Title: Nonlocality, Entanglement Witnesses and Supra-Correlations

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Abstract: While entanglement is believed to underlie the power of quantum computation

and communication, it is not generally well understood for multipartite

systems. Recently, it has been appreciated that there exists proper

no-signaling probability distributions derivable from operators that do not

represent valid quantum states. Such systems exhibit supra-correlations

that are stronger than allowed by quantum mechanics, but less than the

algebraically allowed maximum in Bell-inequalities (in the bipartite case).

Some of these probability distributions are derivable from an entanglement

witness W, which is a non-positive Hermitian operator constructed such that

its expectation value with a separable quantum state (positive density

matrix) rho_sep is non-negative (so that Tr[W rho]< 0 indicates entanglement

in quantum state rho). In the bipartite case, it is known that by a

modification of the local no-signaling measurements by spacelike separated

parties A and B, the supra-correlations exhibited by any W can be modeled as

derivable from a physically realizable quantum state Ï•. However, this result

does not generalize to the n-partite case for n>2. Supra-correlations can

also be exhibited in 2- and 3-qubit systems by explicitly constructing

"states" O (not necessarily positive quantum states) that exhibit PR

correlations for a fixed, but arbitrary number, of measurements available to

each party. In this paper we examine the structure of "states" that exhibit

supra-correlations. In addition, we examine the affect upon the distribution

of the correlations amongst the parties involved when constraints of

positivity and purity are imposed. We investigate circumstances in which

such "states" do and do not represent valid quantum states.

Outline

- Review Bell Inequalities (BI)
- No-Signaling (NS) and Popescu-Rohrlich (PR) supra-quantum correlations (i.e. *stronger* than quantum)
- "States" (operators *O*) reproducing PR correlations
- The Bipartite case: 2-qubits, A and B perform *m* local measurements
  - The form of *O*
  - Some Linear Algebra: existence and uniqueness of solution
  - Numerical investigations (eigenvalues of *O*, correlations, …)
- The Tripartite case: 3-qubits, A,B,C perform *m* local measurements
- Effect of Positivity and Purity constraints on distribution of correlations
- Summary and Conclusion

2

Bell Inequalities and Correlation Distances

Bell Inequalities can be viewed as a violation of a classical quadrilateral inequality

Schumacher, PRA 44, 7047 (1991)

Consider a bipartite system $\mathcal{A}\otimes\mathcal{B}$ with measurement directions A,B$\in\mathcal{A}$ and C,D$\in\mathcal{B}$ taking values {± 1}

Correlations (expectation values) between A$\in\mathcal{A}$ and C$\in\mathcal{B}$

meas. value (*output*)

meas. setting (*input*)

$$E(AC) \equiv \langle AC \rangle = \sum_{a,c=\{\pm 1\}} a\, c\, P(a,c \mid A,C)$$
$$= P(+,+ \mid A,C) + P(-,- \mid A,C) - P(+,- \mid A,C) - P(-,+ \mid A,C)$$

Define *Correlation Distance*: Δ(AC)

$$\Delta(AC) = 1 - E(AC) = P(+,- \mid A,C) + P(-,+ \mid A,C) \geq 0$$

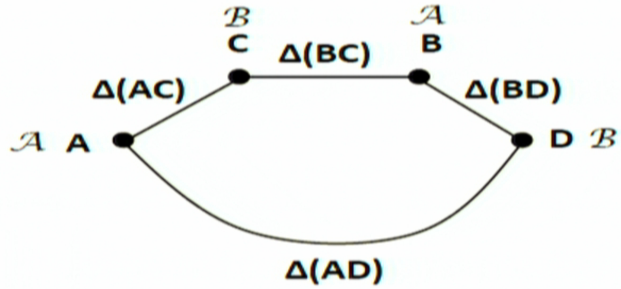
$$(\text{Using: } \sum_{a,c=\{\pm 1\}} P(a,c \mid A,C) = 1,\ \forall A,C)$$

Then $\Delta(AC) + \Delta(BC) + \Delta(BD) \geq \Delta(AD)$

$$\Rightarrow \boxed{S \equiv E(AC) + E(BC) + E(BD) - E(AD) \leq +2}$$

Which the CHSH Bell inequality with bounds

$$\text{Classical: } |S_C| \leq 2; \quad \text{QM: } |S_Q| \leq 2\sqrt{2}; \quad \text{Algebraic Maximum: } |S_{AM}| \leq 4$$



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Classically: Δ(AC) + Δ(BC) + Δ(BD) ≥ Δ(AD)

Quantum Mechanically: this can be violated, e.g. Δ(AC) + Δ(BC) + Δ(BD) < Δ(AD) with singlet state $(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2} = (|01\rangle - |10\rangle)/\sqrt{2}$

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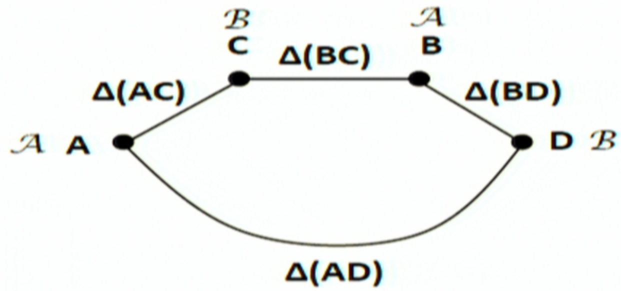

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3

No Signaling (NS) Theories

No Signaling Theories represent valid joint probability distributions (non-local) with valid marginal distributions

Linear constraints on any joint NS prob. distrib.

No Signaling: $P(a_1, a_2, \ldots, a_k \mid x_1, \ldots, x_n)$

$$= \sum_{a_{k+1}, \ldots, a_n \in \{0,1\}} P(a_1, \ldots, a_n \mid x_1, \ldots, x_n)$$

$$= P(a_1, a_2, \ldots, a_k \mid x_1, \ldots, x_k)$$

PR Box:

$$P(a,b \mid x,y) = \begin{cases} 1/2 & \text{if } a \oplus b = x \cdot y \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\sum_{a,b=\{0,1\}} P(a,b \mid x,y)$$

$$= \underbrace{P(0,0 \mid x,y) + P(1,1 \mid x,y)}_{a \oplus b = 0} \underbrace{+ P(0,1 \mid x,y) + P(1,0 \mid x,y)}_{a \oplus b = 1}$$

$$= (1/2 + 1/2)\, \delta_{0,x \cdot y} + (1/2 + 1/2)\, \delta_{1,x \cdot y}$$

$$= \delta_{0,x \cdot y} + \delta_{1,x \cdot y}$$

$$= 1 \quad \forall x, y$$

Normalization: $\sum_{a,b=\{0,1\}} P(a,b \mid x,y) = 1, \ \forall x, y$

No Signaling Principle: (two inputs/outputs)

$$P(a \mid x,y) \equiv \sum_{b=\{0,1\}} P(a,b \mid x,y) = P(a \mid x)\ \forall y$$

$$P(b \mid x,y) \equiv \sum_{a=\{0,1\}} P(a,b \mid x,y) = P(b \mid y)\ \forall x$$

Then $P(a \mid x,y) \equiv \sum_{b=\{0,1\}} P(a,b \mid x,y)$

$$= \underbrace{P(a,0 \mid x,y)}_{a \oplus b = a \oplus 0 = a} + \underbrace{P(a,1 \mid x,y)}_{a \oplus b = a \oplus 1 \equiv \bar{a}}$$

$$= 1/2\ \delta_{a,\, x \cdot y} + 1/2\ \delta_{\bar{a},\, x \cdot y}$$

$$= \begin{cases} 1/2 + 0 \ (\text{if } a = 0 \,\&\, x \cdot y = 0),\ 0 + 1/2 \ (\text{if } a = 0 \,\&\, x \cdot y = 1) \\ 0 + 1/2 \ (\text{if } a = 1 \,\&\, x \cdot y = 0),\ 1/2 + 0 \ (\text{if } a = 1 \,\&\, x \cdot y = 1) \end{cases}$$

$$= 1/2 \quad \forall a, x, y$$

$$= 1/2$$

$$= P(a \mid x)\ \forall a, x$$

(this is *Isotropic*, i.e. $P(a \mid x) = 1/2$ indep of x)

5

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$$\text{Normalization: } \sum_{a,b=\{0,1\}} P(a,b\,|\,x,y) = 1,\ \forall x,y$$

$$\begin{aligned}\text{No Signaling: } & P(a_1,a_2,\ldots,a_k\,|\,x_1,\ldots,x_n) \\ &= \sum_{a_{k+1},\ldots,a_n\in\{0,1\}} P(a_1,\ldots,a_n\,|\,x_1,\ldots,x_n) \\ &= P(a_1,a_2,\ldots,a_k\,|\,x_1,\ldots,x_k)\end{aligned}$$

No Signaling Principle: (two inputs/outputs)

$$P(a\,|\,x,y) \equiv \sum_{b=\{0,1\}} P(a,b\,|\,x,y) = P(a\,|\,x)\ \forall y$$

$$P(b\,|\,x,y) \equiv \sum_{a=\{0,1\}} P(a,b\,|\,x,y) = P(b\,|\,y)\ \forall x$$

PR Box:

$$P(a,b\,|\,x,y) = \begin{cases} 1/2 & \text{if } a\oplus b = x\cdot y \\ 0 & \text{otherwise}\end{cases}$$

Then

$$\begin{aligned}&\sum_{a,b=\{0,1\}} P(a,b\,|\,x,y) \\ &= \underbrace{P(0,0\,|\,x,y)+P(1,1\,|\,x,y)}_{a\oplus b=0} \underbrace{+P(0,1\,|\,x,y)+P(1,0\,|\,x,y)}_{a\oplus b=1} \\ &= (1/2 + 1/2)\,\delta_{0,x\cdot y} + (1/2 + 1/2)\,\delta_{1,x\cdot y} \\ &= \delta_{0,x\cdot y} + \delta_{1,x\cdot y} \\ &= 1\quad \forall x,y\end{aligned}$$

Then

$$\begin{aligned}P(a\,|\,x,y) &\equiv \sum_{b=\{0,1\}} P(a,b\,|\,x,y) \\ &= \underbrace{P(a,0\,|\,x,y)}_{a\oplus b=a\oplus 0=a} + \underbrace{P(a,1\,|\,x,y)}_{a\oplus b=a\oplus 1\equiv\overline{a}} \\ &= 1/2\ \delta_{a,\,x\cdot y} + 1/2\ \delta_{\overline{a},\,x\cdot y} \\ &= \begin{cases} 1/2+0\ (\text{if } a=0\ \&\ x\cdot y=0),\ 0+1/2\ (\text{if } a=0\ \&\ x\cdot y=1) \\ 0+1/2\ (\text{if } a=1\ \&\ x\cdot y=0),\ 1/2+0\ (\text{if } a=1\ \&\ x\cdot y=1)\end{cases} \\ &= 1/2\quad \forall a,x,y \\ &= 1/2 \\ &= P(a\,|\,x)\ \forall a,x\end{aligned}$$

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Structure of NS probability distributions

Consider a set of *n*-spacelike separated measurements

(Acin *et al.*, PRL 104, 140404 (2011)

No-Signaling (local) measurements

$$M_{\text{non-sig}} = M_{a_1}^{x_1} \otimes \cdots \otimes M_{a_n}^{x_n}$$

measurement setting (*input*)

measurement value (*output*)

Probability Distribution from "Trace Rule" for *fixed* set of measurements

$$\boxed{P_O \equiv P_O(a_1,\ldots,a_n \mid x_1,\ldots,x_n) = Tr[O\, M_{a_1}^{x_1} \otimes \cdots \otimes M_{a_n}^{x_n}] \geq 0}$$

projectors: $M_a^x = \Pi_a^x = |a\rangle_x \langle a|$

Probability Distribution from "Trace Rule" for *all* measurements

$$\boxed{P_W \equiv P(a_1,\ldots,a_N \mid x_1,\ldots,x_n) = Tr[W\, M_{a_1}^{x_1} \otimes \cdots \otimes M_{a_n}^{x_n}] \geq 0}$$

(*Gleason correlations*) (W is an *Entanglement Witness*)

$n = 2$ *only*: $P_W(a,b \mid x,y) = Tr[W\, M_a^x \otimes M_b^y] = Tr[\rho_{|\Phi_{BP}\rangle}\, M_a^x \otimes \bar{M}_b^y)]$.

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Entanglement Witness

- A *quantum state* ρ is positive semidefinite (ρ≥0) Hermitian matrix with Tr[ρ]=1
- A *separable state* (classical correlations) has the form $\rho^{sep} = \sum_i p_i \rho_i^{A_1} \otimes \rho_i^{A_2} \otimes \cdots \otimes \rho_i^{A_N}$
- An *entangled state* is a state that cannot be written as separable: $\rho \neq \rho^{sep}$
- An *entanglement witness* W is constructed to be positive on all separable states $\langle a_1, a_2, \ldots | W | a_1, a_2 \ldots \rangle \geq 0$ i.e. $Tr[\rho^{sep} W] \geq 0$ (follows from using $\rho_i^{A_k} = \sum_j p_{ij}^k |\psi_{ij}^k\rangle\langle\psi_{ij}^k|$)
- Hence, if Tr[ρ W] ≤ 0, then the quantum state ρ is entangled
- W is generally not a quantum state ρ≥0, since W can have negative eigenvalues

Current Understanding

1. It is known that for N=2 qubits, one can always write a probability distribution derived from a witness W, from one derived from a valid quantum state using modified measurements. (Barnum, et al. PRL 104, 1040401 (2011))
2. However, this does not generalize to the case of N>2 (Acin, *et al.*, PRL 104, 140404 (2011))

1. Proof: By the Choi-Jamiolkowski isomorphism (CJI), any 2-party witness W can be written as
   CJI $\Rightarrow W = (I_A \otimes \Lambda_B)(\rho_{|\Phi_{BP}\rangle})$, where Λ_B is positive, trace preserving map,
   and $\rho_{|\Phi_{BP}\rangle} = |\Phi_{BP}\rangle\langle\Phi_{BP}|$ is the density matrix for a maximally entangled pure bipartite state.

   acts to right

   Then: $P_W(a,b\,|\,x,y) = Tr[W\, M_a^x \otimes M_b^y] = Tr[(I\otimes\Lambda)(\rho_{|\Phi_{BP}\rangle})\, M_a^x \otimes M_b^y]$

   act to left

   $= Tr[M_a^x \otimes M_b^y (I\otimes\Lambda)(\rho_{|\Phi_{BP}\rangle})] = Tr[M_a^x \otimes \Lambda^*(M_b^y)\,\rho_{|\Phi_{BP}\rangle}] = Tr[\rho_{|\Phi_{BP}\rangle}\, M_a^x \otimes M_b'^y)]$,

   where $M_b'^y = \Lambda^*(M_b^y)$, and Λ^* is the dual map to Λ, i.e Tr[A $\Lambda(B)$]=Tr[$\Lambda^*(A)\,B$].
2. The above made explicit use of the CJI, in particular $W = (I_A \otimes \Lambda_B)(\rho_{|\Phi_{BP}\rangle})$, which does not extend in general to the multipartite case (N>2).
3. The extension of the CJI holds in the N-party case only for those W with the form
   $W = \sum_k p_k \Lambda_{A_1}^x \otimes \ldots \otimes \Lambda_{A_N}^y(\rho_k)$ where ρ_k are N-party quantum states, p_k probs., $\Lambda_{A_i}^x$ pos. maps
   $\Rightarrow P_W(a_1,\ldots,a_N\,|\,x_1,\ldots,x_N) = Tr[W\, M_{a_1}^{x_1} \otimes \ldots \otimes M_{a_N}^{x_N}] = \sum_k p_k Tr[\rho_k \Lambda_{A_1}^x(M_{a_1}^{x_1}) \otimes \ldots \otimes \Lambda_{A_N}^y(M_{a_N}^{x_N})]$ 7

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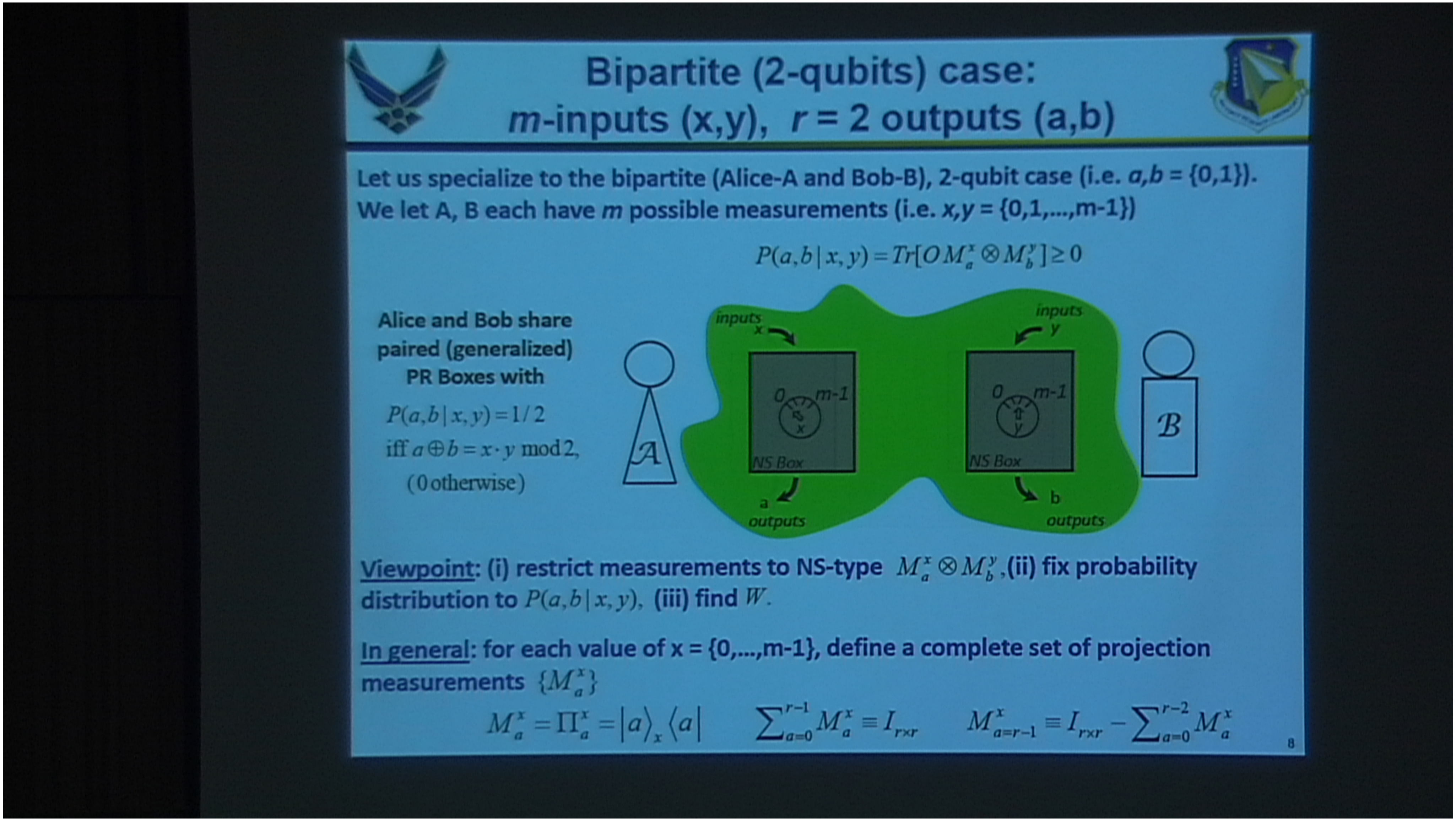
Bipartite (2-qubits) case:
m-inputs (x,y), *r* = 2 outputs (a,b)
Let us specialize to the bipartite (Alice-A and Bob-B), 2-qubit case (i.e. *a,b* = {0,1}).
We let A, B each have *m* possible measurements (i.e. *x,y* = {0,1,...,m-1})
$P(a,b|x,y) = Tr[O\, M_a^x \otimes M_b^y] \geq 0$
Alice and Bob share paired (generalized) PR Boxes with
$P(a,b|x,y)=1/2$
iff $a \oplus b = x \cdot y \mod 2$,
(0 otherwise)
inputs x
inputs y
0
m-1
NS Box
$\mathcal{A}$
$\mathcal{B}$
a outputs
b outputs
Viewpoint: (i) restrict measurements to NS-type $M_a^x \otimes M_b^y$, (ii) fix probability distribution to $P(a,b|x,y)$, (iii) find W.
In general: for each value of x = {0,...,m-1}, define a complete set of projection measurements $\{M_a^x\}$
$M_a^x = \Pi_a^x = |a\rangle_x \langle a|$ $\quad \sum_{a=0}^{r-1} M_a^x \equiv I_{r\times r}$ $\quad M_{a=r-1}^x \equiv I_{r\times r} - \sum_{a=0}^{r-2} M_a^x$
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Bipartite (2-qubits) case: *m*-inputs (x,y), *r* = 2 outputs (a,b)

For qubits: $\boxed{M^x_{a=0} = |0\rangle_x\langle 0| = 1/2\,(I + \vec{m}_x \cdot \vec{\sigma})}$, $\vec{m}_x = m_x(\sin\theta_x \cos\phi_x, \sin\theta_x \cos\phi_x, \cos\theta_x)$

$|\vec{m}_x| \le 1$, (density matrix on Bloch Sphere)

$M^x_{a=1} = |1\rangle_x\langle 1| = 1/2\,(I - \vec{m}_x \cdot \vec{\sigma})$, $\quad (I \equiv I_{2\times 2}$ identity matrix$)$

Solve for dual matrices $\tilde{M}^x_a$ **that satisfy** $\boxed{Tr[M^x_a \tilde{M}^{x'}_{a'}] = \delta_{x,x'}\delta_{a,a'}}$

O:

$$\boxed{\begin{aligned} O &= \sum_{a,b=0}^{r-1=0}\sum_{x,y=0}^{m-1} P(a,b\,|\,x,y)\tilde{M}^x_a \otimes \tilde{M}^y_b + \sum_{a=0}^{r-1=0}\sum_{x=0}^{m-1} P(a\,|\,x)\tilde{M}^x_a \otimes \tilde{I} + \sum_{b=0}^{r-1=0}\sum_{y}^{m-1} P(b\,|\,y)\tilde{I} \otimes \tilde{M}^y_b + \tilde{I} \otimes \tilde{I}, \\ &= \sum_{x,y=0}^{m-1} P(a=0,b=0\,|\,x,y)\tilde{M}^x_0 \otimes \tilde{M}^y_0 + \sum_{x=0}^{m-1} P(a=0\,|\,x)\tilde{M}^x_0 \otimes \tilde{I} + \sum_{y=0}^{m-1} P(b=0\,|\,y)\tilde{I} \otimes \tilde{M}^y_0 + + \tilde{I} \otimes \tilde{I} \end{aligned}}$$

Simplify Notation: let $\{I, M^x_a; a = 0,\ldots,r-2;\ x = 0,\cdots,m-1\} \equiv \{M_0 \equiv I, \{M_{i\ge 1}\} = \{M_1, M_2,\ldots\}\} = \{M_{\alpha=(0,i\ge 1)}\}$

$\{\tilde{M}_{\beta=(0,j\ge 1)}\} \equiv \{\tilde{M}_0 \equiv \tilde{I}, \tilde{M}_1, \tilde{M}_2 \ldots\}$, such that $\underline{\mathrm{Tr}[M_\alpha \tilde{M}_\beta] = \delta_{\alpha,\beta}}$ $\qquad \left(\{I, \{M^B_{j\ge 1}\}\} \to \{N_{\beta=(0,i\ge 1)}\}\right)$

$$\boxed{O = \sum_{i,j=0}^{m-1} P^{0,0}_{i,j}\tilde{M}_i \otimes \tilde{N}_j + \sum_{i=0}^{m-1} P^{0,\bullet}_i \tilde{M}_i \otimes \tilde{I} + \sum_{j=0}^{m-1} P^{\bullet,0}_j \tilde{I} \otimes \tilde{N}_j + \tilde{I} \otimes \tilde{I}}$$

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Bipartite (2-qubits) case:
m-inputs (x,y), *r* = 2 outputs (a,b)

Solution for qubits: (for A; with M,m→N,n for B)

$$M_0 = I \equiv I_{2\times 2}$$
$$M_i = 1/2(I + \vec{m}_i \cdot \vec{\sigma}),\ |\vec{m}_i| \le 1$$

$$\bar{M}_0 \equiv \bar{I} = 1/2(I + \sum_i \vec{\underset{\sim}{m}}_i \cdot \vec{\sigma}) \equiv 1/2(I + \vec{\underset{\sim}{m}} \cdot \vec{\sigma});$$
$$\bar{M}_i = \vec{\underset{\sim}{m}}_i \cdot \vec{\sigma}$$

$$\boxed{\vec{m}_i \cdot \vec{\underset{\sim}{m}}_j = \delta_{i,j}}$$
$$|\vec{m}_i| \le 1,\ |\vec{\underset{\sim}{m}}_i| \ge 1$$

In general

$$O = \frac{1}{4}\Big[\sum_{i,j=0}^{m-1} \underline{(4P^{0,0}_{i,j} - 2(P^{0,\bullet}_i + P^{\bullet,0}_j)} + 1)(\vec{\underset{\sim}{m}}_i \cdot \vec{\sigma}) \otimes (\vec{\underset{\sim}{n}}_j \cdot \vec{\sigma})$$
$$+ \sum_{i=0}^{m-1} (2P^{0,\bullet}_i - 1)\ (\vec{\underset{\sim}{m}}_i \cdot \vec{\sigma}) \otimes I + \sum_{j=0}^{m-1} (2P^{\bullet,0}_j - 1)\ I \otimes (\vec{\underset{\sim}{n}}_j \cdot \vec{\sigma}) + \underline{I \otimes I}\ \Big]$$

Using PR-Box with $P(a,b\,|\,x=i, y=j) = 1/2\ \delta_{a\oplus b,\, i\cdot j \bmod 2} \Rightarrow P^{0,\bullet}_i = P^{\bullet,0}_j = 1/2\ \forall i,j;\ P^{0,0}_{i,j} = 1/2\,\delta_{0,\, i\cdot j \bmod 2}$

***O*: 2-qubits**

$$O = \frac{1}{4}\Big[(\vec{\underset{\sim}{m}}_e \cdot \vec{\sigma}) \otimes (\vec{\underset{\sim}{n}}_e \cdot \vec{\sigma}) + (\vec{\underset{\sim}{m}}_e \cdot \vec{\sigma}) \otimes (\vec{\underset{\sim}{n}}_o \cdot \vec{\sigma}) + (\vec{\underset{\sim}{m}}_o \cdot \vec{\sigma}) \otimes (\vec{\underset{\sim}{n}}_e \cdot \vec{\sigma}) - (\vec{\underset{\sim}{m}}_o \cdot \vec{\sigma}) \otimes (\vec{\underset{\sim}{n}}_o \cdot \vec{\sigma}) + I \otimes I\ \Big]$$

where $\underset{\text{even}}{\vec{\underset{\sim}{m}}_e} = \sum_{i=0,1,2,\ldots} \vec{\underset{\sim}{m}}_{2i}$, $\underset{\text{odd}}{\vec{\underset{\sim}{m}}_o} = \sum_{i=0,1,2,\ldots} \vec{\underset{\sim}{m}}_{2i+1}$, $\underset{\text{even}}{\vec{\underset{\sim}{n}}_e} = \sum_{j=0,1,2,\ldots} \vec{\underset{\sim}{n}}_{2j}$, $\underset{\text{odd}}{\vec{\underset{\sim}{n}}_o} = \sum_{j=0,1,2,\ldots} \vec{\underset{\sim}{n}}_{2j+1}$

Bipartite (2-qubits) case:
m-inputs (x,y), *r* = 2 outputs (a,b)

Solution for qubits: (for A; with M,m→N,n for B)

$$M_0 = I \equiv I_{2\times 2} \qquad M_i = 1/2(I + \vec{m}_i \cdot \vec{\sigma}),\ |\vec{m}_i| \le 1$$

$$\bar{M}_0 \equiv \bar{I} = 1/2(I + \sum_i \underset{\sim}{\vec{m}}_i \cdot \vec{\sigma}) \equiv 1/2(I + \underset{\sim}{\vec{m}} \cdot \vec{\sigma}); \qquad \bar{M}_i = \underset{\sim}{\vec{m}}_i \cdot \vec{\sigma}$$

$$\boxed{\vec{m}_i \cdot \underset{\sim}{\vec{m}}_j = \delta_{i,j}}$$

$$|\vec{m}_i| \le 1,\ |\underset{\sim}{\vec{m}}_i| \ge 1$$

In general

$$O = \frac{1}{4}\Big[\sum_{i,j=0}^{m-1} (4P_{i,j}^{0,0} - 2(P_i^{0,\bullet} + P_j^{\bullet,0}) + 1)(\underset{\sim}{\vec{m}}_i \cdot \vec{\sigma}) \otimes (\underset{\sim}{\vec{n}}_j \cdot \vec{\sigma})$$
$$+ \sum_{i=0}^{m-1} (2P_i^{0,\bullet} - 1)\,(\underset{\sim}{\vec{m}}_i \cdot \vec{\sigma}) \otimes I + \sum_{j=0}^{m-1} (2P_j^{\bullet,0} - 1)\, I \otimes (\underset{\sim}{\vec{n}}_j \cdot \vec{\sigma}) + I \otimes I\ \Big]$$

Using PR-Box with $P(a,b \mid x=i, y=j) = 1/2\ \delta_{a\oplus b,\, i\cdot j \bmod 2} \Rightarrow P_i^{0,\bullet} = P_j^{\bullet,0} = 1/2\ \forall i,j;\ P_{i,j}^{0,0} = 1/2\,\delta_{0,\, i\cdot j \bmod 2}$

***O*: 2-qubits**

$$O = \frac{1}{4}\big[\,(\underset{\sim}{\vec{m}}_e \cdot \vec{\sigma}) \otimes (\underset{\sim}{\vec{n}}_e \cdot \vec{\sigma}) + (\underset{\sim}{\vec{m}}_e \cdot \vec{\sigma}) \otimes (\underset{\sim}{\vec{n}}_o \cdot \vec{\sigma}) + (\underset{\sim}{\vec{m}}_o \cdot \vec{\sigma}) \otimes (\underset{\sim}{\vec{n}}_e \cdot \vec{\sigma}) - (\underset{\sim}{\vec{m}}_o \cdot \vec{\sigma}) \otimes (\underset{\sim}{\vec{n}}_o \cdot \vec{\sigma}) + I \otimes I\ \big]$$

where $\underset{\text{even}}{\underset{\sim}{\vec{m}}_e} = \sum_{i=0,1,2,\dots} \underset{\sim}{\vec{m}}_{2i}$, $\underset{\text{odd}}{\underset{\sim}{\vec{m}}_o} = \sum_{i=0,1,2,\dots} \underset{\sim}{\vec{m}}_{2i+1}$, $\underset{\text{even}}{\underset{\sim}{\vec{n}}_e} = \sum_{j=0,1,2,\dots} \underset{\sim}{\vec{n}}_{2j}$, $\underset{\text{odd}}{\underset{\sim}{\vec{n}}_o} = \sum_{j=0,1,2,\dots} \underset{\sim}{\vec{n}}_{2j+1}$

Bipartite (2-qubits) case:
m-inputs (x,y), *r* = 2 outputs (a,b)

For qubits: $\boxed{M^x_{a=0} = |0\rangle_x\langle 0| = 1/2\,(I + \vec{m}_x\cdot\vec{\sigma})}$, $\vec{m}_x = m_x(\sin\theta_x\cos\phi_x, \sin\theta_x\cos\phi_x, \cos\theta_x)$

$|\vec{m}_x| \le 1$, (density matrix on Bloch Sphere)

$M^x_{a=1} = |1\rangle_x\langle 1| = 1/2\,(I - \vec{m}_x\cdot\vec{\sigma})$, $\quad (I \equiv I_{2\times 2}$ identity matrix)

Solve for dual matrices $\tilde{M}^x_a$ **that satisfy** $\boxed{Tr[M^x_a \tilde{M}^{x'}_{a'}] = \delta_{x,x'}\delta_{a,a'}}$

O:

$$\boxed{\begin{aligned} O &= \sum_{a,b=0}^{r-1=0}\sum_{x,y=0}^{m-1} P(a,b\,|\,x,y)\tilde{M}^x_a \otimes \tilde{M}^y_b + \sum_{a=0}^{r-1=0}\sum_{x=0}^{m-1} P(a\,|\,x)\tilde{M}^x_a \otimes \tilde{I} + \sum_{b=0}^{r-1=0}\sum_{y}^{m-1} P(b\,|\,y)\tilde{I}\otimes\tilde{M}^y_b + \tilde{I}\otimes\tilde{I}, \\ &= \sum_{x,y=0}^{m-1} P(a=0,b=0\,|\,x,y)\tilde{M}^x_0 \otimes \tilde{M}^y_0 + \sum_{x=0}^{m-1} P(a=0\,|\,x)\tilde{M}^x_0 \otimes \tilde{I} + \sum_{y=0}^{m-1} P(b=0\,|\,y)\tilde{I}\otimes\tilde{M}^y_0 + +\tilde{I}\otimes\tilde{I} \end{aligned}}$$

Simplify Notation: let $\{I, M^x_a; a = 0,\ldots,r-2;\ x = 0,\cdots,m-1\} \equiv \{M_0 \equiv I, \{M_{i\ge1}\} = \{M_1, M_2,\ldots\}\} = \{M_{\alpha=(0,i\ge1)}\}$

$\{\tilde{M}_{\beta=(0,j\ge1)}\} \equiv \{\tilde{M}_0 \equiv \tilde{I}, \tilde{M}_1, \tilde{M}_2\ldots\}$, such that $\underline{\mathrm{Tr}[M_\alpha \tilde{M}_\beta] = \delta_{\alpha,\beta}}$ $\qquad \left(\{I, \{M^B_{j\ge1}\}\} \to \{N_{\beta=(0,i\ge1)}\}\right)$

$$\boxed{O = \sum_{i,j=0}^{m-1} P^{0,0}_{i,j}\tilde{M}_i \otimes \tilde{N}_j + \sum_{i=0}^{m-1} P^{0,\bullet}_i \tilde{M}_i \otimes \tilde{I} + \sum_{j=0}^{m-1} P^{\bullet,0}_j \tilde{I}\otimes\tilde{N}_j + \tilde{I}\otimes\tilde{I}}$$

Bipartite (2-qubits) case: *m*-inputs (x,y), *r* = 2 outputs (a,b)

For qubits: $\boxed{M^x_{a=0} = |0\rangle_x\langle 0| = 1/2(I + \vec{m}_x \cdot \vec{\sigma})}$, $\vec{m}_x = m_x(\sin\theta_x \cos\phi_x, \sin\theta_x \cos\phi_x, \cos\theta_x)$

$|\vec{m}_x| \le 1$, (density matrix on Bloch Sphere)

$M^x_{a=1} = |1\rangle_x\langle 1| = 1/2(I - \vec{m}_x \cdot \vec{\sigma})$, $\quad (I \equiv I_{2\times 2}$ identity matrix)

Solve for dual matrices $\tilde{M}^x_a$ that satisfy $\boxed{Tr[M^x_a \tilde{M}^{x'}_{a'}] = \delta_{x,x'}\delta_{a,a'}}$

O:

$$\boxed{\begin{aligned} O &= \sum_{a,b=0}^{r-1=0}\sum_{x,y=0}^{m-1} P(a,b|x,y)\tilde{M}^x_a \otimes \tilde{M}^y_b + \sum_{a=0}^{r-1=0}\sum_{x=0}^{m-1} P(a|x)\tilde{M}^x_a \otimes \tilde{I} + \sum_{b=0}^{r-1=0}\sum_{y}^{m-1} P(b|y)\tilde{I} \otimes \tilde{M}^y_b + \tilde{I} \otimes \tilde{I}, \\ &= \sum_{x,y=0}^{m-1} P(a=0,b=0|x,y)\tilde{M}^x_0 \otimes \tilde{M}^y_0 + \sum_{x=0}^{m-1} P(a=0|x)\tilde{M}^x_0 \otimes \tilde{I} + \sum_{y=0}^{m-1} P(b=0|y)\tilde{I} \otimes \tilde{M}^y_0 + +\tilde{I} \otimes \tilde{I} \end{aligned}}$$

Simplify Notation: let $\{I, M^x_a; a=0,\ldots,r-2; x=0,\cdots,m-1\} \equiv \{M_0 \equiv I, \{M_{i\ge 1}\} = \{M_1, M_2, \ldots\}\} = \{M_{\alpha=(0,i\ge 1)}\}$

$\{\tilde{M}_{\beta=(0,j\ge 1)}\} \equiv \{\tilde{M}_0 \equiv \tilde{I}, \tilde{M}_1, \tilde{M}_2 \ldots\}$, such that $\underline{\mathrm{Tr}[M_\alpha \tilde{M}_\beta] = \delta_{\alpha,\beta}}$ $\qquad (\{I, \{M^B_{j\ge 1}\}\} \to \{N_{\beta=(0,i\ge 1)}\})$

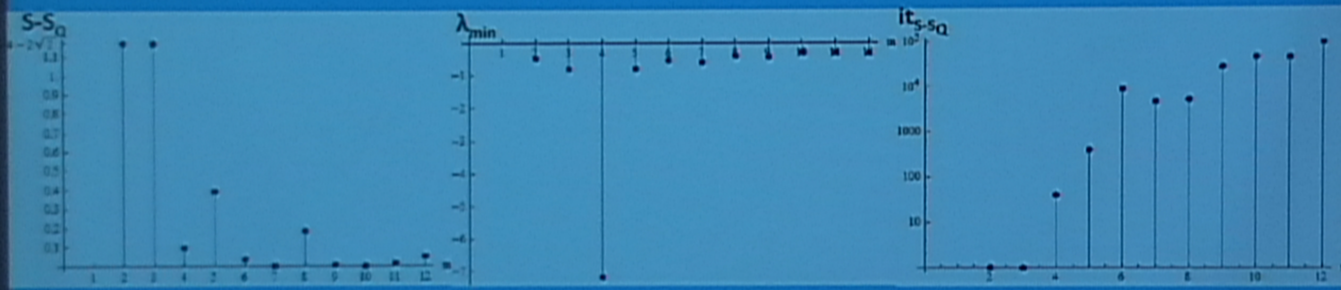
$$\boxed{O = \sum_{i,j=0}^{m-1} P^{0,0}_{i,j}\tilde{M}_i \otimes \tilde{N}_j + \sum_{i=0}^{m-1} P^{0,\bullet}_i \tilde{M}_i \otimes \tilde{I} + \sum_{j=0}^{m-1} P^{\bullet,0}_j \tilde{I} \otimes \tilde{N}_j + \tilde{I} \otimes \tilde{I}}$$

9

Bipartite (2-qubits) case: Properties
m-inputs (x,y), *r* = 2 outputs (a,b)

$$O=\frac{1}{4}\left[\ (\vec{m}_e\cdot\vec{\sigma})\otimes(\vec{n}_e\cdot\vec{\sigma})+(\vec{m}_e\cdot\vec{\sigma})\otimes(\vec{n}_o\cdot\vec{\sigma})+(\vec{m}_o\cdot\vec{\sigma})\otimes(\vec{n}_e\cdot\vec{\sigma})-\underline{(\vec{m}_o\cdot\vec{\sigma})\otimes(\vec{n}_o\cdot\vec{\sigma})}+I\otimes I\ \right]$$

where $\underset{\text{even}}{\vec{m}_e}=\sum_{i=0,1,2,\ldots}\vec{m}_{2i},\quad \underset{\text{odd}}{\vec{m}_o}=\sum_{i=0,1,2,\ldots}\vec{m}_{2i+1},\quad \underset{\text{even}}{\vec{n}_e}=\sum_{j=0,1,2,\ldots}\vec{n}_{2j},\quad \underset{\text{odd}}{\vec{n}_o}=\sum_{j=0,1,2,\ldots}\vec{n}_{2j+1}$



m=3

$$\begin{array}{c} \\ \vec{m}_0\\ \vec{m}_1\\ \vec{m}_2 \end{array}\begin{array}{c} \begin{array}{ccc}\vec{n}_0 & \vec{n}_1 & \vec{n}_2\end{array}\\ \begin{bmatrix}1/2 & 1/2 & 1/2\\ 1/2 & 0 & 1/2\\ 1/2 & 1/2 & 1/2\end{bmatrix}\end{array},\quad \begin{array}{c} \\ \vec{m}_0\\ \vec{m}_1\\ \vec{m}_2 \end{array}\begin{array}{c} \begin{array}{ccc}\vec{n}_0 & \vec{n}_1 & \vec{n}_2\end{array}\\ \begin{bmatrix}0 & 0 & 0\\ 0 & \boxed{1/2} & 0\\ 0 & 0 & 0\end{bmatrix}\end{array},\quad \begin{array}{c} \\ \vec{m}_0\\ \vec{m}_1\\ \vec{m}_2 \end{array}\begin{array}{c} \begin{array}{ccc}\vec{n}_0 & \vec{n}_1 & \vec{n}_2\end{array}\\ \begin{bmatrix}1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & 1\end{bmatrix}\end{array}$$

$P(a=0,b=0\,|\,x=\vec{m}_i,y=\vec{n}_j)$, $P(a=0,b=1\,|\,\vec{m}_i,\vec{n}_j)$, $E(\vec{m}_i,\vec{n}_j)$

$a\oplus b=0,\ P=1/2$ $\quad$ $a\oplus b=1, P=1/2$ $\quad$ correlations

$(x,y)\in\{(e,e),(e,o),(o,e)\}$, $\quad (x,y)\in\{(o,o)\}$.

m=4

$$\begin{array}{c} \\ \vec{m}_0\\ \vec{m}_1\\ \vec{m}_2\\ \vec{m}_3 \end{array}\begin{array}{c} \begin{array}{cccc}\vec{n}_0 & \vec{n}_1 & \vec{n}_2 & \vec{n}_3\end{array}\\ \begin{bmatrix}0.237 & 0.395 & 0.072 & 0.406\\ 0.162 & 0.018 & 0.381 & 0.004\\ 0.469 & 0.449 & 0.341 & 0.457\\ 0.249 & 0.038 & 0.481 & 0.024\end{bmatrix}\end{array},\quad \begin{array}{c} \\ \vec{m}_0\\ \vec{m}_1\\ \vec{m}_2\\ \vec{m}_3 \end{array}\begin{array}{c} \begin{array}{cccc}\vec{n}_0 & \vec{n}_1 & \vec{n}_2 & \vec{n}_3\end{array}\\ \begin{bmatrix}-0.052 & 0.581 & -0.710 & 0.623\\ -0.350 & -0.927 & 0.522 & -0.982\\ 0.875 & 0.796 & 0.365 & 0.829\\ -0.004 & -0.847 & 0.923 & -0.905\end{bmatrix}\end{array}$$

$P(a=0,b=0\,|\,x=\vec{m}_i,y=\vec{n}_j)$, $\quad E(\vec{m}_i,\vec{n}_j)$

probabilities $\quad$ correlations

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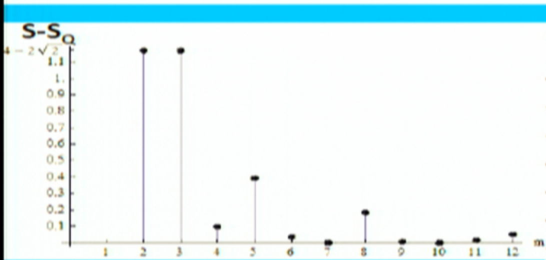
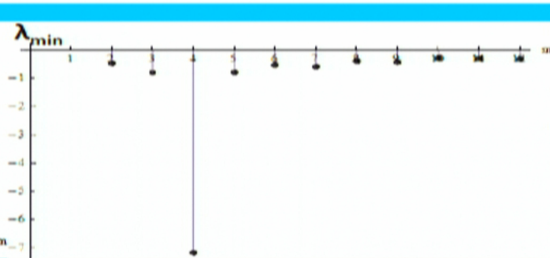
Bipartite (2-qubits) case: Properties
m-inputs (x,y), $r = 2$ outputs (a,b)

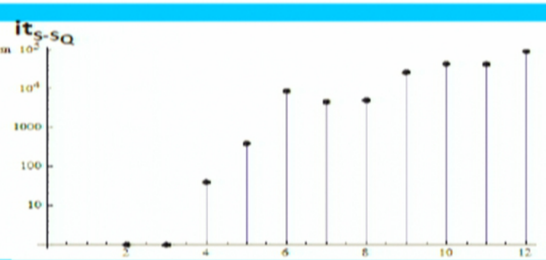
$$O = \frac{1}{4}\left[\, (\underset{\sim}{\vec{m}}_e \cdot \vec{\sigma}) \otimes (\underset{\sim}{\vec{n}}_e \cdot \vec{\sigma}) + (\underset{\sim}{\vec{m}}_e \cdot \vec{\sigma}) \otimes (\underset{\sim}{\vec{n}}_o \cdot \vec{\sigma}) + (\underset{\sim}{\vec{m}}_o \cdot \vec{\sigma}) \otimes (\underset{\sim}{\vec{n}}_e \cdot \vec{\sigma}) - (\underset{\sim}{\vec{m}}_o \cdot \vec{\sigma}) \otimes (\underset{\sim}{\vec{n}}_o \cdot \vec{\sigma}) + I \otimes I \,\right]$$

where $\underset{\sim}{\vec{m}}_e = \sum_{i=0,1,2,\ldots} \underset{\sim}{\vec{m}}_{2i}$, $\underset{\sim}{\vec{m}}_o = \sum_{i=0,1,2,\ldots} \underset{\sim}{\vec{m}}_{2i+1}$, $\underset{\sim}{\vec{n}}_e = \sum_{j=0,1,2,\ldots} \underset{\sim}{\vec{n}}_{2j}$, $\underset{\sim}{\vec{n}}_o = \sum_{j=0,1,2,\ldots} \underset{\sim}{\vec{n}}_{2j+1}$

even   odd   even   odd







m=3

$$\begin{array}{c|ccc} & \vec{n}_0 & \vec{n}_1 & \vec{n}_2 \\ \hline \vec{m}_0 & 1/2 & 1/2 & 1/2 \\ \vec{m}_1 & 1/2 & 0 & 1/2 \\ \vec{m}_2 & 1/2 & 1/2 & 1/2 \end{array}, \quad \begin{array}{c|ccc} & \vec{n}_0 & \vec{n}_1 & \vec{n}_2 \\ \hline \vec{m}_0 & 0 & 0 & 0 \\ \vec{m}_1 & 0 & \boxed{1/2} & 0 \\ \vec{m}_2 & 0 & 0 & 0 \end{array}, \quad \begin{array}{c|ccc} & \vec{n}_0 & \vec{n}_1 & \vec{n}_2 \\ \hline \vec{m}_0 & 1 & 1 & 1 \\ \vec{m}_1 & 1 & -1 & 1 \\ \vec{m}_2 & 1 & 1 & 1 \end{array}$$

$P(a=0, b=0 \mid x=\vec{m}_i, y=\vec{n}_j)$, $a \oplus b = 0$, $P = 1/2$, $(x,y) \in \{(e,e),(e,o),(o,e)\}$.

$P(a=0, b=1 \mid \vec{m}_i, \vec{n}_j)$, $a \oplus b = 1$, $P = 1/2$, $(x,y) \in \{(o,o)\}$.

$E(\vec{m}_i, \vec{n}_j)$ correlations

m=4

$$\begin{array}{c|cccc} & \vec{n}_0 & \vec{n}_1 & \vec{n}_2 & \vec{n}_3 \\ \hline \vec{m}_0 & 0.237 & 0.395 & 0.072 & 0.406 \\ \vec{m}_1 & 0.162 & 0.018 & 0.381 & 0.004 \\ \vec{m}_2 & 0.469 & 0.449 & 0.341 & 0.457 \\ \vec{m}_3 & 0.249 & 0.038 & 0.481 & 0.024 \end{array}, \quad \begin{array}{c|cccc} & \vec{n}_0 & \vec{n}_1 & \vec{n}_2 & \vec{n}_3 \\ \hline \vec{m}_0 & -0.052 & 0.581 & -0.710 & 0.623 \\ \vec{m}_1 & -0.350 & -0.927 & 0.522 & -0.982 \\ \vec{m}_2 & 0.875 & 0.796 & 0.365 & 0.829 \\ \vec{m}_3 & -0.004 & -0.847 & 0.923 & -0.905 \end{array}$$

$P(a=0, b=0 \mid x=\vec{m}_i, y=\vec{n}_j)$, probabilities

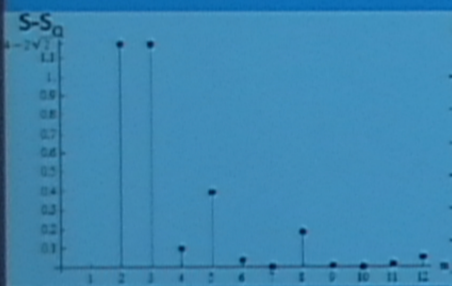
$E(\vec{m}_i, \vec{n}_j)$ correlations

Bipartite (2-qubits) case: Properties
m-inputs (x,y), *r* = 2 outputs (a,b)

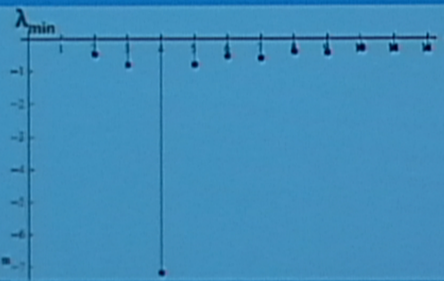
$$O=\frac{1}{4}\left[\ (\underset{\sim}{\vec{m}}_e\cdot\vec{\sigma})\otimes(\underset{\sim}{\vec{n}}_e\cdot\vec{\sigma})+(\underset{\sim}{\vec{m}}_e\cdot\vec{\sigma})\otimes(\underset{\sim}{\vec{n}}_o\cdot\vec{\sigma})+(\underset{\sim}{\vec{m}}_o\cdot\vec{\sigma})\otimes(\underset{\sim}{\vec{n}}_e\cdot\vec{\sigma})-\underline{(\underset{\sim}{\vec{m}}_o\cdot\vec{\sigma})\otimes(\underset{\sim}{\vec{n}}_o\cdot\vec{\sigma})}+I\otimes I\ \right]$$

where $\underset{\sim}{\vec{m}}_e=\sum_{i=0,1,2,\ldots}\underset{\sim}{\vec{m}}_{2i}$, $\underset{\sim}{\vec{m}}_o=\sum_{i=0,1,2,\ldots}\underset{\sim}{\vec{m}}_{2i+1}$, $\underset{\sim}{\vec{n}}_e=\sum_{j=0,1,2,\ldots}\underset{\sim}{\vec{n}}_{2j}$, $\underset{\sim}{\vec{n}}_o=\sum_{j=0,1,2,\ldots}\underset{\sim}{\vec{n}}_{2j+1}$

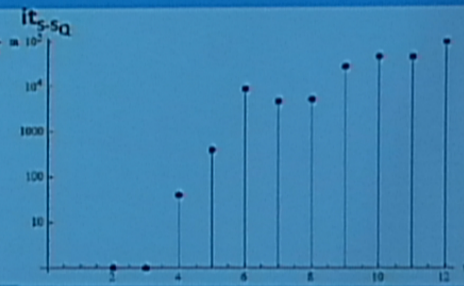
even odd even odd



m



m



m

m=3

$$\begin{array}{c|ccc} & \vec{n}_0 & \vec{n}_1 & \vec{n}_2 \\ \hline \vec{m}_0 & 1/2 & 1/2 & 1/2 \\ \vec{m}_1 & 1/2 & 0 & 1/2 \\ \vec{m}_2 & 1/2 & 1/2 & 1/2 \end{array},\qquad \begin{array}{c|ccc} & \vec{n}_0 & \vec{n}_1 & \vec{n}_2 \\ \hline \vec{m}_0 & 0 & 0 & 0 \\ \vec{m}_1 & 0 & \boxed{1/2} & 0 \\ \vec{m}_2 & 0 & 0 & 0 \end{array},\qquad \begin{array}{c|ccc} & \vec{n}_0 & \vec{n}_1 & \vec{n}_2 \\ \hline \vec{m}_0 & 1 & 1 & 1 \\ \vec{m}_1 & 1 & -1 & 1 \\ \vec{m}_2 & 1 & 1 & 1 \end{array}$$

$P(a=0,b=0\,|\,x=\vec{m}_i,y=\vec{n}_j)$, $a\oplus b=0$, $P=1/2$, $(x,y)\in\{(e,e),(e,o),(o,e)\}$.

$P(a=0,b=1\,|\,\vec{m}_i,\vec{n}_j)$, $a\oplus b=1$, $P=1/2$, $(x,y)\in\{(o,o)\}$.

$E(\vec{m}_i,\vec{n}_j)$ correlations

m=4

$$\begin{array}{c|cccc} & \vec{n}_0 & \vec{n}_1 & \vec{n}_2 & \vec{n}_3 \\ \hline \vec{m}_0 & 0.237 & 0.395 & 0.072 & 0.406 \\ \vec{m}_1 & 0.162 & 0.018 & 0.381 & 0.004 \\ \vec{m}_2 & 0.469 & 0.449 & 0.341 & 0.457 \\ \vec{m}_3 & 0.249 & 0.038 & 0.481 & 0.024 \end{array},\qquad \begin{array}{c|cccc} & \vec{n}_0 & \vec{n}_1 & \vec{n}_2 & \vec{n}_3 \\ \hline \vec{m}_0 & -0.052 & 0.581 & -0.710 & 0.623 \\ \vec{m}_1 & -0.350 & -0.927 & 0.522 & -0.982 \\ \vec{m}_2 & 0.875 & 0.796 & 0.365 & 0.829 \\ \vec{m}_3 & -0.004 & -0.847 & 0.923 & -0.905 \end{array}$$

$P(a=0,b=0\,|\,x=\vec{m}_i,y=\vec{n}_j)$, probabilities

$E(\vec{m}_i,\vec{n}_j)$ correlations

Bipartite (2-qubits) case: m-inputs (x,y), $r = 2$ outputs (a,b)

PR Box state is of the form:

$$O_{PR} = \frac{1}{4}\left[I \otimes I + \sum_{\alpha,\beta=1}^{3} C^{PR}_{\alpha\beta}\, \sigma_\alpha \otimes \sigma_\beta \right], \quad \boxed{C^{PR}_{\alpha\beta} = \begin{array}{c|ccc} & e & o & e \\ \hline e & 1 & 1 & 1 \\ o & 1 & -1 & 1 \\ e & 1 & 1 & 1 \end{array}}, \quad \Rightarrow \boxed{E(\vec{m},\vec{n}) = \sum_{\alpha,\beta=1}^{3} m_\alpha C^{PR}_{\alpha\beta} \otimes n_\beta \equiv \vec{m} \cdot \mathbf{C}^{PR} \cdot \vec{n}}$$

$$P(a,b \mid x,y) = 1/2 \quad \text{if } a \oplus b = x \cdot y \bmod 2$$

Most general 2-qubit state:

$$O^{(2)} = \frac{1}{4}\left[\underbrace{I \otimes I}_{\text{max. mixed}} + (\underbrace{\vec{s}^{(1)}}_{\text{Alice Bloch vector}} \cdot \vec{\sigma}) \otimes I + I \otimes (\underbrace{\vec{s}^{(2)}}_{\text{Bob Bloch vector}} \cdot \vec{\sigma}) + \sum_{\alpha,\beta=1}^{3} \underbrace{C_{\alpha\beta}}_{\text{2-party Alice-Bob correlations}} \sigma_\alpha \otimes \sigma_\beta \right]$$

Quantum Singlet (Pure) State (Bell) $\quad \left|\psi_{\text{singlet}}\right\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$

$$\rho^{\text{singlet}} = \frac{1}{4}\left[I \otimes I + \sum_{\alpha,\beta=1}^{3} C^{\text{singlet}}_{\alpha\beta}\, \sigma_\alpha \otimes \sigma_\beta \right], \quad \boxed{C^{\text{singlet}}_{\alpha\beta} = -\delta_{\alpha\beta} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}, \quad \Rightarrow \boxed{E^{\text{singlet}}(\vec{m},\vec{n}) = -\vec{m}\cdot\vec{n}}$$

Bell states $\quad C^{\text{Bell States}}_{\alpha\beta} = \left\{ \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\}$

What determines the structure of $\vec{s}^{(1)}, \vec{s}^{(2)}, C_{\alpha\beta}$?

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Bipartite (2-qubits) case: *m*-inputs (*x*,*y*), *r* = 2 outputs (*a*,*b*)

PR Box state is of the form:

$$O_{PR} = \frac{1}{4}\left[I\otimes I + \sum_{\alpha,\beta=1}^{3} C_{\alpha\beta}^{PR}\,\sigma_\alpha \otimes \sigma_\beta \right], \quad \boxed{\begin{array}{cc} & \begin{array}{ccc} e & o & e \end{array} \\ C_{\alpha\beta}^{PR} = \begin{array}{c} e \\ o \\ e \end{array} & \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \end{array}}, \quad \Rightarrow \boxed{E(\vec{m},\vec{n}) = \sum_{\alpha,\beta=1}^{3} m_\alpha C_{\alpha\beta}^{PR} \otimes n_\beta \equiv \vec{m}\cdot \mathbf{C}^{PR}\cdot \vec{n}}$$

$$P(a,b\,|\,x,y) = 1/2 \quad \text{if}\ a\oplus b = x\cdot y \bmod 2$$

Most general 2-qubit state:

max. mixed — Alice Bloch vector — Bob Bloch vector — 2-party Alice-Bob correlations

$$O^{(2)} = \frac{1}{4}\left[I\otimes I + (\vec{s}^{(1)}\cdot\vec{\sigma})\otimes I + I\otimes(\vec{s}^{(2)}\cdot\vec{\sigma}) + \sum_{\alpha,\beta=1}^{3} C_{\alpha\beta}\,\sigma_\alpha\otimes\sigma_\beta \right]$$

Quantum Singlet (Pure) State (Bell) $\quad |\psi_{\text{singlet}}\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$

$$\rho^{\text{singlet}} = \frac{1}{4}\left[I\otimes I + \sum_{\alpha,\beta=1}^{3} C_{\alpha\beta}^{\text{singlet}}\,\sigma_\alpha\otimes\sigma_\beta \right], \quad \boxed{C_{\alpha\beta}^{\text{singlet}} = -\delta_{\alpha\beta} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}, \quad \Rightarrow \boxed{E^{\text{singlet}}(\vec{m},\vec{n}) = -\vec{m}\cdot\vec{n}}$$

Bell states $\quad C_{\alpha\beta}^{\text{Bell States}} = \left\{ \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\}$

What determines the structure of $\vec{s}^{(1)}, \vec{s}^{(2)}, C_{\alpha\beta}$?

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No Signaling (NS) Theories
Extension to Tripartite case: 3-qubits

- **No Signaling Theories can be extended to tripartite systems of 3-qubits, each with *m*-inputs (measurements) to measure pure tripartite entanglement.**
- **The generalization of the bipartite CHSH inequality was given by Svetlichny**

Svetlichny, PRD 35, 3066 (1987)

First, consider correlations E(a,b,c|x,y,z) between 3 observers A,B,C with

inputs: $x, y, z \in \{0,1\}$, and outputs: $a, b, c \in \{0,1\}$

The relevant inequality to compute is the *Svetlichny inequality* (SI)

$$\begin{aligned} S \equiv |\, & E(a,b,c\,|\,0,0,0) + E(a,b,c\,|\,0,1,0) + E(a,b,c\,|\,1,0,0) - E(a,b,c\,|\,1,1,0) \\ & + E(a,b,c\,|\,0,0,1) - E(a,b,c\,|\,0,1,1) - E(a,b,c\,|\,1,0,1) - E(a,b,c\,|\,1,1,1)\,| \end{aligned}$$

The bounds on the Svetlichny inequality are

Classical: $|S_C| \le 4$; QM: $|S_Q| \le 4\sqrt{2}$; Algebraic Maximum: $|S_{AM}| \le 8$

A Tripartite No Signaling (TNS) Box yielding the algebraic maximum of S is given by probabilities

TPR Box:

$$P(a,b,c\,|\,x,y,z) = \begin{cases} 1/4 & \text{if } a \oplus b \oplus c = x \cdot y \oplus y \cdot z \oplus x \cdot z \\ 0 & \text{otherwise} \end{cases}$$

with measurement settings x, y, z and outcomes a, b, c as *bits*,

i.e $a,b,c,x,y,z \in \{0,1\}$ (Note: $\{0,1\} \leftrightarrow \{+1,-1\}$)

With a TPR Box:

$$\begin{aligned} S = \; & 1 + 1 + 1 - (-1) \\ & + 1 - (-1) - (-1) - (-1) \\ & = 8 \end{aligned}$$

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Tripartite (3-qubits) case:
m-inputs (*x*,*y*), *r* = 2 outputs (*a*,*b*)

We generalize this to *m* measurements settings (inputs), still with binary outputs

TPR Box:

$$P(a,b,c\,|\,x,y,z)=\begin{cases}1/4 & \text{if } a\oplus b\oplus c = x\cdot y\oplus y\cdot z\oplus x\cdot z\\ 0 & \text{otherwise}\end{cases}$$

with m measurement settings x, y, z and binary outcomes a,b,c, (i.e. qubits)

$\Rightarrow\ a,b,c,\in\{0,1\}, x,y,z\in\{0,1,\ldots,m-1\}$

and find the 3-qubit entanglement witness exhibiting TPR correlations:

O_{PR} 2-qubit term

$$O_{TPR}=\frac{1}{8}[I\otimes I\otimes I+$$
$$+\{(\vec{m}_e\cdot\vec{\sigma})\otimes(\vec{n}_e\cdot\vec{\sigma})+(\vec{m}_e\cdot\vec{\sigma})\otimes(\vec{n}_o\cdot\vec{\sigma})+(\vec{m}_o\cdot\vec{\sigma})\otimes(\vec{n}_e\cdot\vec{\sigma})-(\vec{m}_o\cdot\vec{\sigma})\otimes(\vec{n}_o\cdot\vec{\sigma})\}\otimes(\vec{r}_e\cdot\vec{\sigma})$$
$$-\{(\vec{m}_o\cdot\vec{\sigma})\otimes(\vec{n}_o\cdot\vec{\sigma})+(\vec{m}_o\cdot\vec{\sigma})\otimes(\vec{n}_e\cdot\vec{\sigma})+(\vec{m}_e\cdot\vec{\sigma})\otimes(\vec{n}_o\cdot\vec{\sigma})-(\vec{m}_e\cdot\vec{\sigma})\otimes(\vec{n}_e\cdot\vec{\sigma})\}\otimes(\vec{r}_o\cdot\vec{\sigma})]$$

where $\underset{\text{even}}{\vec{q}_e}=\sum_{i=0,1,2,\ldots}\vec{q}_{2i}$, $\underset{\text{odd}}{\vec{q}_o}=\sum_{i=0,1,2,\ldots}\vec{q}_{2i+1}$, $\vec{q}=\{\vec{m},\vec{n},\vec{r}\}$

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Tripartite (3-qubits) case:
m-inputs (*x*,*y*), *r* = 2 outputs (*a*,*b*)

We generalize this to *m* measurements settings (inputs), still with binary outputs

TPR Box:

$$P(a,b,c\,|\,x,y,z)=\begin{cases}1/4 & \text{if } a\oplus b\oplus c = x\cdot y\oplus y\cdot z\oplus x\cdot z\\ 0 & \text{otherwise}\end{cases}$$

with m measurement settings x, y, z and binary outcomes a, b, c, (i.e. qubits)

$\Rightarrow\ a,b,c,\in\{0,1\}, x,y,z\in\{0,1,\ldots,m-1\}$

and find the 3-qubit entanglement witness exhibiting TPR correlations:

O_{PR} **2-qubit term**

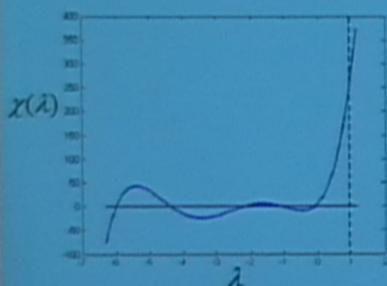
$$\begin{aligned}O_{TPR}=\frac{1}{8}[\,I\otimes I\otimes I &+\{(\vec{\underset{\sim}{m}}_e\cdot\vec{\sigma})\otimes(\vec{\underset{\sim}{n}}_e\cdot\vec{\sigma})+(\vec{\underset{\sim}{m}}_e\cdot\vec{\sigma})\otimes(\vec{\underset{\sim}{n}}_o\cdot\vec{\sigma})+(\vec{\underset{\sim}{m}}_o\cdot\vec{\sigma})\otimes(\vec{\underset{\sim}{n}}_e\cdot\vec{\sigma})-(\vec{\underset{\sim}{m}}_o\cdot\vec{\sigma})\otimes(\vec{\underset{\sim}{n}}_o\cdot\vec{\sigma})\}\otimes(\vec{\underset{\sim}{r}}_e\cdot\vec{\sigma})\\ &-\{(\vec{\underset{\sim}{m}}_o\cdot\vec{\sigma})\otimes(\vec{\underset{\sim}{n}}_o\cdot\vec{\sigma})+(\vec{\underset{\sim}{m}}_o\cdot\vec{\sigma})\otimes(\vec{\underset{\sim}{n}}_e\cdot\vec{\sigma})+(\vec{\underset{\sim}{m}}_e\cdot\vec{\sigma})\otimes(\vec{\underset{\sim}{n}}_o\cdot\vec{\sigma})-(\vec{\underset{\sim}{m}}_e\cdot\vec{\sigma})\otimes(\vec{\underset{\sim}{n}}_e\cdot\vec{\sigma})\}\otimes(\vec{\underset{\sim}{r}}_o\cdot\vec{\sigma})\,]\end{aligned}$$

$$\text{where } \underset{\textbf{even}}{\vec{\underset{\sim}{q}}_e}=\sum_{i=0,1,2,\ldots}\vec{\underset{\sim}{q}}_{2i},\quad \underset{\textbf{odd}}{\vec{\underset{\sim}{q}}_o}=\sum_{i=0,1,2,\ldots}\vec{\underset{\sim}{q}}_{2i+1},\quad \vec{\underset{\sim}{q}}=\{\vec{\underset{\sim}{m}},\vec{\underset{\sim}{n}},\vec{\underset{\sim}{r}}\}$$

Positivity Constraints

Traceless part of state $O^{(n)} = \frac{1}{N}[I_N - K]$ $\boxed{\text{Define: } K \equiv [I_N - N\,O^{(n)}]. \quad \text{If } O^{(n)} \geq 0 \Rightarrow K \leq 1.}$

Characteristic Polynomial $\chi_A(\lambda) = \prod_{i=1}^{N}(\lambda - \lambda_i), \quad \{\lambda_{i=1,\ldots,N} \leq 1\}$ **Cayley-Hamilton Theorem** $\chi_A(K) = 0$



$$\chi_A(\lambda) = \prod_{i=1}^{N}(\lambda - \lambda_i)\Big|_{\lambda \geq 1} \geq 0 \;\Rightarrow\; \frac{d^k \chi(\lambda)}{d\lambda^k}\bigg|_{\lambda \geq 1} \geq 0 \Rightarrow \frac{d^k \chi(\lambda)}{d\lambda^k}\bigg|_{\lambda = 1} \geq 0$$

General: Characteristic Polynomial in terms of Traces of Matrix A: $\boxed{T_m \equiv Tr[A^m]}$

$$\boxed{\begin{aligned} &\chi_A(\lambda) = \lambda^N + D_1\lambda^{N-1} + \ldots + D_{N-1}\lambda + D_N \\ \Rightarrow\; &mD_m + D_{m-1}T_1 + D_{m-2}T_2 + \ldots + D_1T_{m-1} + T_m = 0,\; m = 1,2,\ldots,N. \end{aligned}}$$

$$\boxed{\sum_{r=1}^{k}\binom{N-r}{N-k}(-D_r) \leq \binom{N}{k} = \frac{N!}{k!(N-k)!},}\; \boxed{T_2 \leq N(N-1),}\; \boxed{(N-2)(T_2/2) + (T_3/3) \leq \binom{N}{3},}\; \boxed{\frac{(N-2)(N-3)}{2}(T_2/2) + (N-3)(T_3/3) + 1/4(T_4 - 1/2T_2^2) \leq \binom{N}{4}}$$

Results:
2 & 3 Qubits

$$T_2 = Tr[K^2] = N\left[\sum_{i=1}^{n}\left(\bar{s}^{(i)}\right)^2 + \sum_{i<j}^{n}\sum_{\alpha_1,\alpha_2=1}^{3}\left(C^{(i,j)}_{\alpha_1\alpha_2}\right)^2 + \ldots + \sum_{\alpha_1,\alpha_2,\ldots,\alpha_n=1}^{3}\left(C^{(1,2,\ldots,n)}_{\alpha_1,\alpha_2,\ldots,\alpha_n}\right)^2\right] \leq N(N-1)$$

$$\Rightarrow \boxed{n = 2:\; \bar{s}^{2(1)} + \bar{s}^{2(2)} + \sum_{\alpha_1,\alpha_2=1}^{3} C^2_{\alpha_1\alpha_2} \leq (N-1)_{N=2^2} = 3,}$$

$$\Rightarrow \boxed{n = 3:\; \bar{s}^{2(1)} + \bar{s}^{2(2)} + \bar{s}^{2(3)} + \sum_{\alpha_1,\alpha_2=1}^{3} A^2_{\alpha_1\alpha_2} + \sum_{\alpha_1,\alpha_2=1}^{3} B^2_{\alpha_1\alpha_2} + \sum_{\alpha_1,\alpha_2=1}^{3} C^2_{\alpha_1\alpha_2} + \sum_{\alpha_1,\alpha_2,\alpha_3=1}^{3} C^2_{\alpha_1\alpha_2\alpha_3} \leq (N-1)_{N=2^3} = 7,}$$

Positivity Constraints

Traceless part of state $O^{(n)} = \frac{1}{N}[I_N - K]$ $\boxed{\text{Define: } K \equiv [I_N - N O^{(n)}]. \quad \text{If } O^{(n)} \geq 0 \Rightarrow K \leq 1.}$

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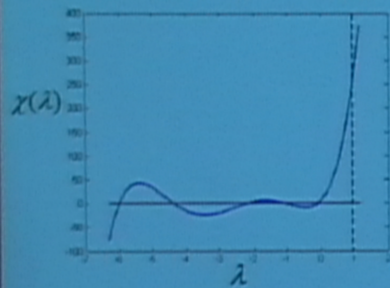

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$$\boxed{\sum_{r=2}^{k} \binom{N-r}{N-k} (-D_r) \leq \binom{N}{k} = \frac{N!}{k!(N-k)!},} \ \boxed{T_2 \leq N(N-1),} \ \boxed{(N-2)(T_2/2) + (T_3/3) \leq \binom{N}{3},} \ \boxed{\frac{(N-2)(N-3)}{2}(T_2/2) + (N-3)(T_3/3) + 1/4(T_4 - 1/2 T_2^2) \leq \binom{N}{4}}$$

Results:
2 & 3 Qubits

$$T_2 = Tr[K^2] = N\left[\sum_{i=1}^{n} \left(\vec{s}^{(i)}\right)^2 + \sum_{i<j}^{n} \sum_{\alpha_1,\alpha_2=1}^{3} \left(C^{(i,j)}_{\alpha_1\alpha_2}\right)^2 + \ldots + \sum_{\alpha_1,\alpha_2,\ldots,\alpha_n=1}^{3} \left(C^{(1,2,\ldots,n)}_{\alpha_1,\alpha_2,\ldots,\alpha_n}\right)^2\right] \leq N(N-1)$$

$$\Rightarrow \boxed{n = 2:\ \vec{s}^{2(1)} + \vec{s}^{2(2)} + \sum_{\alpha_1,\alpha_2=1}^{3} C^2_{\alpha_1\alpha_2} \leq (N-1)_{N=2^2} = 3,}$$

$$\Rightarrow \boxed{n = 3:\ \vec{s}^{2(1)} + \vec{s}^{2(2)} + \vec{s}^{2(3)} + \sum_{\alpha_1,\alpha_2=1}^{3} A^2_{\alpha_1\alpha_2} + \sum_{\alpha_1,\alpha_2=1}^{3} B^2_{\alpha_1\alpha_2} + \sum_{\alpha_1,\alpha_2=1}^{3} C^2_{\alpha_1\alpha_2} + \sum_{\alpha_1,\alpha_2,\alpha_3=1}^{3} C^2_{\alpha_1\alpha_2\alpha_3} \leq (N-1)_{N=2^3} = 7,}$$

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Positivity Constraints

Traceless part of state $O^{(n)} = \frac{1}{N}[I_N - K]$ $\boxed{\text{Define: } K \equiv [I_N - N O^{(n)}]. \quad \text{If } O^{(n)} \geq 0 \Rightarrow K \leq 1.}$

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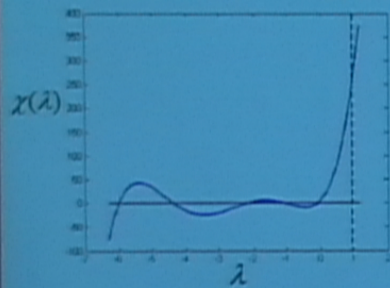

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$$\boxed{\sum_{r=2}^{k}\binom{N-r}{N-k}(-D_r) \leq \binom{N}{k} = \frac{N!}{k!(N-k)!},} \quad \boxed{T_2 \leq N(N-1),} \quad \boxed{(N-2)(T_2/2) + (T_3/3) \leq \binom{N}{3},} \quad \boxed{\frac{(N-2)(N-3)}{2}(T_2/2) + (N-3)(T_3/3) + 1/4(T_4 - 1/2 T_2^2) \leq \binom{N}{4}}$$

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$$\Rightarrow \boxed{n = 2:\ \bar{s}^{2(1)} + \bar{s}^{2(2)} + \sum_{a_1,a_2=1}^{3} C_{a_1 a_2}^2 \leq (N-1)_{N=2^2} = 3,}$$

$$\Rightarrow \boxed{n = 3:\ \bar{s}^{2(1)} + \bar{s}^{2(2)} + \bar{s}^{2(3)} + \sum_{a_1,a_2=1}^{3} A_{a_1 a_2}^2 + \sum_{a_1,a_2=1}^{3} B_{a_1 a_2}^2 + \sum_{a_1,a_2=1}^{3} C_{a_1 a_2}^2 + \sum_{a_1,a_2,a_3=1}^{3} C_{a_1 a_2 a_3}^2 \leq (N-1)_{N=2^3} = 7,}$$

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Positivity Constraints

3-Qubit states $O^{(3)} = \frac{1}{8}[I\otimes I\otimes I + (\vec{s}^{(1)}\cdot\vec{\sigma})\otimes I\otimes I + I\otimes(\vec{s}^{(2)}\cdot\vec{\sigma})\otimes I + I\otimes I\otimes(\vec{s}^{(3)}\cdot\vec{\sigma})$

$$+\sum_{\alpha,\beta=1}^{3} A_{\alpha\beta}\,\sigma_\alpha\otimes\sigma_\beta\otimes I + \sum_{\alpha,\beta=1}^{3} B_{\alpha\beta}\,\sigma_\alpha\otimes I\otimes\sigma_\beta + \sum_{\alpha,\beta=1}^{3} C_{\alpha\beta}\,I\otimes\sigma_\alpha\otimes\sigma_\beta + \sum_{\alpha,\beta,\gamma=1}^{3} C_{\alpha\beta\gamma}\,\sigma_\alpha\otimes\sigma_\beta\otimes\sigma_\gamma].$$

T_2 Positivity Constraint

$$\boxed{T_2 = Tr[K^2];\ n=3:\ \vec{s}^{2(1)} + \vec{s}^{2(2)} + \vec{s}^{2(3)} + \sum_{\alpha_1,\alpha_2=1}^{3} A^2_{\alpha_1\alpha_2} + \sum_{\alpha_1,\alpha_2=1}^{3} B^2_{\alpha_1\alpha_2} + \sum_{\alpha_1,\alpha_2=1}^{3} C^2_{\alpha_1\alpha_2} + \sum_{\alpha_1,\alpha_2,\alpha_3=1}^{3} C^2_{\alpha_1\alpha_2\alpha_3} \le (N-1)_{N=2^3} = 7}$$

Tripartite PR Box States are of the form:

$$O_{TPR} = \frac{1}{8}\left[I\otimes I\otimes I + \sum_{\alpha,\beta,\gamma=1}^{3} C^{TPR}_{\alpha\beta\gamma}\,\sigma_\alpha\otimes\sigma_\beta\otimes\sigma_\gamma\right],$$

Example: $C^{TPR;\varepsilon}_{\alpha\beta\gamma} \equiv \varepsilon_{\alpha\beta\gamma}\ (\varepsilon_{123}=1)$

(i) **NOT a quantum state**

(ii) **NOT a pure state**

But odd *n*-qubit states of solely *n*=2*n*'+1-party correlations have $T_{2n'+1} \equiv 0$ ($T_3 \equiv 0$); Consider D_3 constraint:

$$(N-2)(T_2/2) + (T_3/3) \le \binom{N}{3} \Rightarrow \boxed{T_2 \le N(N-1)/3}$$

$$\boxed{\Rightarrow \sum^{3}_{\alpha,\beta,\gamma=1}\left(C_{\alpha\beta\gamma}\right)^2 \le 7/3,\ \text{but}\ \sum^{3}_{\alpha,\beta,\gamma=1}\left(C^{TPR;\varepsilon}_{\alpha\beta\gamma}\right)^2 = 6 > 7/3}$$

$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$

$\varepsilon_{213} = \varepsilon_{321} = \varepsilon_{132} = -1$

$\boxed{T_2 = 6 < 7}$

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Positivity Constraints

3-Qubit states $O^{(3)} = \frac{1}{8}[I\otimes I\otimes I + (\vec{s}^{(1)}\cdot\vec{\sigma})\otimes I\otimes I + I\otimes(\vec{s}^{(2)}\cdot\vec{\sigma})\otimes I + I\otimes I\otimes(\vec{s}^{(3)}\cdot\vec{\sigma})$

$$+\sum_{\alpha,\beta=1}^{3} A_{\alpha\beta}\,\sigma_\alpha\otimes\sigma_\beta\otimes I + \sum_{\alpha,\beta=1}^{3} B_{\alpha\beta}\,\sigma_\alpha\otimes I\otimes\sigma_\beta + \sum_{\alpha,\beta=1}^{3} C_{\alpha\beta}\, I\otimes\sigma_\alpha\otimes\sigma_\beta + \sum_{\alpha,\beta,\gamma=1}^{3} C_{\alpha\beta\gamma}\,\sigma_\alpha\otimes\sigma_\beta\otimes\sigma_\gamma].$$

T_2 Positivity Constraint

$$\boxed{T_2 = Tr[K^2];\ n=3:\ \vec{s}^{2(1)}+\vec{s}^{2(2)}+\vec{s}^{2(3)} + \sum_{\alpha_1,\alpha_2=1}^{3} A^2_{\alpha_1\alpha_2} + \sum_{\alpha_1,\alpha_2=1}^{3} B^2_{\alpha_1\alpha_2} + \sum_{\alpha_1,\alpha_2=1}^{3} C^2_{\alpha_1\alpha_2} + \sum_{\alpha_1,\alpha_2,\alpha_3=1}^{3} C^2_{\alpha_1\alpha_2\alpha_3} \le (N-1)_{N=2^3} = 7}$$

Tripartite PR Box States are of the form: $O_{TPR} = \frac{1}{8}\left[I\otimes I\otimes I + \sum_{\alpha,\beta,\gamma=1}^{3} C^{TPR}_{\alpha\beta\gamma}\,\sigma_\alpha\otimes\sigma_\beta\otimes\sigma_\gamma\right],$

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$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$

$\varepsilon_{213} = \varepsilon_{321} = \varepsilon_{132} = -1$

$\boxed{T_2 = 6 < 7}$

18

Purity Constraints

Pure State $O_{pure}^{(n)} = |\Psi_N\rangle\langle\Psi_N| \quad O_{pure}^{2(n)} = O_{pure}^{(n)} \quad O_{pure}^{(n)}(O_{pure}^{(n)} - I_N) = 0. \quad (N = 2^n)$

Pure State Constraint $O^{(n)} = 1/N[I_N - K]: \; O_{pure}^{(2)}(O_{pure}^{(2)} - I_N) = 0 \;\Rightarrow\; K^2 = (N-1)I_N - (N-2)K$

$$\Rightarrow T_{m+2} = (N-1)T_m - (N-2)T_{m+1}: \quad \boxed{T_3 = -N(N-1)(N-2),} \quad T_4 = N(N-1)^2 + N(N-1)(N-2)^2,$$

An Immediate Consequence

States of solely odd $n=2n'+1$-party correlations cannot be pure due to $T_{2n'+1} \equiv 0$ ($T_3 \equiv 0$)

$$O^{(2n'+1)} = \frac{1}{N}[I \otimes I \otimes \cdots \otimes I + \sum_{\alpha_1,\alpha_2,\ldots,\alpha_{2n'+1}=1}^{3} C_{\alpha_1,\alpha_2,\ldots,\alpha_{2n'+1}}^{(1,2,\ldots,n)} \sigma_{\alpha_1} \otimes \sigma_{\alpha_2} \otimes \cdots \otimes \sigma_{\alpha_{2n'+1}}]$$

Tripartite PR Box States are not pure states:

$$O_{TPR} = \frac{1}{8}\left[I \otimes I \otimes I + \sum_{\alpha,\beta,\gamma=1}^{3} C_{\alpha\beta\gamma}^{TPR} \sigma_\alpha \otimes \sigma_\beta \otimes \sigma_\gamma\right],$$

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Purity Constraints: 2-Qubits

Most general 2-qubit state:

max. mixed — Alice Bloch vector — Bob Bloch vector — 2-party Alice-Bob correlations

$$O^{(2)} = \frac{1}{4}\left[I \otimes I + (\vec{s}^{(1)} \cdot \vec{\sigma}) \otimes I + I \otimes (\vec{s}^{(2)} \cdot \vec{\sigma}) + \sum_{\alpha,\beta=1}^{3} C_{\alpha\beta}\, \sigma_\alpha \otimes \sigma_\beta \right]$$

Pure State $O^{(n)} = 1/N[I_N - K]: \;\; O^{(2)}_{pure}(O^{(2)}_{pure} - I_N) = 0 \;\; \Rightarrow \;\; K^2 = (N-1)I_N - (N-2)K$

2-Qubit Pure State

$$\boxed{\rho^{(2)}_{pure} = 1/4[I \otimes I + p\,\sigma_1 \otimes I + p\,I \otimes \sigma_1 - \sigma_1 \otimes \sigma_1 - q\,\sigma_2 \otimes \sigma_2 - \sigma_3 \otimes \sigma_3] \geq 0}$$

$0 \leq p \leq 1$ ($p = 0$: max ent. Bell States), $\quad q = Tr[\rho^{(2)*}_{pure}\rho^{(2)}_{pure}] = \sqrt{1-p^2} \geq 0$ (concurrence)

Purity Constraint

$$\vec{s}^{(1)} = \mathbf{C} \cdot \vec{s}^{(2)}, \qquad \vec{s}^{(2)} = \vec{s}^{(1)} \cdot \mathbf{C},$$

$$C_{\alpha\beta} = s^{(1)}_\alpha s^{(2)}_\beta - C^{(sub)}_{\alpha\beta} \qquad C^{(sub)}_{\alpha\beta} \equiv 1/2 \sum_{\mu\nu\mu'\nu'=1}^{3} C_{\mu\nu} C_{\mu'\nu'} \varepsilon_{\mu\mu'\alpha} \varepsilon_{\nu\nu'\beta}$$

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Purity Constraints: 3-Qubits (cont)

General 3-Qubit $O^{(3)} = \frac{1}{8}[I\otimes I\otimes I + (\vec{s}^{(1)}\cdot\vec{\sigma})\otimes I\otimes I + I\otimes(\vec{s}^{(2)}\cdot\vec{\sigma})\otimes I + I\otimes I\otimes(\vec{s}^{(3)}\cdot\vec{\sigma})$

$$+\sum_{\alpha,\beta=1}^{3} A_{\alpha\beta}\,\sigma_\alpha\otimes\sigma_\beta\otimes I + \sum_{\alpha,\beta=1}^{3} B_{\alpha\beta}\,\sigma_\alpha\otimes I\otimes\sigma_\beta + \sum_{\alpha,\beta=1}^{3} C_{\alpha\beta}\, I\otimes\sigma_\alpha\otimes\sigma_\beta + \sum_{\alpha,\beta,\gamma=1}^{3} C_{\alpha\beta\gamma}\,\sigma_\alpha\otimes\sigma_\beta\otimes\sigma_\gamma].$$

notation: $s_i^2 = \vec{s}^{2(i)},\; C_{12}^2 = \sum_{\alpha\beta=1}^{3} A_{\alpha\beta}^2 = Tr[A^T A],\; C_{13}^2 = Tr[B^T B],\; C_{23}^2 = Tr[C^T C],\; C_{123}^2 = \sum_{\alpha\beta\gamma=1}^{3} C_{\alpha\beta\gamma}^2$

Maximal Slice States $S_{\max}^{(MS)} = 4\sqrt{1+\tau^{(MS)}}$

$$|MS_2\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + \cos\theta_2|101\rangle + \sin\theta_2|111\rangle\right),\quad |MS_3\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + \cos\theta_3|110\rangle + \sin\theta_3|111\rangle\right)$$

	$s_1^2 = s_2^2 = 0,\; s_3^2 = 1-\tau,$	$C_{12}^2 = 3-2\tau,\; C_{13}^2 = C_{23}^2 = \tau,$	$C_{123}^2 = 3+\tau,$
Max tangle $\tau = 1$ on GHZ state	$s_1^2 = s_2^2 = s_3^2 = 0,$	$C_{12}^2 = C_{13}^2 = C_{23}^2 = 1$	$C_{123}^2 = 4$

Generalized GHZ State $S_{\max}^{(GGHZ)} = 4\sqrt{1-\tau^{(GGHZ)}}$ for $\tau \le 1/3$ $\quad S_{\max}^{(GGHZ)} = 4\sqrt{2\tau^{(GGHZ)}}$ for $\tau \ge 1/3$.

$|GGHZ\rangle = \cos\theta_1|000\rangle + \sin\theta_1|111\rangle$, with $\tau_{MS_2}^2 = \sin^2\theta_2,\; \tau_{MS_3}^2 = \sin^2\theta_3,\; \tau_{GGHZ}^2 = \sin^2 2\theta_1.$

$\boxed{s_3^{(1)} = s_3^{(2)} = s_3^{(3)} = \sqrt{1-\tau},\; s_1^2+s_2^2+s_3^2 = 3-3\tau,\; C_{12} = C_{13} = C_{23} = diag(0,0,1),\; C_{12}^2+C_{13}^2+C_{23}^2 = 3,\; C_{123}^2 = 1+3\tau.}$

$\boxed{\text{cusp at } \tau = 1/3 \text{ occurs at } s_1^2+s_2^2+s_3^2 = C_{123}^2.}$

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