

Title: Algebra - Lecture 2

Date: Aug 15, 2012 10:30 AM

URL: <http://pirsa.org/12080016>

Abstract:

$$\text{Tr}(ABC) = \text{Tr}(BCA) = \text{Tr}(CAB)$$

$$\text{Det}(ABC) = (\text{Det } A) (\text{Det } B) (\text{Det } C)$$

$$\text{Tr}(ABC) = \text{Tr}(BCA) = \text{Tr}(CAB)$$

$$\text{Det}(ABC) = (\text{Det } A) (\text{Det } B) (\text{Det } C)$$

- Traces of Matrices
- Determinants
- Diagonalization

Jogla

$$\text{Tr}(ABC) = \text{Tr}(BCA) = \text{Tr}(CAB)$$

$$\text{Det}(ABC) = (\text{Det } A) (\text{Det } B) (\text{Det } C)$$

- Traces of Matrices
- Determinants
- Diagonalization

Jogle

$i=1, \dots, n$

$|e_i\rangle$ $\langle e_i|$ $\langle e^i|$

$\langle e_i|$: covector dual to $|e_i\rangle$

$\langle e^i|$ covector defined st. $\langle e^i|(|e_j\rangle) = \delta^i_j$

Diagonalization

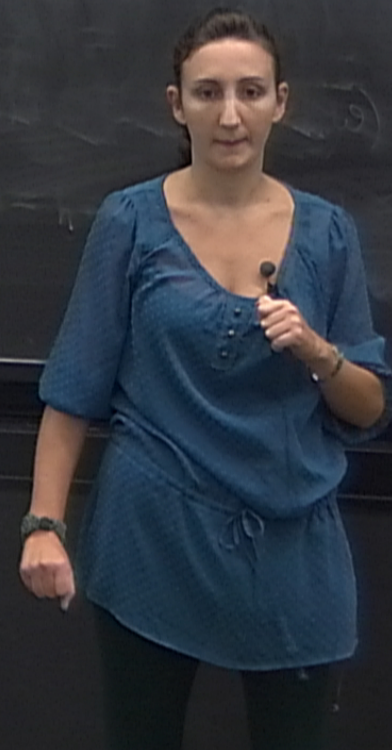
$i=1, \dots, n$

$$|e_i\rangle \quad \langle e_i| \quad \langle e^i|$$

$\langle e_i|$: covector dual to $|e_i\rangle$

$\langle e^i|$: covector defined st $\langle e^i|(e_j\rangle) = \delta^i_j$

$i=1, \dots, n$ $|e_i\rangle$ $\langle e_i|$ $\langle e^i|$
 $\langle e_i|$: covector dual to $|e_i\rangle$ $\langle e_i|(e_j\rangle) = g_{ij}$
 $\langle e^i|$: covector defined st $\langle e^i|(e_j\rangle) = \delta^i_j$



$\langle e_i |$

Group Theory

- Poincaré Group (SR)
- $SU(2)$, $SU(3)$ (Particle Physics, Quantum Information)
- Exceptional Groups (String Theory)

$\langle e_i |$

Group Theory

- Poincaré Group (SR), 3d Rotations
- SU(2), SU(3) (Particle Physics, Quantum Information)
- Exceptional Groups (String)
- Symmetries of Crystals

} - Renormalization

$\langle e_i |$

Group Theory

- Poincaré Group (SR), 3d Rotations
- $SU(2)$, $SU(3)$ (Particle Physics, Quantum Information)
- Exceptional Groups (String Theory)
- Symmetries of fields

} - Renormalization

Group Theory

- Poincaré Group (SR), 3d Rotations
- $SU(2)$, $SU(3)$ (Particle Physics, Quantum Information)
- Exceptional Groups (String Theory)
- Symmetries of Crystals

{ - Renormalization

Group Theory

- Poincaré Group (SR), 3d Rotations
- $SU(2)$, $SU(3)$ (Particle Physics, Quantum Information)
- Exceptional Groups (String Theory)
- Symmetries of Crystals

{ - Renormalizat

Group Theory

- Poincaré Group (SR), 3d Rotations
- $SU(2)$, $SU(3)$ (Particle Physics, Quantum Information)
- Exceptional Groups (String Theory)
- Symmetries of Crystals | - Symplectic Groups Sp

{ - Renormalizati

Group Theory

- Poincaré Group (SR), 3d Rotations
- $SU(2)$, $SU(3)$ (Particle Physics, Quantum Information)
- Exceptional Groups (String Theory)
- Symmetries of Crystals
- Symplectic Groups Sp

{ - Renormalizati

Group Theory

CONTINUOUS

- Poincaré Group (SR), 3d Rotations
- $SU(2)$, $SU(3)$ (Particle Physics, Quantum Information)
- Exceptional Groups (String Theory)

$\left\{ \begin{array}{l} - \text{Renormalizati} \end{array} \right.$

FINITE

- Symmetries of Crystals
- Symplectic Groups Sp

Group Theory

- CONTINUOUS
- Poincaré Group (SR), 3d Rotations
 - $SU(2)$, $SU(3)$ (Particle Physics, Quantum Information)
 - Exceptional Groups (String Theory)

} - Renormalizat

Symmetries of Crystals

- Symplectic Groups Sp

Group Theory

CONTINUOUS

- Poincaré Group (SR), 3d Rotations
- SU(2), SU(3) (Particle Physics, Quantum Information)
- Exceptional Groups (String Theory)

} - Renormalizati

FINITE

- Symmetries of Crystals
- Symplectic Groups Sp

Group G

$$g_1, g_2, \dots \in G$$

$$* g_1 (g_2 g_3) = (g_1 g_2) g_3$$

$$* e \cdot g = g \cdot e = g$$

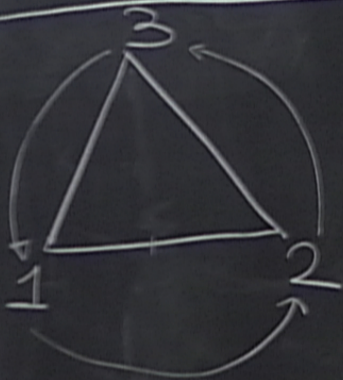
$$* g^{-1} \cdot g = g \cdot g^{-1} = e$$

Order # of elements

Finite Groups

Order: # of elements

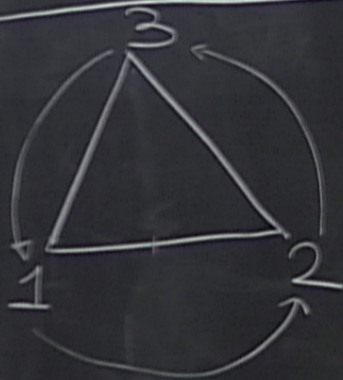
Finite Groups



rotations $\rightarrow$ Cyclic Group $C_3 = \{e$

Order: # of elements

Finite Groups

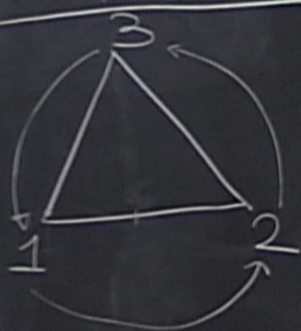


rotations $\rightarrow$ Cyclic Group C_3

Cyclic Group C_3

Order: # of elements

Finite Groups

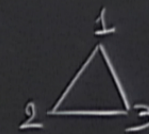


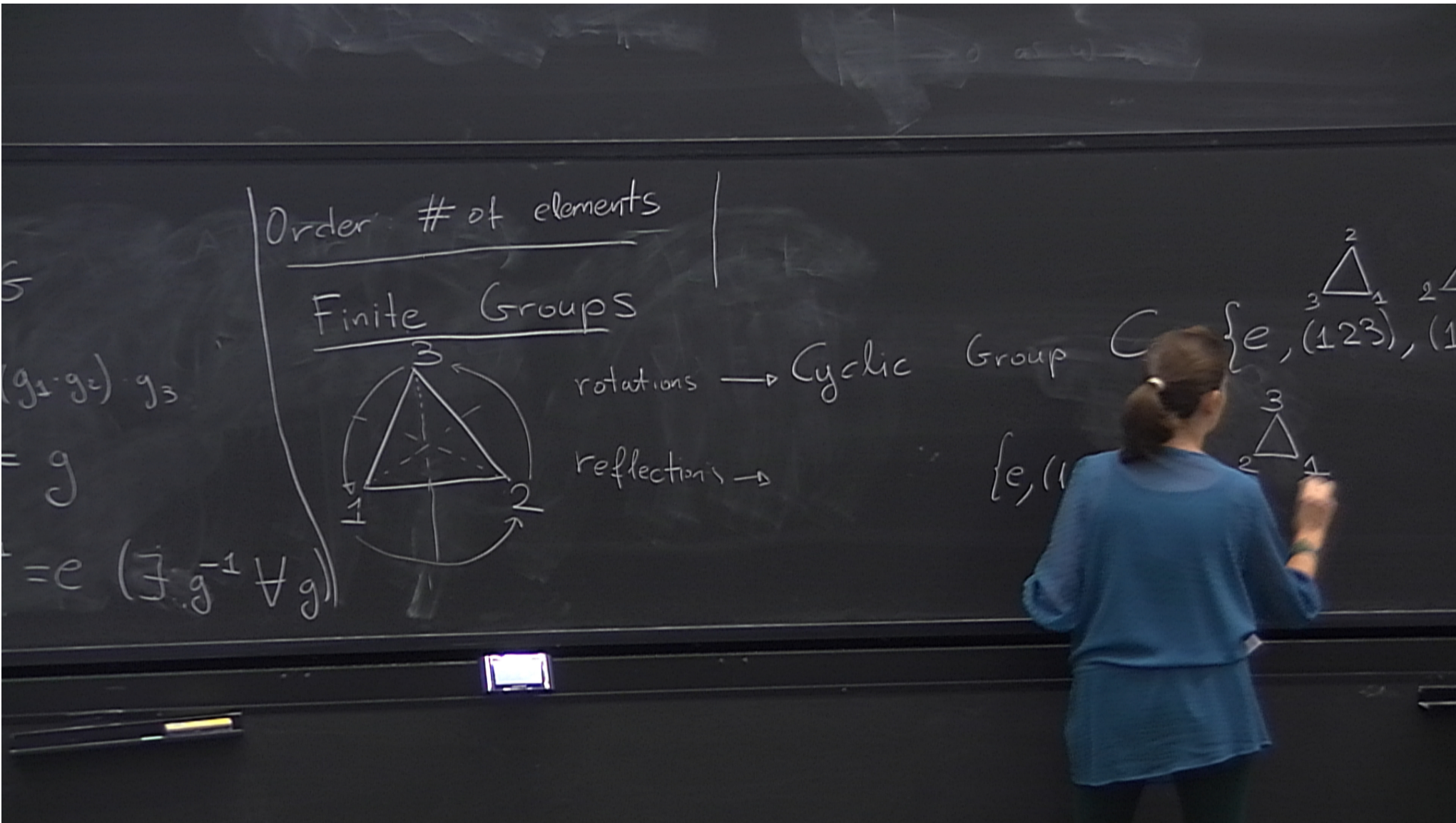
rotations $\rightarrow$

Cyclic

Group

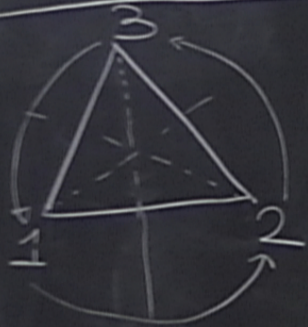
$$C_3 = \{e, (123), (132)\}$$





der: # of elements

Finite Groups



rotations $\rightarrow$ Cyclic

Group

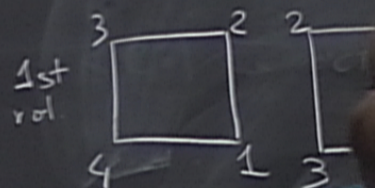
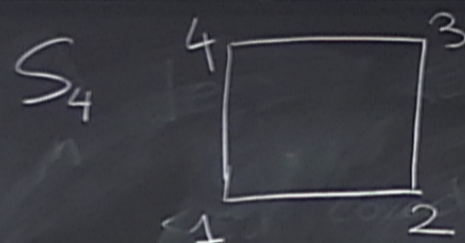
$$C_3 = \{e, (123), (132)\}$$

reflections $\rightarrow$ Permutation
Group S_3

$$\{e, (123), (132), (12), (13), (23)\}$$

$$* g^{-1} \cdot g = g \cdot g^{-1} = e \quad (\exists g^{-1} \forall g)$$

Group



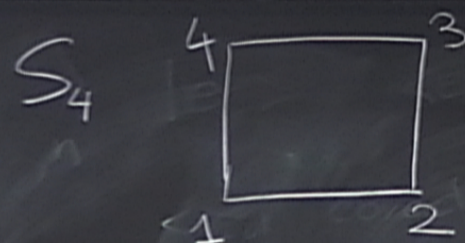
$$C_4 = \{e, (1234), (13)(24), (1432)\}$$

$$S_4 =$$

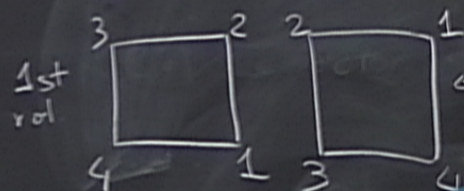
2nd

$$* g^{-1} \cdot g = g \cdot g^{-1} = e \quad (\exists g^{-1} \forall g)$$

Group



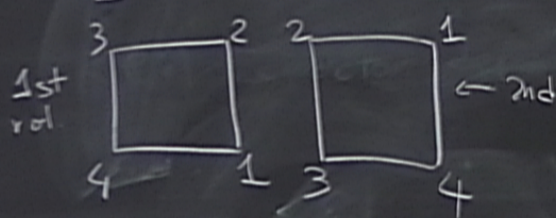
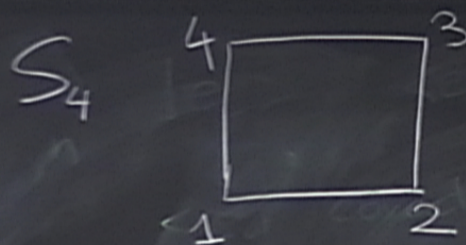
$$C_4 = \{e, (1234), (13)(24), (1432)\}$$



$$S_4 = \{ \dots \}$$

$$g^{-1} \cdot g = g \cdot g^{-1} = e \quad (\exists g^{-1} \forall g)$$

Group S_3

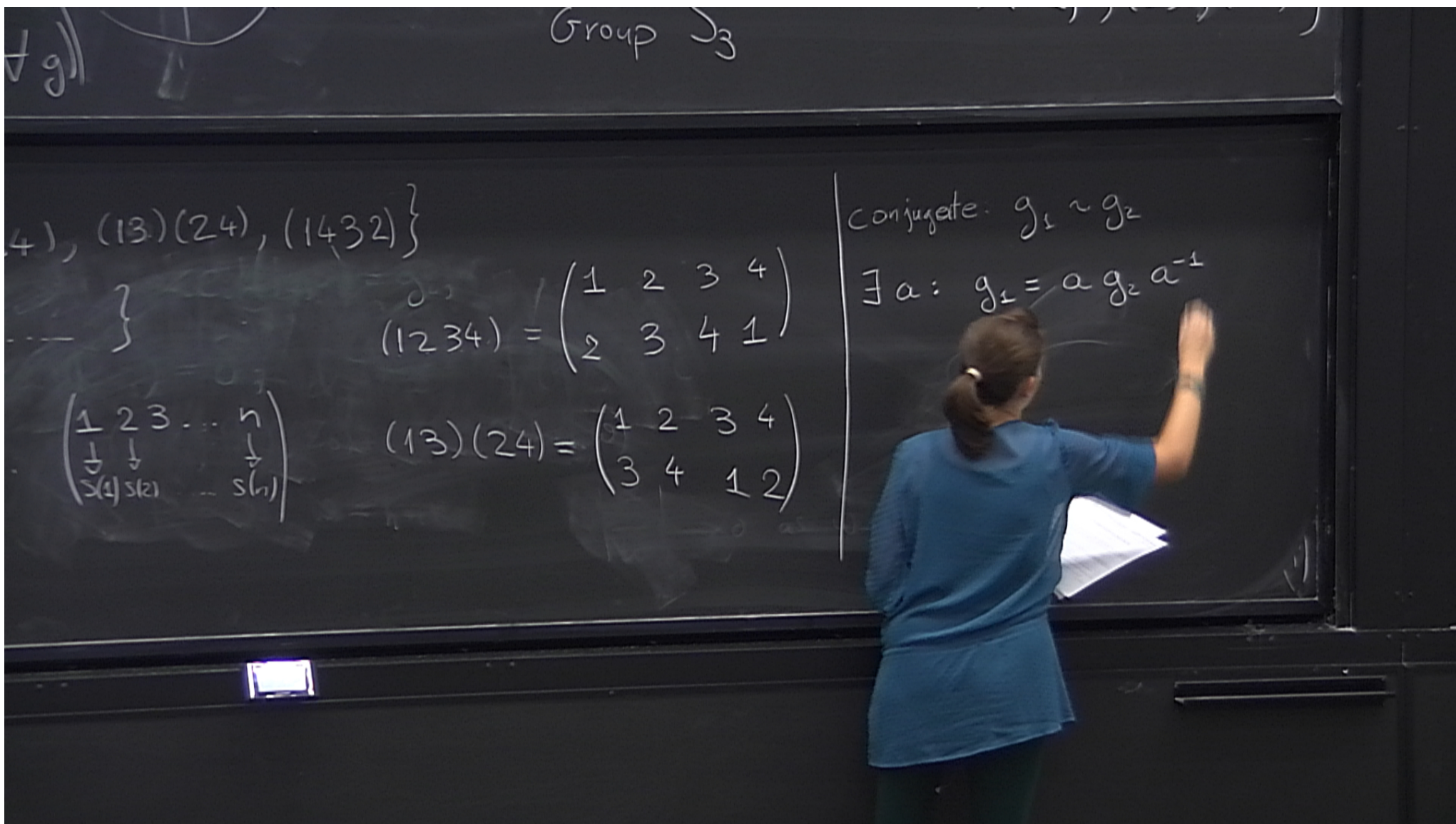


$$C_4 = \{e, (1234), (13)(24), (1432)\}$$

$$S_4 = \{ \dots \}$$

$$S_n = \left(\begin{array}{cccc} 1 & 2 & 3 & \dots & n \\ \downarrow & \downarrow & & & \downarrow \\ S(1) & S(2) & & & S(n) \end{array} \right)$$

$$(1234) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$$



Group S_3

$$\{ (1234), (13)(24), (1432) \}$$

}

$$(1234) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & \dots & n \\ \downarrow & \downarrow & & & \downarrow \\ s(1) & s(2) & & & s(n) \end{pmatrix}$$

$$(1234) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

conjugate: $g_1 \sim g_2$

$$\exists a: g_1 = a g_2 a^{-1}$$

Conjugacy class of S_n

elements that have the same cycle structure

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 4 & 1 & 2 \end{pmatrix} \in S_5$$
$$= (1$$

$$\begin{pmatrix} 5 \\ 2 \end{pmatrix} \in S_5$$

S_n = Group of all possible permutations
of n element

$$\begin{pmatrix} 4 & 5 \\ 5 & 4 \end{pmatrix} = (132)(45)$$

$$\begin{pmatrix} 5 \\ 2 \end{pmatrix} \in S_5$$

S_n = Group of all possible permutations
of n elements

$$\begin{pmatrix} 4 & 5 \\ 5 & 4 \end{pmatrix} = (132)(45)$$

$$\sigma \in S_5$$

S_n = Group of all possible permutations
of n elements.

$$\begin{pmatrix} 5 \\ 4 \end{pmatrix} = (132)(45) \quad \left| \quad \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ \downarrow & \downarrow & & & \downarrow \\ n & n-1 & & & 1 \end{pmatrix} \in S_n \right. \quad \# \text{ of elements}$$

$$\sigma \in S_5$$

S_n = Group of all possible permutations
of n elements

$$\begin{pmatrix} 5 \\ 4 \end{pmatrix} = (132)(45) \quad \left| \quad \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ \downarrow & \downarrow & & & \downarrow \\ n & n-1 & & & 1 \end{pmatrix} \in S_n \right.$$

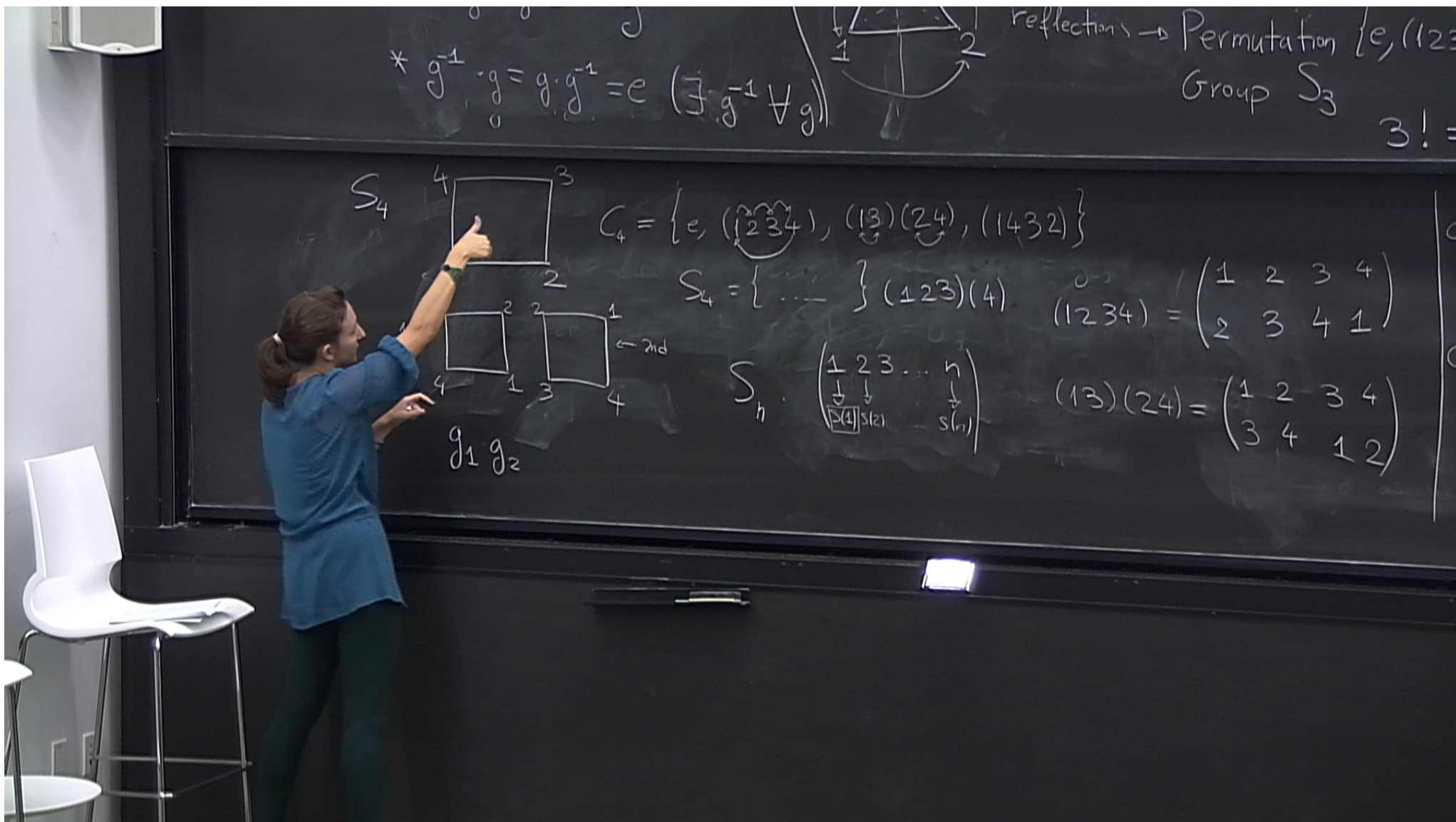
of elements

$$(3 \ 1 \ 2 \ 5 \ 4) = (132)(45) \quad \left| \quad \begin{array}{c} \downarrow \quad \downarrow \\ n \quad n-1 \end{array} \right. \quad \begin{array}{c} \downarrow \\ 1 \end{array} \quad \# \text{ of elem}$$

C_4 is homomorphic to S_4

C_4 is isomorphic to a subgroup of S_4

Abelian Group: $g_1 g_2 = g_2 g_1$



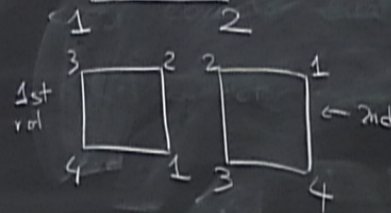
$$* g^{-1} \cdot g = g \cdot g^{-1} = e \quad (\exists g^{-1} \forall g)$$



reflections $\rightarrow$ Permutation $\{e, (1234), (1324), (1432)\}$
Group S_3 $3! = 6$



$$C_4 = \{e, (1234), (13)(24), (1432)\}$$



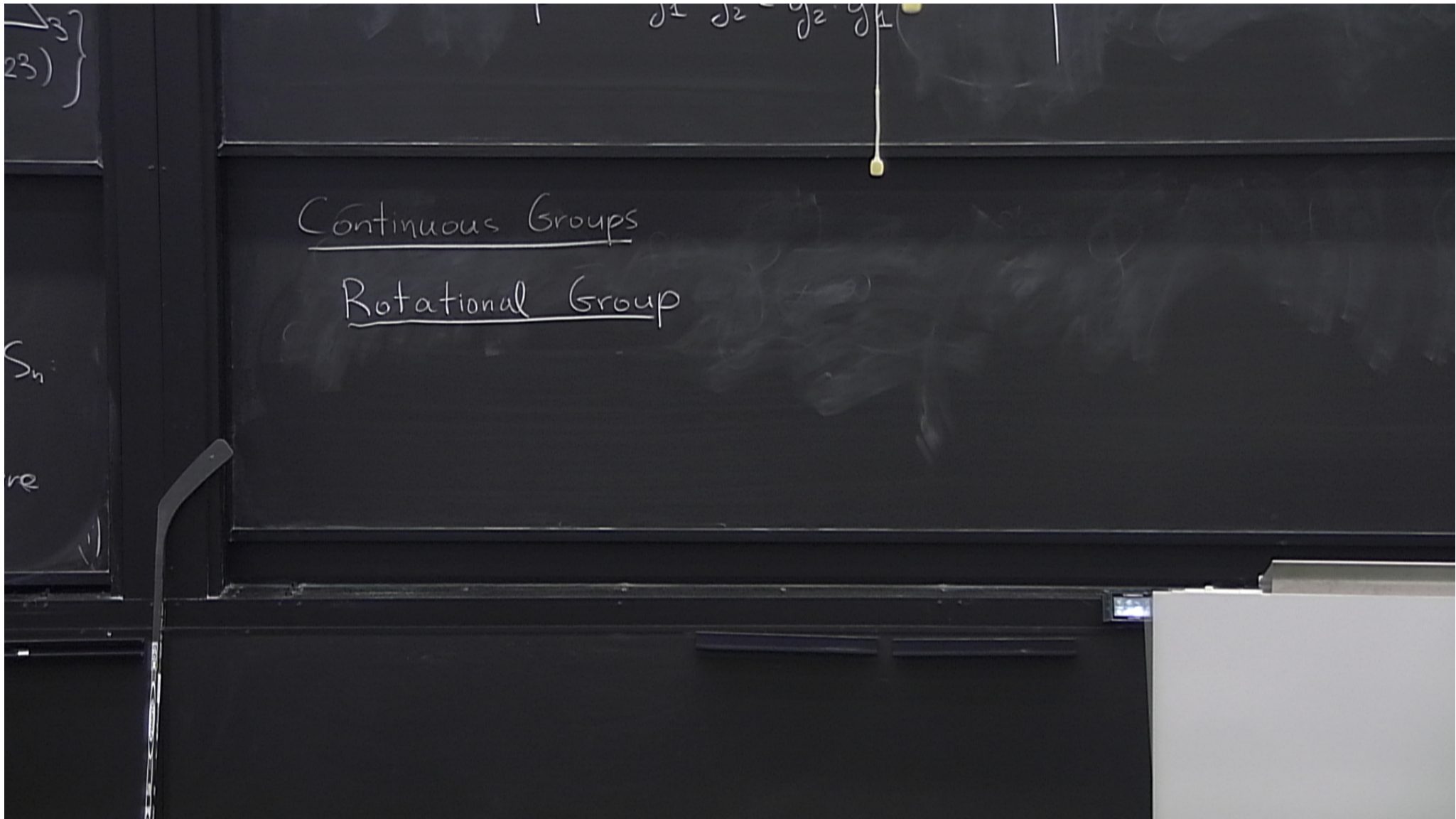
$g_1 g_1$

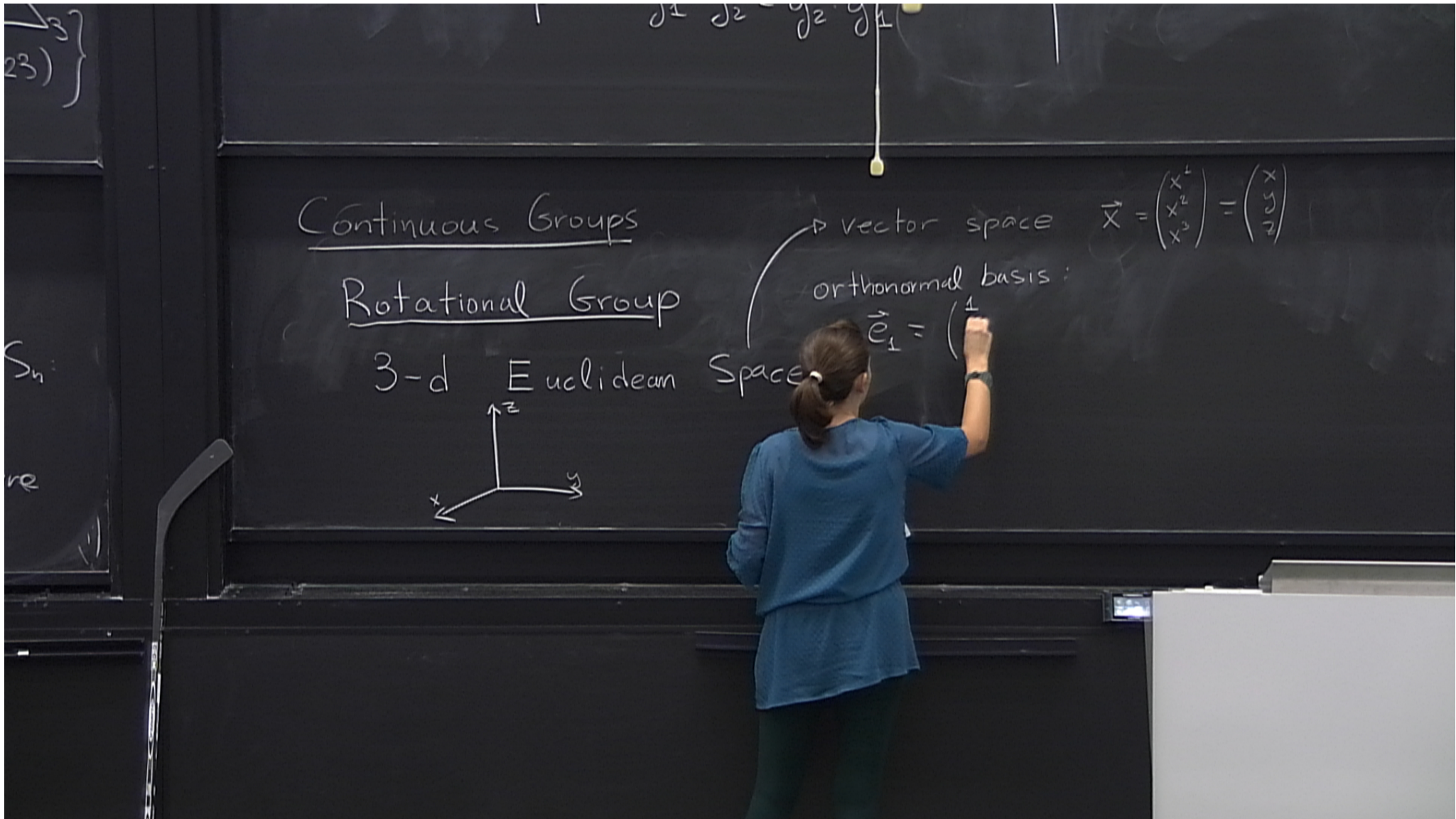
$$S_4 = \{ \dots \} (123)(4)$$

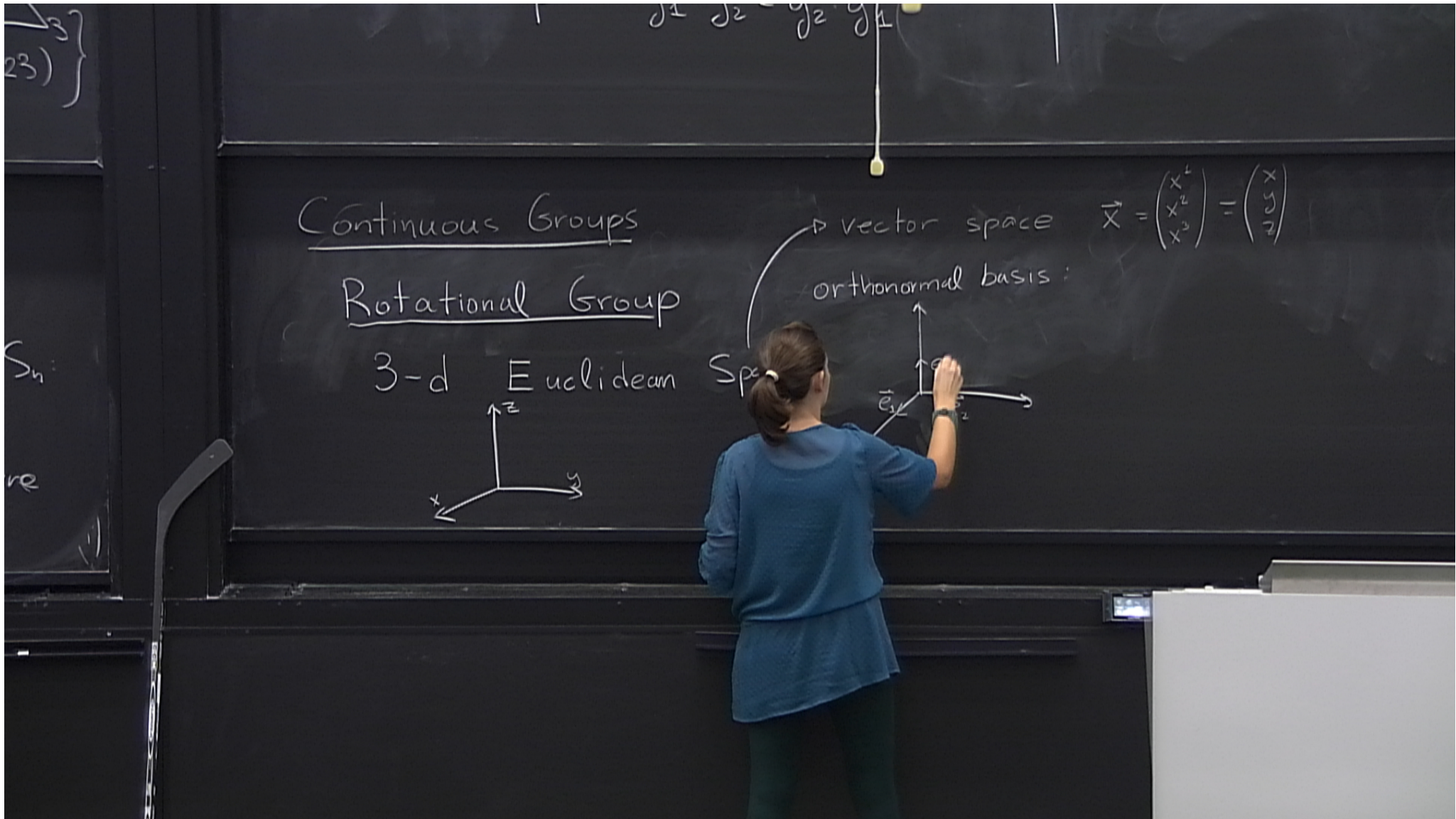
$$S_n = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ \downarrow & \downarrow & \downarrow & & \downarrow \\ s(1) & s(2) & s(3) & & s(n) \end{pmatrix}$$

$$(1234) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

$$(13)(24) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$





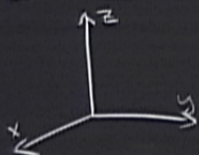


Abelian Group: $g_1 g_2 = g_2 g_1$

Continuous Groups

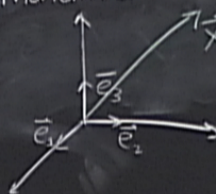
Rotational Group

3-d Euclidean Space



vector space

orthonormal basis:



$$\vec{x} \leftrightarrow \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\vec{x} = x^i \vec{e}_i$$

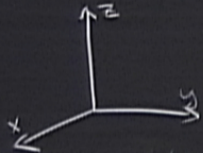
Scalar product $(\vec{x}, \vec{y}) = \vec{x} \cdot \vec{y}$

Abelian Group: $g_1 g_2 = g_2 g_1$

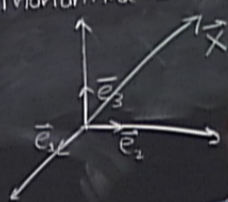
Continuous Groups

Rotational Group

3-d Euclidean Space



vector space
orthonormal basis:



$$\vec{x} \leftrightarrow \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\vec{x} = x^i \vec{e}_i$$

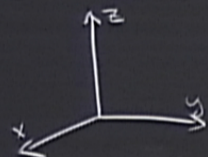
Scalar product

$$(\vec{x}, \vec{y}) = \vec{x}^T \cdot \vec{y} = (x^1 \ x^2 \ x^3) \begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix} = x^1 y^1 + x^2 y^2 + x^3 y^3$$

covector

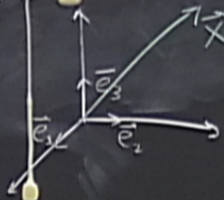
Rotational Group

3-d Euclidean Space



$$x_i = x^i$$

orthonormal basis:



Scalar product

$$(\vec{x}, \vec{y}) = \vec{x}^T \cdot \vec{y} = (x^1 \ x^2 \ x^3) \begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix} \\ = x^1 y^1 + x^2 y^2 + x^3 y^3$$

$$\vec{x}^T \equiv (x_1, x_2, x_3) = (x^1, x^2, x^3)$$

$$g_{ij} = \vec{e}_i \cdot \vec{e}_j = \delta_{ij} \quad \text{flat Euclidean metric}$$

Rotations transformations that leave the distance from origin invariant

$$* g^{-1} \cdot g = g \cdot g^{-1} = e \quad (\exists g^{-1} \forall g)$$

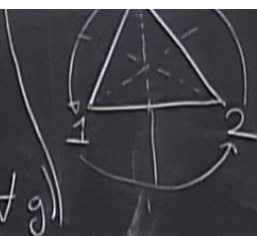
Group S_3

Rotations $[O(3)]$: preserve the Euclidean metric

$$R^T g R = g$$

$$* e \cdot g = g \cdot e = g$$

$$* g^{-1} \cdot g = g \cdot g^{-1} = e \quad (\exists g^{-1} \forall g)$$



reflections $\rightarrow$ Permutation $\{e, (123), (132), \overset{3}{\underset{2}{\triangle}} \overset{1}{\underset{3}{\triangle}}\}$
Group S_3 $3! = 6$ elements

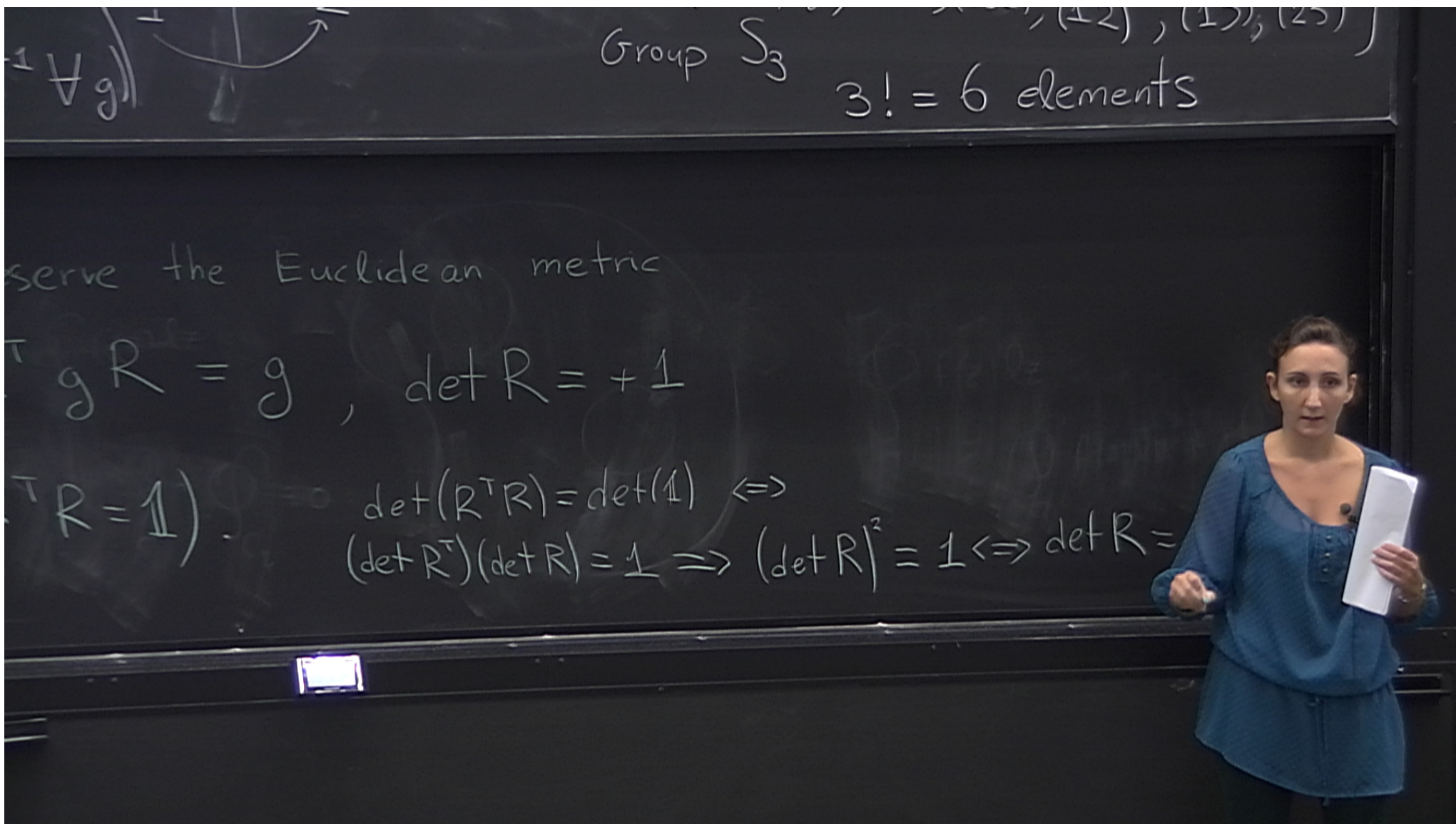
Rotations $[O(3)]$: preserve the Euclidean metric

$$R^T g R = g, \quad \det R = +1$$

$$(R^T R = \mathbb{1})$$

$$\Rightarrow \det(R^T R) = \det(\mathbb{1}) \Leftrightarrow$$

$$(\det R^T)(\det R) = 1 \Rightarrow (\det R)^2 = 1 \Leftrightarrow$$



Group 3

$3! = 6$ elements

Euclidean metric

$$\det R = +1$$

Exclude reflections

$$\vec{x} \rightarrow -\vec{x}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} -x^1 \\ -x^2 \\ -x^3 \end{pmatrix}$$

$$\hookrightarrow \det = -1$$

P Parity

$$\det(R^T R) = \det(1) \Leftrightarrow$$

$$(\det R^T)(\det R) = 1 \Rightarrow (\det R)^2 = 1 \Leftrightarrow \det R = \pm 1$$