

Title: Uncertainty Relations on a Planck Lattice and Black Hole Temperature

Date: Jun 27, 2012 04:40 PM

URL: <http://pirsa.org/12060063>

Abstract: <span>After an introduction to generalized uncertainty principle(s), we study uncertainty relations as formulated in a crystal-like universe, whose lattice spacing is of order of  
&nbsp;Planck length. For Planckian energies, the uncertainty relation for position and momenta has a lower bound equal to zero. Connections of this result with 't Hooft's deterministic quantization proposal, and with double special relativity are briefly presented. We then apply our formulae to

(micro) black holes, we derive a new mass-temperature relation for Schwarzschild black holes, and we discuss the new thermodynamic entropy and heat capacity.

In contrast to standard results based on Heisenberg and stringy uncertainty relations, we obtain both a finite Hawking's temperature and a zero rest-mass remnant at the end of the (micro) black hole evaporation.

[Ref.Paper: PRD 81, 084030 (2010). arXiv:0912.2253]</span>

# UNCERTAINTY RELATIONS ON A PLANCK LATTICE AND BLACK HOLES TEMPERATURE

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collaboration with *H. Kleinert* and *P. Jizba* [PRD 81, 084030, 2010]

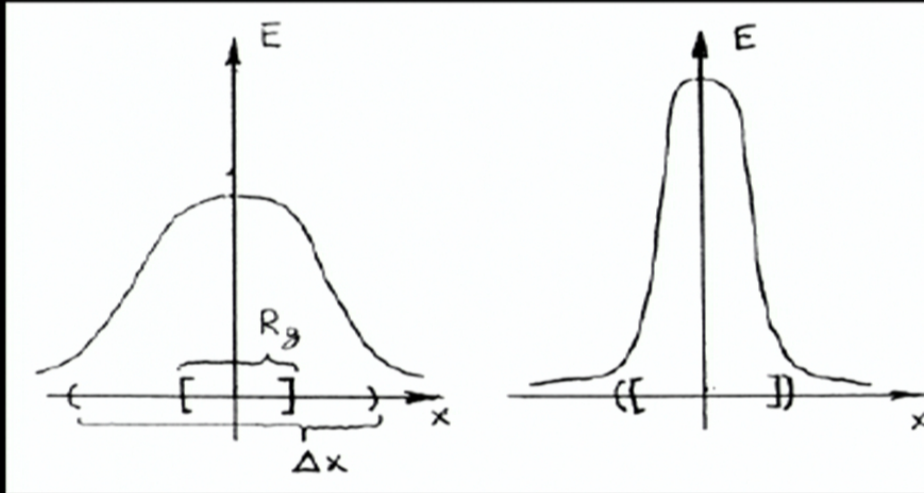
Perimeter Institute, Waterloo, Ontario, Canada , 27 June 2012

# Generalized Uncertainty Principles (GUPs)

- Research on generalizations of the Heisenberg uncertainty principle has several decades of history (C.N. Yang, 1947 - Snyder, 1947 - F. Karolyhazy, 1966).
- Last 20 years: **string theory** (Veneziano 1987, Gross 1987) suggests that, in gedanken experiments involving **Gravity** at high energy strings scattering, the uncertainty relation should read

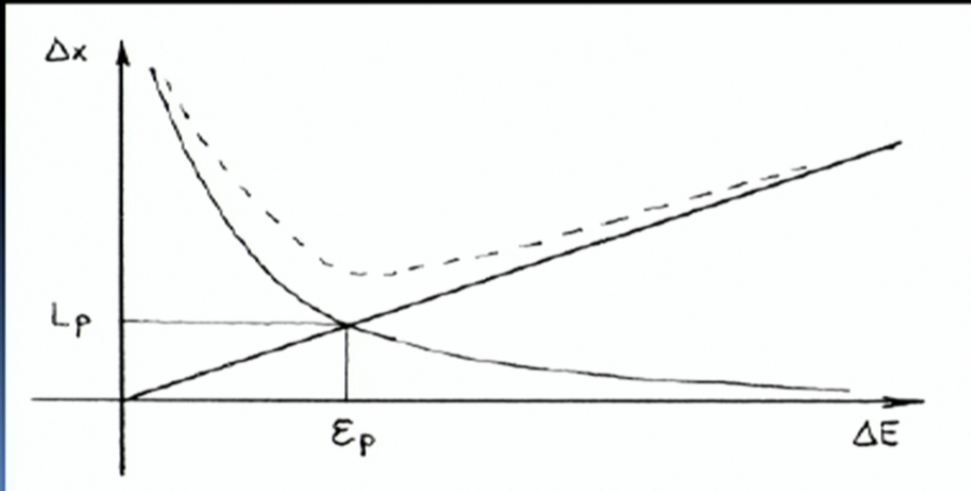
$$\Delta x \geq \frac{\hbar}{2\Delta p} + 2\beta \ell_{4n}^2 \frac{\Delta p}{\hbar},$$

Gedanken Experiment on scatterings involving **formation of MicroBlack Holes** (Scardigli, Adler 1999) yields similar relation



$$\Delta x \geq \begin{cases} \frac{\hbar c}{2 \Delta E} & \text{for } \Delta E \leq \epsilon_p \\ \frac{2 G \Delta E}{c^4} & \text{for } \Delta E > \epsilon_p \end{cases}$$

$$\Delta x \geq \frac{\hbar c}{2 \Delta E} + \frac{2 G \Delta E}{c^4}$$



# Quantum Mechanics and GUP on a Planck Lattice

In order to reconcile GR and QM a dramatic conceptual shift is required in our understanding of a *spacetime*.  $\Rightarrow$

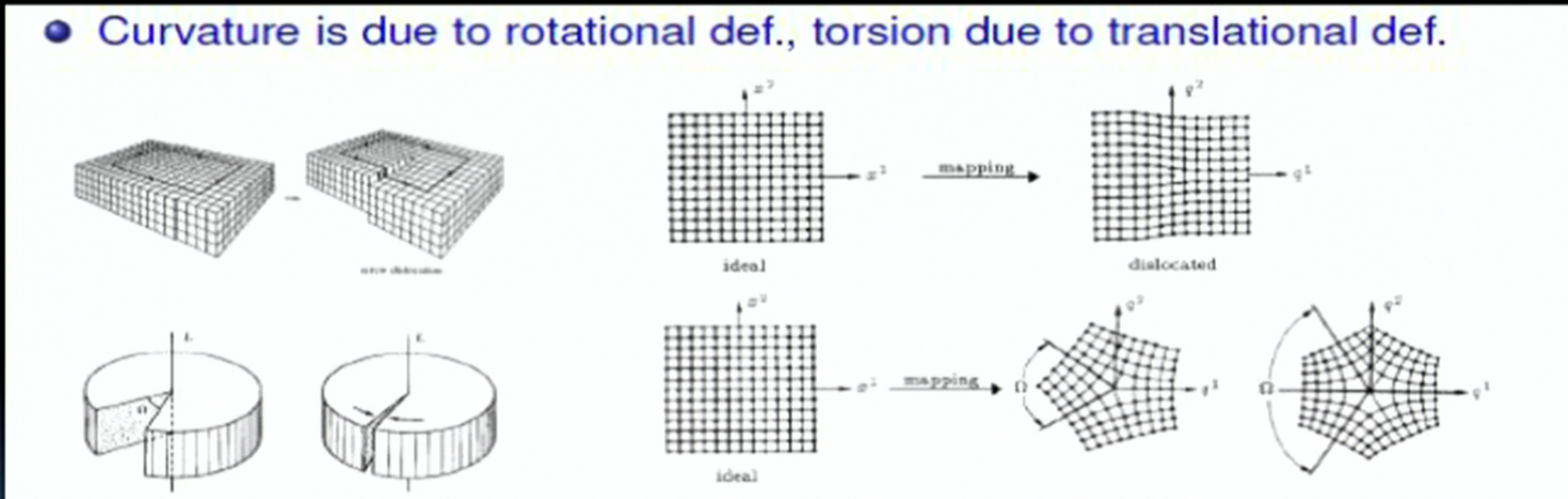
Revival of the idea of spacetime as a discrete coarse-grained structure at Plackian lengths  $\ell_p \approx 10^{-35}\text{m}$   $\Rightarrow$

## Quantum-gravity models:

- space-time foam (John Wheeler - 1955)
- loop quantum gravity
- non-commutative geometry
- black-hole physics
- cosmic cellular automata (Stephen Wolfram - 2004)

\*A **Discrete LATTICE** seems a good TOY MODEL for Planck Physics.  
 (Lattices are also numerical regulator in QFT, or in GR)  
 \* EXAMPLE: The **defect structure of a crystal (Kleinert 1989)**,  
 (lattice spacing of about a Planck length) the so called  
**WORLD CRYSTAL**  
 can reproduce the geometry of Einstein(-Cartan) spaces

• Curvature is due to rotational def., torsion due to translational def.



\* Formulate Quantum Mechanics on a **Planck Lattice**, and study the associated **Generalized Uncertainty Principle (GUP)**



Consequences for **Black Hole physics (LHC?)**

## DIFFERENTIAL CALCULUS ON A LATTICE (1D Lattice)

The lattice sites are at  $x_n = n\epsilon$ , with  $n \in \mathcal{Z}$ .

There are two fundamental derivatives of a function  $f(x)$ :

$$\begin{aligned}(\nabla f)(x) &= \frac{1}{\epsilon} [f(x + \epsilon) - f(x)] , \\ (\bar{\nabla} f)(x) &= \frac{1}{\epsilon} [f(x) - f(x - \epsilon)] .\end{aligned}$$

They obey the generalized Leibnitz rule

$$\begin{aligned}(\nabla fg)(x) &= (\nabla f)(x)g(x) + f(x + \epsilon)(\nabla g)(x) , \\ (\bar{\nabla} fg)(x) &= (\bar{\nabla} f)(x)g(x) + f(x - \epsilon)(\bar{\nabla} g)(x) .\end{aligned}$$

On a lattice, integration is performed as a summation:

$$\int dx f(x) \equiv \epsilon \sum_x f(x) ,$$

where  $x$  runs over all  $x_n$ .

Integration by parts:

$$\sum_x f(x) \nabla g(x) = - \sum_x g(x) \bar{\nabla} f(x)$$

One can also define the **lattice Laplacian** as

$$\nabla \bar{\nabla} f(x) = \bar{\nabla} \nabla f(x) = \frac{1}{\epsilon^2} [f(x + \epsilon) - 2f(x) + f(x - \epsilon)]$$

which reduces in the continuum limit to Laplace operator  $\partial_x^2$ .

The above calculus can be easily extended to any number of dims.



- *POSITION and MOMENTUM OPERATORS on a LATTICE*
- Quantum Mechanics on a 1-D lattice:
- Scalar Product

$$\langle \psi_1 | \psi_2 \rangle = \epsilon \sum_x \psi_1^*(x) \psi_2(x).$$

This implies that

$$\langle f | \nabla g \rangle = -\langle \bar{\nabla} f | g \rangle$$

so that  $(i\nabla)^\dagger = i\bar{\nabla}$ , and neither  $i\nabla$  nor  $i\bar{\nabla}$  are hermitian operators.

The lattice Laplacian  $\nabla\bar{\nabla} = \bar{\nabla}\nabla$  is HERMITIAN.

The position operator  $\hat{X}_\epsilon$  acting on wave functions of  $x$  is defined by a simple multiplication with  $x$ :

$$(\hat{X}_\epsilon f)(x) = xf(x)$$

The **lattice momentum operator**  $\hat{P}_\epsilon$ : To ensure hermiticity we relate  $\hat{P}_\epsilon$  to symmetric lattice derivative, i.e.,

$$(\hat{P}_\epsilon f)(x) = \frac{\hbar}{2i} [(\nabla f)(x) + (\bar{\nabla} f)(x)] = \frac{\hbar}{2i\epsilon} [f(x + \epsilon) - f(x - \epsilon)]$$

For small  $\epsilon$ , this reduces to momentum operator  $\hat{p} \equiv -i\hbar\partial_x$ :

$$\hat{P}_\epsilon = \hat{p} + \mathcal{O}(\epsilon^2)$$

The **“canonical” commutator** between  $\hat{X}_\epsilon$  and  $\hat{P}_\epsilon$  on the lattice:

$$([\hat{X}_\epsilon, \hat{P}_\epsilon]f)(x) = \frac{i\hbar}{2} [f(x + \epsilon) + f(x - \epsilon)] \equiv i\hbar(\hat{l}_\epsilon f)(x)$$

Operator  $\hat{l}_\epsilon$  is a lattice-version of unit operator (average over neighboring sites)  
Operators  $\hat{X}_\epsilon, \hat{P}_\epsilon, \hat{l}_\epsilon$  are hermitian under the defined scalar product

$\hat{X}_\epsilon, \hat{P}_\epsilon$ , and  $\hat{l}_\epsilon$  form  $E(2)$  algebra, which contracts to the standard Weyl–Heisenberg algebra in the limit  $\epsilon \rightarrow 0$ :  $\hat{X}_\epsilon \rightarrow \hat{x}, \hat{P}_\epsilon \rightarrow \hat{p}, \hat{l}_\epsilon \rightarrow \hat{1}$ .

$\Rightarrow$  ordinary QM is obtained from lattice QM by a contraction of the  $E(2)$  algebra via the limit  $\epsilon \rightarrow 0$

• Fourier-decomposition with wave numbers in the Brillouin zone:

$$f(x) = \int_{-\pi/\epsilon}^{\pi/\epsilon} \frac{dk}{2\pi} \tilde{f}(k) e^{ikx},$$

with the coefficients

$$\tilde{f}(k) = \epsilon \sum_x f(x) e^{-ikx}.$$

This implies the good-old de Broglie relation

$$(\hat{p}\tilde{f})(k) = \hbar k \tilde{f}(k),$$

and its lattice version

$$(-i\hat{\nabla}\tilde{f})(k) = K\tilde{f}(k), \quad (-i\bar{\nabla}\tilde{f})(k) = \bar{K}\tilde{f}(k),$$

with the eigenvalues

$$K \equiv (e^{ik\epsilon} - 1)/i\epsilon = \bar{K}^*.$$

the Fourier transforms of the operators  $\hat{X}_\epsilon, \hat{P}_\epsilon, \hat{I}_\epsilon$ :

$$(\hat{X}_\epsilon \tilde{f})(k) = i \frac{d}{dk} \tilde{f}(k),$$

$$(\hat{P}_\epsilon \tilde{f})(k) = \frac{\hbar}{\epsilon} \sin(k\epsilon) \tilde{f}(k),$$

$$(\hat{I}_\epsilon \tilde{f})(k) = \cos(k\epsilon) \tilde{f}(k),$$

we can rewrite the  
commutation  
relation

as

$$([\hat{X}_\epsilon, \hat{P}_\epsilon] f)(x) = i\hbar \cos(\epsilon \hat{p}/\hbar) f(x).$$

Thus the lattice unit operator

$\hat{\lambda}_\epsilon$  is  $\cos(\epsilon \hat{p}/\hbar)$ .  $\hat{\lambda}_\epsilon = \hat{\mathbf{1}}$  on all lattice nodes.

## UNCERTAINTY RELATIONS ON A LATTICE

Uncertainty of an observable  $A$  in a state  $\psi$  defined by the standard deviation

$$(\Delta A)_\psi \equiv \sqrt{\langle \psi | (\hat{A} - \langle \psi | \hat{A} | \psi \rangle)^2 | \psi \rangle}$$

Via the Schwarz inequality we get the Uncertainty Relation on Lattice

$$\Delta X_\epsilon \Delta P_\epsilon \geq \frac{1}{2} |\langle \psi | [\hat{X}_\epsilon, \hat{P}_\epsilon] | \psi \rangle| = \frac{\hbar}{2} |\langle \psi | \hat{I}_\epsilon | \psi \rangle| = \frac{\hbar}{2} |\langle \psi | \cos(\epsilon \hat{p} / \hbar) | \psi \rangle|$$

### TWO CRITICAL REGIMES of the GUP

- I) long-wavelengths regime where  $\langle \hat{p} \rangle_\psi \rightarrow 0$
- II) regime near boundary of Brillouin zone where  $\langle \hat{p} \rangle_\psi \rightarrow \pi \hbar / 2\epsilon$

For mirror-symmetric states where  $\langle \hat{p} \rangle_\psi = 0$  this implies

$$\Delta X_\epsilon \Delta P_\epsilon \geq \frac{\hbar}{2} \left( 1 - \frac{\epsilon^2}{2\hbar^2} (\Delta p)^2 \right)$$

Here we have substituted  $|\dots|$  with  $(\dots)$  since we assume:  $\epsilon \simeq \ell_p$  (Planckian lattice) and  $\Delta p \simeq 0$ . Therefore  $\epsilon^2 (\Delta p)^2 / 2\hbar^2 \ll 1$ .

For Planckian lattices we can neglect in higher orders in  $\epsilon$  and write

$$\Delta X_\epsilon \Delta P_\epsilon \geq \frac{\hbar}{2} \left( 1 - \frac{\epsilon^2}{2\hbar^2} (\Delta P_\epsilon)^2 \right)$$

II) At the border of the first Brillouin zone

$$\langle \hat{p} \rangle_\psi = \frac{\pi \hbar}{2\epsilon}$$

Use the expansion for  $\cos[\pi/2 + (\epsilon \hat{p}/\hbar - \pi/2)]_\psi$ :

$$\sum_{n=0}^{\infty} \int_0^{\infty} dp \varrho(p) (-1)^n \frac{(\pi/2 - \epsilon p/\hbar)^{2n+1}}{(2n+1)!}$$

$\varrho(p)$  is peaked around  $p \simeq \pi \hbar / 2\epsilon \Rightarrow$  dominant contribution gives

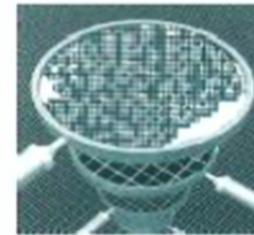
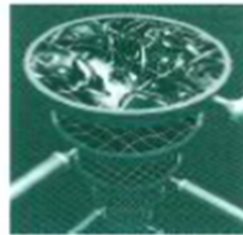
$$\Delta X_\epsilon \Delta P_\epsilon \geq \frac{\hbar}{2} \left| \frac{\pi}{2} - \frac{\epsilon}{\hbar} \langle \hat{p} \rangle_\psi \right|$$

Since  $k$  is always inside Brillouin zone,  $\langle \hat{p} \rangle_\psi \leq \pi \hbar / 2\epsilon$  and  $|\dots| \rightarrow (\dots)$ .

Up to order  $\mathcal{O}(\epsilon)$  GUP close to boundary of Brillouin zone is

$$\Delta X_\epsilon \Delta P_\epsilon \geq \frac{\hbar}{2} \left( \frac{\pi}{2} - \frac{\epsilon}{\hbar} \langle \hat{P}_\epsilon \rangle_\psi \right)$$

- As momentum reaches boundary of **Brillouin zone** RHS vanishes so that lattice QM at short wavelengths **can exhibit classical behavior.** G. 't Hooft, *Class. Quant. Grav.* 16 (1999); *Int. J. Theor. Phys.* 42 (2003)



- GUP leads to physical conclusions analogous to those found by Magueijo and Smolin in **DSR.** J. Magueijo and L. Smolin, *Phys. Rev. D* 67 (2003)

In this model **the world can become "classical"** for energies close to the Brillouin zone, i.e., for Planckian energies.  
(as in 't Hooft's "deterministic" quantum mechanics).

## DISPERSION RELATION for PHOTONS

Vector potential of a photon in the Lorentz gauge in (1 + 1)D satisfies

$$\frac{1}{c^2} \partial_t^2 A^\mu(x, t) = \partial_x^2 A^\mu(x, t)$$

Plane wave  $A^\mu(x) = \epsilon^\mu \exp[i(kx - \omega(k)t)]$  exhibits linear disp. rel.

$$\omega(k) = c|k|$$

On a 1D lattice  $\partial_x^2 \rightarrow \bar{\nabla} \nabla$ , and the spectrum becomes

$$\frac{\omega(k)}{c} = \sqrt{K\bar{K}} = \frac{\sqrt{[2 - 2\cos(k\epsilon)]}}{\epsilon} = \frac{2}{\epsilon} \left| \sin\left(\frac{k\epsilon}{2}\right) \right|$$

## GUP for PHOTONS

Denoting the energy on the lattice  $\hbar\omega$  by  $E_\epsilon$ , we obtain the disp. rel.

$$\frac{E_\epsilon}{\hbar c} = \frac{2}{\epsilon} \left| \sin \left( \frac{p\epsilon}{2\hbar} \right) \right|$$

For small momenta ( $p \ll \hbar/\epsilon$ ) this has the expansion

$$E_\epsilon = \left| cp - c \frac{\epsilon^2 p^3}{24\hbar^2} + \mathcal{O}(p^5) \right|$$

Up to order  $\mathcal{O}(\epsilon^2)$  this allows us to rephrase GUP as

$$\Delta X_\epsilon \Delta E_\epsilon \geq \frac{\hbar c}{2} \left[ 1 - \frac{\epsilon^2}{2\hbar^2 c^2} (\Delta E_\epsilon)^2 \right]$$

This relation will be our starting point for applications of the GUP to micro black holes.



## APPLICATIONS TO (MICRO) BLACK HOLES PHYSICS

The mass-temperature relation for Black Holes strongly depends on the actual form of the energy-position uncertainty relation

- Heisenberg microscope argument: **the smallest resolvable detail  $\delta x$  of an object goes roughly as the wavelength of the probing photons.** If  $E$  is the (average) energy of the photons

Heisenberg

$$\delta x = \frac{\hbar c}{2E}$$

The **Lattice Version** of this standard Heisenberg formula is

$$\delta X_\epsilon \simeq \frac{\hbar c}{2E_\epsilon} \left[ 1 - \frac{\epsilon^2}{2\hbar^2 c^2} (E_\epsilon)^2 \right]$$

$\delta X$  (average) wavelength a photon and  $E$  its energy

For a lattice spacing  $\epsilon = a\ell_p$  and denoting the Planck energy as  $\mathcal{E}_p = \hbar c/2\ell_p$ , we have

$$\delta X_\epsilon \simeq \frac{\hbar c}{2E_\epsilon} - \frac{a^2 \ell_p E_\epsilon}{8\mathcal{E}_p}$$

## COMPARISON WITH HEISENBERG AND STRINGY UNCERTAINTY PRINCIPLES

In continuum limit  $\epsilon$ ,  $a \rightarrow 0$  and GUP reduces to Heisenberg UP

$$\Rightarrow m = \frac{1}{4\pi\Theta} \Leftrightarrow T_H = \frac{\hbar c^3}{8\pi G k_B M} = \frac{\hbar c}{4\pi k_B R_S}$$

which is the dimensionless version of Hawking's formula.

Lattice  $m - \Theta$  relation can be compared with the one coming from stringy uncertainty relation:

$$2m = \frac{1}{2\pi\Theta} + \zeta^2 2\pi\Theta$$

The phenomenological consequences of the lattice relation are quite different due to the *opposite* sign in front of the deformation term.

- Consider an ensemble of un-polarized photons of Hawking radiation just outside the event horizon. **Average wavelength of the Hawking radiation  $\approx$  geometrical size of the hole.**

photon position  
uncertainty

$$\delta X_\epsilon \simeq 2\mu R_S = 2\mu \ell_p m$$

with  $R_S = \ell_p m$ , where  $m = M/M_p$  is the black hole mass in Planck units ( $M_p = \mathcal{E}_p/c^2$ ), and  $\mu$  is a free parameter of  $\mathcal{O}(1)$ . In this regime

$$2\mu m = \frac{\mathcal{E}_p}{E_\epsilon} - \frac{a^2 E_\epsilon}{8 \mathcal{E}_p}$$

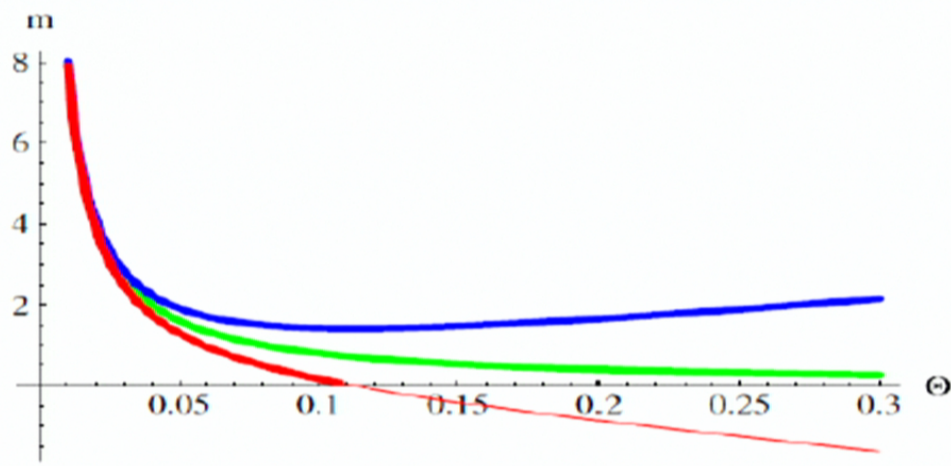
**Equipartition law:** energy of un-polarized photons of outgoing Hawking radiation

$$E_\epsilon \simeq k_B T.$$

Defining  $T_p = 2\mathcal{E}_p/k_B \approx 10^{32}$  K and  $\Theta = T/T_p$ , we can rewrite  $m - \Theta$  formula as

$$2m = \frac{1}{2\pi\Theta} - 2\pi\zeta^2\Theta$$

where  $\zeta = a/(2\sqrt{2}\pi)$  and  $\mu = \pi$ , in order to agree with Hawking's formula in cont. limit.



Three  $m - \Theta$  relations, **lattice**, **Hawking's**, and **stringy GUP**, with  $\zeta = \sqrt{2}$ .

For the **stringy GUP**, the **blue** line predicts a maximum temperature

$$\Theta_{\max} = \frac{1}{2\pi\zeta}$$

and minimum rest mass

$$m_{\min} = \zeta$$

- **Stringy GUP:** The end of the evaporation process is reached **after a finite time, the final  $\Theta$  is finite**, and there is a **REMNANT** of a **finite rest mass**.
- From the standard Heisenberg UP we find **the green curve**, representing the Hawking formula. Here **the evaporation process ends, after a finite time, with a zero mass and a worrisome infinite temperature**.

Stringy GUP → finite mass remnants

Pro: Candidates for dark matter  
 Contra: detectability issue, excessive production in the early universe.

In contrast, our lattice GUP predicts the red curve. This yields a finite end-temperature

$$\Theta_{\max} = \frac{1}{2\pi\zeta}$$

with a zero-mass remnant.

**The analysis of the short wave limit fully confirms the previous result.**

## Conclusions

- We derived the GUP on a cubic lattice.
- **Cubic-Lattice GUP allows for "classical" behavior at energies near the border of the Brillouin zone (Planck energies).**
- We derived a new mass-temperature relation for Schwarzschild (micro) black holes.
- Phenomenological consequences of this formula are:
- The final Hawking temperature of a decaying micro black hole remains finite, in contrast to the infinite temperature of the standard result obtained by Heisenberg's uncertainty principle.
- **The final mass of the evaporation process is ZERO, → NO massive black hole remnants.**
- **Avenues for Future Investigations**
- Flexible Lattice
- WHAT KIND OF LATTICE (microstructure) IS REQUIRED IN ORDER TO OBTAIN A STRINGY GUP? (i.e. related to the existence of BH REMNANTS?)

October 14th to 18th, 2012  
Taipei, Taiwan

International Workshop

# Horizons of Quantum Physics

## from Foundations to Quantum-Enabled Technologies

### KEYNOTE SPEAKERS



**JACOB BEKENSTEIN**  
Hebrew University of Jerusalem  
Jerusalem, Israel  
Wolf Prize 2012



**GERARD 'T HOOFT**  
Universiteit Utrecht  
Utrecht, the Netherlands  
Nobel Prize for Physics 1999



**HAGEN KLEINERT**  
Freie Universität Berlin  
Berlin, Germany  
Max Born Prize 2008



**WILLIAM UNRUH**  
University of British Columbia  
Vancouver, Canada  
Rutherford Memorial Medal 1982



**ANTON ZEILINGER**  
Universität Wien  
Vienna, Austria  
Wolf Prize 2010

From its very early years, Quantum Physics has puzzled the minds of human beings. Echoes of heated debates among founding fathers, from Einstein and Bohr, Schrödinger, to Dirac and Heisenberg are still inspiring discussions today.

With time passing, the strength of the explanatory power of Quantum Theory has manifested in any sector of theoretical or applied physics, in chemistry, even in biology, opening windows on possibilities otherwise simply unthinkable.

It has also inspired an unprecedented growth in a world market of devices that since the late 1980s to today relies heavily on working principles based on quantum effects (microprocessors, lasers, special glasses, DVD, computers...), many of which have been the fortune of IT industry and especially Taiwanese society.

Nevertheless, the debate on the logical and physical foundations of Quantum Theory has never faded: pillar concepts like entropy, information, space-time, gravity, wave function collapse, entanglement, and holography have still seen in recent years relevant discussions on their deep meaning and their relationship. Foundational aspects questions find today the most concrete answers and a new impetus with the very possibility to design and build quantum devices.

In particular, the last two decades have witnessed the appearance of Quantum Information as a Science, and the passage to revolutionary how we handle information, while a recent, fortunate convergence of technologies and experimental techniques has made building quantum computers and quantum simulators a very real possibility. These quantum devices will not only allow us to perform tasks not amenable to classical computers, but also enable to probe more deeply into the quantum world, opening up perspectives and pathways into the understanding of foundations of modern physics.

Our workshop sits at the crossroads of cutting edge Theoretical Quantum Physics, Quantum Information Processing and Industrial Research. Its scope is to create a top level environment for discussion and confrontation where a common language between theorists, experimental physicists and technologists might start to develop, with the ultimate goal to explore and make the fascinating knowledge that will shape the world of tomorrow.

### INVITED SPEAKERS

- JEREMY O'BRIEN**  
University of Bristol, UK
- LAJOS DIOSI**  
Wigner Research Centre for Physics, Hungary
- HSI-SHERM GOAN**  
National Taiwan University, Taiwan
- THOMAS BREKENHOF**  
University of Waterloo, Canada

- ACHIN KOPPE**  
Protonic Institute, Canada
- GERALD MILBERG**  
University of Queensland, Australia
- KEIJI OHMORI**  
National Institutes of Natural Sciences, Japan
- JONATHAN OPPENHEIM**  
University College London, UK
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Tsinghua University, China

- ROBERTA RAMPODI**  
Politecnico di Milano, Italy
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Center for Quantum Technologies, Singapore
- FABIO SCARDIOLI**  
Accademia Sinica, Taiwan
- DANIEL TERNI**  
Monash University, Australia

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- SAM LEE** HP, Senior Manager, Office of the CTO
- MATTEO NERAPPÀ** Academia Sinica, National Taiwan University
- FABIO SCARDIOLI** Academia Sinica

### INFORMATION

#### DEADLINES

- Early Bird: 1<sup>st</sup> September 2012
- Registration: 1<sup>st</sup> October 2012
- Poster Submission: 1<sup>st</sup> October 2012

Website: [www.quantumhorizons.org](http://www.quantumhorizons.org)



PHONE

#### CONFERENCE TOPICS

- Foundations of Quantum Physics
- Quantum Physics and Gravity
- Entropy, Information and Holography
- Relativistic Quantum Information Theory
- Experimental Quantum Information
- Nonlocality
- Quantum Optics and Photonics
- Quantum Simulation and Quantum Algorithms