Title: Topological Order in Spin Systems

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Abstract: Much effort has been devoted to the study of systems with topological order, motivated by practical issues as well as more field theoretical and mathematical concerns. This talk will give an overview of some of the field, describing abelian systems relevant to the search for spin liquids, and non-abelian systems relevant to topological quantum computation. I will focus in particular on problems not reducible to free-fermion ones; examples include the RVB state of electrons as well as models of quantum loops and nets.


Topological order
in spin systems
Paul Fendley

Topological order in spin systems

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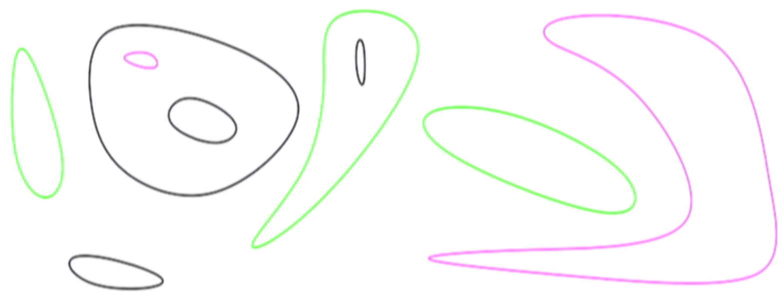
What is topological order?

My favorite definition: the number of ground states depends on the topology (e.g. genus) of the surface.

A topological order parameter is non-local, and so is effectively intermediate between order and disorder.

Wen introduced this idea to characterize the order in the fractional quantum Hall effect.

Models with loops as degrees of freedom



are conducive to topological order

if the loops are deconfined.

Why do we care?

Systems with topological order typically have fractionalization.

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If that's not motivation enough:

In 2+1 dimensions, non-abelian anyons can change state under exchange. Thus one could flip the state of a qubit by exchanging distant anyons. This would provide quantum computing hardware robust against local decoherence.

How do we make progress in such strongly coupled problems?

In many interesting cases, there are strong connections between 2d quantum (i.e. 2+1d) problems and 2d classical (i.e. 2+0d) problems.

This can be exploited qualitatively and quantitatively by using both field theory and integrability.

Physical systems:	Some relevant theory:
fractional quantum Hall effect	Chern-Simons
non-abelian FQHE	knot theory/TQFT
spin liquids	RVB model
topological insulators	band structure from K theory; group cohomology
toric codes/ discrete gauge theory	Dijkgraaf-Witten TQFT
BCS superconductor	BF theory

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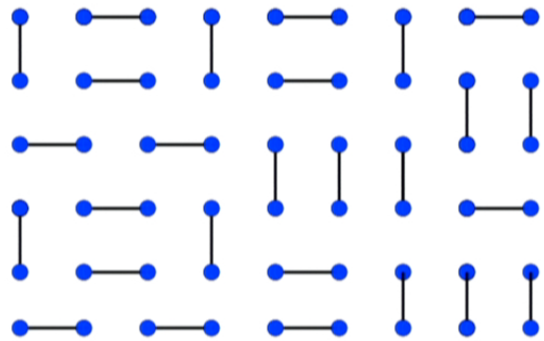
The RVB wave function

Consider one electron pinned to each site of some lattice.
Antiferromagnetic interactions encourage valence bonds:

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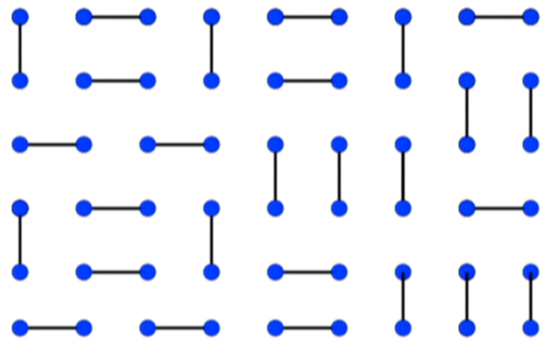
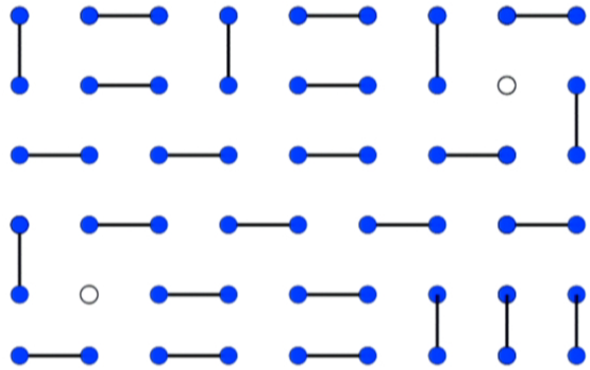


 $= |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$

The (short-range) "resonating valence bond" state is a linear superposition of all such "dimer" configurations.
Anderson

If instead you remove two electrons



the resulting state has spin 0 and charge 2e. With deconfinement, each “holon” has spin zero and charge -*e*.

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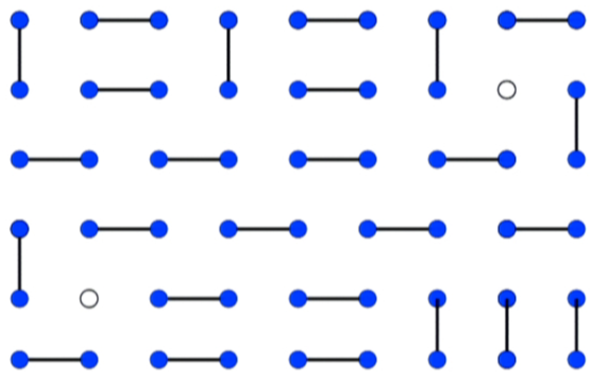

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The electron has “separated” into a spinon and holon. This sounds exotic, but in an ordinary BCS superconductor, the excitations are essentially spinons!

Spin-charge separation is familiar in 1+1 dimensional conformal field theory from non-abelian bosonization.
Polyakov-Wiegmann; Witten

Restrict an electron to live in one spatial dimension. The two components of spin result in two Dirac fermions:

$$\bar{\psi}\sigma^{\mu}\psi \quad = (\bar{\psi}\vec{\sigma}\psi, \bar{\psi}\psi)$$

spin charge

$$U(2) \quad = SU(2)_1 \times U(1)$$

Spin and charge are described by distinct conformal field theories.

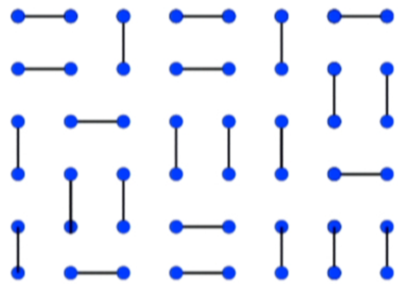
How do we make this concrete in 2+1 dimensions?

Rokhsar and Kivelson had a clever idea:

- Study the much-simpler model of quantum dimers instead of spins, i.e. treat the dimers as the degrees of freedom. Distinct dimer configurations are orthogonal.
- Construct a Hamiltonian that has the RVB dimer state as its exact ground state.

Quantum dimers
= 2+1d lattice gauge theory

Rewrite the dimers in terms of loops:



reference configuration:

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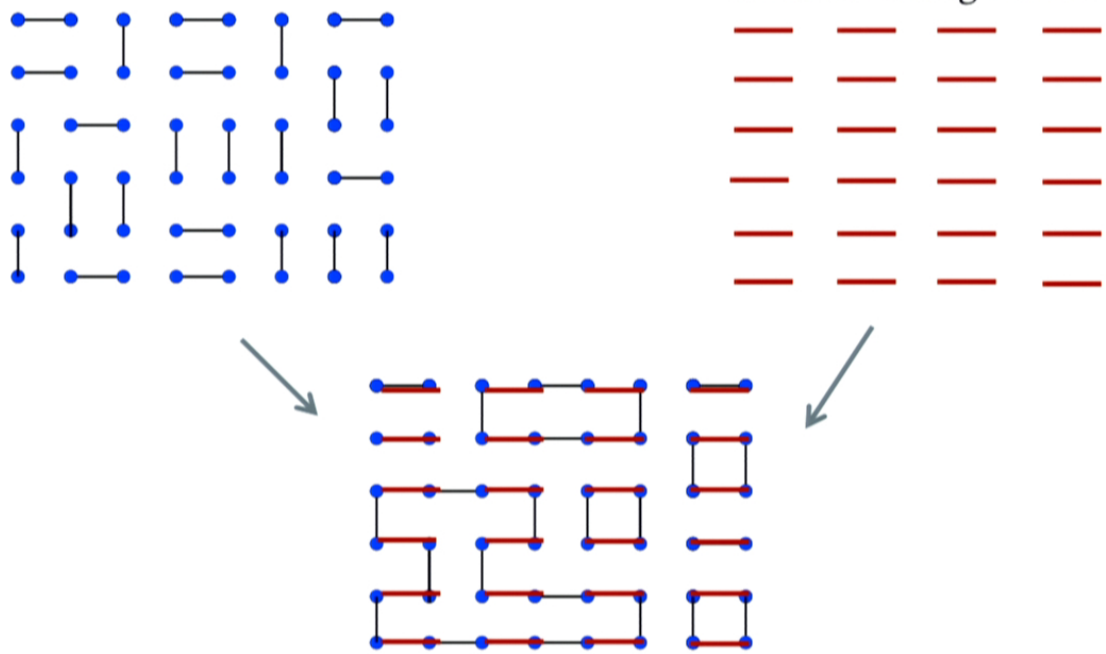
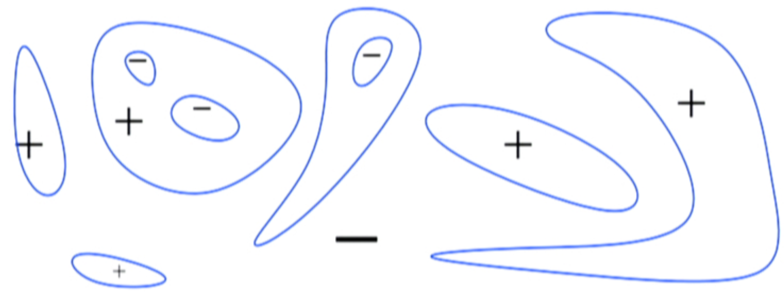


Each dimer configuration is equivalently a loop configuration:



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Fradkin, Kivelson; Moessner, Sondhi, Fradkin

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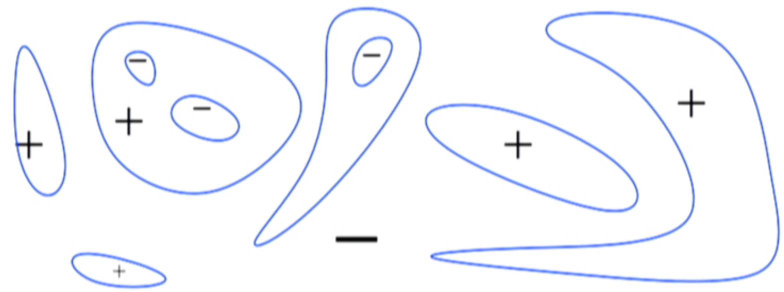

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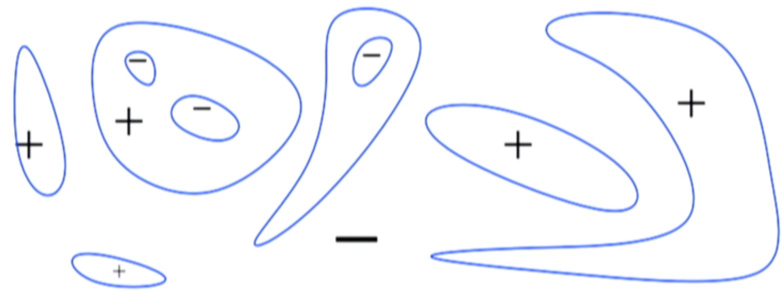

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The dual of a loop with ends is a string with weight

$$(-1)^{\text{dimers crossed}}$$

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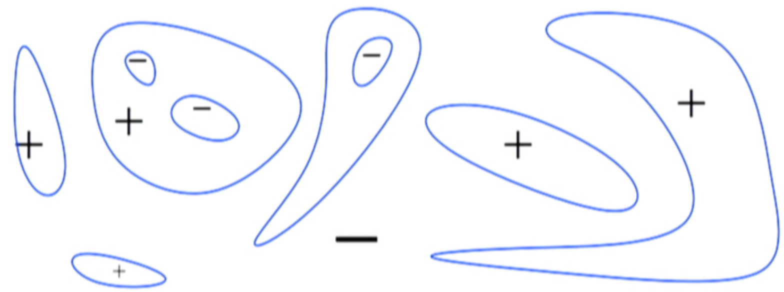

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In all languages, when you take this around a monomer/holon, you pick up a minus sign!

Freak Hamiltonians

An RK Hamiltonian is the sum of local projectors:

$$H = \sum_i P_i \,, \qquad P_i^2 \propto P_i$$

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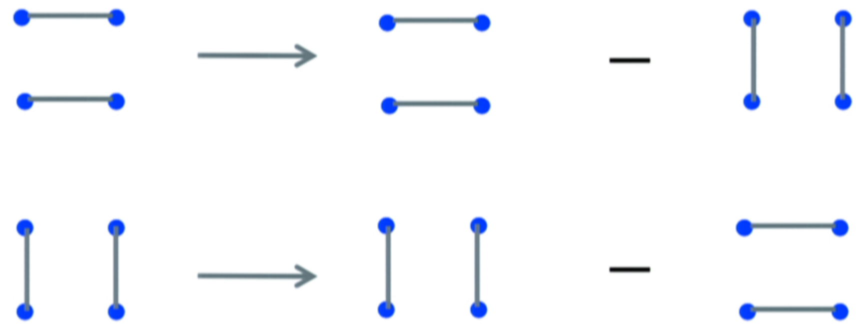
FFRK?

For quantum dimers, the projectors annihilate everything save the "flippable" plaquettes containing two dimers:

P_i :



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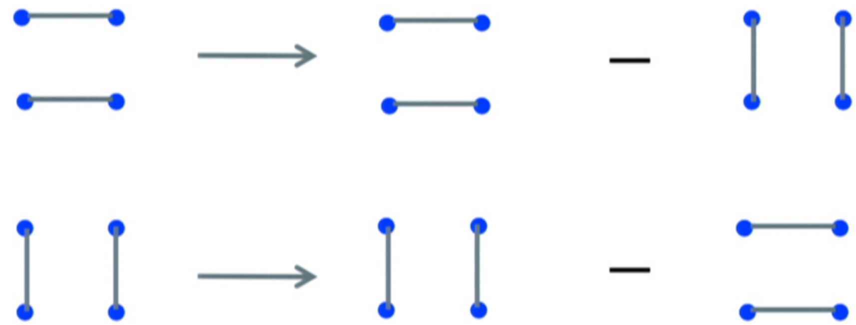
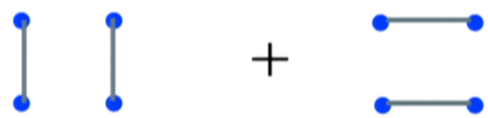


This annihilates the linear combination



so the exact ground state is the equal amplitude sum over all dimer configurations!

Equal-time zero-temperature correlators in the ground state of freak Hamiltonians are those of classical models.

The ground state is $|\Psi\rangle = \sum_D |D\rangle$, the sum over all dimer configurations.

For a diagonal operator O,

$$\langle O \rangle_{\text{quantum}} = \frac{\langle \Psi | O | \Psi \rangle}{\langle \Psi | \Psi \rangle} = \frac{\sum_D \langle D | O | D \rangle}{\sum_D \langle D | D \rangle} = \langle O \rangle_{\text{classical}}$$

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Classical dimer correlators can be found from free fermions!

Kasteleyn, Temperley and Fisher, Fisher and Stephenson

Major complication:

the classical dimer model on the square lattice is critical!

Correlators of local objects decay algebraically, so a theorem of Hastings requires that the quantum Hamiltonian be gapless.

Thus the quantum dimer model on the square lattice is quantum critical, so the interesting topological behavior is unprotected. There is algebraic confinement.

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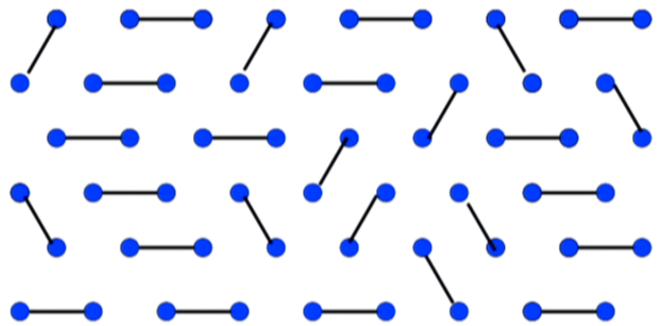
Major insight:

the quantum dimer model with RK Hamiltonian on the triangular lattice has topological order with a gap!

Moessner and Sondhi



With topological order, the holons and spinons are deconfined: their two-point functions must fall off to constant values.

We computed the monomer two-point function in the classical dimer model on the triangular lattice to check.

This is complicated, because this is non-local in terms of the free fermions (like the spin correlators in Ising)

This correlator is a horrible mess, given as a determinant of a Toeplitz+Hankel matrix, whose long-distance asymptotic is

0.14942924536134225401731517482693…

Fendley, Moessner and Sondhi

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$$0.1494292453613422540173151748269\ldots$$
$$= \sin(\pi/12)/\sqrt{3}$$

Fendley, Moessner and Sondhi

When interpolating between square and triangular lattices, the gap appears immediately. Analysis of the asymptotics gives

$$\frac{1}{2}\sqrt{\frac{t}{2t(2+t^2)+(1+2t^2)\sqrt{2+t^2}}}$$

where t=0 is the square lattice and t=1 is the triangular.

Basor and Ehrhardt

Is there a spin liquid on the square lattice in the RVB wave function?

Let's treat go back to treating the dimers as spin singlets.

Different "dimer" configurations are not orthogonal: their inner product is now given by overlapping the two and counting the number of loops:

$$\langle D' | D \rangle \propto 2^{\#\,\text{loops}}$$

Sutherland

This inner product is thus not only not orthogonal but non-local in the dimer basis, making life tricky.

"Dimer" correlators in the RVB state algebraically decay, as in the quantum dimer model. Hastings' theorem says any local Hamiltonian with the RVB state as a ground state must be gapless.

However, the spin-spin correlator exponentially decays.
Liang, Doucot and Anderson

Moreover, recent numerical work clearly shows there is no evidence of spin order.
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Is there a spin gap? Does the RVB state describe a spin liquid?

There are two kinds of terms, all written in terms of spin-S projectors $P^{(S)}$

$$H = \sum_s K_s + \sum_p F_p$$

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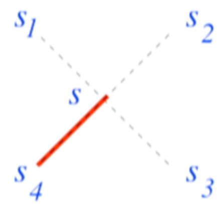
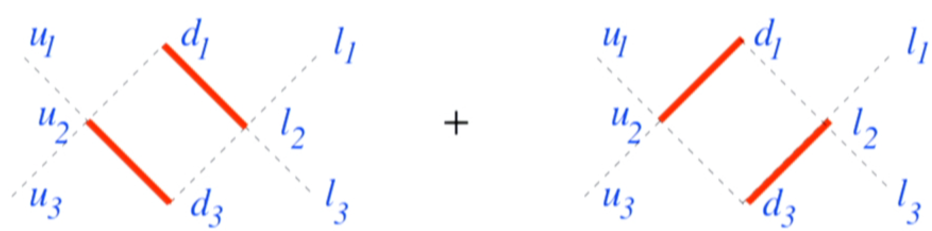
The first is called a Klein term, and is Gauss' Law at site s. It annihilates dimer configurations, while giving an energy to others.

$$K_s = P_s^{(5/2)}(\{s\})$$



The second is the plaquette flip term, which forces zero flux. It annihilates the desired linear combination of dimer configurations:

$$F_p = P_u^{(3/2)} P_d^{(0)} P_l^{(3/2)}$$



So far there remains only indirect evidence for the square-lattice RVB state being a gapped spin liquid.

However, we have found good evidence that its gapless sector is in the same universality class as a (generalized) quantum dimer model.
Stephan, Ju, Fendley and Melko

Since the square-lattice QDM turns into a liquid by perturbing to a triangular lattice, maybe it can turn into a spin liquid by allowing dimers to break into spins?

Equal-time correlators in the RVB state are those of a classical 2d model, as with dimers.

There is strong evidence that they are described by a Coulomb gas, whose field theory description is a free boson.
Tang, Sandvik and Henley; Albuquerque and Alet

We provided further evidence for universality, by:

- Interpolating between dimers and RVB
- Measuring and computing the entanglement entropy

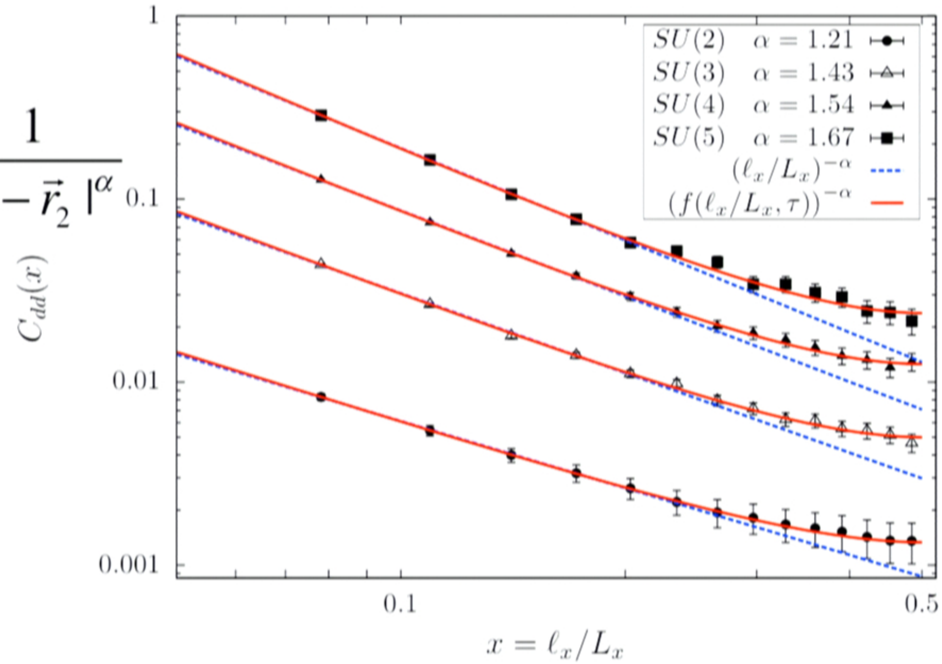
Interpolating between dimers and RVB

- Use SU(N) instead of SU(2) spin singlets to make "dimers".
- The inner product changes, making the weight per loop go up. As N increases, more and shorter loops are favored. Thus as $N \to \infty$, only dimers are left!

Measuring the “dimer”-“dimer” correlator:

at long distances

$$C_{dd}(\vec{r}_1-\vec{r}_2) \sim \frac{1}{|\vec{r}_1-\vec{r}_2|^{\alpha}}$$



Stephan, Ju, Fendley and Melko

The long-distance decay is algebraic for all N, with exponent increasing with N toward the dimer value $\alpha = 2$.

One can develop an expansion around the dimer case.

Damle, Dhar, and Ramola

Numerics:		Leading term, with coefficient fixed by SU(2) case	
N=2:	1.21(7)	N=2:	1.22
N=3:	1.43(7)	N=3:	1.40
N=4:	1.54(8)	N=4:	1.52
N=5:	1.67(9)	N=5:	1.6

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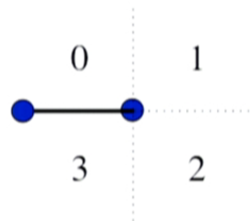
study the nicest of all field theories….

Classical dimers as a free boson

On the square lattice, the classical dimer model can be rewritten in terms of heights, integer-valued variables on the dual lattice:



In the continuum limit, the suitably-averaged height turns into a scalar field ϕ with action

$$S = \frac{\kappa}{4\pi} \int d^2x \; (\nabla\phi)^2$$

The critical exponents vary with the "stiffness" κ .
The basic two-point functions are of the form

$$C(\vec{r}_1 - \vec{r}_2) \sim \frac{1}{|\vec{r}_1 - \vec{r}_2|^{1/\kappa}}$$

The dimer creation operator has $\alpha = \frac{1}{\kappa} = 2$.

So $\kappa = 1/2$ for dimers, and is natural to assume that all singlet correlators are described by the same theory, with κ depending on N.

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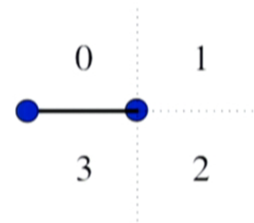

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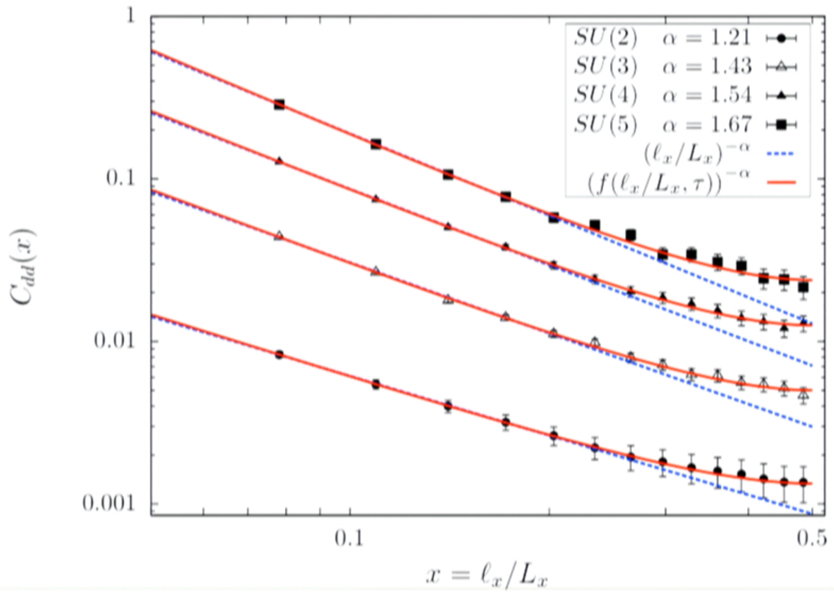
One check is that the finite-size effects of the correlators are also known exactly, in agreement with the numerics.

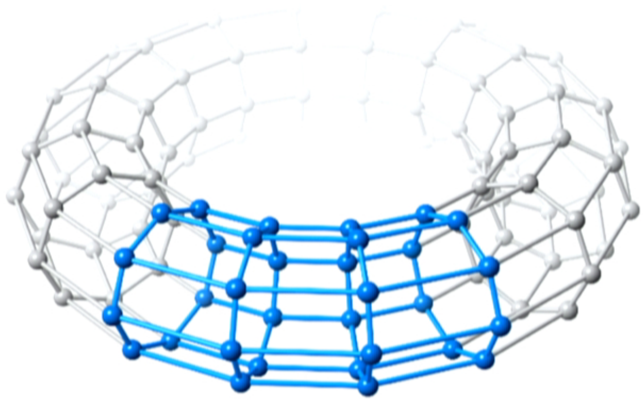


So let's move on another interesting quantity….

By means of a clever trick, the Renyi entanglement entropy for some freak 2d quantum Hamiltonians, including quantum dimers (but not RVB), can be reduced to a 2d classical computation of the Shannon entropy.
Stephan, Misguich, and Pasquier

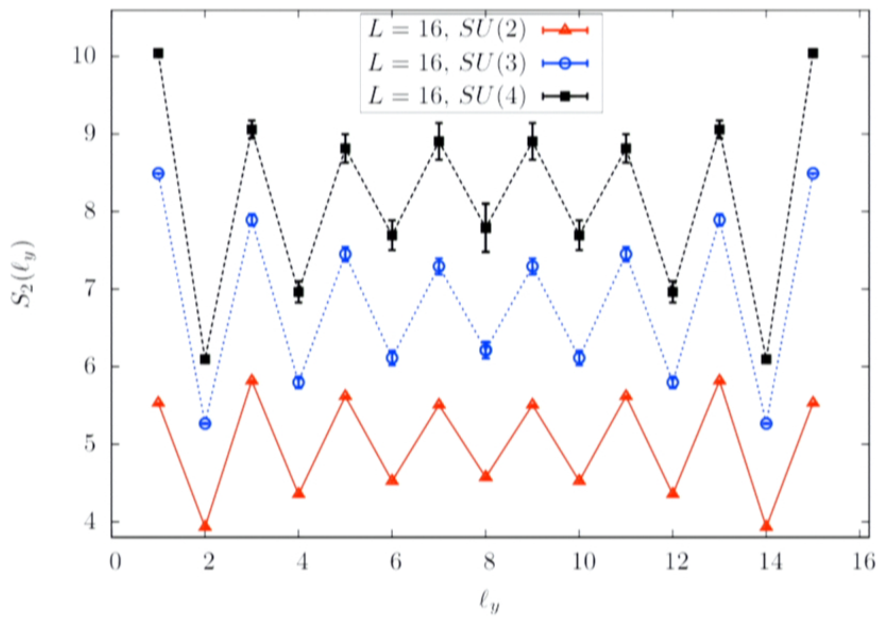
In the square-lattice quantum dimer and RVB states, we find some fascinating features…

A very useful geometry for probing gapless behavior is the entanglement from cutting a torus into two cylinders.



The length of the boundary between the two cylinders is independent of the area of the cylinders. Varying the size of the cylinder directly probes the gapless physics!

For SU(N) RVB, we find a very pronounced even-odd effect!



Ju, Kallin, Fendley, Hastings and Melko; Stephan, Ju, Fendley and Melko

A violation of strong subadditivity (only a theorem for von Neumann; this is S_2).

For quantum dimers, we computed it using the Shannon trick!

$$s_n^{(\mathrm{even})}(y,\tau) = \frac{n}{1-n}\ln\left(\frac{\eta(\tau)^2}{\theta_3(2\tau)\theta_3(\tau/2)}\times\frac{\theta_3(2y\tau)\theta_3(2(1-y)\tau)}{\eta(2y\tau)\eta(2(1-y)\tau)}\right)$$

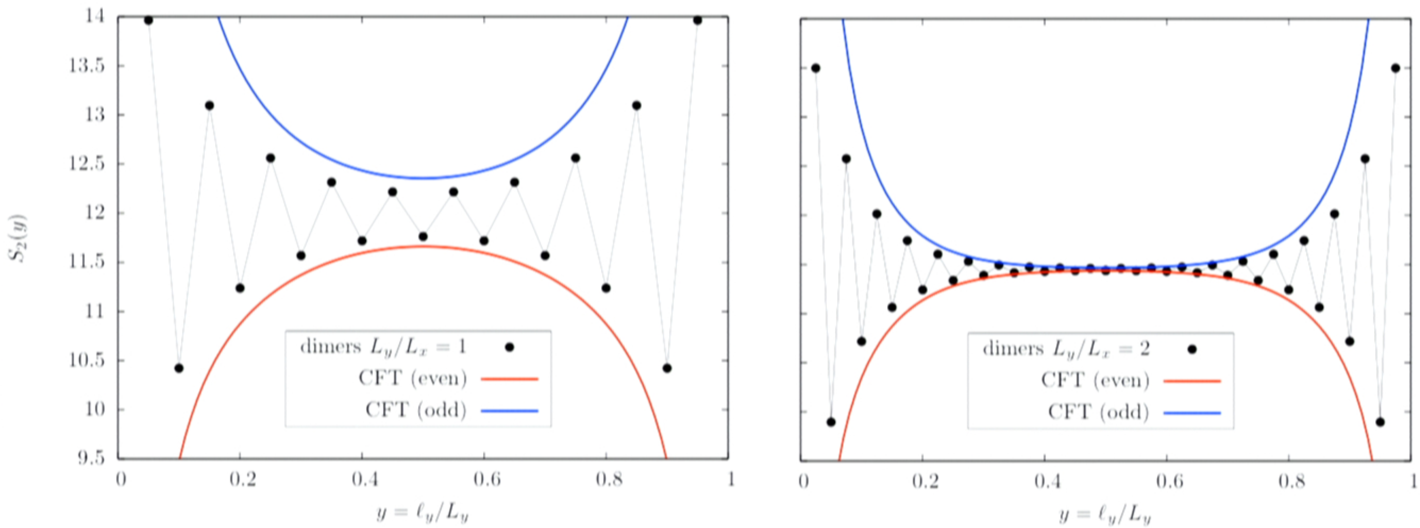
$$s_n^{(\mathrm{odd})}(y,\tau) = \frac{n}{1-n}\ln\left(\frac{\eta(\tau)^2}{\theta_3(2\tau)\theta_3(\tau/2)}\times\frac{\theta_4(2y\tau)\theta_4(2(1-y)\tau)}{\eta(2y\tau)\eta(2(1-y)\tau)}\right)$$

y is the ratio of the length of the cylinder to the length of the torus
τ is the aspect ratio of the torus
Θ_3, Θ_4 are the Jacobi theta functions
η is the Dedekind eta function

The odd curve is different because in the mapping to heights, the boundary conditions across the cylinder are twisted.

Stephan, Ju, Fendley and Melko

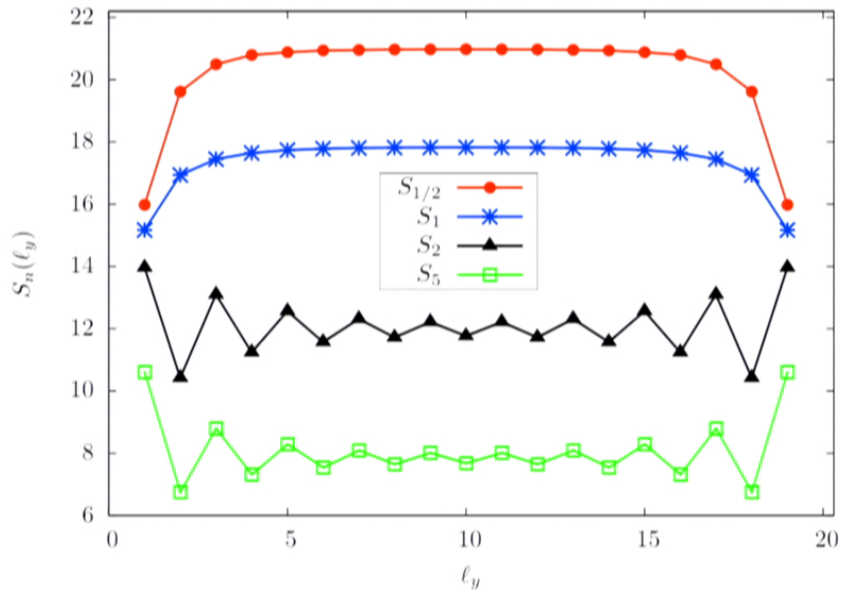
Finite-size effects are large; for a 20 by 20 torus:



The agreement is perfect for large enough systems.

There is a phase transition in the Renyi parameter n:

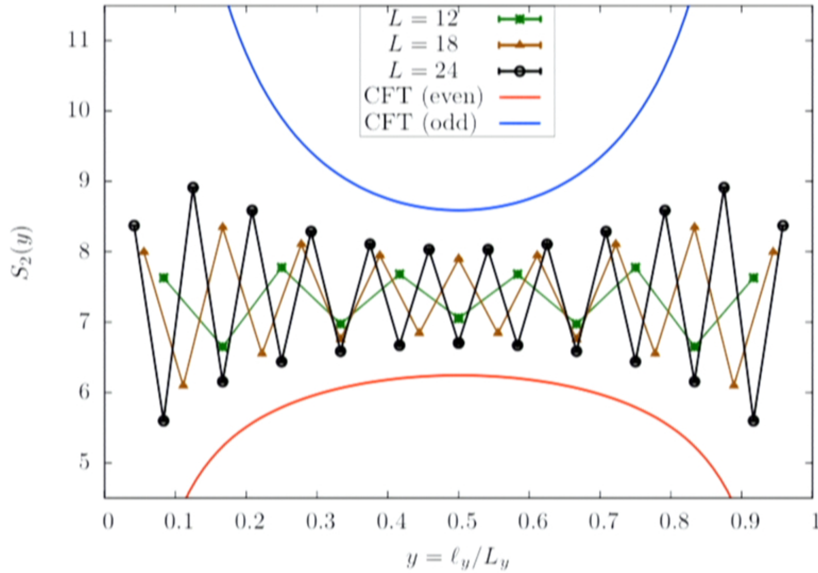
$$S_n = \frac{1}{1-n} \ln \mathrm{Tr}(\rho_A)^n$$



The critical value of n is not universal; for dimers on the square lattice it is 1; on the honeycomb it is 9.

This result can be generalized to all κ, and the universality assumption means it should apply then to SU(N) RVB.

The strong finite size effects make only qualitative comparison possible. For SU(2),



Free-boson theory applies beyond the RVB ground state.

We found a lattice model that describes not just the critical line, but also the neighboring phase with topological order.

Phase diagram of the quantum eight-vertex model