Title: Any Quantum State Can be Cloned in the Presence of a Closed Timelike Curve

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Abstract: Using the Deutsch approach, we show that the no-cloning theorem can be circumvented in the presence of closed timelike curves, allowing the perfect cloning of a quantum state chosen randomly from a finite alphabet of states. Further, we show that a universal cloner can be constructed that when acting on a completely arbitrary qubit state, exceeds the no-cloning bound on fidelity. Since the â€œno cloning theoremâ€• has played a central role in the development of quantum information science, it is clear that the existence of closed timelike curves would radically change the rules for quantum information technology.

Cloning Quantum States Using Closed Timelike Curves

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&

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eQus

Australian Government
Australian Research Council

Quantum Cloning

- Suppose we have an operation that maps an arbitrary state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, to:

$$|\psi\rangle \rightarrow |\psi\rangle \otimes |\psi\rangle$$

- Then the arbitrary state $|\phi\rangle$ is mapped by:

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- By the linearity of QM, we expect:

$$|\psi\rangle + |\phi\rangle \rightarrow |\psi\rangle \otimes |\psi\rangle + |\phi\rangle \otimes |\phi\rangle$$

- However:

$$|\psi\rangle \otimes |\psi\rangle + |\phi\rangle \otimes |\phi\rangle \neq (|\psi\rangle + |\phi\rangle) \otimes (|\psi\rangle + |\phi\rangle)$$

- So we have failed to copy $|\psi\rangle + |\phi\rangle$
- Wooters & Zurek, Nat. 1982; Gottesman quant-ph/9705052

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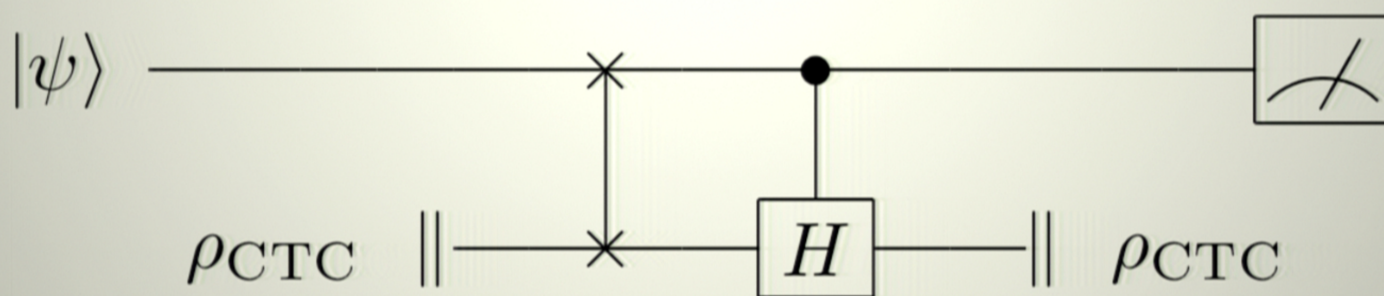
Cloning with CTC's

- Brun et al. have conjectured a CTC-assisted quantum cloner could exist with Fid ➔ 1 at the cost of increasing the available dimensions of the CTC ancilla.
- This conjecture is based on their results showing that CTC's can perfectly distinguish any quantum states. For example: distinguish $|0\rangle$ from $|-\rangle$.



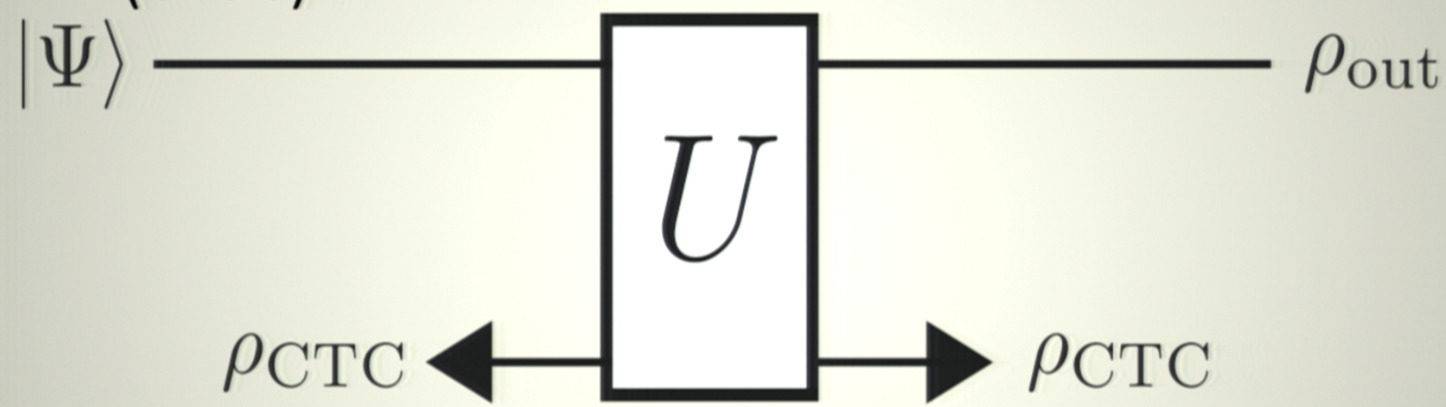
*Brun, Harringtom & Wilde PRL 2009.

Outline

- In this presentation, we consider the cloning of an arbitrary qubit state.
  - We consider building a quantum cloner from a quantum broadcasting circuit.
  - We test the broadcasting circuit's ability to clone as a function of the number of states broadcast.
  - We find that we can violate the No Cloning bound.

Background

- We consider Deutsch type closed time curves (CTC's).



- Must satisfy the consistency conditions:

$$\rho_{\mathrm{CTC}} = \mathrm{Tr}_1[U(|\Psi\rangle\langle\Psi| \otimes \rho_{\mathrm{CTC}})U^\dagger] \quad (1)$$

$$\rho_{\mathrm{out}} = \mathrm{Tr}_2[U(|\Psi\rangle\langle\Psi| \otimes \rho_{\mathrm{CTC}})U^\dagger] \quad (2)$$

D. Deutsch PRD 1991

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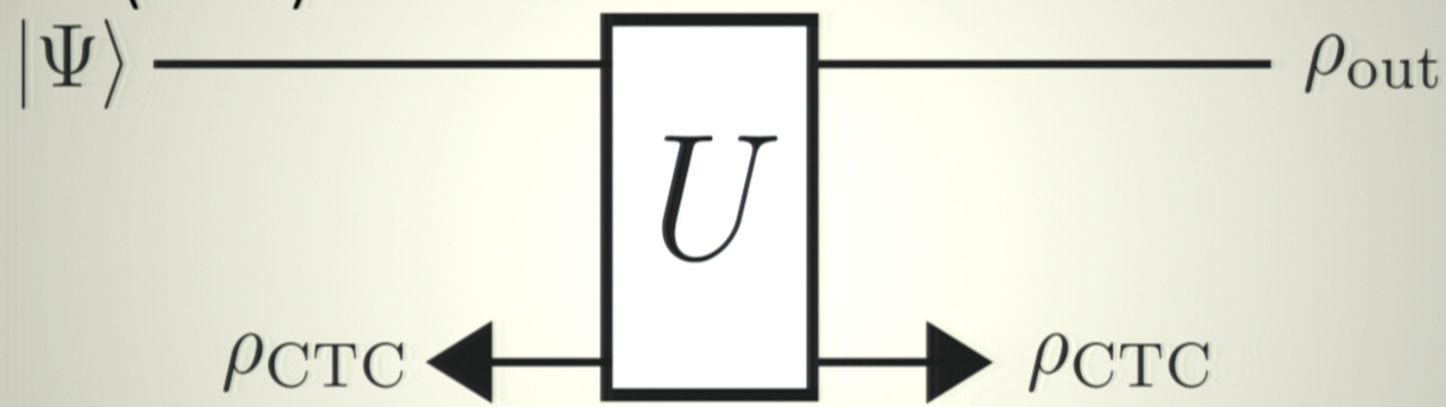

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D. Deutsch PRD 1991

State Broadcasting

- Generalisation of quantum cloning for mixed states.
- There is no physical process, consistent with the laws of quantum mechanics, that allows

$$\rho_s \otimes \rho_{\mathrm{anc}} \rightarrow \tilde{\rho}_s$$

where $\rho_s \in \{\rho_j\}_{j=0}^{N-1}$ is one of N possible states and

$$\mathrm{Tr}_1[\tilde{\rho}_s] = \rho_s$$
$$\mathrm{Tr}_2[\tilde{\rho}_s] = \rho_s$$

Barnum et al. PRL 1996

Pure State Broadcasting

- This is also true for pure state broadcasting.
- That is, quantum mechanics does not allow the broadcasting of N distinct states $|\psi_j\rangle$ in a space of dimension N , where $|\psi_j\rangle \in \{|\psi_j\rangle\}_{j=0}^{N-1}$.
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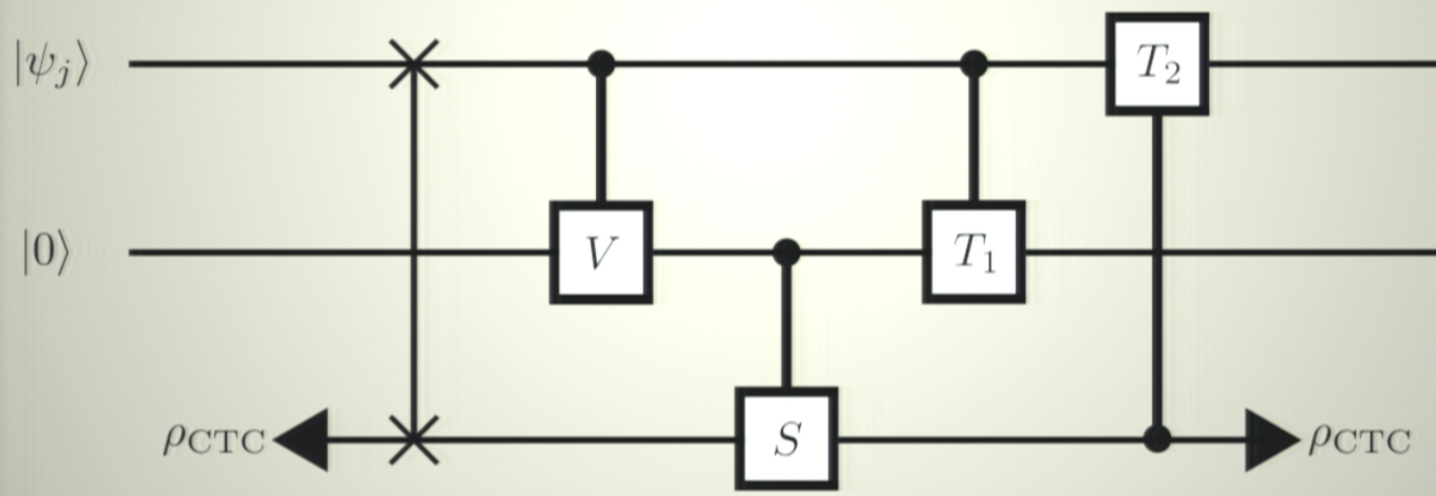
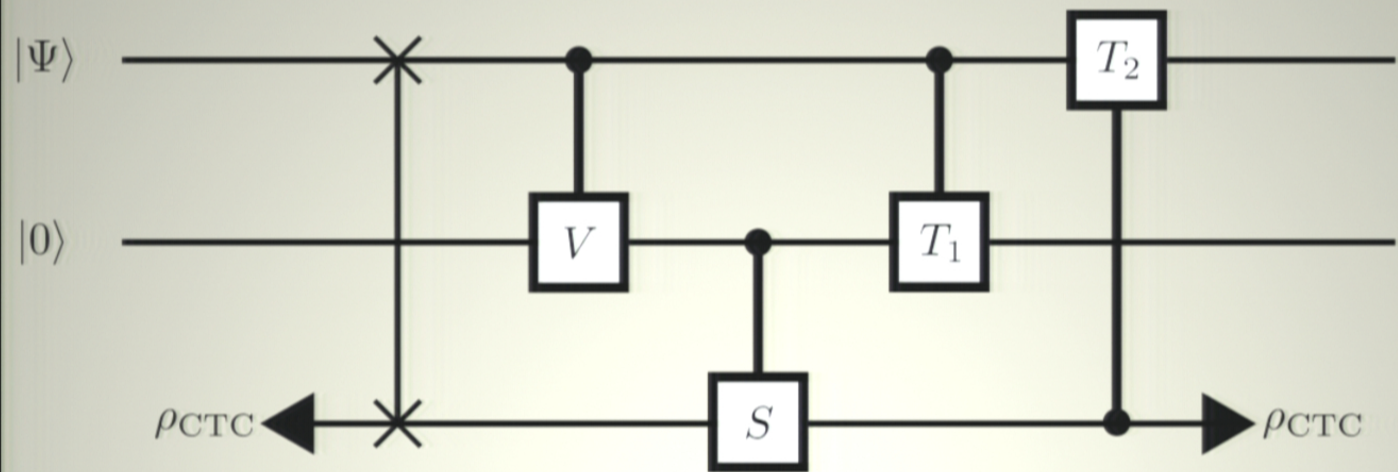
Pure State Broadcasting with a CTC

- If we allow a CTC ancilla, pure state broadcasting is possible



Ahn, Ralph & Mann arXiv:1008.0221v2



Controlled-V: $CSUM\,(|i\rangle \otimes |j\rangle) = |i\rangle \otimes |j + i(\mathrm{mod}\, N)\rangle$

Controlled-S: $\mathbb{1} \otimes \sum_k |k\rangle\langle k| \otimes U_k$

Controlled-T$_1$: $\sum_l |l\rangle\langle l| \otimes U_l^\dagger \otimes \mathbb{1}$

Controlled-T$_2$: $\sum_m U_m^\dagger \otimes \mathbb{1} \otimes |m\rangle\langle m|$

- Initial state: $\rho_{in} = |\Psi\rangle\langle\Psi| \otimes |0\rangle\langle 0| \otimes \rho_{\mathrm{CTC}}$

where $\rho_{\mathrm{CTC}} = \sum_{m,n=0}^{N-1} \lambda_{mn} |m\rangle\langle n|$

with $\sum_n \lambda_{nn} = 1$ and $\lambda_{mn} = \lambda_{nm}^*$.

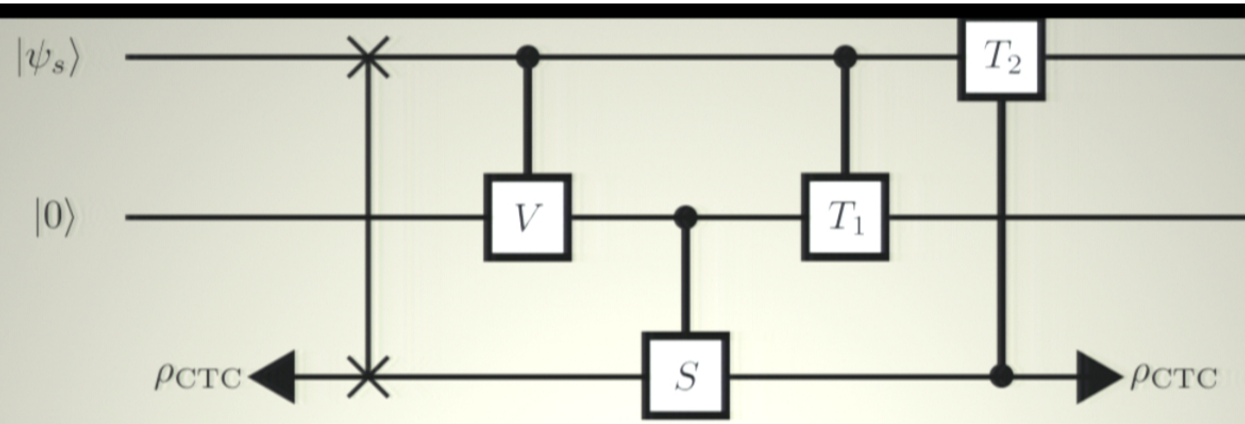
- Given the Brun et al. conditions*:

$$U_k|\psi_k\rangle = |k\rangle, U_k^\dagger|k\rangle = |\psi_k\rangle$$

$$\langle j|U_k|\psi_j\rangle \neq 0 \quad \forall j, k.$$

- The first consistency condition gives:

*Brun, Harringtom & Wilde PRL 2009.



$$\lambda_{a,b} = \sum_{m,n} \lambda_{m,n} \langle n|U_b U_a^\dagger|m\rangle\langle\psi_n|\psi_m\rangle\langle a|U_m|\Psi\rangle\langle\Psi|U_n^\dagger|b\rangle$$

Setting $|\Psi\rangle = |\psi_s\rangle$

We find: $$\lambda_{a,b} = \sum_{m,n\neq s} \lambda_{m,n} \langle n|U_b U_a^\dagger|m\rangle\langle\psi_n|\psi_m\rangle\langle a|U_m|\Psi\rangle\langle\Psi|U_n^\dagger|b\rangle + \lambda_{ss}\delta_{as}\delta_{bs}$$

But since $\sum_n \lambda_{nn} = 1$ and $\lambda_{mn} = \lambda_{nm}^*$

We have $\rho_{\mathrm{CTC}} = |s\rangle\langle s|$

- In general, the second consistency condition gives:

$$\rho_{out} = \sum_{i,m,n} \left(\lambda_{mn} \langle i|U_m|\Psi\rangle\langle\Psi|U_n^\dagger|i\rangle\right) U_i^\dagger|m\rangle\langle n|U_i \otimes |\psi_m\rangle\langle\psi_n|.$$

- Since $\rho_{\mathrm{CTC}} = |s\rangle\langle s|$

- We have $\rho_{out} = |\psi_s\rangle\langle\psi_s| \otimes |\psi_s\rangle\langle\psi_s|$

- Perfect pure state broadcasting.

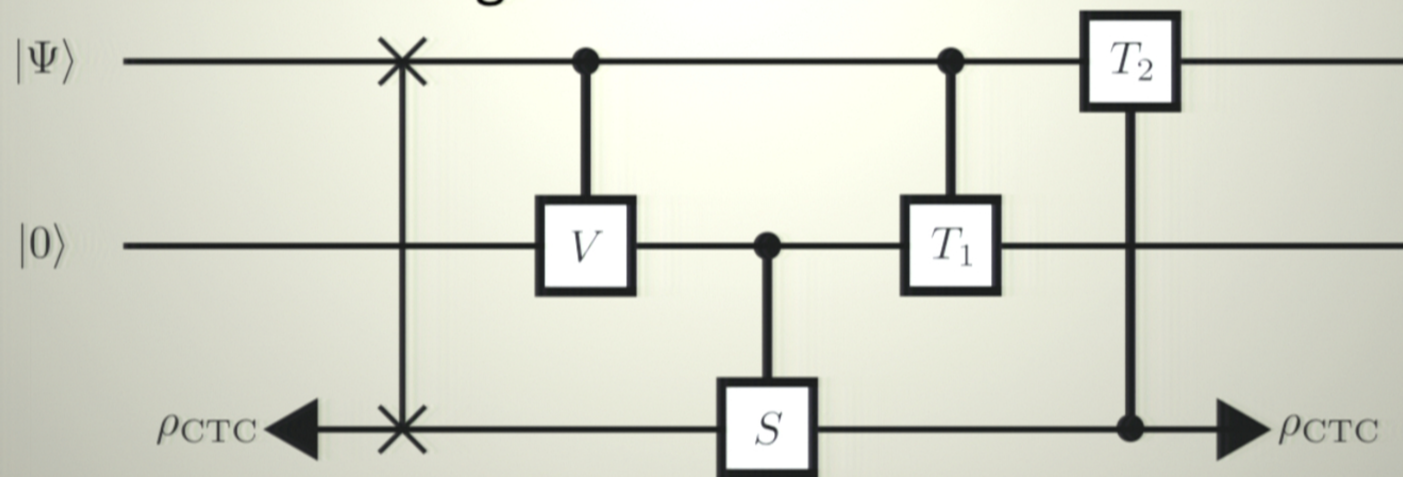
Cloning with Pure State Broadcasting

- If we can broadcast the N states $|\psi_j\rangle \in \{|\psi_j\rangle\}_{j=0}^{N-1}$

can we use this to clone an arbitrary qubit state:

$$|\Psi\rangle = a|0\rangle + be^{i\phi}|1\rangle \text{ ?}$$

- If the answer is yes, what value of N breaks the no-cloning bound?



- We proceed by encoding our qubit state $|\Psi\rangle$ into an N dimensional space.
- We now need to construct the unitaries U_k explicitly.
- Remember, the requirement is they must satisfy the Brun et al. condition*:

$$U_k|\psi_k\rangle = |k\rangle,\ U_k^\dagger|k\rangle = |\psi_k\rangle$$
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- Using the Brun et al. recipe*, we find one possible (non-unique) solution for the unitaries to be:

$$U_k = |k\rangle\langle\psi_k| + \frac{1}{\mathcal{N}_k\sqrt{N-1}}\hat{\Pi}_k \sum_{j=1}^{N-1} |j\rangle \left[\langle\psi_{(k+1)\bmod N}| - \langle\, \psi_{(k+1)\bmod N}|\psi_k\rangle\langle\psi_k|\right]$$
$$+ \sum_{m=1}^{N-2} \frac{1}{\sqrt{m(m+1)}}\hat{\Pi}_k \left[-m|m+1\rangle + \sum_{j=1}^{m}|j\rangle\right] \langle m+1|$$

where $\hat{\Pi}_k = \mathbb{1}_N - (|0\rangle - |k\rangle)(\langle 0| - \langle\, k|)$

and $\mathcal{N}_k = \sqrt{1 - |\langle\psi_{(k+1)\bmod N}|\psi_k\rangle|^2}$

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- As before, after the first consistency condition, we have:

$$\lambda_{a,b} = \sum_{m,n} \lambda_{m,n} \langle n|U_b U_a^\dagger|m\rangle \langle \psi_n|\psi_m\rangle \langle a|U_m|\Psi\rangle\langle\Psi|U_n^\dagger|b\rangle$$

where $|\Psi\rangle = a|0\rangle + be^{i\phi}|1\rangle$ is the arbitrary qubit state to be cloned.

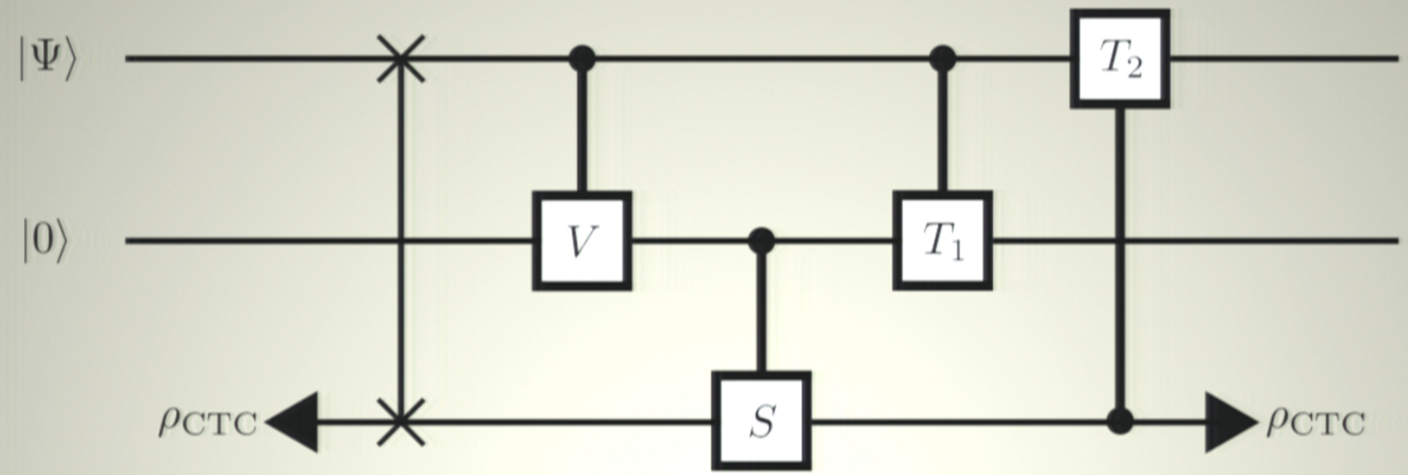
- After the second consistency condition, the fidelities with respect to $|\Psi\rangle \otimes |\Psi\rangle$:

$$F_T = \sqrt{\langle\Psi| \otimes \langle\, \Psi|\rho_{out}|\Psi\rangle \otimes |\Psi\rangle},$$

$$F_1 = \sqrt{\langle\Psi|\mathrm{Tr}_B[\rho_{out}]|\Psi\rangle},$$

$$F_2 = \sqrt{\langle\Psi|\mathrm{Tr}_A[\rho_{out}]|\Psi\rangle}.$$

are given by:



$$F_T = \Big[\sum_{i,m,n} \lambda_{mn} \langle i|U_m|\Psi\rangle\langle\Psi|U_n^\dagger|i\rangle\langle\Psi|U_i^\dagger|m\rangle\langle n|U_i|\Psi\rangle\langle\Psi|\psi_m\rangle\langle\psi_n|\Psi\rangle\Big]^{\frac{1}{2}},$$

$$F_1 = \Big[\sum_{i,m,n} \lambda_{mn} \langle i|U_m|\Psi\rangle\langle\Psi|U_n^\dagger|i\rangle\langle\psi_n|\psi_m\rangle\langle\Psi|U_i^\dagger|m\rangle\langle n|U_i|\Psi\rangle\Big]^{\frac{1}{2}},$$

$$F_2 = \Big[\sum_{i,n} \lambda_{nn} \langle i|U_n|\Psi\rangle\langle\Psi|U_n^\dagger|i\rangle\langle\Psi|\psi_n\rangle\langle\psi_n|\Psi\rangle\Big]^{\frac{1}{2}},$$

Choosing the N states $|\psi_j\rangle \in \{|\psi_j\rangle\}_{j=0}^{N-1}$

- Before we investigate how well we clone, we need to first choose the N states to broadcast.
- Logical choice: choose N equidistant points on the Bloch sphere.
- Choosing N equally spaced points on a sphere is a very difficult mathematical problem, with no analytical solution.
- We instead consider two of the best approximations to having N equally spaced points on the Bloch sphere.

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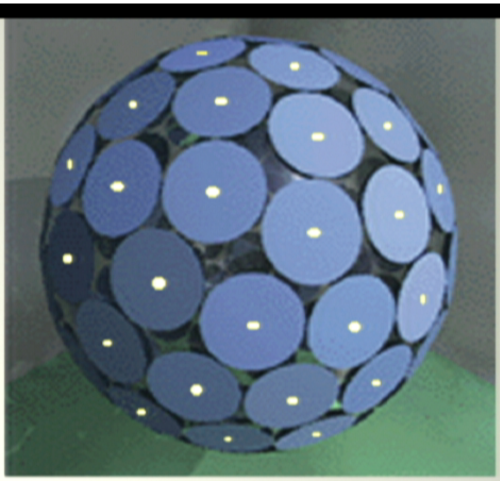
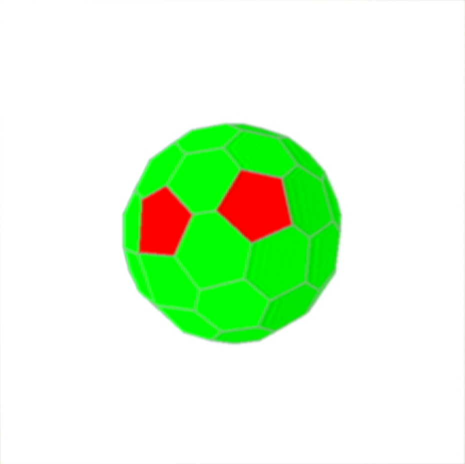
Tammes problem



• Numerical solutions[1] to the Tammes problem, where we consider packing a given number of circles on the surface of a sphere such that the minimum distance between circles is maximised.

1: N.J.A.Sloane, R.H.Hardin and W.D.Smith, ``Spherical Codes'', http://www2.research.att.com/~njas/packings/

Thomson problem

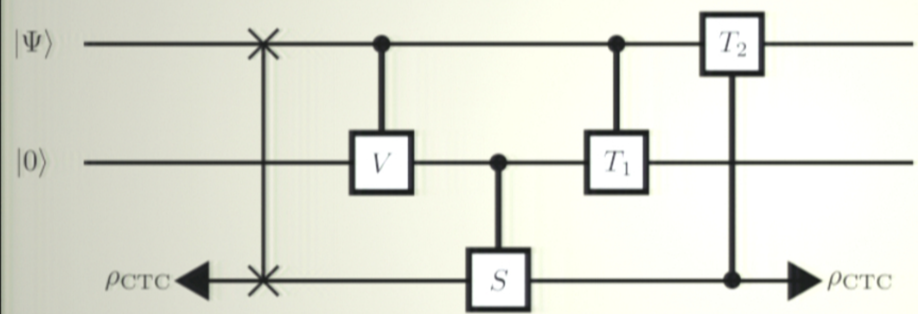


• Numerical solutions[2] to the Thomson problem, where we instead consider the equilibrium configuration of N electrons on the surface of a sphere, such that the potential energy is minimised.

2: N.J.A.Sloane, R.H.Hardin and W.D.Smith, ``Minimal Energy Arrangements of Points on a Sphere'', http://www2.research.att.com/~njas/electrons/

How to test the broadcasting circuit

- We generate 10,000 random points on the Bloch sphere and calculate the fidelity of cloning each of these points as a function of N.



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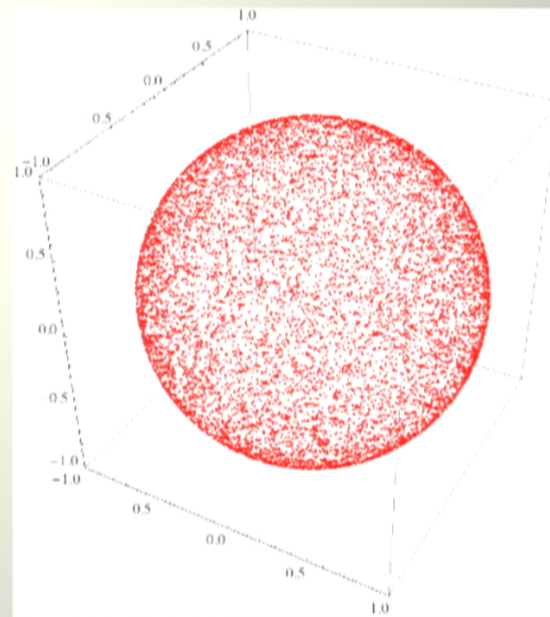

The output is both asymmetric and state dependent:

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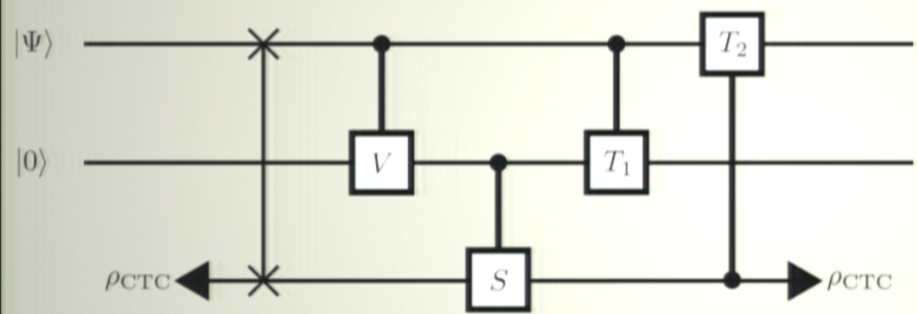

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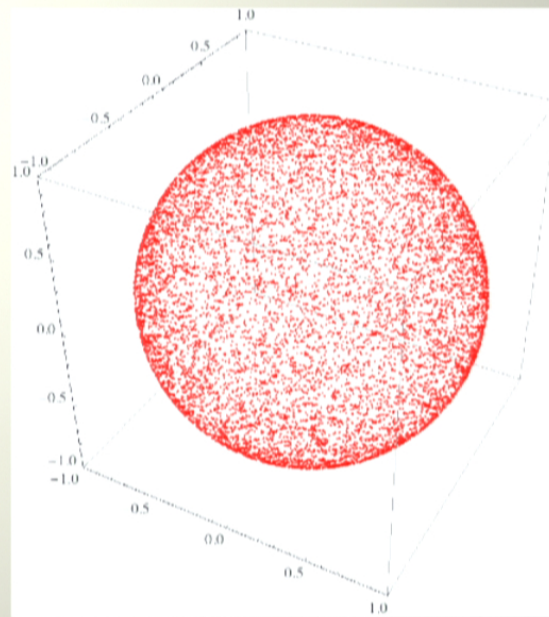

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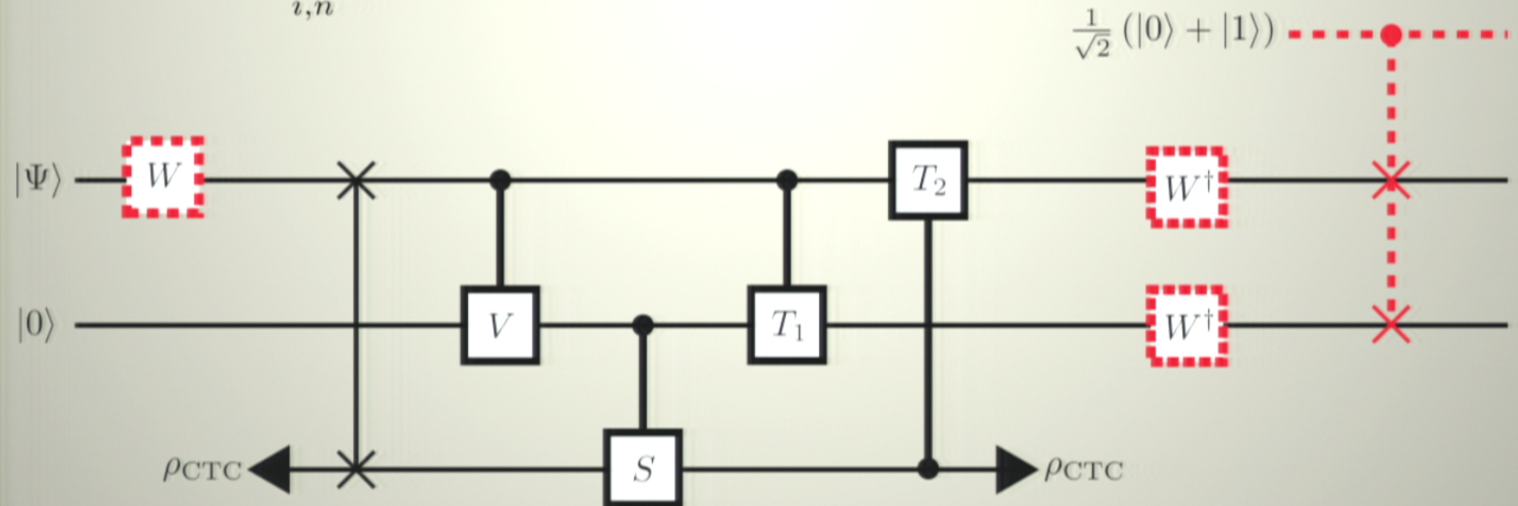

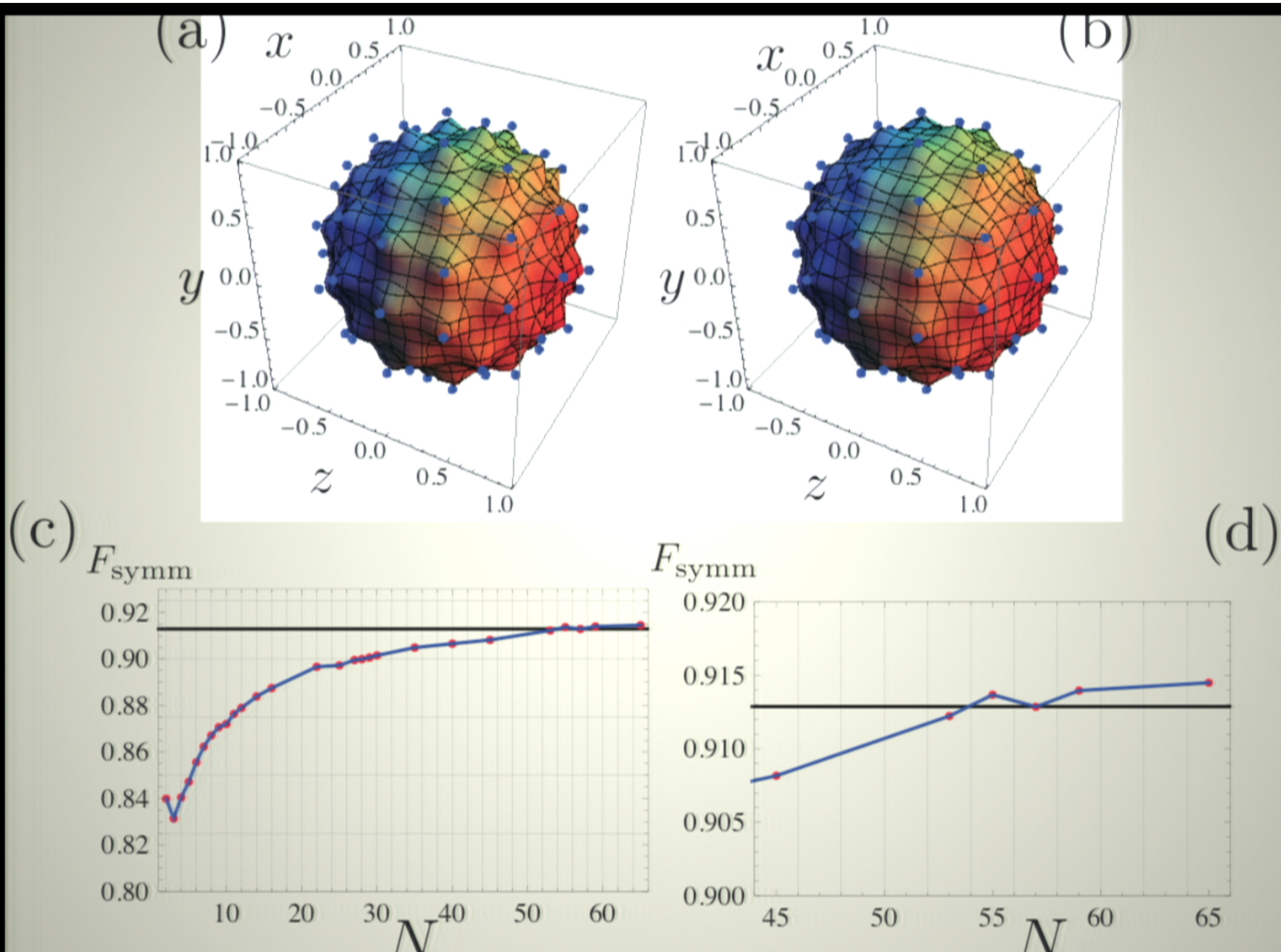
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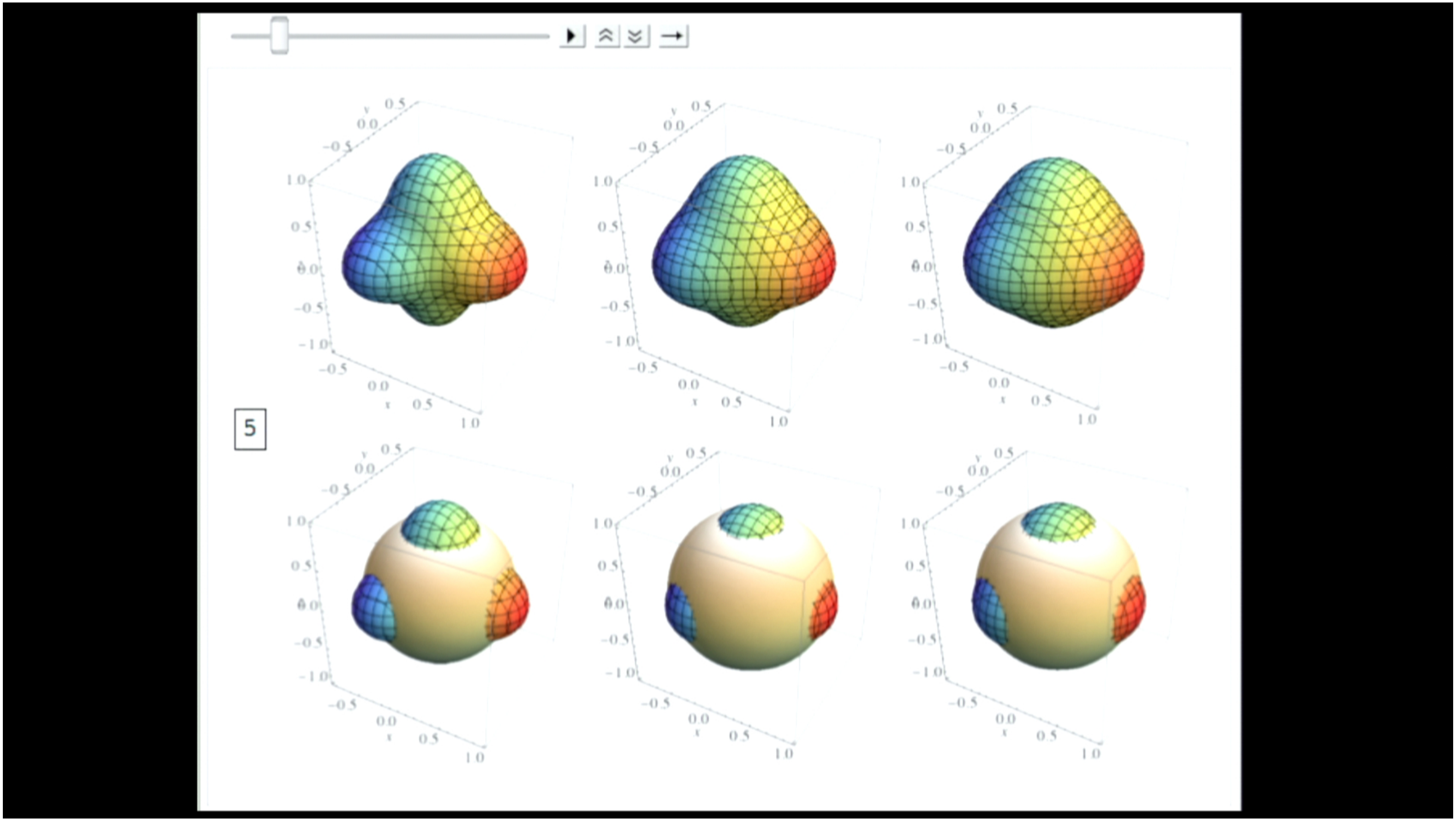
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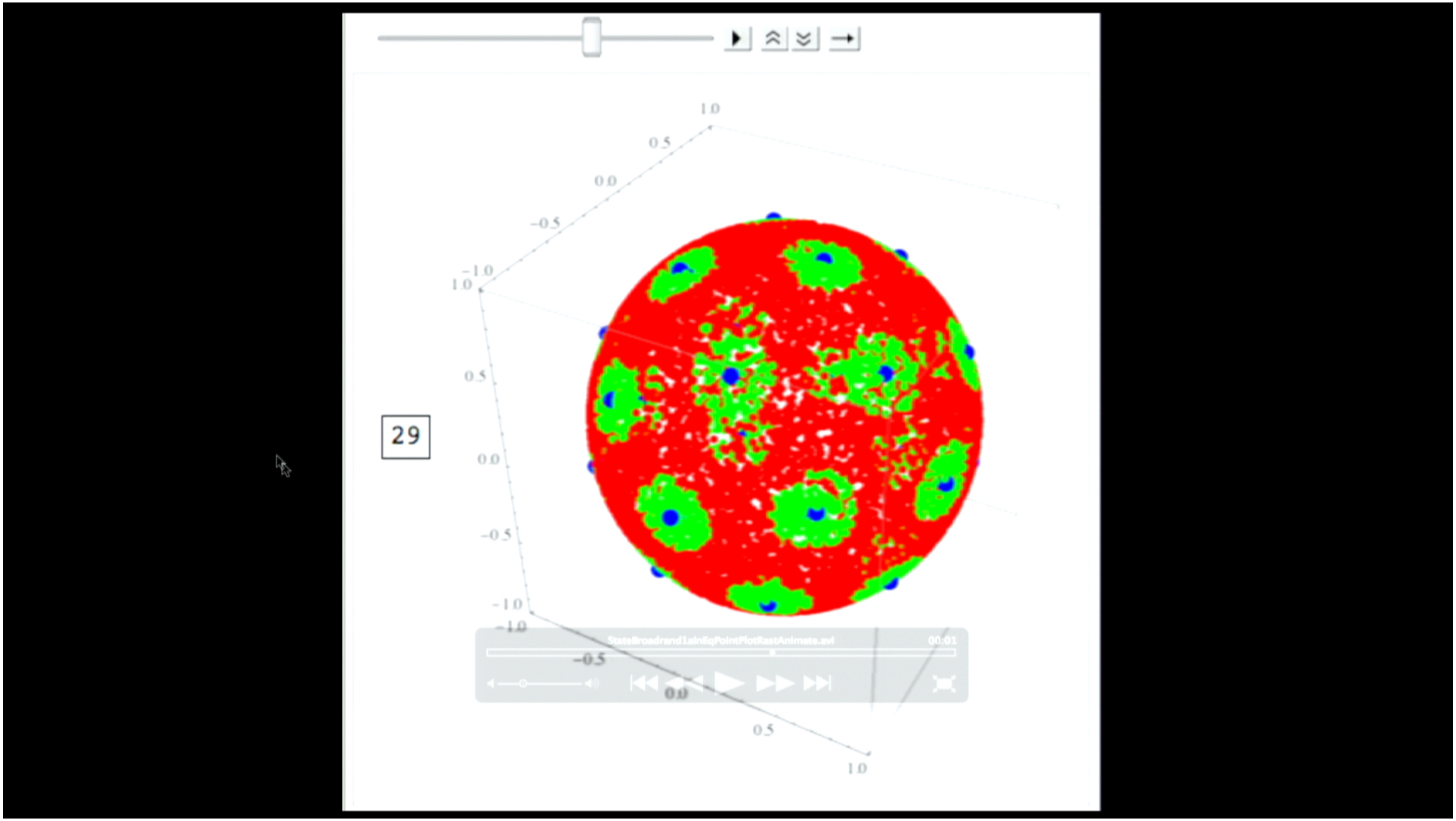
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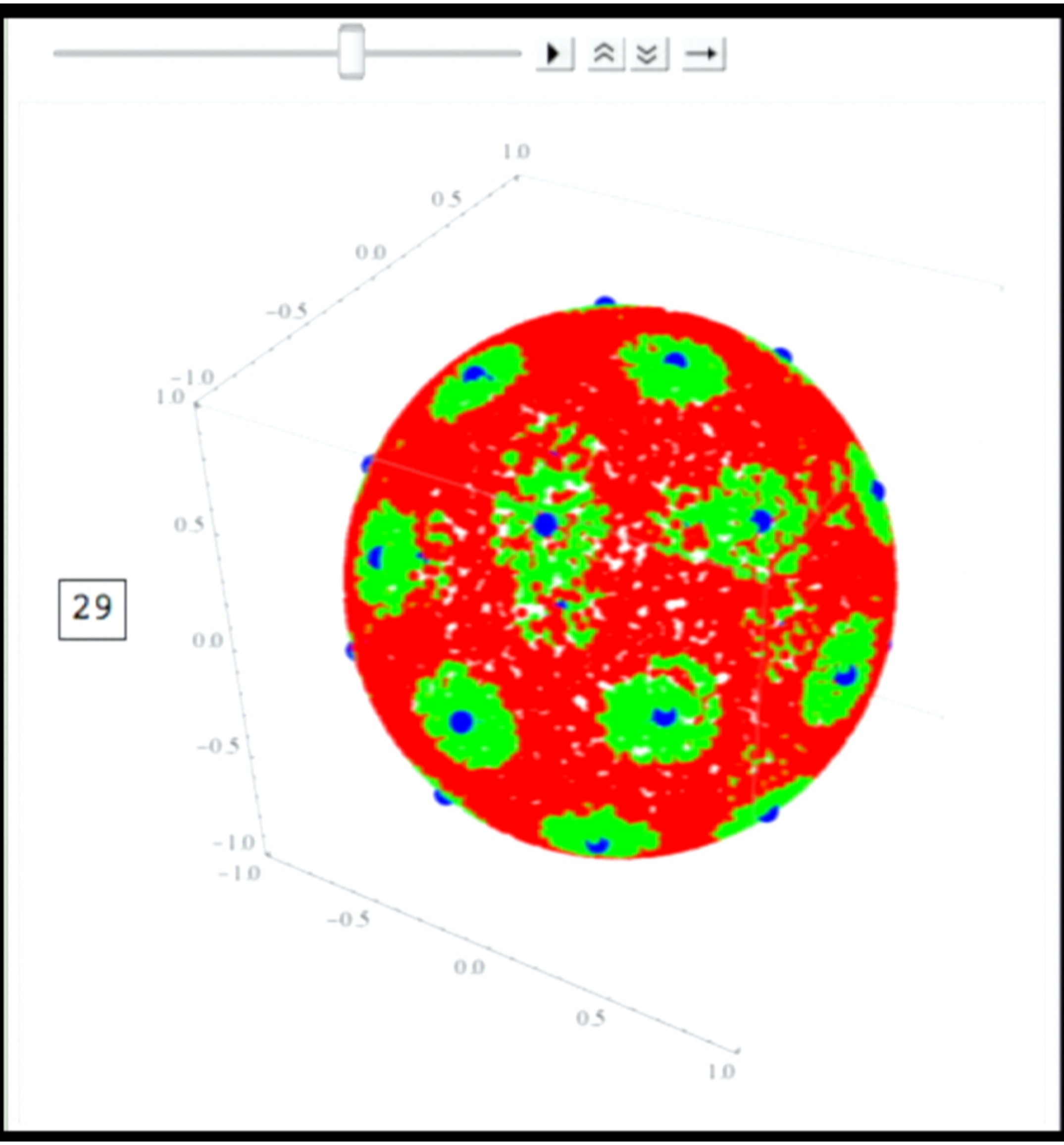
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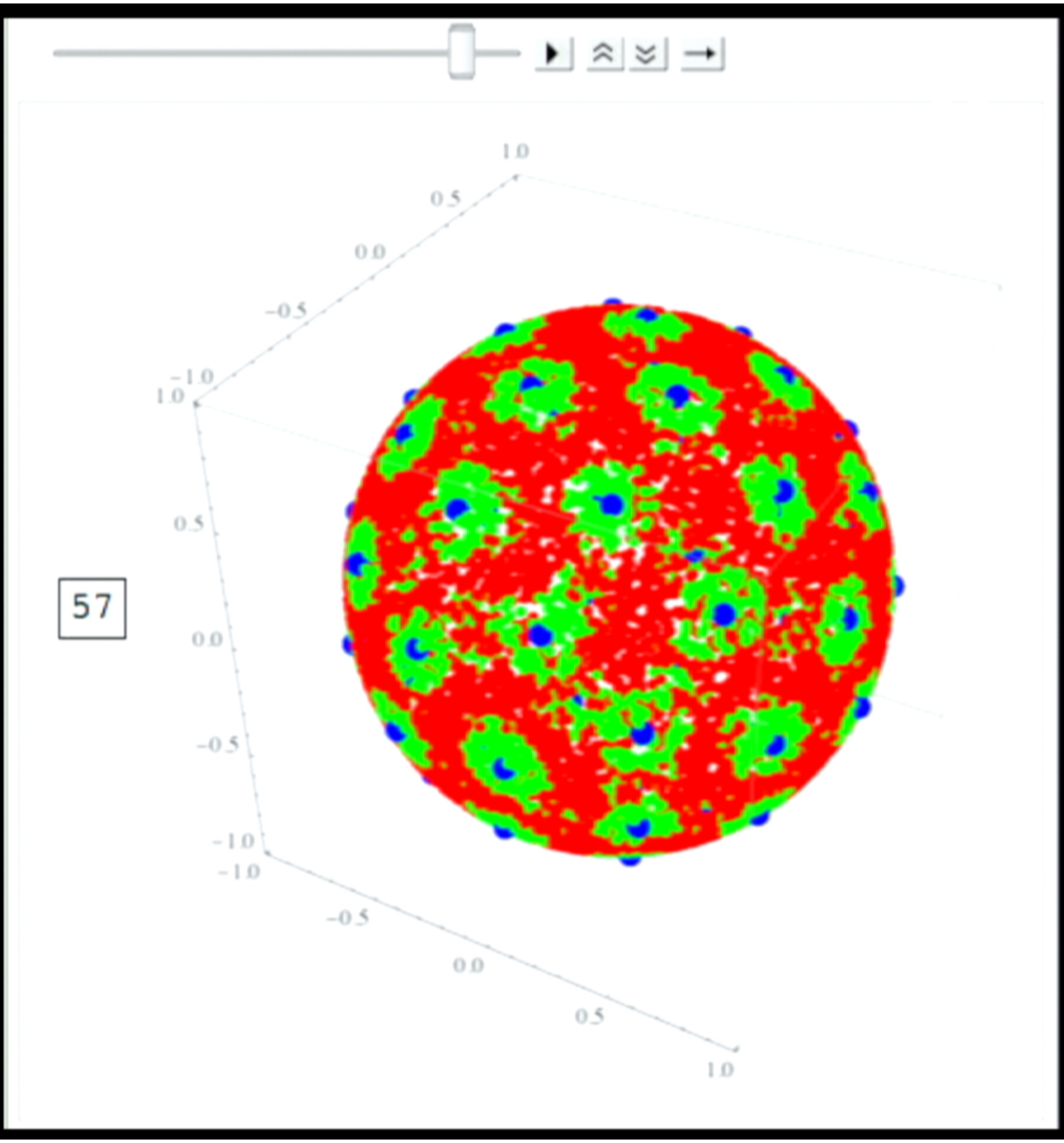
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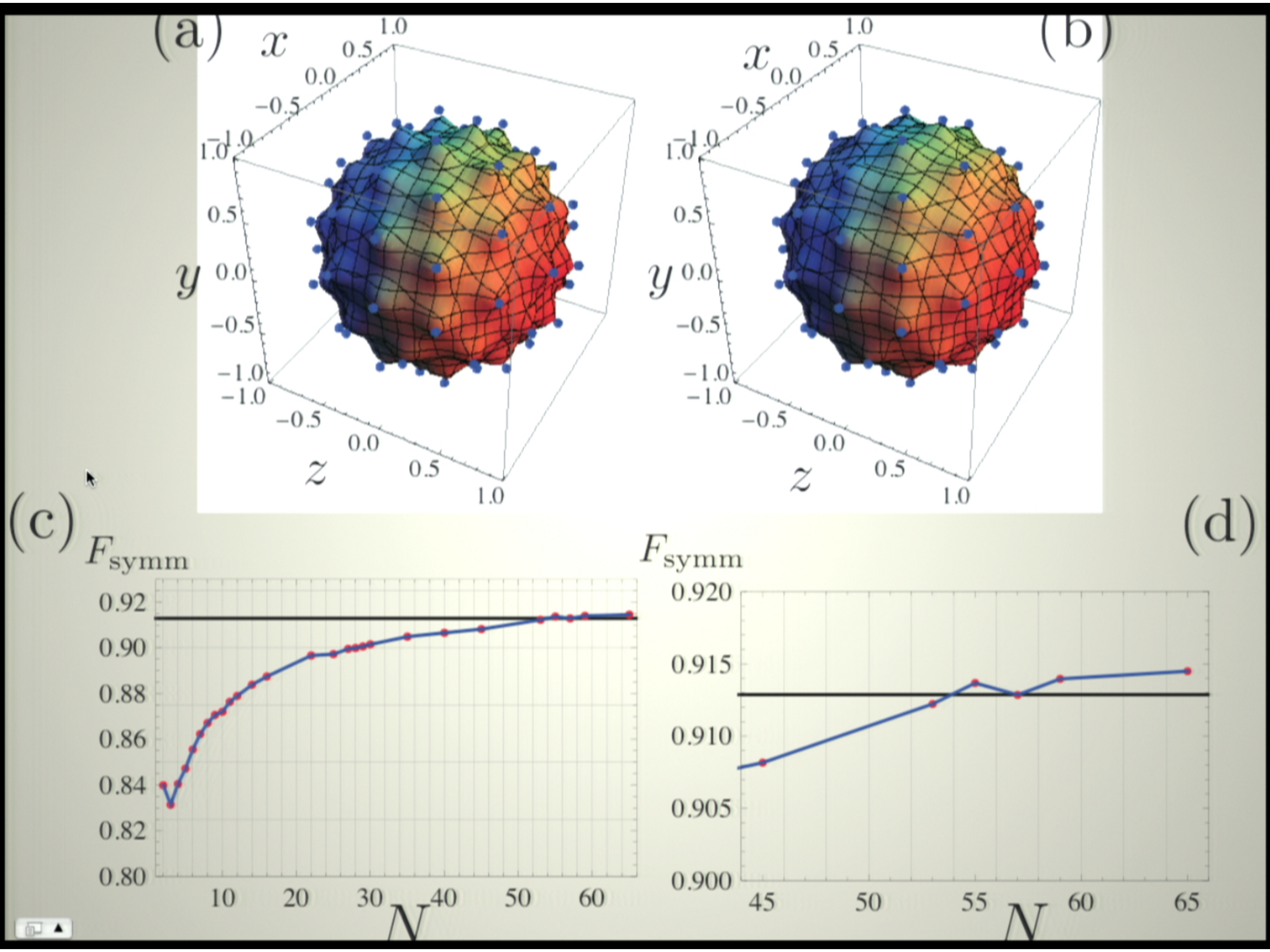
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- The Brun et al. condition*:

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where $\hat{\Pi}_k = \mathbb{1}_N - (|0\rangle - |k\rangle)(\langle 0| - \langle\, k|)$

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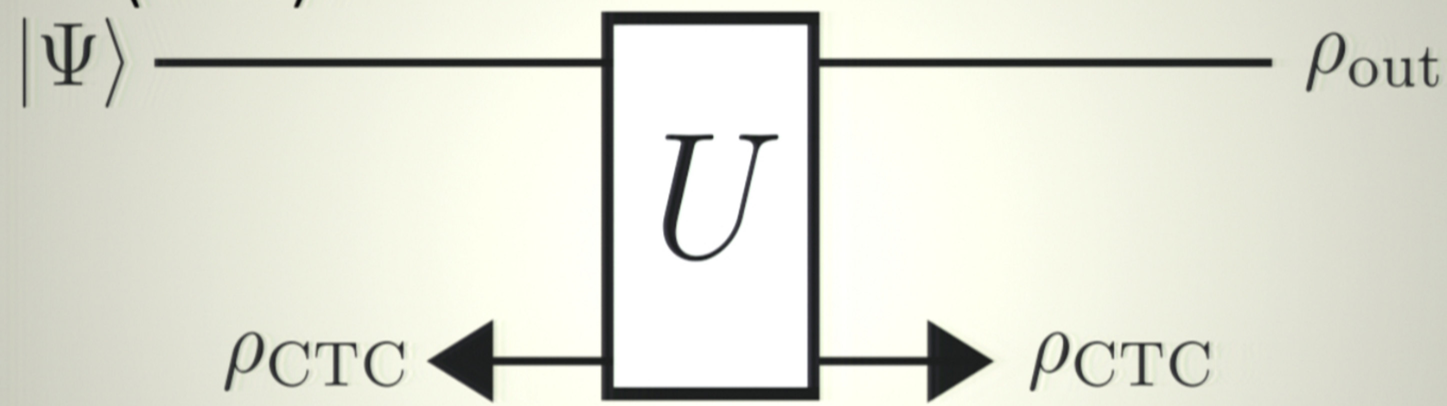
Conclusions

- We have shown that we can violate the no-cloning bound given we broadcast at least $N = 54$ states and have access to an N dimensional CTC.
- The scaling of the circuit required to clone qubit scales linearly with N.
- There are still some things to investigate:
  - There is freedom when defining U_k
  - There is also freedom when choosing the N states $|\psi_j\rangle \in \{|\psi_j\rangle\}_{j=0}^{N-1}$. Could just consider random points.
- This work only provides numerical evidence for quantum cloning. Next step would be to analytical investigate the perfect cloning of an arbitrary qubit state $|\Psi\rangle$

Background

- We consider Deutsch type closed time curves (CTC's).



- Must satisfy the consistency conditions:

$$\rho_{\mathrm{CTC}} = \mathrm{Tr}_1[U(|\Psi\rangle\langle\Psi| \otimes \rho_{\mathrm{CTC}})U^\dagger] \quad (1)$$

$$\rho_{\mathrm{out}} = \mathrm{Tr}_2[U(|\Psi\rangle\langle\Psi| \otimes \rho_{\mathrm{CTC}})U^\dagger] \quad (2)$$

D. Deutsch PRD 1991