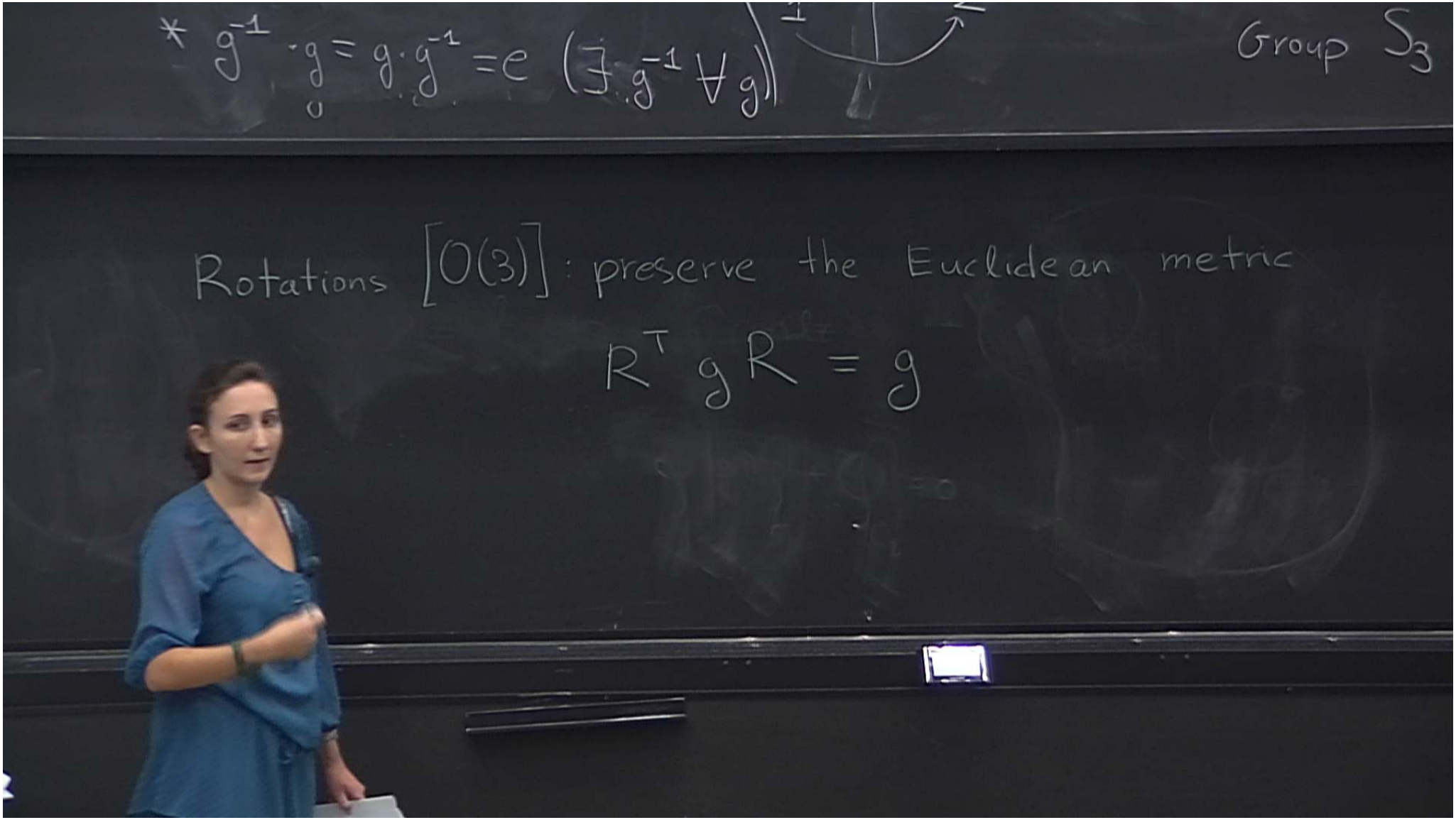
Scalar product $(\vec{x}, \vec{y}) = \vec{x}^T \cdot \vec{y} = (x^1\ x^2\ x^3) \begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix}$

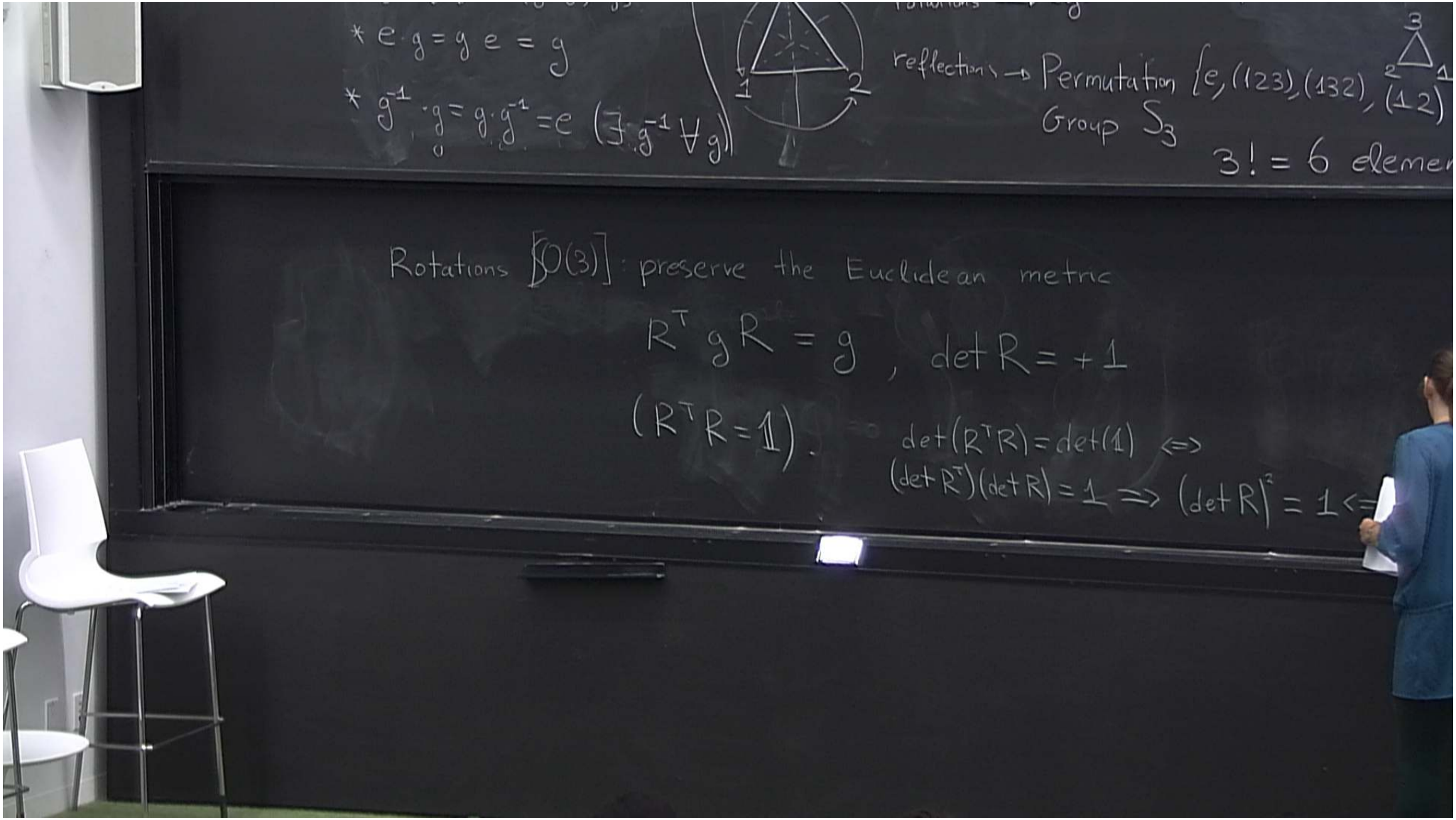
$= x^1 y^1 + x^2 y^2 + x^3 y^3$

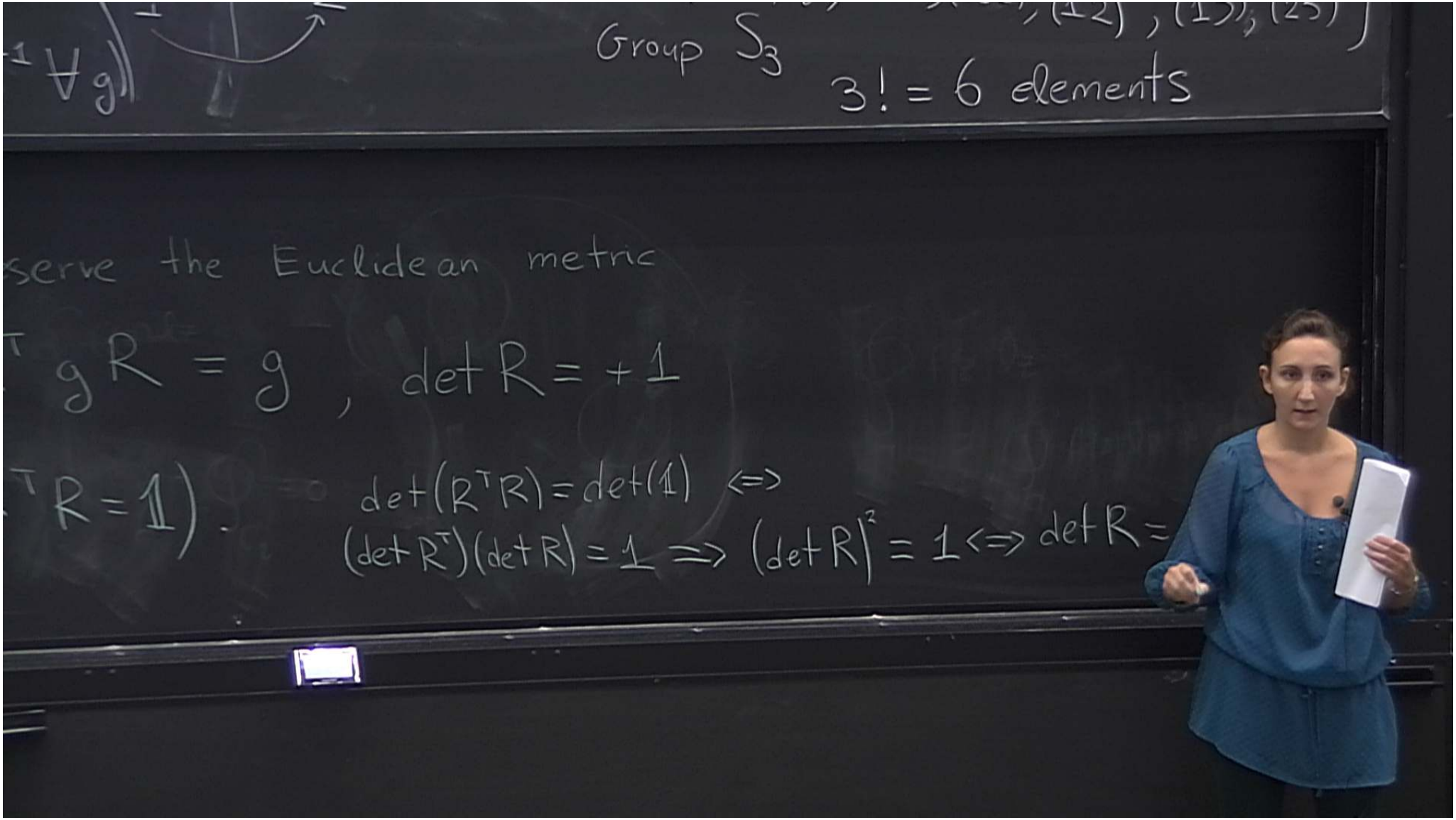
$\vec{x}^T \equiv (x_1, x_2, x_3) = (x^1, x^2, x^3)$

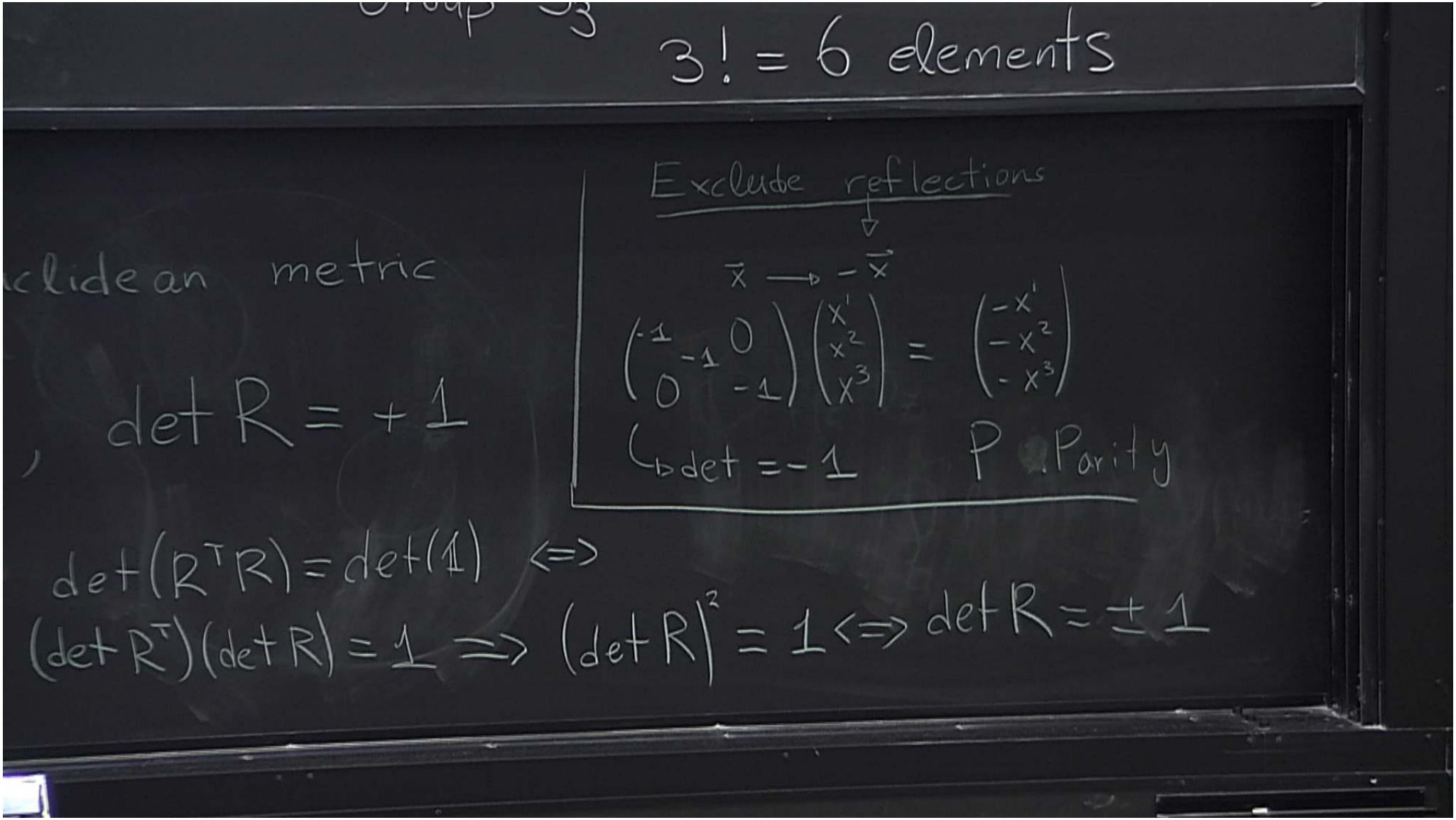
$g_{ij} = \vec{e}_i \cdot \vec{e}_j = \delta_{ij}$ flat Euclidean metric

Rotations transformations that leave the distance from origin invariant


$* \; g^{-1} \cdot g = g \cdot g^{-1} = e \quad (\exists \, g^{-1} \; \forall \, g)$
Group S_3
Rotations $[O(3)]$: preserve the Euclidean metric
$R^T g R = g$


* e·g = g·e = g
* g⁻¹·g = g·g⁻¹ = e (∃ g⁻¹ ∀ g)
reflections → Permutation {e, (123), (132), (12)
Group S3
3! = 6 elemen
Rotations [O(3)]: preserve the Euclidean metric
RᵀgR = g , det R = +1
(RᵀR = 𝟙).
det(RᵀR) = det(𝟙) ⇔
(det Rᵀ)(det R) = 1 ⇒ (det R)² = 1 ⇐


Group S_3
$3! = 6$ elements
serve the Euclidean metric
$gR = g$, $\det R = +1$
$R = \mathbb{1}$).
$\det(R^T R) = \det(\mathbb{1}) \Leftrightarrow$
$(\det R^T)(\det R) = 1 \Rightarrow (\det R)^2 = 1 \Leftrightarrow \det R =$


Group S_3
$3! = 6$ elements
Euclidean metric
$\det R = +1$
Exclude reflections
$\vec{x} \longrightarrow -\vec{x}$
$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} -x^1 \\ -x^2 \\ -x^3 \end{pmatrix}$
$\det = -1$ P: Parity
$\det(R^T R) = \det(\mathbb{1}) \iff$
$(\det R^T)(\det R) = 1 \Rightarrow (\det R)^2 = 1 \iff \det R = \pm 1$