Title: Uncertainty Relations on a Planck Lattice and Black Hole Temperature

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Abstract: After an introduction to generalized uncertainty principle(s), we study uncertainty relations as formulated in a crystal-like universe, whose lattice spacing is of order of
Planck length. For Planckian energies, the uncertainty relation for position and momenta has a lower bound equal to zero. Connections of this result with 't Hooft's deterministic quantization proposal, and with double special relativity are briefly presented. We then apply our formulae to

(micro) black holes, we derive a new mass-temperature
relation for Schwarzschild black holes, and we discuss the new thermodynamic entropy and heat capacity.

In contrast to standard results based on Heisenberg and
stringy uncertainty relations, we obtain both a finite Hawking's temperature and a zero rest-mass remnant at the end of the (micro) black hole evaporation.

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UNCERTAINTY RELATIONS ON A PLANCK LATTICE AND BLACK HOLES TEMPERATURE

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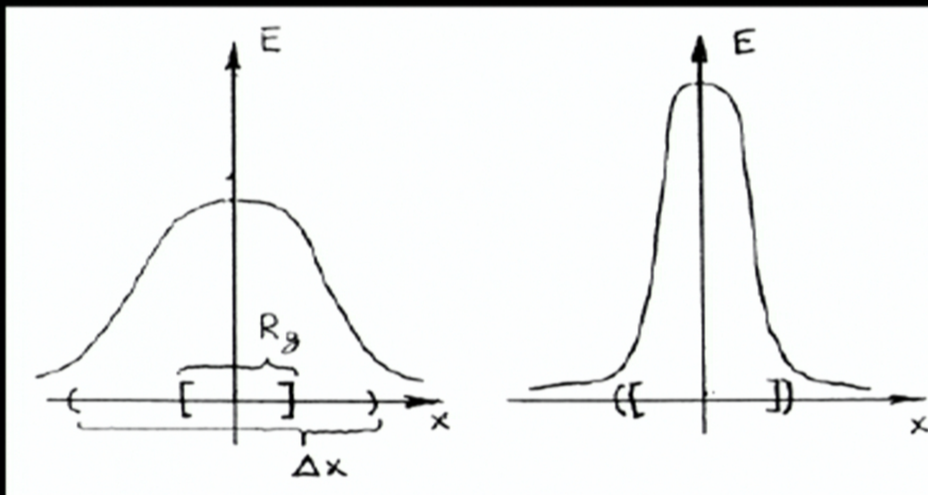
Generalized Uncertainty Principles (GUPs)

- Research on generalizations of the Heisenberg uncertainty principle has several decades of history (C.N. Yang, 1947 - Snyder, 1947 - F. Karolyhazy, 1966).

- Last 20 years: string theory (Veneziano 1987, Gross 1987) suggests that, in gedanken experiments involving Gravity at high energy strings scattering, the uncertainty relation should read

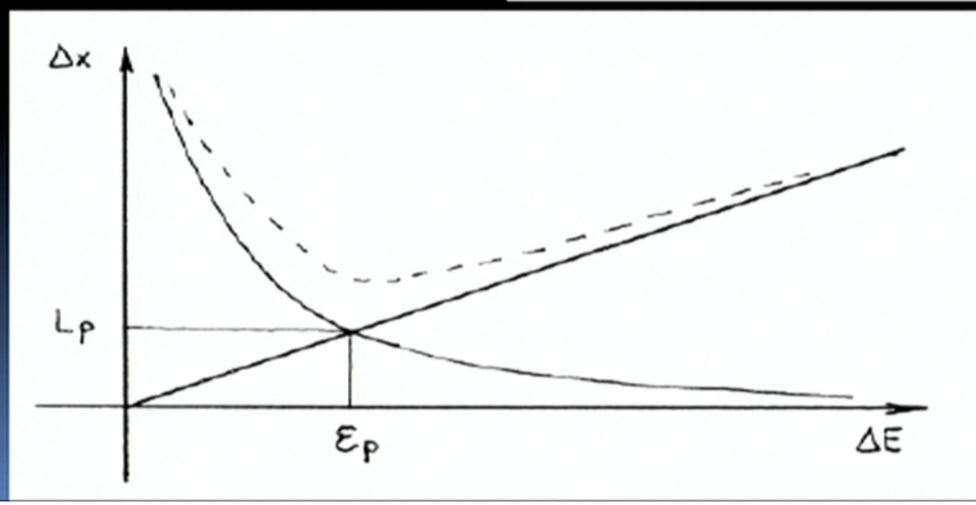
$$\Delta x \geq \frac{\hbar}{2\Delta p} + 2\beta\,\ell_{4n}^2\,\frac{\Delta p}{\hbar},$$

Gedanken Experiment on scatterings involving formation of MicroBlack Holes (Scardigli, Adler 1999) yields similar relation



$$\Delta x \geq \begin{cases} \dfrac{\hbar c}{2\,\Delta E} & \text{for } \Delta E \leq \epsilon_P \\ \dfrac{2\,G\Delta E}{c^4} & \text{for } \Delta E > \epsilon_P \end{cases}$$

$$\Delta x \geq \frac{\hbar c}{2\,\Delta E} + \frac{2\,G\Delta E}{c^4}$$



Quantum Mechanics and GUP on a Planck Lattice

In order to reconcile GR and QM a dramatic conceptual shift is required in our understanding of a *spacetime*. $\Rightarrow$

Revival of the idea of spacetime as a discrete coarse-grained structure at Plackian lengths $\ell_p \approx 10^{-35}$m $\Rightarrow$

Quantum-gravity models:

- space-time foam (John Wheeler - 1955)
- loop quantum gravity
- non-commutative geometry
- black-hole physics
- cosmic cellular automata (Stephen Wolfram - 2004)

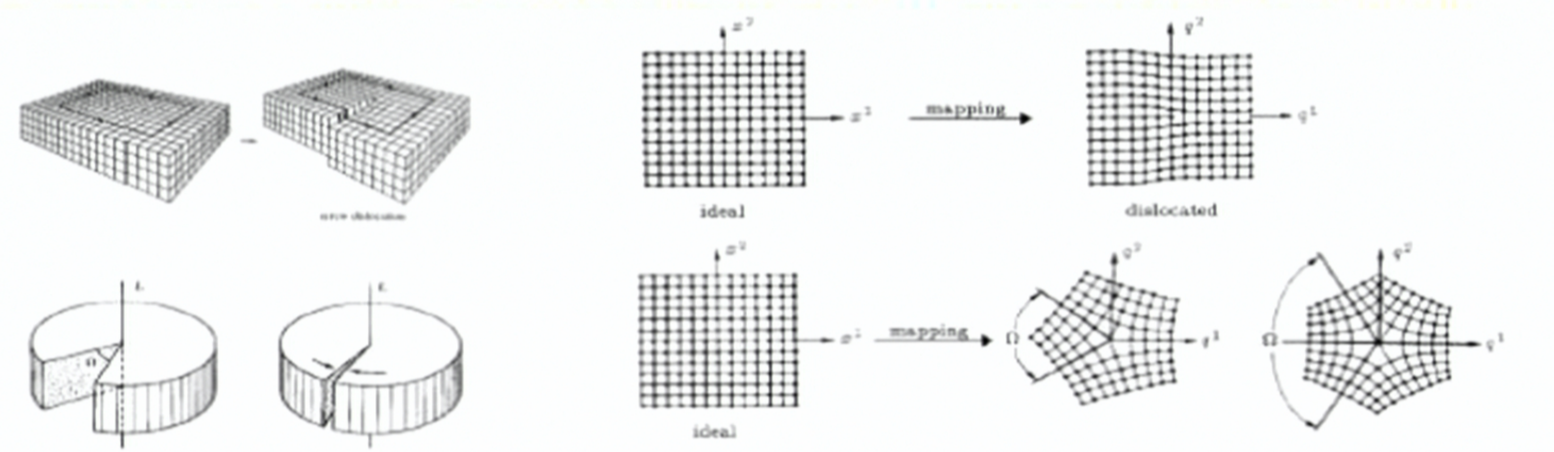
*A Discrete LATTICE seems a good TOY MODEL for Planck Physics.
(Lattices are also numerical regulator in QFT, or in GR)

* EXAMPLE: The defect structure of a crystal (Kleinert 1989), (lattice spacing of about a Planck length) the so called

WORLD CRYSTAL

can reproduce the geometry of Einstein(-Cartan) spaces

- Curvature is due to rotational def., torsion due to translational def.



* Formulate Quantum Mechanics on a Planck Lattice, and study the associated Generalized Uncertainty Principle (GUP)

→ Consequences for Black Hole physics (LHC?)

DIFFERENTIAL CALCULUS ON A LATTICE (1D lattice)

The lattice sites are at $x_n = n\epsilon$, with $n \in \mathcal{Z}$.
There are two fundamental derivatives of a function $f(x)$:

$$
\begin{aligned}
(\nabla f)(x) &= \frac{1}{\epsilon}[f(x+\epsilon) - f(x)] \,, \\
(\bar{\nabla} f)(x) &= \frac{1}{\epsilon}[f(x) - f(x-\epsilon)] \,.
\end{aligned}
$$

They obey the generalized Leibnitz rule

$$
\begin{aligned}
(\nabla fg)(x) &= (\nabla f)(x)g(x) + f(x+\epsilon)(\nabla g)(x) \,, \\
(\bar{\nabla} fg)(x) &= (\bar{\nabla} f)(x)g(x) + f(x-\epsilon)(\bar{\nabla} g)(x) \,.
\end{aligned}
$$

On a lattice, integration is performed as a summation:

$$
\int dx\, f(x) \equiv \epsilon \sum_x f(x) \,,
$$

where x *runs over all* x_n.

Integration by parts:

$$\sum_x f(x)\nabla g(x) = -\sum_x g(x)\bar{\nabla} f(x)$$

One can also define the lattice Laplacian as

$$\nabla\bar{\nabla} f(x) = \bar{\nabla}\nabla f(x) = \frac{1}{\epsilon^2}[f(x+\epsilon) - 2f(x) + f(x-\epsilon)]$$

which reduces in the continuum limit to Laplace operator ∂_x^2.

The above calculus can be easily extended to any number of dims.

- *POSITION and MOMENTUM OPERATORS on a LATTICE*
- Quantum Mechanics on a 1-D lattice:
- Scalar Product

$$\langle \psi_1 | \psi_2 \rangle \; = \; \epsilon \sum_x \psi_1^*(x) \psi_2(x) \, .$$

This implies that

$$\langle f | \nabla g \rangle \; = \; -\langle \bar{\nabla} f | g \rangle$$

so that $(i\nabla)^\dagger = i\bar{\nabla}$, and neither $i\nabla$ nor $i\bar{\nabla}$ are hermitian operators.

The lattice Laplacian $\nabla\bar{\nabla} = \bar{\nabla}\nabla$ is HERMITIAN.

The position operator $\hat{X}_\epsilon$ acting on wave functions of x is defined by a simple multiplication with x:

$$(\hat{X}_\epsilon f)(x) \; = \; x f(x)$$

The lattice momentum operator $\hat{P}_\epsilon$: To ensure hermiticity we relate $\hat{P}_\epsilon$ to symmetric lattice derivative, i.e.,

$$(\hat{P}_\epsilon f)(x) = \frac{\hbar}{2i}[(\nabla f)(x) + (\bar{\nabla} f)(x)] = \frac{\hbar}{2i\epsilon}[f(x+\epsilon) - f(x-\epsilon)]$$

For small ϵ, this reduces to momentum operator $\hat{p} \equiv -i\hbar\partial_x$:

$$\hat{P}_\epsilon = \hat{p} + \mathcal{O}(\epsilon^2)$$

The "canonical" commutator between $\hat{X}_\epsilon$ and $\hat{P}_\epsilon$ on the lattice:

$$\left([\hat{X}_\epsilon, \hat{P}_\epsilon] f\right)(x) = \frac{i\hbar}{2}[f(x+\epsilon) + f(x-\epsilon)] \equiv i\hbar(\hat{I}_\epsilon f)(x)$$

Operator **I** is a lattice-version of unit operator (average over neighboring sites)
Operators X, P, I, are hermitian under the defined scalar product

$\hat{X}_\epsilon$, $\hat{P}_\epsilon$, and $\hat{I}_\epsilon$ form $E(2)$ algebra, which contracts to the standard Weyl–Heisenberg algebra in the limit $\epsilon \to 0$: $\hat{X}_\epsilon \to \hat{x}$, $\hat{P}_\epsilon \to \hat{p}$, $\hat{I}_\epsilon \to \hat{\mathbf{1}}$.

$\Rightarrow$ ordinary QM is obtained from lattice QM by a contraction of the $E(2)$ algebra via the limit $\epsilon \to 0$

•Fourier-decomposition with wave numbers in the Brillouin zone:

$$f(x) = \int_{-\pi/\epsilon}^{\pi/\epsilon} \frac{dk}{2\pi} \tilde{f}(k) e^{ikx},$$

with the coefficients

$$\tilde{f}(k) = \epsilon \sum_x f(x) e^{-ikx}.$$

This implies the good-old de Broglie relation

$$(\hat{p}\tilde{f})(k) = \hbar k \tilde{f}(k),$$

and its lattice version

$$(-i\nabla\tilde{f})(k) = K\tilde{f}(k), \quad (-i\bar{\nabla}\tilde{f})(k) = \bar{K}\tilde{f}(k),$$

with the eigenvalues

$$K \equiv (e^{ik\epsilon} - 1)/i\epsilon = \bar{K}^*.$$

the Fourier transforms of the operators $\hat{X}_\epsilon, \hat{P}_\epsilon, \hat{I}_\epsilon$:

$$(\hat{X}_\epsilon\tilde{f})(k) = i\frac{d}{dk}\tilde{f}(k),$$

$$(\hat{P}_\epsilon\tilde{f})(k) = \frac{\hbar}{\epsilon}\sin(k\epsilon)\tilde{f}(k),$$

$$(\hat{I}_\epsilon\tilde{f})(k) = \cos(k\epsilon)\tilde{f}(k),$$

we can rewrite the **commutation relation** as

$$\left([\hat{X}_\epsilon, \hat{P}_\epsilon] f\right)(x) = i\hbar \cos(\epsilon\hat{p}/\hbar)\, f(x).$$

Thus the lattice unit operator $\hat{I}_\epsilon$ is $\cos(\epsilon\hat{p}/\hbar)$. $\hat{I}_\epsilon = \hat{\mathbf{1}}$ on all lattice nodes.

UNCERTAINTY RELATIONS ON A LATTICE

Uncertainty of an observable *A in a state ψ* defined by the standard deviation

$$(\Delta A)_\psi \equiv \sqrt{\langle\psi|(\hat{A} - \langle\psi|\hat{A}|\psi\rangle)^2|\psi\rangle}$$

Via the Schwarz inequality we get the Uncertainty Relation on Lattice

$$\Delta X_\epsilon \Delta P_\epsilon \geq \frac{1}{2}\left|\langle\psi|[\hat{X}_\epsilon, \hat{P}_\epsilon]|\psi\rangle\right| = \frac{\hbar}{2}\left|\langle\psi|\hat{I}_\epsilon|\psi\rangle\right| = \frac{\hbar}{2}\left|\langle\psi|\cos\left(\epsilon\hat{p}/\hbar\right)|\psi\rangle\right|$$

TWO CRITICAL REGIMES of the GUP

I) long-wavelengths regime where $\langle\hat{p}\rangle_\psi \to 0$

II) regime near boundary of Brillouin zone where $\langle\hat{p}\rangle_\psi \to \pi\hbar/2\epsilon$

For mirror-symmetric states where $\langle\hat{p}\rangle_\psi = 0$ this implies

$$\Delta X_\epsilon \Delta P_\epsilon \;\geq\; \frac{\hbar}{2}\left(1 - \frac{\epsilon^2}{2\hbar^2}(\Delta p)^2\right)$$

Here we have substituted $|...|$ with $(...)$ since we assume: $\epsilon \simeq \ell_p$ (Planckian lattice) and $\Delta p \simeq 0$. Therefore $\epsilon^2(\Delta p)^2/2\hbar^2 \ll 1$.

For Planckian lattices we can neglect in higher orders in ϵ and write

$$\Delta X_\epsilon \Delta P_\epsilon \;\geq\; \frac{\hbar}{2}\left(1 - \frac{\epsilon^2}{2\hbar^2}(\Delta P_\epsilon)^2\right)$$

II) At the border of the first Brillouin zone

$$\langle\hat{p}\rangle_\psi \;=\; \frac{\pi\hbar}{2\epsilon}$$

Use the expansion for $\cos[\pi/2 + (\epsilon\hat{p}/\hbar - \pi/2)]\rangle_\psi$:

$$\sum_{n=0}^{\infty}\int_0^{\infty} dp\, \varrho(p)\,(-1)^n \frac{(\pi/2 - \epsilon p/\hbar)^{2n+1}}{(2n+1)!}$$

$\varrho(p)$ is peaked around $p \simeq \pi\hbar/2\epsilon \;\Rightarrow\;$ dominant contribution gives

$$\Delta X_\epsilon \Delta P_\epsilon \;\geq\; \frac{\hbar}{2}\left|\,\frac{\pi}{2} - \frac{\epsilon}{\hbar}\langle\hat{p}\rangle_\psi\,\right|$$

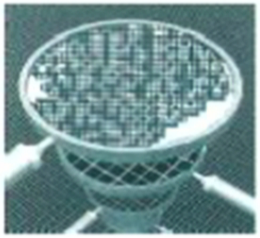
Since k is always inside Brillouin zone, $\langle\hat{p}\rangle_\psi \leq \pi\hbar/2\epsilon$ and $|...| \to (...)$.

Up to order $\mathcal{O}(\epsilon)$ GUP close to boundary of Brillouin zone is

$$\Delta X_\epsilon \Delta P_\epsilon \geq \frac{\hbar}{2}\left(\frac{\pi}{2} - \frac{\epsilon}{\hbar}\langle \hat{P}_\epsilon \rangle_\psi\right)$$

- As momentum reaches boundary of **Brillouin zone** RHS vanishes so that lattice QM at short wavelengths **can** exhibit **classical behavior.** G. 't Hooft, Class. Quant. Grav. 16 (1999),; Int. J. Theor. Phys. 42 (2003)

$\longrightarrow$



- GUP leads to physical conclusions analogous to those found by Magueijo and Smolin in **DSR.** J. Magueijo and L. Smolin, Phys. Rev. D 67 (2003)

In this model the world can become "classical" for energies close to the Brillouin zone, i.e., for Planckian energies.
(as in 't Hooft's "deterministic" quantum mechanics).

DISPERSION RELATION for PHOTONS

Vector potential of a photon in the Lorentz gauge in $(1+1)$D satisfies

$$\frac{1}{c^2}\partial_t^2 A^\mu(x,t) \; = \; \partial_x^2 A^\mu(x,t)$$

Plane wave $A^\mu(x) = \epsilon^\mu \exp[i(kx - \omega(k)t)]$ exhibits linear disp. rel.

$$\omega(k) \; = \; c\,|k|$$

On a 1D lattice $\partial_x^2 \to \bar{\nabla}\nabla$, and the spectrum becomes

$$\frac{\omega(k)}{c} \; = \; \sqrt{K\bar{K}} \; = \; \frac{\sqrt{[2-2\cos(k\epsilon)]}}{\epsilon} \; = \; \frac{2}{\epsilon}\left|\sin\left(\frac{k\epsilon}{2}\right)\right|$$

GUP for PHOTONS

Denoting the energy on the lattice $\hbar\omega$ by E_ϵ, we obtain the disp. rel.

$$\frac{E_\epsilon}{\hbar c} = \frac{2}{\epsilon}\left|\sin\left(\frac{p\,\epsilon}{2\,\hbar}\right)\right|$$

For small momenta ($p \ll \hbar/\epsilon$) this has the expansion

$$E_\epsilon = \left| c\,p - c\frac{\epsilon^2 p^3}{24\,\hbar^2} + \mathcal{O}(p^5) \right|$$

Up to order $\mathcal{O}(\epsilon^2)$ this allows us to rephrase GUP as

$$\Delta X_\epsilon \Delta E_\epsilon \geq \frac{\hbar c}{2}\left[1 - \frac{\epsilon^2}{2\hbar^2 c^2}(\Delta E_\epsilon)^2\right]$$

This relation will be our starting point for applications of the GUP to micro black holes.

APPLICATIONS TO (MICRO) BLACK HOLES PHYSICS

The mass-temperature relation for Black Holes strongly depends on the actual form of the energy-position uncertainty relation

- Heisenberg microscope argument: the smallest resolvable detail δx of an object goes roughly as the wavelength of the probing photons. If E is the (average) energy of the photons

Heisenberg

$$\delta x = \frac{\hbar c}{2E}.$$

The Lattice Version of this standard Heisenberg formula is

$$\delta X_\epsilon \simeq \frac{\hbar c}{2E_\epsilon}\left[1 - \frac{\epsilon^2}{2\hbar^2 c^2}(E_\epsilon)^2\right]$$

δX (average) wavelength a photon and E its energy

For a lattice spacing $\epsilon = a\ell_p$ and denoting the Planck energy as $\mathcal{E}_p = \hbar c/2\ell_p$, we have

$$\delta X_\epsilon \simeq \frac{\hbar c}{2E_\epsilon} - \frac{a^2 \ell_p E_\epsilon}{8\mathcal{E}_p}$$

COMPARISON WITH HEISENBERG AND STRINGY UNCERTAINTY PRINCIPLES

In continuum limit $\epsilon, \; a \to 0$ and GUP reduces to Heisenberg UP

$$\Rightarrow \quad m = \frac{1}{4\pi\Theta} \quad \Leftrightarrow \quad T_H = \frac{\hbar c^3}{8\pi G k_B M} = \frac{\hbar c}{4\pi k_B R_S}$$

which is the dimensionless version of Hawking's formula.

Lattice $m - \Theta$ relation can be compared with the one coming from stringy uncertainty relation:

$$2m = \frac{1}{2\pi\Theta} + \zeta^2 \, 2\pi\Theta$$

The phenomenological consequences of the lattice relation are quite different due to the *opposite* sign in front of the deformation term.

- Consider an ensemble of un-polarized photons of Hawking radiation just outside the event horizon. Average wavelength of the Hawking radiation ≈ geometrical size of the hole.

photon position uncertainty

$$\delta X_\epsilon \simeq 2\mu R_S = 2\mu \ell_p m$$

with $R_S = \ell_p m$, where $m = M/M_p$ is the black hole mass in Planck units ($M_p = \mathcal{E}_p/c^2$), and μ is a free parameter of $\mathcal{O}(1)$. In this regime

$$2\mu m = \frac{\mathcal{E}_p}{E_\epsilon} - \frac{a^2}{8}\frac{E_\epsilon}{\mathcal{E}_p}$$

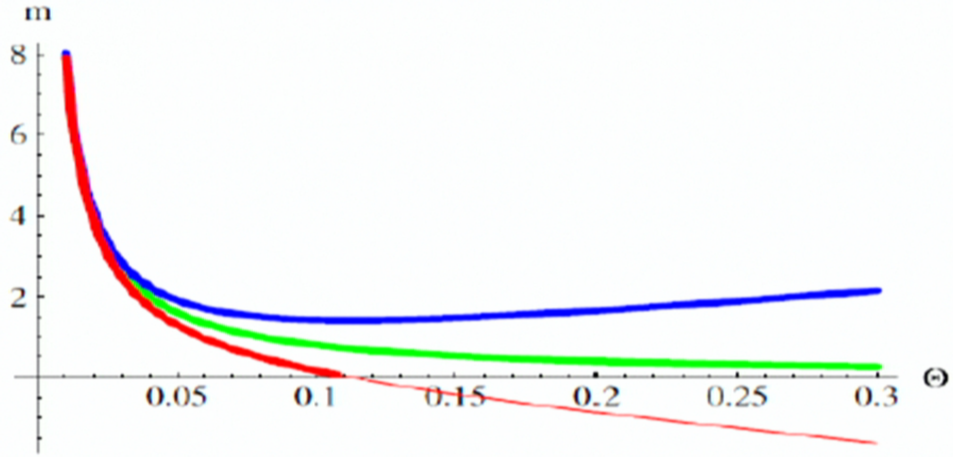
Equipartition law: energy of un-polarized photons of outgoing Hawking radiation

$$E_\epsilon \simeq k_B T .$$

Defining $T_p = 2\mathcal{E}_p/k_B \approx 10^{32}$ K and $\Theta = T/T_p$, we can rewrite $m - \Theta$ formula as

$$2m = \frac{1}{2\pi\Theta} - 2\pi\zeta^2\Theta$$

where $\zeta = a/(2\sqrt{2}\pi)$ and $\mu = \pi$, in order to agree with Hawking's formula in cont. limit.



Three $m - \Theta$ relations, lattice, Hawking's, and stringy GUP, with $\zeta = \sqrt{2}$.

For the stringy GUP, the blue line predicts a maximum temperature

$$\Theta_{\max} = \frac{1}{2\pi\zeta}$$

and minimum rest mass

$$m_{\min} = \zeta$$

- Stringy GUP: The end of the evaporation process is reached after a finite time, the final Θ is finite, and there is a **REMNANT** of a finite rest mass.
- From the standard Heisenberg UP we find the green curve, representing the Hawking formula. Here the evaporation process ends, after a finite time, with a zero mass and a worrisome infinite temperature.

Stringy GUP ➔ finite mass remnants

Pro: Candidates for dark matter
Contra: detectability issue, excessive production in the early universe.

In contrast, our lattice GUP predicts the red curve. This yields a finite end-temperature

$$\Theta_{\max} = \frac{1}{2\pi\zeta}$$

with a zero-mass remnant.

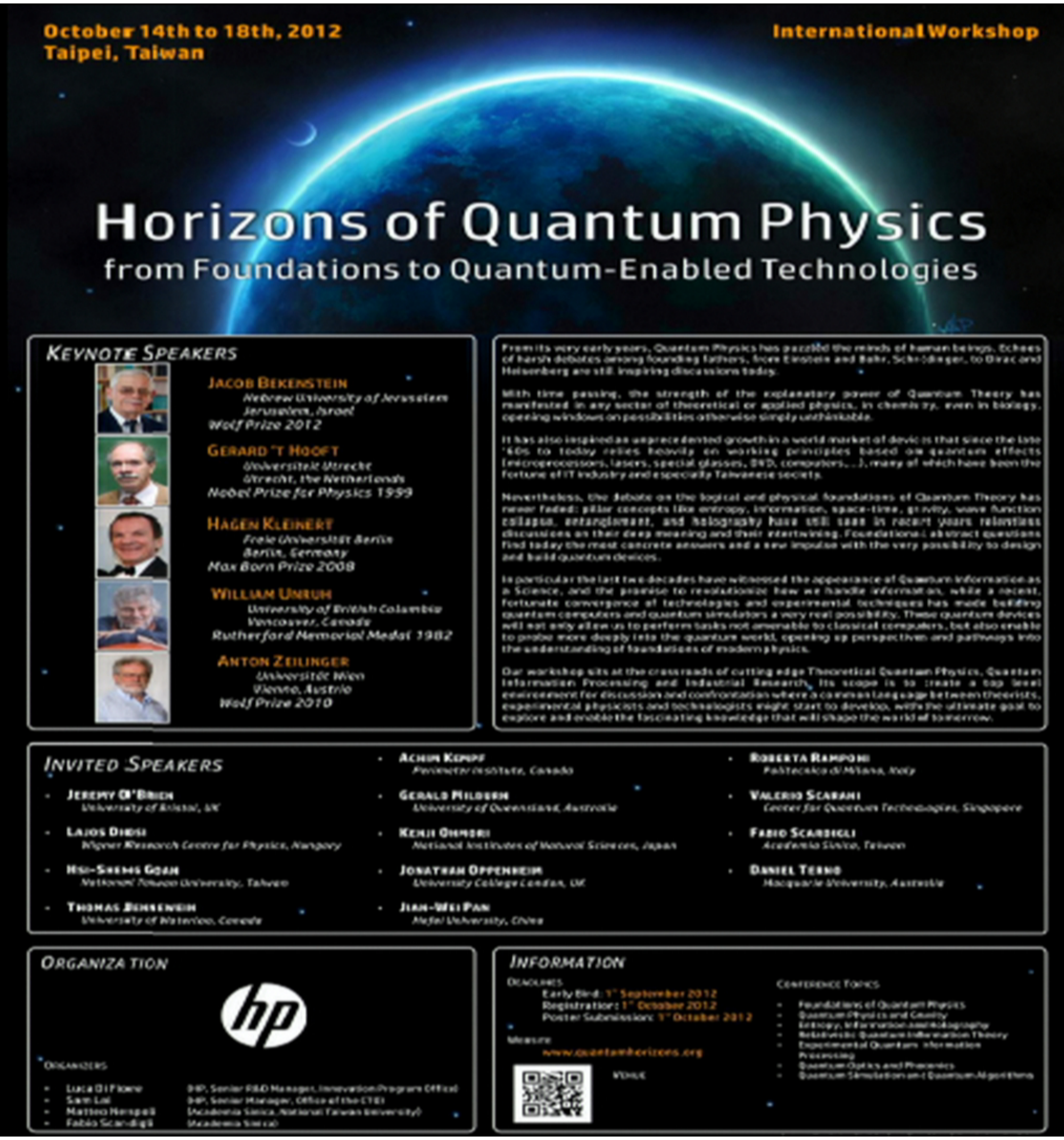
The analysis of the short wave limit fully confirms the previous result.

Conclusions

- We derived the GUP on a cubic lattice.
- Cubic-Lattice GUP allows for "classical" behavior at energies near the border of the Brillouin zone (Planck energies).
- We derived a new mass-temperature relation for Schwarzschild (micro) black holes.
- Phenomenological consequences of this formula are:
- The final Hawking temperature of a decaying micro black hole remains finite, in contrast to the infinite temperature of the standard result obtained by Heisenberg's uncertainty principle.
- **The final mass of the evaporation process is ZERO,**

  ➔ NO massive black hole remnants.

- **Avenues for Future Investigations**
- Flexible Lattice
- WHAT KIND OF LATTICE (microstructure) IS REQUIRED IN ORDER TO OBTAIN A STRINGY GUP? (i.e. related to the existence of BH REMNANTS?)



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