Title: Vacuum Entanglement and Gauge Symmetry

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Abstract:

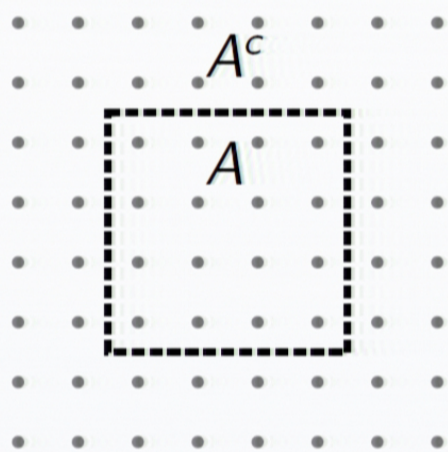
Entanglement entropy

In a quantum field theory, every region of space has entanglement entropy.

This quantity arises:

- As a quantum correction to the Bekenstein-Hawking entropy.
- In efficient representations of the ground state.
- As a probe of phases.
  - Confinement/deconfinement.
  - Topological phases.
- In AdS/CFT.

Entanglement entropy



Consider a lattice with nodes N.

For each $A \subset N$ there is a Hilbert space:

$$\mathcal{H}_A = \bigotimes_{n \in A} \mathcal{H}_n.$$

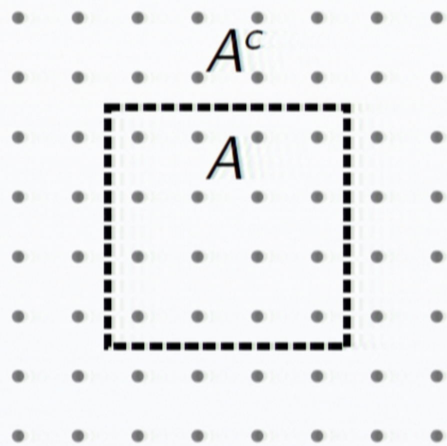
The full Hilbert space splits as a tensor product

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{A^c}.$$

For each region there is a density matrix and an entanglement entropy:

$$\rho_A = \mathrm{tr}_{A^c}(|\psi\rangle\langle\psi|) \qquad S = -\mathrm{tr}\, \rho_A \ln \rho_A.$$

Entanglement entropy



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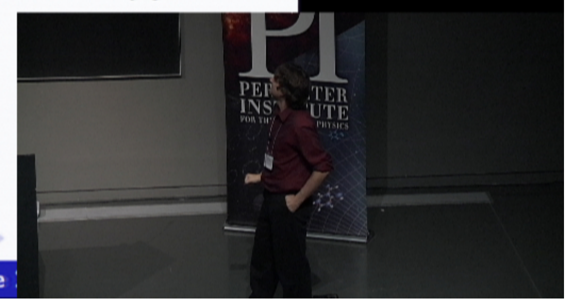

Entanglement entropy in gauge theory

In gauge theory, states are **gauge-invariant** functionals,

$$\mathcal{H} = L^2(\mathcal{A}/\mathcal{G}), \qquad \frac{\mathcal{A}}{\mathcal{G}} = \frac{\text{Vector potentials}}{\text{Gauge transformations}}.$$

The vector potentials split as a tensor product,

$$L^2(\mathcal{A}) = L^2(\mathcal{A}_A) \otimes L^2(\mathcal{A}_{A^c}).$$

But gauge symmetry implies constraints, e.g. Gauss' law:

$$\nabla \cdot E = 0.$$

This constraint is **nonlocal**, so breaks the tensor product struc

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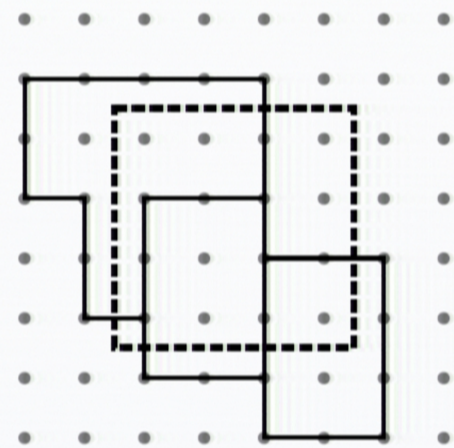
Lattice gauge theory

In **lattice gauge theory**, degrees of freedom are group elements on links.

Wilson loops are gauge invariant:

$$\mathrm{tr}_r \left(\prod_{l \in \square} u_l \right)$$



Extend to an orthonormal basis: (generalized) spin networks

$$S = (R, I) \qquad \text{spin network}$$
$$R = \text{an irreducible representation for each link}$$
$$I = \text{an intertwining operator for each node}$$

ex: If $G = \mathrm{U}(1)$, $r \in \mathbb{Z}$ are electric flux. Spin networks $\approx$ E-field basis.

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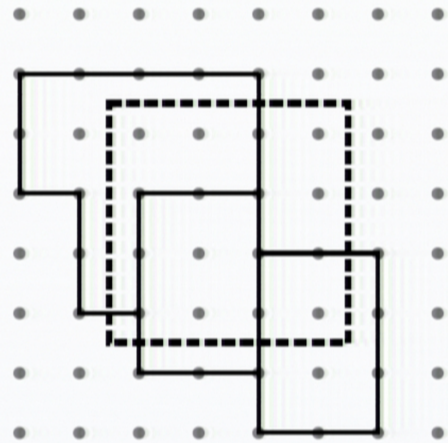

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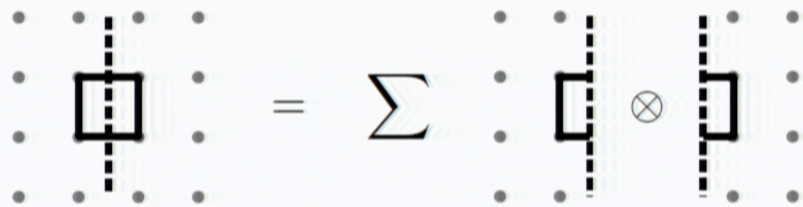
Localization of states

Issue: States are not localized!

Consider partial trace of a Wilson loop state:



Reduced state is an electric string with an endpoint:

Not allowed by Gauss' law.

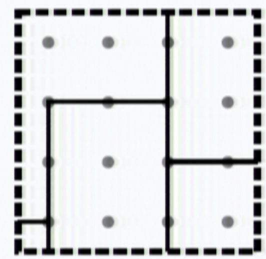
Edge states

Solution: Relax gauge-invariance along the boundary ∂A.

Reducing the gauge group gives new degrees of freedom: **edge states**.
c.f. $2+1$ quantum gravity, quantum Hall effect.

An orthonormal basis of $\mathcal{H}_A$ is given by open spin networks:

$$S_A = (R_A, I_A, \underbrace{R_\partial, M_\partial}_{\text{Edge DOF}}).$$



Instead of equality have an embedding:

$$\mathcal{H} \neq \mathcal{H}_A \otimes \mathcal{H}_{A^c},$$
$$\mathcal{H} \subset \mathcal{H}_A \otimes \mathcal{H}_{A^c}.$$

But this is sufficient: first embed, then compute entanglement.

Decomposition of the reduced density matrix

Edge states transform under $G^{|\partial A|}$; this restricts the form of ρ_A:

$$\rho_A = \bigoplus_{R_\partial} p(R_\partial) \left[\left(\bigotimes_{l \in L_\partial} \frac{\mathbb{1}_{r_l}}{\dim(r_l)} \right) \otimes \rho_A(R_\partial) \right].$$

Where: $p(R_\partial) =$ probability of a given set of representations crossing ∂A.

$\rho_A(R_\partial) =$ matrix elements of ρ_A with fixed R_∂.

Note that:

- Different boundary representations R_∂ cannot be in superposition.
- The M vectors are maximally mixed (c.f. singlet of $j \times j$).

Using properties of von Neumann entropy under $\oplus$ and $\otimes$,

$$S = -\sum p(R_\partial) \ln p(R_\partial) + \sum p(R_\partial) \sum_{l \in \partial A} \ln \dim r_l + \sum p(R_\partial) S(\rho_A(R_\partial)).$$

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Three kinds of entropy

The Shannon entropy of boundary representations:

$$-\sum p(R_\partial) \ln p(R_\partial)$$

A correction for non-abelian G:

$$\sum p(R_\partial) \sum_{l \in \partial A} \ln \dim r_l.$$

Both of these are local to the boundary: **area law is automatic**.

There is also an entropy associated to non-local correlations

$$\sum p(R_\partial) S(\rho_A(R_\partial)).$$

This seems to be subdominant.

Example: Electric string ansatz

Consider $\mathbb{Z}_2$ gauge theory: two irreps (trivial and alternating).

Let $L(S)$ be the total length of "electric strings" (nontrivial irreps)

$$|\alpha\rangle = \frac{1}{\mathcal{N}} \sum_S e^{-\frac{\alpha}{2} L(S)} |S\rangle$$

Small $\alpha \Rightarrow$ long strings, large $\alpha \Rightarrow$ short strings.

Only Shannon entropy term is nonzero: $S = -\sum p(R_\partial) \ln p(R_\partial)$.

As $\alpha \to 0$, get an equal superposition of all string states,

$$S = (|\partial A| - \#\ \text{components}(\partial A)) \ln 2.$$

This is a topological phase, topological entanglement entropy $2 \ln 2$.

Example: Strong coupling

Consider ground state of SU(2) Kogut-Susskind Hamiltonian

$$H = \sum_{l \in L} j_l(j_l + 1) + 3\lambda \sum_{\square} [\mathrm{tr}(u_\square) + \mathrm{h.c.}]$$

First term is diagonal in spin network basis, second creates/destroys loops.

For strong coupling ($\lambda \ll 1$), the ground state is single loops:

$$|\Omega\rangle = \left(1 - \tfrac{1}{2} N_\square \lambda^2\right) |0\rangle + \lambda \sum_{\square} |\square\rangle + O(\lambda^2)$$

Only local terms contribute at order λ^2:

$$S = |\partial A| \, (d-1)\lambda^2(-\ln \lambda^2 + 1 + 2\ln 2) + O(\lambda^3).$$

Area law at strong coupling, with non-analytic coefficient.

Conclusion

- Definition of entanglement entropy requires care in gauge theory.
- Localizing degrees of freedom leads to edge states.
- Boundary gauge symmetry restricts the form of ρ_A.
- Entropy splits as a sum of three terms:
  - The Shannon entropy of spin network endpoints,
  - A correction for non-abelian theories,
  - A non-local term.
- In some interesting cases, the dominant contribution is local.

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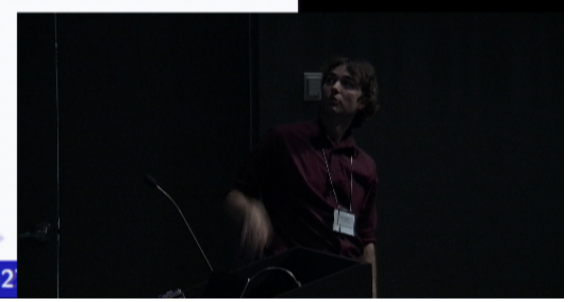

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