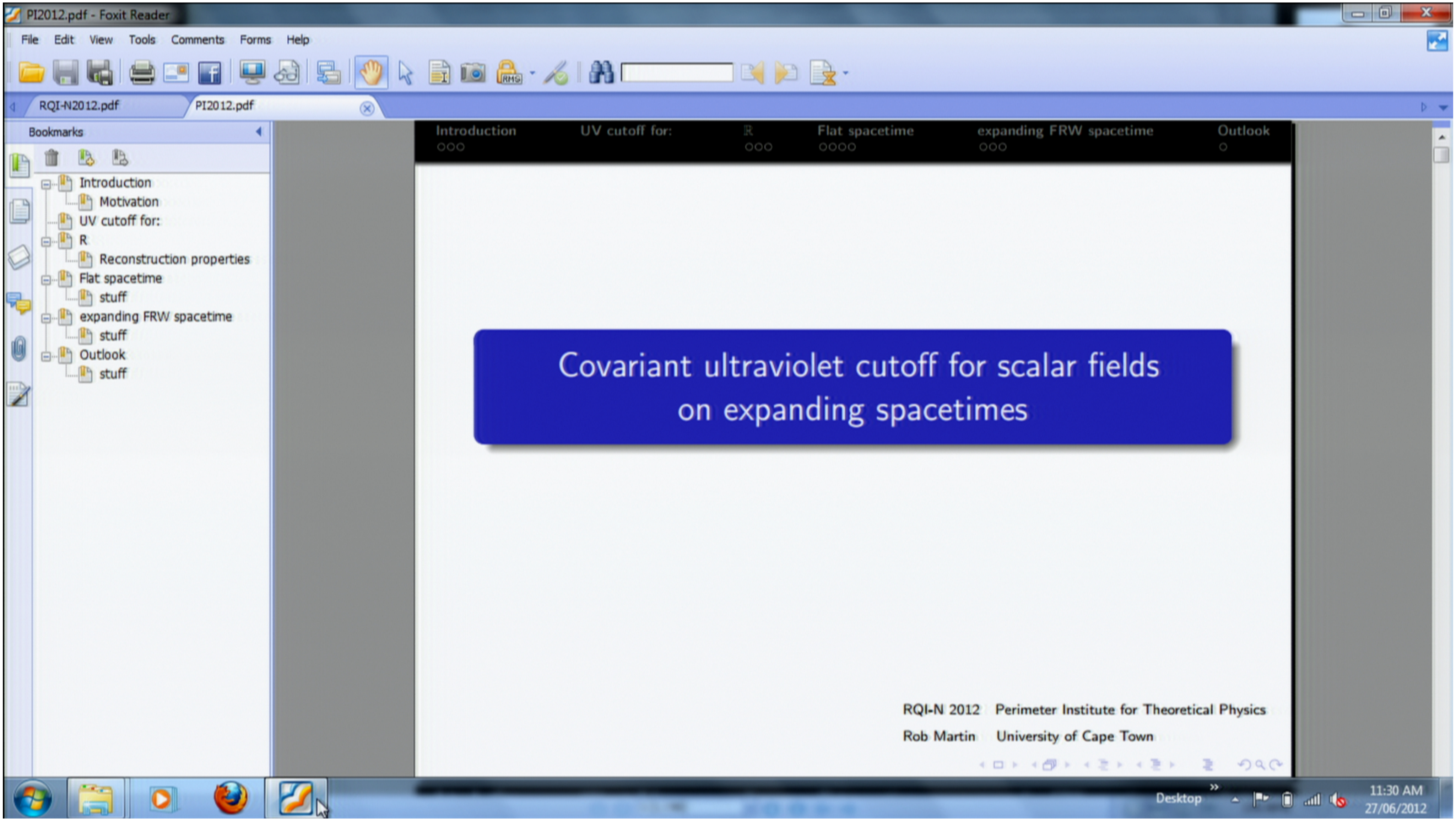
Title: A Fully Covariant Information Theoretic Ultraviolet Cutoff for Fields on Expanding FRW Spacetimes

Date: Jun 27, 2012 11:30 AM

URL: http://pirsa.org/12060060

Abstract: A covariant ultra-violet cutoff on the modes of physical fields on a given space-time can be achieved by cutting off the spectrum of the D'Alembertian of the manifold. This cutoff is a natural generalization of the naive ultra-violet cutoff inEuclidean space which is obtained by simply projecting out frequencies greater in magnitude than a given maximum frequency. Here it is shown that for flat spacetime and expanding FRW spacetimes thiscutoff manifests itself as a decrease in temporal degrees of freedom for large spatial modes. In a large class of expanding FRW spacetimes where the proper time co-ordinate ends at a finite value, it is shown how the numberof temporal degrees of freedom of a fixed spatial mode depends on the magnitude of the spatial mode. We further indicate how the effects of this ultra-violet cutoff on the dynamics of field theories can be studied, and how the resulting modifications to inflationary predictions of the CMB spectrum could be calculated. This talk is based on ongoing joint work with Prof. Achim Kempf (University of Waterloo) and Aidan Chatwin-Davies (UW).


PI2012.pdf - Foxit Reader
File Edit View Tools Comments Forms Help
RQI-N2012.pdf
PI2012.pdf
Bookmarks
Introduction
Motivation
UV cutoff for:
R
Reconstruction properties
Flat spacetime
stuff
expanding FRW spacetime
stuff
Outlook
stuff
Introduction UV cutoff for: ℝ Flat spacetime expanding FRW spacetime Outlook
Covariant ultraviolet cutoff for scalar fields on expanding spacetimes
RQI-N 2012 Perimeter Institute for Theoretical Physics
Rob Martin University of Cape Town
Desktop
11:30 AM
27/06/2012

Outline

In this talk:

- A covariant ultraviolet (UV) cutoff for physical scalar fields on spacetimes

Outline

In this talk:

- A covariant ultraviolet (UV) cutoff for physical scalar fields on spacetimes
- We will apply this to fields on expanding FRW spacetimes and discuss some of its basic consequences.

For intuition we will first discuss this UV cutoff for the simple cases:

1. $\mathbb{R}$ - the real line
2. Minkowski spacetime.

Motivation

Motivation:

- A cutoff on the degrees of freedom of large modes of physical fields in nature is generally expected.

Motivation

Motivation:

- A cutoff on the degrees of freedom of large modes of physical fields in nature is generally expected.
- Laws of physics are independent of co-ordinate system - such an ultraviolet (UV) cutoff must be covariant.
    - There have been many attempts to introduce a 'smallest length' or UV cutoff in the literature, most are not covariant.

An ultra-violet cutoff on $\mathbb{R}$

UV cutoff for scalar fields on $\mathbb{R}$:

$$\text{Project onto } B(A) := \mathcal{F}^{-1}L^2[-A, A]$$

$$\mathcal{F} := \text{Fourier transform, } A > 0.$$

An ultra-violet cutoff on $\mathbb{R}$

UV cutoff for scalar fields on $\mathbb{R}$:

$$\text{Project onto } B(A) := \mathcal{F}^{-1}L^2[-A, A]$$

$$\mathcal{F} := \text{Fourier transform, } A > 0.$$

- $B(A)$ = the set of all fields which have no frequencies $> A$ in magnitude.
- Any $f \in B(A)$ is called A–bandlimited.

$$f(x) = \int_{-A}^{A} F(w)e^{iwx}dw, \quad F \in L^2[-A, A]. \qquad (1)$$

An ultra-violet cutoff on $\mathbb{R}$

UV cutoff for scalar fields on $\mathbb{R}$:

$$\text{Project onto } B(A) := \mathcal{F}^{-1}L^2[-A, A]$$

$$\mathcal{F} := \text{Fourier transform}, \; A > 0.$$

- $B(A)$ = the set of all fields which have no frequencies $> A$ in magnitude.
- Any $f \in B(A)$ is called $A-$bandlimited.

$$f(x) = \int_{-A}^{A} F(w)e^{iwx}dw, \quad F \in L^2[-A, A]. \tag{1}$$

- There is a natural generalization to manifolds, but first: properties of $B(A)$.

Bandlimited functions on $\mathbb{R}$

Any A–bandlimited function has a finite density of degrees of freedom:

- *e.g.* if $t_n := \frac{n\pi}{A}$

$$f \in B(A) \;\Rightarrow\; f(t) = \sum_{n\in\mathbb{Z}} f(t_n)\frac{\sin\left(A(t-t_n)\right)}{A(t-t_n)}. \quad (2)$$

Bandlimited functions on $\mathbb{R}$

Any $A-$bandlimited function has a finite density of degrees of freedom:

- *e.g.* if $t_n := \frac{n\pi}{A}$

$$f \in B(A) \;\Rightarrow\; f(t) = \sum_{n\in\mathbb{Z}} f(t_n)\frac{\sin\left(A(t-t_n)\right)}{A(t-t_n)}. \tag{2}$$

- More generally: if $\Lambda := (t_n) \subset \mathbb{R}$ is strictly increasing and 'dense enough' then any $f \in B(A)$ is perfectly reconstructible from the 'samples' $\{f(t_n)\}$.

Bandlimited functions on $\mathbb{R}$

Any A−bandlimited function has a finite density of degrees of freedom:

- *e.g.* if $t_n := \frac{n\pi}{A}$

$$f \in B(A) \;\Rightarrow\; f(t) = \sum_{n\in\mathbb{Z}} f(t_n) \frac{\sin\left(A(t-t_n)\right)}{A(t-t_n)}. \quad (2)$$

- More generally: if $\Lambda := (t_n) \subset \mathbb{R}$ is strictly increasing and 'dense enough' then any $f \in B(A)$ is perfectly reconstructible from the 'samples' $\{f(t_n)\}$.
  - Such Λ called a set of sampling for $B(A)$.

Reconstruction properties

More precisely:

- Let $\Lambda := (t_n) \subset \mathbb{R}$ be a strictly increasing sequence.
- Beurling density:
    - $n(r)$:= minimum number of points of Λ in any subinterval of length r.
    - $D(\Lambda) := lim_{r\to\infty} \frac{n(r)}{r}$.

Bandlimited functions on $\mathbb{R}$

Any $A-$bandlimited function has a finite density of degrees of freedom:

- *e.g.* if $t_n := \frac{n\pi}{A}$

$$f \in B(A) \;\Rightarrow\; f(t) = \sum_{n\in\mathbb{Z}} f(t_n)\frac{\sin\left(A(t-t_n)\right)}{A(t-t_n)}. \qquad (2)$$

- More generally: if $\Lambda := (t_n) \subset \mathbb{R}$ is strictly increasing and 'dense enough' then any $f \in B(A)$ is perfectly reconstructible from the 'samples' $\{f(t_n)\}$.
  - Such Λ called a set of sampling for $B(A)$.

Reconstruction properties

More precisely:

- Let $\Lambda := (t_n) \subset \mathbb{R}$ be a strictly increasing sequence.
- Beurling density:
  - $n(r)$:= minimum number of points of Λ in any subinterval of length r.
  - $D(\Lambda) := lim_{r\to\infty} \frac{n(r)}{r}$.

Reconstruction properties

More precisely:

- Let $\Lambda := (t_n) \subset \mathbb{R}$ be a strictly increasing sequence.
- Beurling density:
  - $n(r) :=$ minimum number of points of Λ in any subinterval of length r.
  - $D(\Lambda) := lim_{r\to\infty} \frac{n(r)}{r}$.

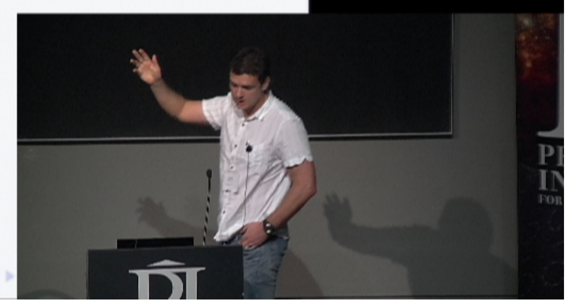
Theorem (Beurling)

1. *If Λ is sampling then $D(\Lambda) \geq \frac{A}{\pi}$.*
2. *Λ is sampling for $B(A)$ if $D(\Lambda) > \frac{A}{\pi}$.*

Frequency limited functions

More generally if $K \subset \mathbb{R}$ is compact:

- $B(K) := \mathcal{F}^{-1}L^2(K)$ frequency limited functions.



Frequency limited functions

More generally if $K \subset \mathbb{R}$ is compact:

- $B(K) := \mathcal{F}^{-1}L^2(K)$ frequency limited functions.
- Necessity part of previous theorem generalizes:

Theorem (Landau)

If Λ is a set of sampling for $B(K)$ then $D(\Lambda) \geq \frac{m(K)}{(2\pi)}$.

$m :=$ Lebesgue measure.

Frequency limited functions

More generally if $K \subset \mathbb{R}$ is compact:

- $B(K) := \mathcal{F}^{-1}L^2(K)$ frequency limited functions.
- Necessity part of previous theorem generalizes:

Theorem (Landau)

If Λ is a set of sampling for $B(K)$ then $D(\Lambda) \geq \frac{m(K)}{(2\pi)}$.

$m :=$ Lebesgue measure.

- The minimum density of a set of sampling is proportional to the bandwidth volume $m(K)$.

Covariant UV cutoff

Q: How to generalize this UV cutoff to fields on manifolds in a covariant way?

- Note: If $\Delta := \frac{-d^2}{dx^2}$ is the Laplacian on $\mathbb{R}$, then:
  - eigenfunctions: $\Delta e^{iwx} = w^2 e^{iwx}$

Covariant UV cutoff

Q: How to generalize this UV cutoff to fields on manifolds in a covariant way?

- Note: If $\Delta := \frac{-d^2}{dx^2}$ is the Laplacian on $\mathbb{R}$, then:
  - eigenfunctions: $\Delta e^{iwx} = w^2 e^{iwx}$
  - $B(A) :=$ subspace spanned by e^{iwx} for $|w|^2 \leq A^2$.

$$f \in B(A) \quad \Leftrightarrow \quad f(x) = \int_{-A}^{A} F(w) e^{iwx} dw. \tag{3}$$

Covariant UV cutoff

Q: How to generalize this UV cutoff to fields on manifolds in a covariant way?

- Note: If $\Delta := \frac{-d^2}{dx^2}$ is the Laplacian on $\mathbb{R}$, then:
  - eigenfunctions: $\Delta e^{iwx} = w^2 e^{iwx}$
  - $B(A) :=$ subspace spanned by e^{iwx} for $|w|^2 \leq A^2$.

$$f \in B(A) \quad \Leftrightarrow \quad f(x) = \int_{-A}^{A} F(w) e^{iwx} dw. \qquad (3)$$

- Definition: Given a manifold M with D'Alembertian $\Box$, $B(M, A) :=$ subspace of $L^2(M)$ spanned by eigenfunctions of $\Box$ with eigenvalues in $[-A^2, A^2]$.

Covariant UV cutoff

- Conjecture: The set of physical fields on a given spacetime manifold M is $B(M, A)$ for some $A > 0$ (on the order of the Planck frequency).

Covariant UV cutoff

- Conjecture: The set of physical fields on a given spacetime manifold M is $B(M, A)$ for some $A > 0$ (on the order of the Planck frequency).
  - This is a fully covariant UV cutoff.

Covariant UV cutoff

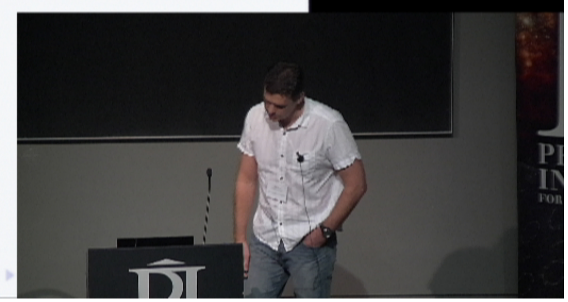
- Conjecture: The set of physical fields on a given spacetime manifold M is $B(M, A)$ for some $A > 0$ (on the order of the Planck frequency).
  - This is a fully covariant UV cutoff.
  - We will now investigate some of the basic sampling/reconstruction properties of $B(M, A)$.

Covariant UV cutoff

- Conjecture: The set of physical fields on a given spacetime manifold M is $B(M, A)$ for some $A > 0$ (on the order of the Planck frequency).
  - This is a fully covariant UV cutoff.
  - We will now investigate some of the basic sampling/reconstruction properties of $B(M, A)$.

Flat Spacetime:

- Let $M := 1 + 3\text{D}$ flat spacetime, $\Box := -\frac{d^2}{dt^2} + \Delta$,



Covariant UV cutoff

- Conjecture: The set of physical fields on a given spacetime manifold M is $B(M, A)$ for some $A > 0$ (on the order of the Planck frequency).
  - This is a fully covariant UV cutoff.
  - We will now investigate some of the basic sampling/reconstruction properties of $B(M, A)$.

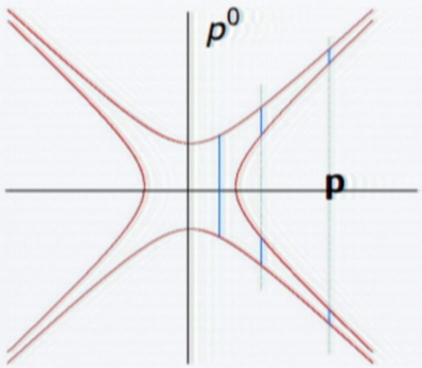
Flat Spacetime:

- Let $M := 1 + 3D$ flat spacetime, $\Box := -\frac{d^2}{dt^2} + \Delta$,
  - $B(M, A)$:=set of all fields whose frequencies obey:

$$|(p^0)^2 - \mathbf{p}^2| \leq A^2$$

Flat spacetime: Sampling of fixed spatial modes

$$|(p^0)^2 - \mathbf{p}^2| \leq A^2$$

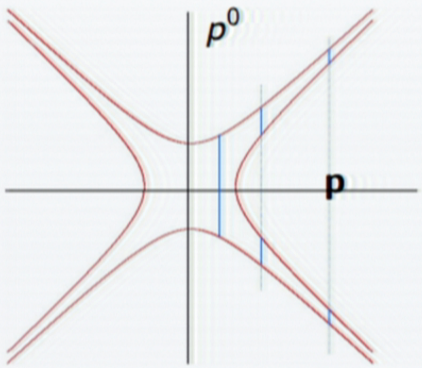


Let $\varphi_{\mathbf{p}}(t)$:=spatial Fourier transform of $\phi(t, x) \in B(M, A)$ for fixed spatial mode $\mathbf{p}$.

$$\varphi_{\mathbf{p}}(t) := \int_{\mathbb{R}^3} e^{i\mathbf{p}\cdot\mathbf{x}} \phi(t, \mathbf{x}) d\mathbf{x}.$$

Flat spacetime: Sampling of fixed spatial modes

$$|(p^0)^2 - \mathbf{p}^2| \leq A^2$$



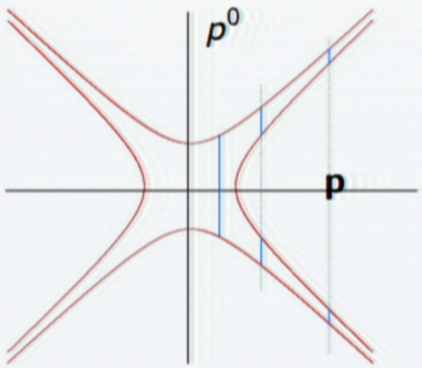
Let $\varphi_{\mathbf{p}}(t)$:=spatial Fourier transform of $\phi(t,x) \in B(M,A)$ for fixed spatial mode **p**.

$$\varphi_{\mathbf{p}}(t) := \int_{\mathbb{R}^3} e^{i\mathbf{p}\cdot\mathbf{x}} \phi(t,\mathbf{x}) d\mathbf{x}.$$

- $\varphi_{\mathbf{p}}$ has finite temporal bandwidth volume $V(p)$ ($p := |\mathbf{p}|$)

Flat spacetime: Sampling of fixed spatial modes

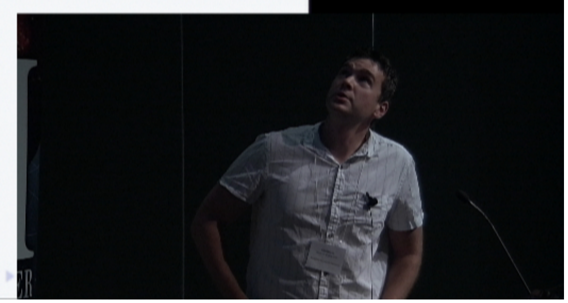
$$|(p^0)^2 - \mathbf{p}^2| \leq A^2$$



Let $\varphi_{\mathbf{p}}(t)$:=spatial Fourier transform of $\phi(t,x) \in B(M,A)$ for fixed spatial mode $\mathbf{p}$.

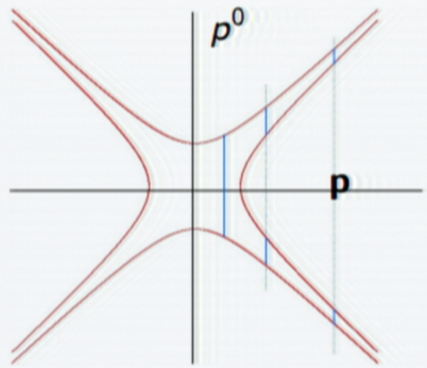
$$\varphi_{\mathbf{p}}(t) := \int_{\mathbb{R}^3} e^{i\mathbf{p}\cdot\mathbf{x}} \phi(t,\mathbf{x}) d\mathbf{x}.$$

- $\varphi_{\mathbf{p}}$ has finite temporal bandwidth volume $V(p)$ ($p := |\mathbf{p}|$)
- can be reconstructed from samples taken on some $\Lambda = (t_n)$ with finite density.



Flat spacetime: Sampling of fixed spatial modes

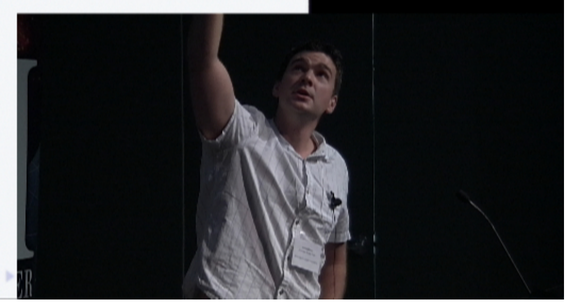
$$|(p^0)^2 - \mathbf{p}^2| \leq A^2$$



Let $\varphi_{\mathbf{p}}(t)$:=spatial Fourier transform of $\phi(t,x) \in B(M,A)$ for fixed spatial mode **p**.

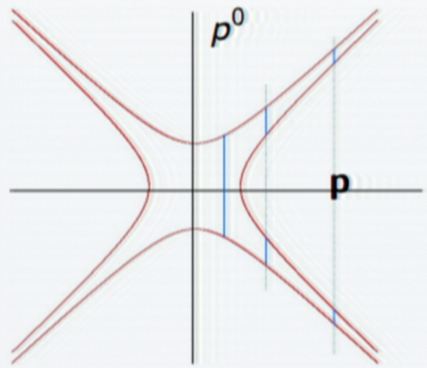
$$\varphi_{\mathbf{p}}(t) := \int_{\mathbb{R}^3} e^{i\mathbf{p}\cdot\mathbf{x}} \phi(t,\mathbf{x}) d\mathbf{x}.$$

- $\varphi_{\mathbf{p}}$ has finite temporal bandwidth volume $V(p)$ ($p := |\mathbf{p}|$)
- can be reconstructed from samples taken on some $\Lambda = (t_n)$ with finite density.
- $V(p)$ vanishes as $p \to \infty$.



Flat spacetime: Sampling of fixed spatial modes

$$|(p^0)^2 - \mathbf{p}^2| \leq A^2$$



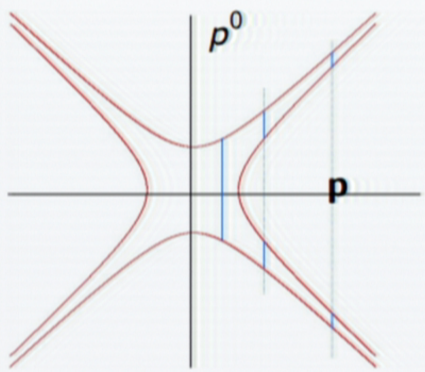
Let $\varphi_{\mathbf{p}}(t)$:=spatial Fourier transform of $\phi(t,x) \in B(M,A)$ for fixed spatial mode $\mathbf{p}$.

$$\varphi_{\mathbf{p}}(t) := \int_{\mathbb{R}^3} e^{i\mathbf{p}\cdot\mathbf{x}}\phi(t,\mathbf{x})d\mathbf{x}.$$

- $\varphi_{\mathbf{p}}$ has finite temporal bandwidth volume $V(p)$ ($p := |\mathbf{p}|$)
- can be reconstructed from samples taken on some $\Lambda = (t_n)$ with finite density.
- $V(p)$ vanishes as $p \to \infty$.
- Fixed spatial modes have a finite density of degrees of freedom in time

Flat spacetime: fixed spatial modes II

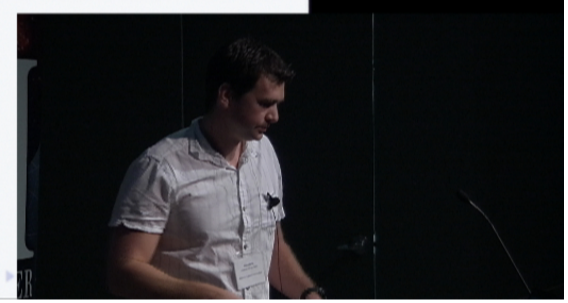
$$|(p^0)^2 - \mathbf{p}^2| \leq A^2$$



Let $\varphi_{\mathbf{p}}(t)$:=spatial Fourier transform of $\phi(t,x) \in B(M,A)$ for fixed spatial mode $\mathbf{p}$.

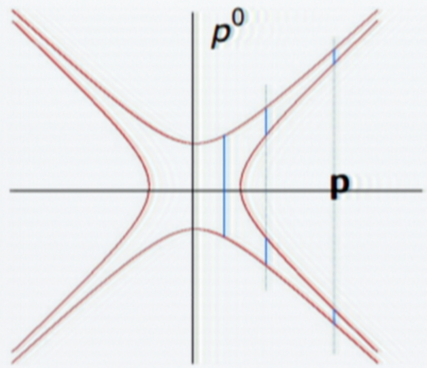
$$\varphi_{\mathbf{p}}(t) := \int_{\mathbb{R}^3} e^{i\mathbf{p}\cdot\mathbf{x}}\phi(t,\mathbf{x})d\mathbf{x}.$$

- can be reconstructed from samples taken on some $\Lambda = (t_n)$ with finite density.
- This is consistent with Lorentz invariance:
- Inertial observer with rest frame $(t', \mathbf{x}')$:
  - sees $\Lambda = (t'_n)$ as having lower density due to time dilation



Flat spacetime: fixed spatial modes II

$$|(p^0)^2 - \mathbf{p}^2| \leq A^2$$



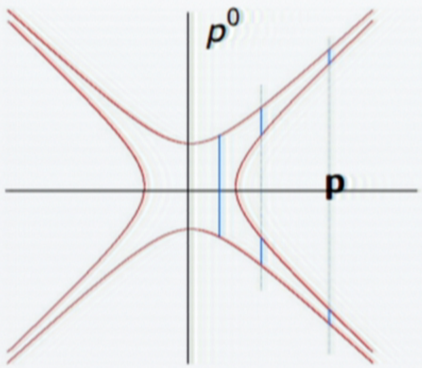
Let $\varphi_{\mathbf{p}}(t)$:=spatial Fourier transform of $\phi(t,x) \in B(M,A)$ for fixed spatial mode $\mathbf{p}$.

$$\varphi_{\mathbf{p}}(t) := \int_{\mathbb{R}^3} e^{i\mathbf{p}\cdot\mathbf{x}} \phi(t,\mathbf{x}) d\mathbf{x}.$$

- can be reconstructed from samples taken on some $\Lambda = (t_n)$ with finite density.
- This is consistent with Lorentz invariance:
- Inertial observer with rest frame $(t', \mathbf{x}')$:
  - sees $\Lambda = (t'_n)$ as having lower density due to time dilation
  - also sees $\varphi_{\mathbf{p}'}$ as having higher spatial frequency $\mathbf{p}'$ by length contraction.

Flat spacetime: fixed spatial modes II

$|(p^0)^2 - \mathbf{p}^2| \leq A^2$



Let $\varphi_{\mathbf{p}}(t)$:=spatial Fourier transform of $\phi(t,x) \in B(M,A)$ for fixed spatial mode **p**.

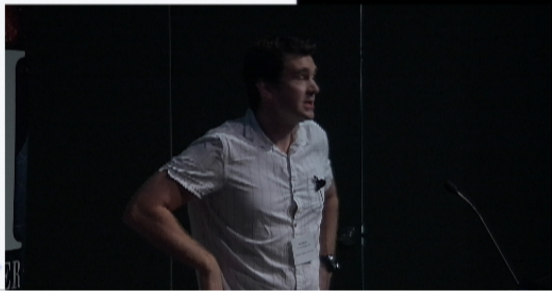
$$\varphi_{\mathbf{p}}(t) := \int_{\mathbb{R}^3} e^{i\mathbf{p}\cdot\mathbf{x}} \phi(t,\mathbf{x}) d\mathbf{x}.$$

- can be reconstructed from samples taken on some $\Lambda = (t_n)$ with finite density.
- This is consistent with Lorentz invariance:
- Inertial observer with rest frame $(t', \mathbf{x}')$:
  - sees $\Lambda = (t'_n)$ as having lower density due to time dilation
  - also sees $\varphi_{\mathbf{p}'}$ as having higher spatial frequency $\mathbf{p}'$ by length contraction.
  - Lower density of (t'_n) suffices as necessary density is proportional to $V(\mathbf{p}') < V(\mathbf{p})$.

UV cutoff in expanding FRW spacetimes

Let $M :=$ a $1+3$D expanding FRW spacetime with scale factor $a(t)$.

$$ds^2 := -dt^2 + a^2(t)d\mathbf{x}^2 \qquad \Box := a^{-3}(t)\left(\frac{d}{dt}a^3(t)\frac{d}{dt} - a(t)\Delta\right)$$

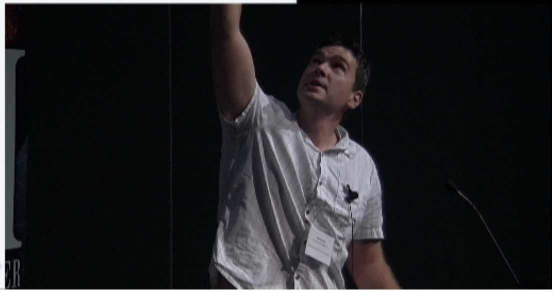


UV cutoff in expanding FRW spacetimes

Let $M :=$ a $1+3$D expanding FRW spacetime with scale factor $a(t)$.

$$ds^2 := -dt^2 + a^2(t)d\mathbf{x}^2 \qquad \Box := a^{-3}(t)\left(\frac{d}{dt}a^3(t)\frac{d}{dt} - a(t)\Delta\right)$$

- Given $\phi \in B(M, A)$, $\varphi_k :=$ be the spatial Fourier transform of ϕ for fixed co-moving momentum $\mathbf{k}$.
  - $B_k(M, A) :=$ the space of all fixed co-moving spatial modes $\varphi_{\mathbf{k}}$ of fields $\phi \in B(M, A)$ where $k = |\mathbf{k}|$.



UV cutoff in expanding FRW spacetimes

Let $M :=$ a $1+3$D expanding FRW spacetime with scale factor $a(t)$.

$$ds^2 := -dt^2 + a^2(t)d\mathbf{x}^2 \qquad \Box := a^{-3}(t)\left(\frac{d}{dt}a^3(t)\frac{d}{dt} - a(t)\Delta\right)$$

- Given $\phi \in B(M,A)$, $\varphi_k :=$ be the spatial Fourier transform of ϕ for fixed co-moving momentum $\mathbf{k}$.
  - $B_k(M,A) :=$ the space of all fixed co-moving spatial modes $\varphi_{\mathbf{k}}$ of fields $\phi \in B(M,A)$ where $k = |\mathbf{k}|$.
- After spatial Fourier transform: $\Box_k := a^{-3}(t)\left(\frac{d}{dt}a^3(t)\frac{d}{dt} + a^2(t)k^2\right)$ acting on $L^2\left([t_i,t_f]; a^3(t)dt\right)$, for any fixed spatial mode $k := |\mathbf{k}|$.

UV cutoff in expanding FRW spacetimes

Let $M :=$ a $1+3$D expanding FRW spacetime with scale factor $a(t)$.

$$ds^2 := -dt^2 + a^2(t)d\mathbf{x}^2 \qquad \Box := a^{-3}(t)\left(\frac{d}{dt}a^3(t)\frac{d}{dt} - a(t)\Delta\right)$$

- Given $\phi \in B(M,A)$, $\varphi_k :=$ be the spatial Fourier transform of ϕ for fixed co-moving momentum $\mathbf{k}$.
  - $B_k(M,A) :=$ the space of all fixed co-moving spatial modes $\varphi_\mathbf{k}$ of fields $\phi \in B(M,A)$ where $k = |\mathbf{k}|$.
- After spatial Fourier transform: $\Box_k := a^{-3}(t)\left(\frac{d}{dt}a^3(t)\frac{d}{dt} + a^2(t)k^2\right)$ acting on $L^2\left([t_i, t_f]; a^3(t)dt\right)$, for any fixed spatial mode $k := |\mathbf{k}|$.
- Studying the spaces $B_k(M,A)$ amounts to studying the Sturm-Liouville differential operators $\Box_k$.

Fixed Assumptions

$M :=$ expanding FRW spacetime $\quad t \in [t_i, t_f]$

Assumptions:

1. $t_f < \infty$ finite end-time
2. $a(t) > 0$ a.e. is differentiable.
3. One of the following two conditions holds
   - $t_i > -\infty$
   - $\int_{-\infty}^{t_f} a(t)dt < \infty$. *e.g.* de Sitter $a(t) = e^{Ht}$.

Fixed Assumptions

$M :=$ expanding FRW spacetime $\quad t \in [t_i, t_f]$

Assumptions:

1. $t_f < \infty$ finite end-time
2. $a(t) > 0$ a.e. is differentiable.
3. One of the following two conditions holds
   - $t_i > -\infty$
   - $\int_{-\infty}^{t_f} a(t)dt < \infty$. *e.g.* de Sitter $a(t) = e^{Ht}$.

These imply $\varphi_{\mathbf{k}}$ has a **finite** number N_k of degrees of freedom in time, $k := |\mathbf{k}|$.

- *i.e.* $\dim(B_k(M, A)) = N_k$.

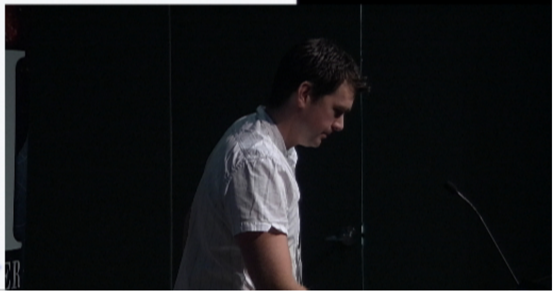
Spatial mode dependence of N_k

Behaviour of $N_k = \dim(B_k(M,A))$ (the number of temporal degrees of freedom of a fixed spatial co-moving mode of a field obeying our UV cutoff).

- there is a value $K > 0$ such that $0 \leq N_k = c \leq 2$ is constant for all $k \geq K$.
  - *i.e.* As in the case of flat spacetime we see that larger spatial modes have fewer degrees of freedom in time.

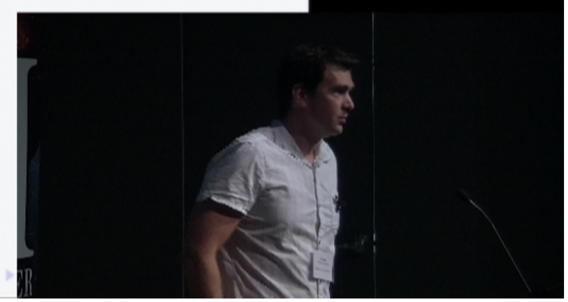
Outlook

- We have discussed properties of scalar fields on expanding FRW spacetimes which obey a certain covariant UV cutoff
    - Large fixed spatial comoving modes have fewer temporal degrees of freedom, and are 'frozen out'.



Outlook

- We have discussed properties of scalar fields on expanding FRW spacetimes which obey a certain covariant UV cutoff
  - Large fixed spatial comoving modes have fewer temporal degrees of freedom, and are 'frozen out'.
- Q: How does this cutoff affect the dynamics of scalar field theories? Predictions for the CMB spectrum ?



Outlook

- We have discussed properties of scalar fields on expanding FRW spacetimes which obey a certain covariant UV cutoff
  - Large fixed spatial comoving modes have fewer temporal degrees of freedom, and are 'frozen out'.
- Q: How does this cutoff affect the dynamics of scalar field theories? Predictions for the CMB spectrum ?
  - This is now being calculated with Aidan Chatwin-Davies and Achim Kempf at the University of Waterloo.