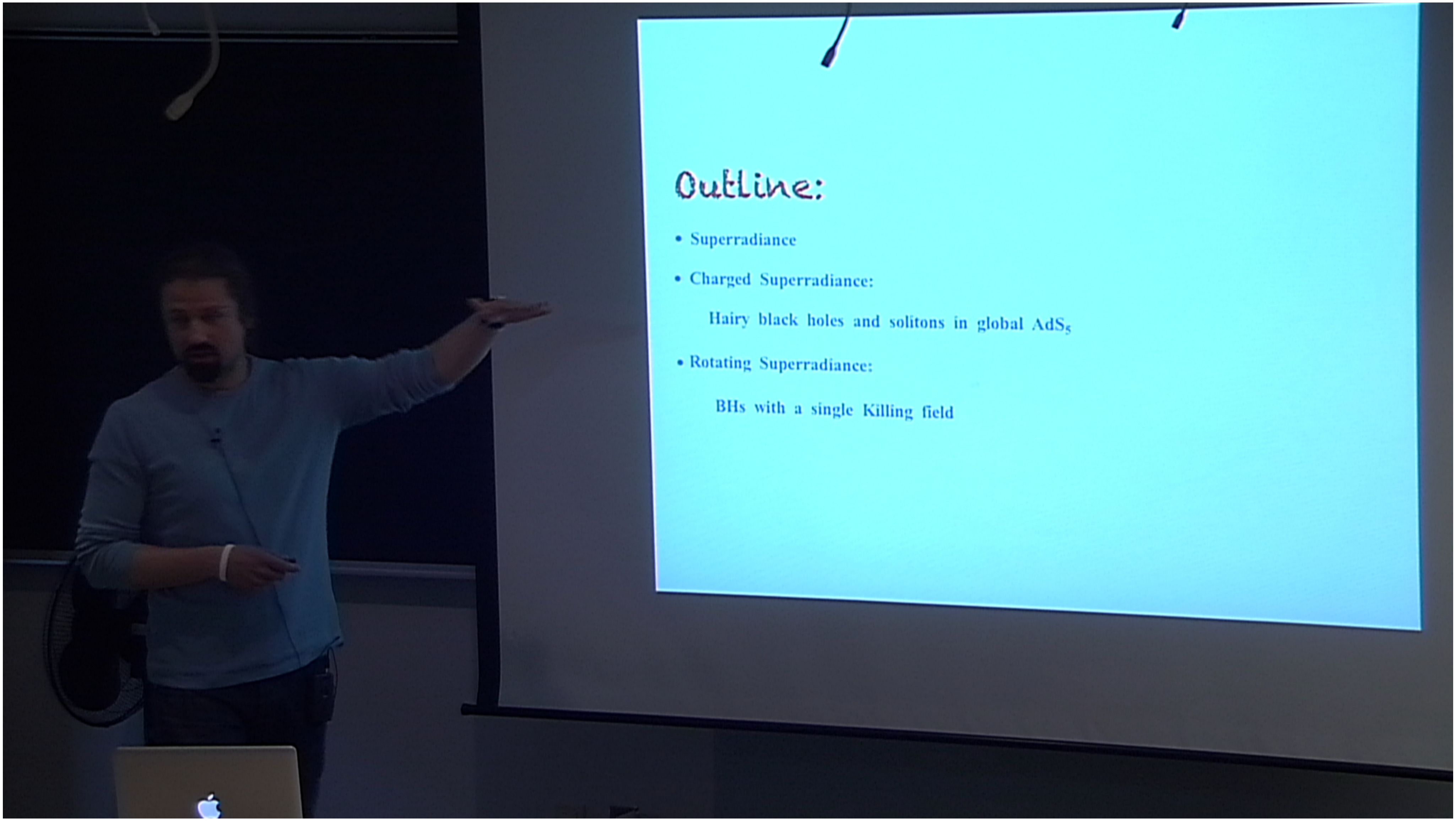
Title: Superradiance and Black Holes with a Single Killing Field

Date: Jun 06, 2012 09:00 AM

URL: http://pirsa.org/12060010

Abstract: It is well known that superradiance can extract energy from a black hole and, in an asymptotically global AdS background, it drives the black hole unstable. The onset of superradiance also signals a bifurcation to a new family of AdS black holes in a phase diagram of stationary solutions. We construct non-linearly the hairy black holes, solitons and boson stars associated to scalar superradiance. We present both charged and rotating solutions with scalar hair. In the charged case, the structure of phase diagram varies considerably, depending on the charge of the condensate. In the rotating case, the hairy solutions give the first examples of black holes with only a Killing field: the black holes are neither stationary nor axisymmetric, but are invariant under a single Killing field which is tangent to the null generators of the horizon.
We discuss the role of these solutions in a full time evolution of the superradiant instability. We emphasize how scarce is our knowledge of the rotating superradiant instability endpoint, and that this instability will compete with the turbulent instability of AdS.


Outline:
• Superradiance
• Charged Superradiance:
Hairy black holes and solitons in global AdS5
• Rotating Superradiance:
BHs with a single Killing field

Superradiance & Black holes with a single Killing field

Óscar Dias

energie atomique • energies alternatives

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CEA - Saclay

Based on: 1105.4167, 1112.4447, 1007.3745

Gary Horowitz, Jorge Santos

Pau Figueras, Shiraz Minwalla, Prahar Mitra, Ricardo Monteiro, Jorge Santos

Ricardo Monteiro, Harvey Reall, Jorge Santos

See also: Gentle, Rangamani, Withers [1112.3979] and Stotyn, Park, McGrath, Mann [1110.2223]

Numerical applications of AdS/CFT,
PI Canada, June 2012

Outline:

- **Superradiance**
- **Charged Superradiance:**

  Hairy black holes and solitons in global AdS_5

- **Rotating Superradiance:**

  BHs with a single Killing field

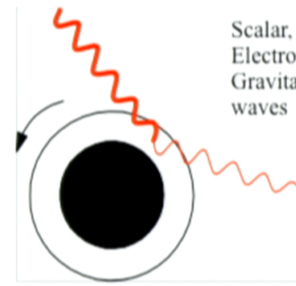
→ Superradiant Scattering

Zel'dovich 71', Storobinsky, 73', Unruh 76'

• **Superradiant scattering** on a **rotating BH** or **charged BH:**

Waves incident upon a BH
with **angular velocity** Ω_H or **chemical potential** μ
are amplified by superradiant scattering if $\omega \leq m\,\Omega_H$ or $\omega \leq e\,\mu$

$$\Phi = F(r,\theta)\, e^{-i\,\omega\, t} e^{i\,m\,\phi}$$



• In the **ergoregion**, Killing vector that defines energy measured by asymptotic observers becomes spacelike.
So, we can have **negative energy excitations** (**absorbed by horizon**) that,
asymptotically look like **positive outward flux**.
Energy extraction occurs classically and **BH spins-down.**

• **Why can we have superradiance only for $\omega \leq m\,\Omega_H$?**

First law applied to emission process from BH with $\delta E = -\omega$ and $\delta J = -m$:

$$\frac{\kappa}{8\pi G}\delta\mathcal{A}_H = -(\omega - m\Omega_H)$$

Superradiance of modes with $\omega > m\,\Omega_H$ would **violate** the **second law of thermodynamics**

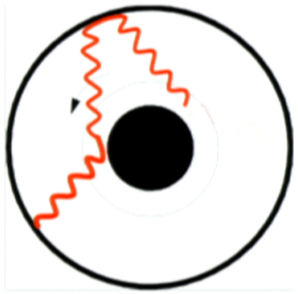
→ Superradiant Instabilities

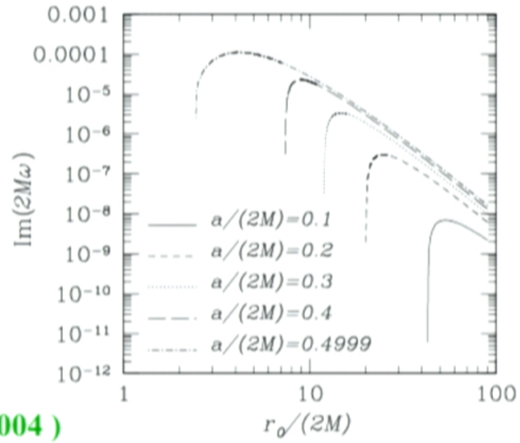
- Insert a ***mirror*** around a **rotating** ***absorbing*** **cylinder:** **(Zel'dovich, 1972)**

  Multiple reflection & amplification → Instability

- Insert a ***mirror*** around a **rotating black hole (BH):**

  Make a **black hole bomb!** **(Press, Teukolsky, 1972)**





(Cardoso, OD, Lemos, Yoshida, 2004)

- **Natural mirrors** around a **rotating** or **charged** BH:

  Global AdS box **(Cardoso, OD, 2004; Uchikata, Yoshida 2010 ...)**

  Massive scalar field **(Detweiller; Dolan; Kodama, Yoshino ...)**

  KK momentum **(Cardoso, Lemos 2005; OD, 2006 ...)**

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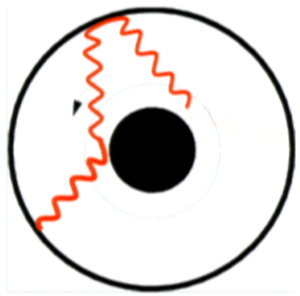

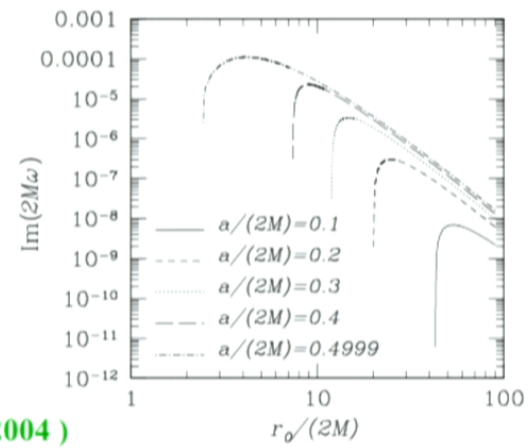

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→ Microscopic description of Superradiance

0712.0791,
OD, Emparan, Maccarrone

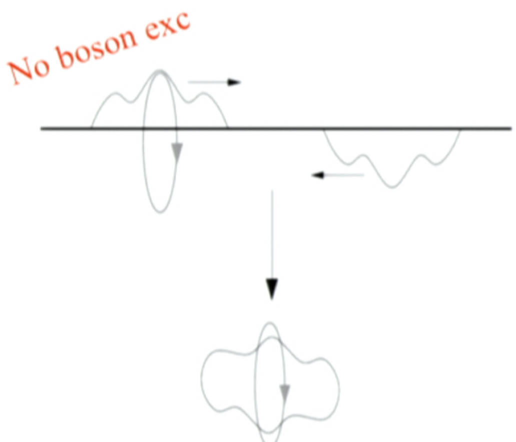
- Superradiant scattering ⟷ ***Stimulated*** emission
- But, semiclassically there is also ***spontaneous*** superradiant emission

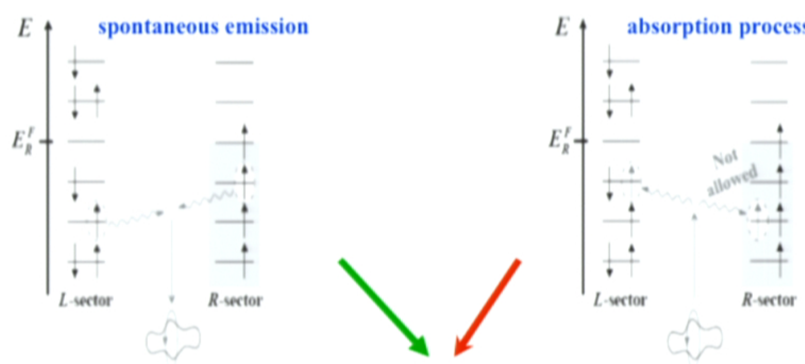
- To isolate superradiance, we need an extremal rotating BH

$T = 0$ ⇒ Absence of Hawking emission
Rotating ⇒ Presence of ergoregion
} Ergo-cold BH with only superradiant emission

- **Rotation** in SUGRA solution ⟷ **Fermionic excitations** charged under R-symmetry group on **CFT**

***L*- sector** is **thermally excited**: provides for the entropy.
***R*-sector** ($T_R \to 0$, $S_R \to 0$) populated by **polarized fermions filling up energy levels** up to the **Fermi level**.
***T* =0 but** L,R-movers are still **available to annihilate** and emmit a closed string to the bulk.





Net result of competition: superradiant scattering

Hairy black holes and solitons in global AdS_5

1112.4447, OD, Figueras, Minwalla, Mitra, Monteiro, Santos
See also 1112.3979, Gentle, Rangamani, Withers

- *AdS* Abelian Higgs model: *AdS* Einstein - Maxwell gravity interacting with a charged massless scalar field

$$S = \frac{1}{8\pi G_5}\int d^5x\sqrt{-g}\left[\frac{1}{2}\left(\mathcal{R}[g]+12\right)-\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}-|D_\mu\phi|^2\right] \qquad D_\mu = \nabla_\mu - ieA_\mu, \quad \ell\equiv 1$$

- Field content: gravity, Maxwell field and a charged complex scalar.

- Static and spherically symmetric solutions:
  expect a three parameter family of solutions parametrized by $\{M,Q,e\}$.

$$ds^2 = -f(r)\,dt^2 + g(r)\,dr^2 + r^2\,d\Omega_{(3)}^2\,, \qquad A_\mu\,dx^\mu = A(r)\,dt\,, \qquad \phi=\phi(r)$$

- AdS Reissner-Nordstrom BH: $E,Q = E,Q\,(R,\mu)$

$$\phi(r)=0,\; f=g$$

$$f(r)=\left(\frac{r^2}{\ell^2}-\frac{R^2}{\ell^2}\right)\left(1+\frac{R^2+\ell^2}{r^2}-\frac{2}{3}\frac{R^2\ell^2\mu^2}{r^4}\right), \quad \text{and} \quad \mathcal{A}_t=\mu\left(1-\frac{R^2}{r^2}\right)$$

Regular extremal limit, with near horizon geometry $AdS_2\times S^3$, with $S_{ext}\neq 0$:

$$0\le\mu\le\mu_{\rm ext} \qquad \text{with} \qquad \mu_{\rm ext}=\sqrt{\frac{3}{2}}\sqrt{1+\frac{2R^2}{\ell^2}}$$

• **AdS Reissner-Nordstrom BH has two instabilities:**

1) Superradiant Instability:

If a wave $e^{-i\omega t}$ scatters off a charged black hole with $\mathbf{0 < \omega \leq e\,\mu}$,

it returns with a larger amplitude: superradiant scattering.

In AdS, the outgoing wave reflects-off infinity: Multiple Superradiance / Reflection leads to instability.

Can we estimate the instability onset?

- The scalar modes that can propagate in RN-AdS, for $R << \ell$,

  are effectively the normal modes of global AdS: $\omega\ell = 4 + 2p$. Lowest mode has $p = 0$.

- On the other hand, small extremal black holes require $\mu \leq \mu_{\rm ext}\big|_{R\to 0} = \sqrt{\frac{3}{2}}$

- Assuming instability 1st appears at extremality, we get the superradiance condition ($\mathbf{0 < \omega \leq e\,\mu}$):

$$\frac{4}{\ell} < e\sqrt{\frac{3}{2}}$$

→ **Arbitrarily *small* extremal RN-AdS black holes are superradiant unstable for** $e^2\ell^2 > \frac{32}{2}$

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2) Near-Horizon scalar condensation instability:

- Consider charged massive scalar field: $\Box\phi - m_s^2\,\phi = 0$

  Normalizable modes → scalar field must obey the BF bound: $m_s^2 \geq m_s^2\big|_{\rm BF} = -\dfrac{(d-1)^2}{4\,\ell^2}$

- Take ***any*** **extreme**, **AdS$_d$ BH** whose **near-horizon** geometry contains an **AdS$_2$** factor w/ radius $l_{\rm AdS_2}$:

  the BF bound associated to this AdS$_2$, $m_s^2\big|_{\rm NH\ BF} = -\dfrac{1}{4\,l^2_{AdS_2}}$, is different from the BF of AdS$_d$

In particular if: $$m_s^2\big|_{\rm BF} \leq m_s^2 \leq m_s^2\big|_{\rm NH\ BF}$$

then the asymptotic **AdS$_d$ space** will be **stable**, but the **near-horizon geometry** is **unstable**.

→ This **suggests** that the **AdS$_d$ BH** will be **unstable to scalar condensation of scalar field !**

→ Confirmed in **1007.3745** , OD, Ricardo Monteiro, Harvey Reall, Jorge Santos,
A scalar field condensation instability of rotating AdS BHs

ANY extreme BH with AdS$_2$ NH geometry has this instability. Includes:

- Charged BHs (e.g. planar RN-AdS (holographic superconductors) where it was 1st found)
- Rotating (uncharged or charged) BHs
- Static and uncharged BHs: hyperbolic Schwarzschild-AdS with spatial horizon topology H^{d-2}

2) Near-Horizon scalar condensation instability:

- Return to the particular RN-AdS case where we **start** with **massless** scalar.
  Linearized eq for charged ϕ on **NH** RN-AdS reduces to eq for a massive scalar with **effective mass:**

$$m_s^2 l_{AdS_2}^2 = -\frac{3e^2R^2}{8}\frac{\ell^2+2R^2}{(\ell^2+3R^2)^2}$$

- AdS_2 is unstable whenever it violates the 2d BF bound: $m_s^2 l_{AdS_2}^2 < -\frac{1}{4}$

$\rightarrow$ extremal RN-AdS is unstable whenever $e^2\ell^2 \geq \frac{2(\ell^2+3R^2)^2}{3R^2(\ell^2+2R^2)}$

- The RHS is a monotonically decreasing function of R. At large R, this reduces to

$$e^2\ell^2 \geq \frac{2(\ell^2+3R^2)^2}{3R^2(\ell^2+2R^2)} \geq 3+\mathcal{O}(\ell^2/R^2)$$

It follows that *large* extremal RN-AdS BHs are unstable when $e^2\ell^2 > 3$

Note that when $R\to 0$ the NH instability requires $e\to\infty$.

So **NH instability $\neq$ Superradiant instability**

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→ **Heuristics (conclusion):** RN-AdS BHs (apparently) **stable for $e^2 l^2 < 3$**

Very ***large* extremal RN-AdS** BHs are NH unstable when $e^2 l^2 > 3$.

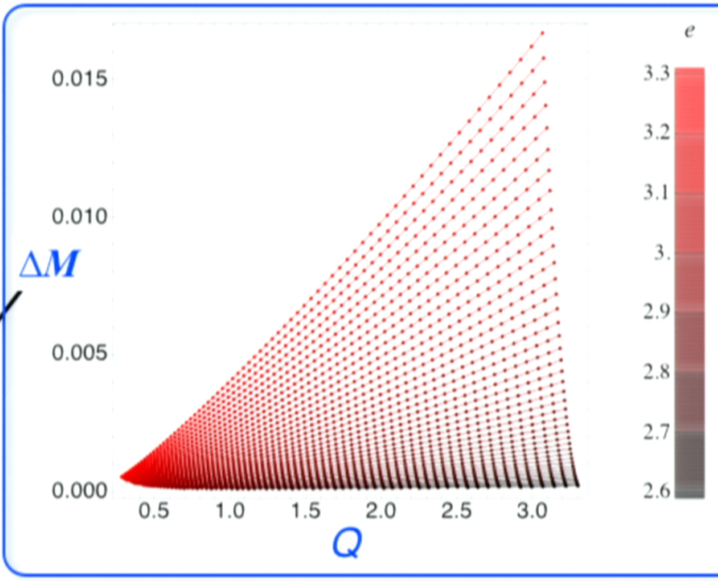
Arbitrarily ***small* extremal BHs** are superradiant unstable when $e^2 l^2 > 32/3$

→ **Linear instability analysis of zero-modes ($\omega = 0$) confirms expectations:**

Critical value of scalar charge $e(M,Q)$ for instability:

- For given R, **minimum** value of e^2 is for **extremal BHs**
- e^2_{min} monotonically decreases from 32/3 to 3 as BH size ↗
- For $e^2 < 3$ **all BHs** are **stable** under scalar condensation.
- For $e^2 > 32/3$, **all extremal** BHs are **unstable**.

$\Delta M = M - M_{ext}$,
M_{ext} is the mass of the extremal RN AdS BH with the same charge Q



→ Assuming that **hairy BHs** bifurcate from RN-AdS family at the **onset of the instability**, this suggests we should **look** into **3 regimes**:

$$e^2\ell^2 < 3$$

$$3 < e^2\ell^2 < \frac{32}{3}$$

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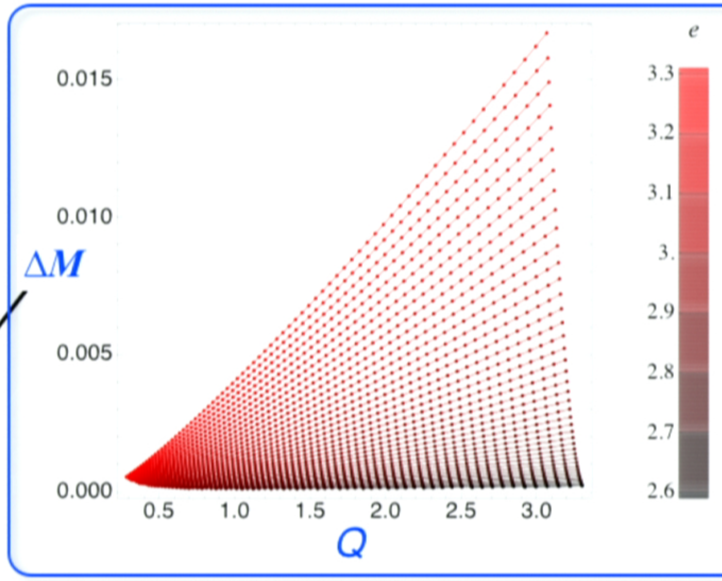

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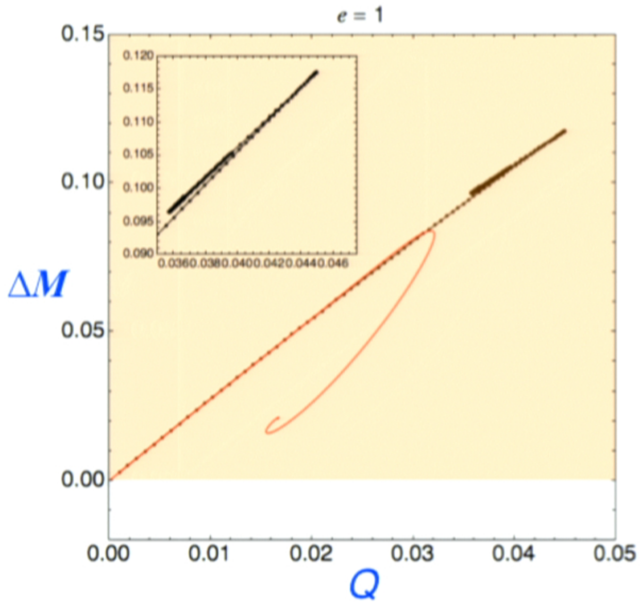

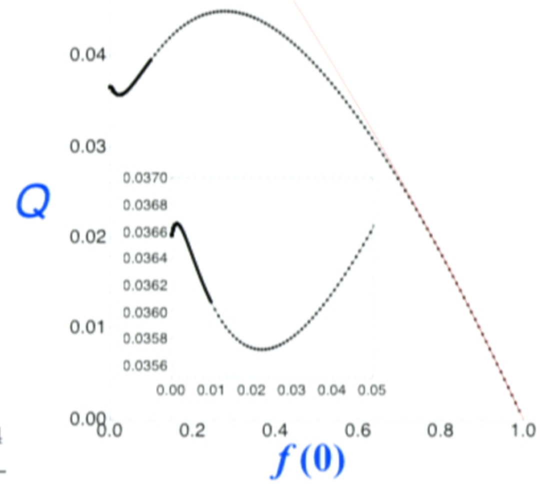

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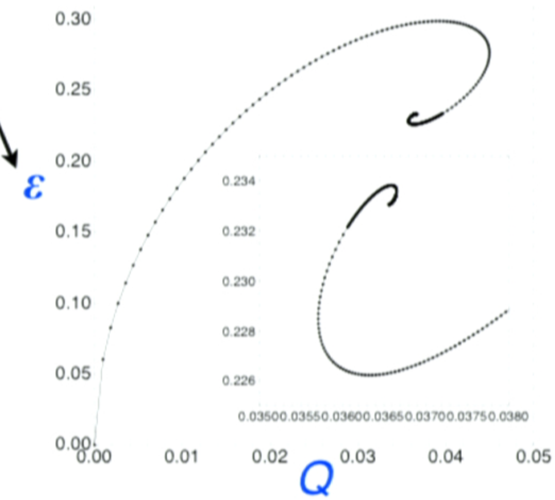
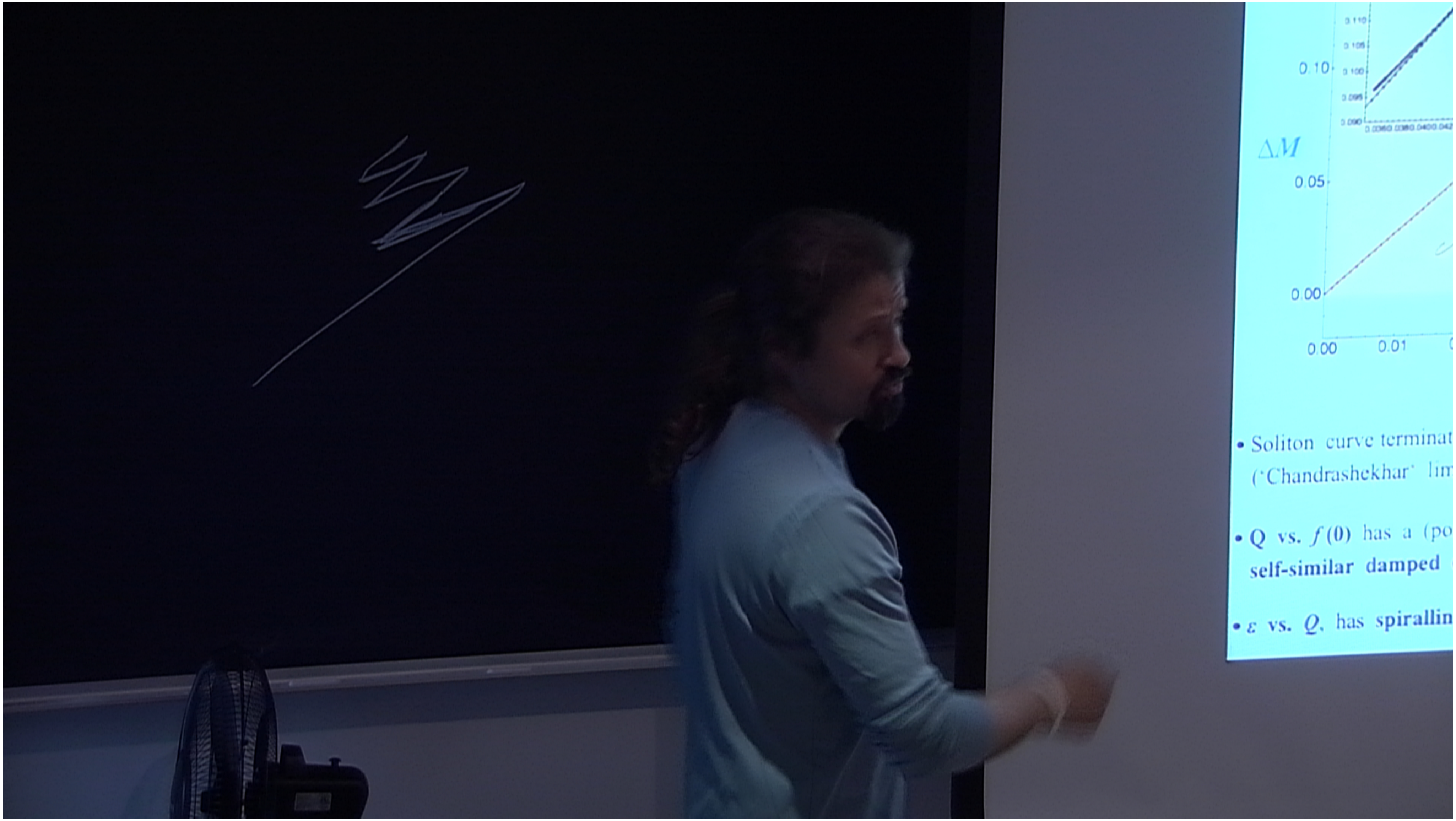


- Soliton curve terminates at a naked singularity at some finite Q ('Chandrashekhar' limit) where $K|_{r=0} \to \infty$ & $f(0) = 0$.

- **Q vs.** $f(0)$ has a (possibly infinite) series of **self-similar damped oscillations** as we approach $f(0) = 0$

- ε **vs.** Q, has **spiralling behavior** towards the singular solution.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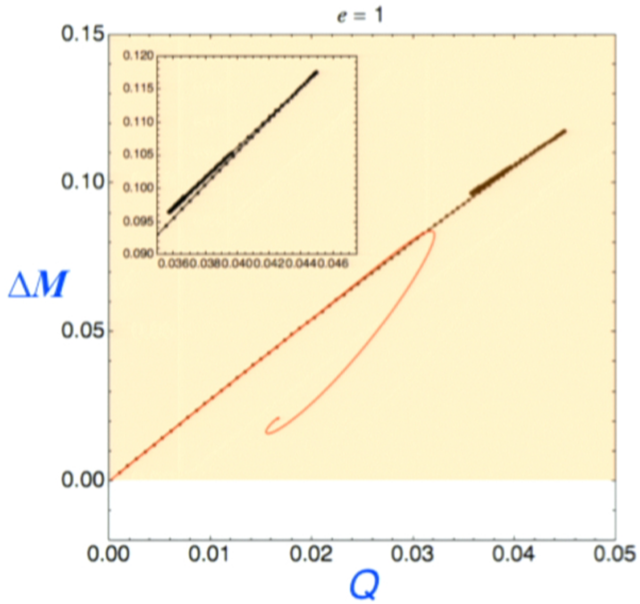

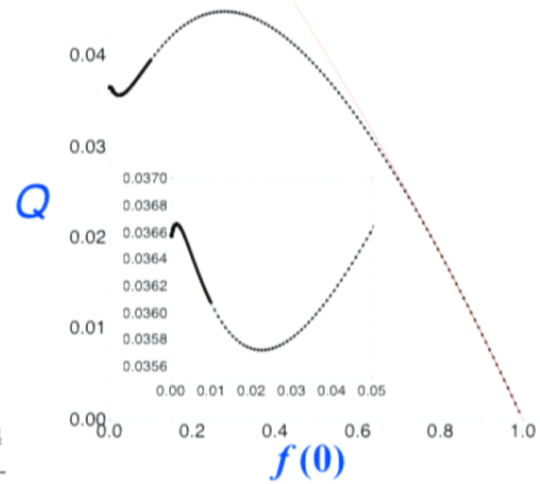

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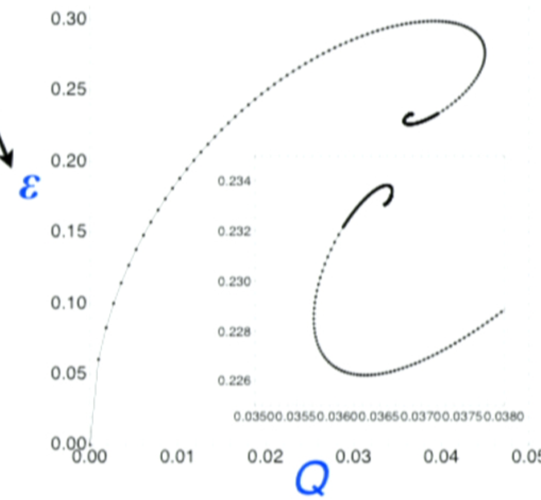

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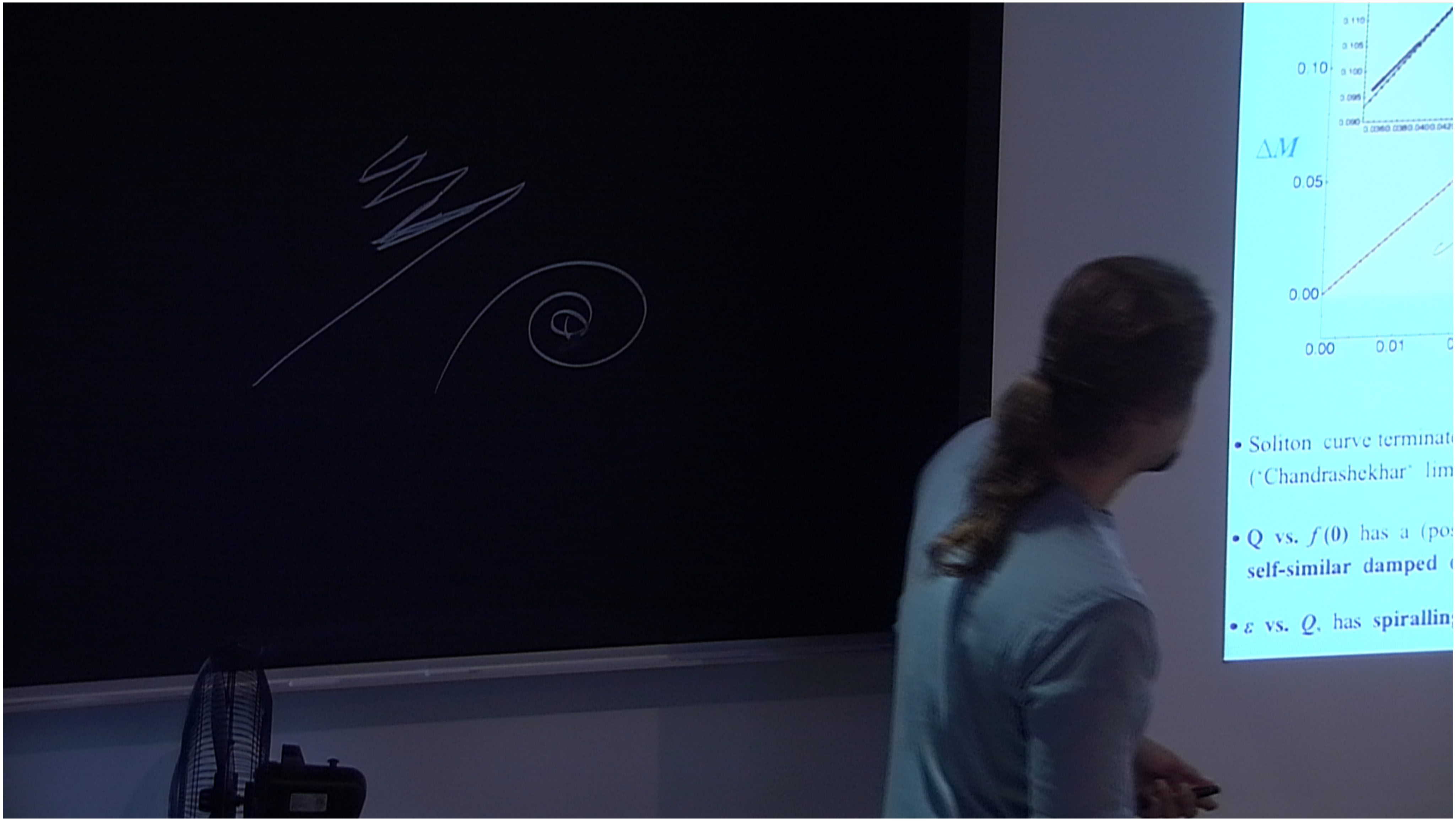

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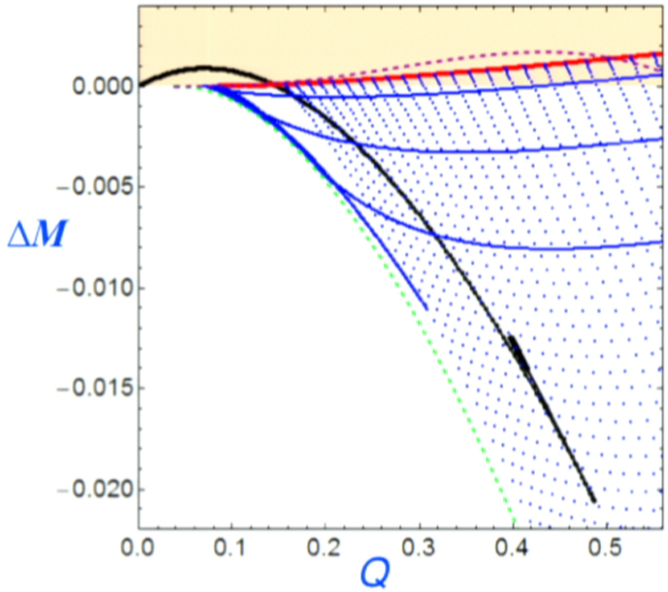
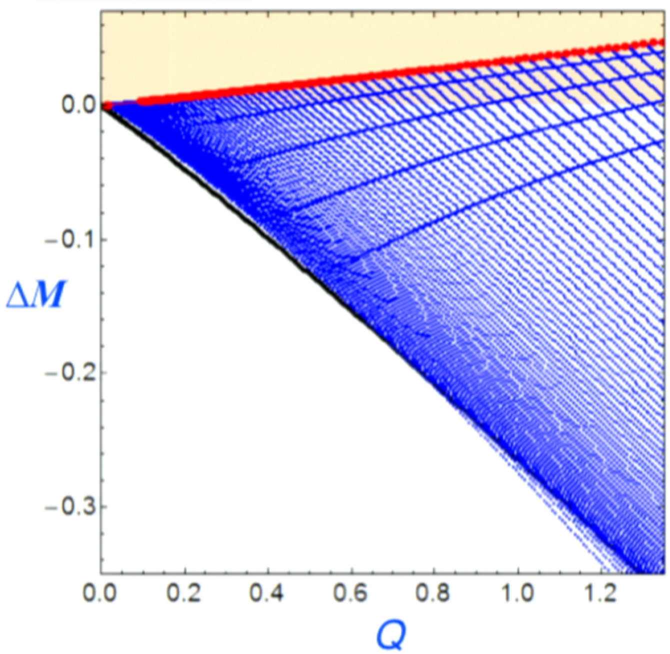


- **Shaded region:** **RN AdS** BHs above extremality ($\Delta M = 0$).
- **Black curve:** **soliton** branch with cusps that ends in singularity.
- **Blue region:** **hairy BHs**
  ('horizontal' lines: fixed values of ε, and $R \searrow$ to the left;
  'diagonal' blue lines near green curve are segments of a hairy BH with fixed R, and $\varepsilon \nearrow$ to the right).
- **Lower mass bound** of **hairy BHs** is well described at small Q by the dashed **green line** (**perturbative prediction**).
- **Red curve:** line of **marginal modes** of the **linear problem**; agrees w/ dashed **magenta line** for small Q (**perturbative prediction**).

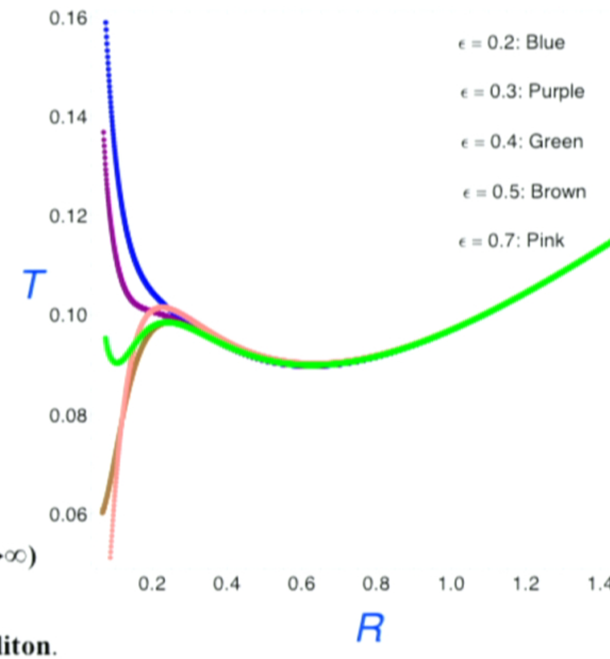
- **Soliton** curve and the **hairy BHs** surface are **NOT related** in the range $3 < e^2 l^2 < 32/3$.
  In particular, **soliton family** does **not** arise as a **zero size limit** of the **hairy BH.**

- We **find** **large hairy BHs**, but **NO small hairy BHs** in agreement with:
  large extremal RN-AdS are NH unstable when $e^2 l^2 > 3$,
  but small extremal RN-AdS are superradiant unstable only when $e^2 l^2 > 32/3$.
- Keeping ε **fixed** as $R \searrow$, we **approach lower mass bound** of hairy BH: $T \to 0$ & $K_H \to \infty$ as $R \to R_{min}$.
- These results suggest that, for $3 < e^2 l^2 < 32/3$, **hairy BHs** have an **extremal singular limit.**
- **Solitons** are **more massive** than the **extremal hairy BHs** of the same charge

$$e^2\ell^2 > \frac{32}{3}$$



- **Shaded region:** **RN AdS** BHs above extremality ($\Delta M = 0$).
- **Black curve: soliton** branch (no cusps; no end).
- **Blue region: hairy BHs**
  ('horizontal' lines: fixed values of ε, and $R \searrow$ to the left; 'diagonal' blue lines near green curve are segments of a hairy BH with fixed R, and $\varepsilon \nearrow$ to the right).
- **Red curve:** line of **marginal modes** of the **linear problem**; agrees w/ non-linear hairy BHs in limit $\varepsilon \to 0$.



- **Soliton curve** now lies entirely **below RN AdS region**, & it continues for arbitrarily large values of Q.
- **Soliton** curve and **hairy BHs** surface are **now related:**
  For $Q < Q_c \sim$ **0.75, soliton** is **zero size limit** of **hairy BH** ($T \to \infty$)
  However, for $Q > Q_c$, **lower mass bound** of **hairy BHs** is an **extremal singular** ($T \to 0$ & $K_H \to \infty$) solution **below** the **soliton**.

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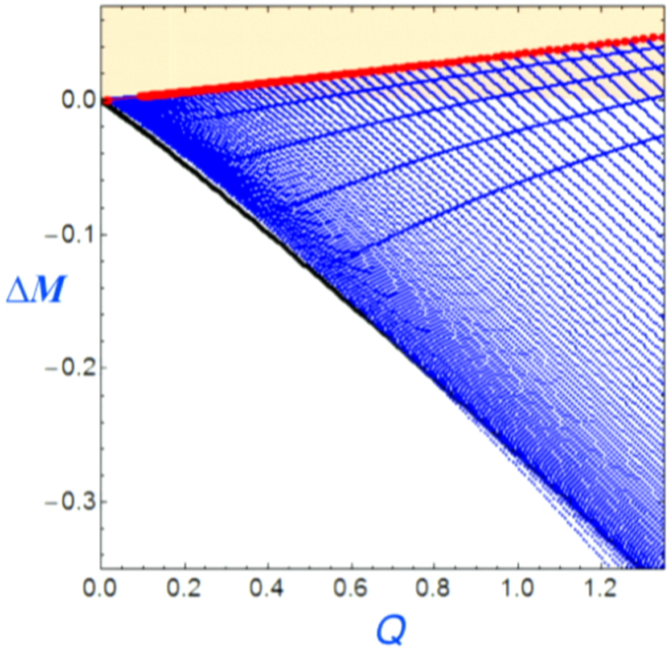

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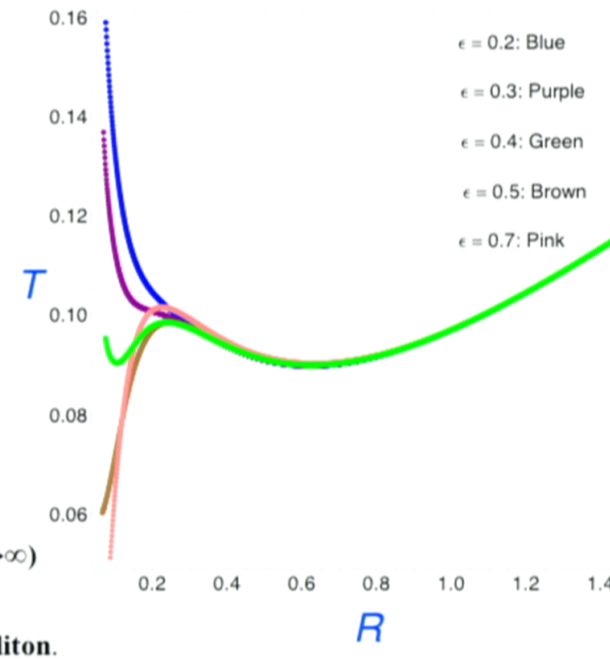

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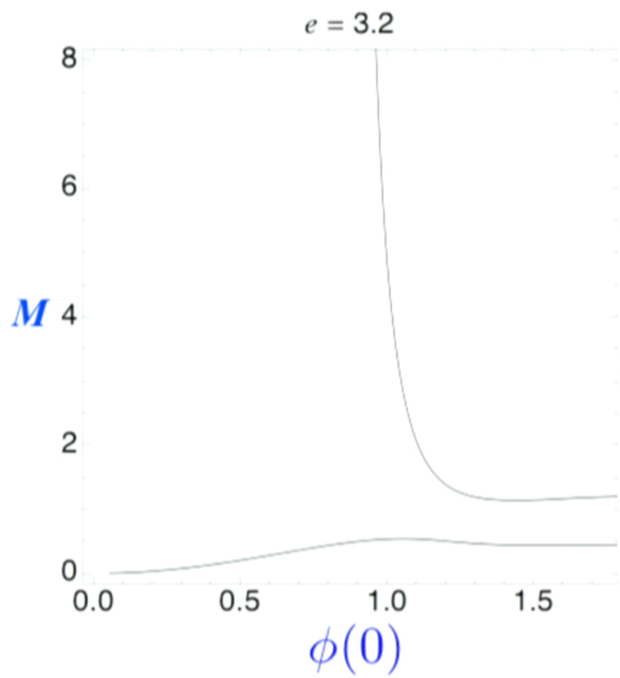
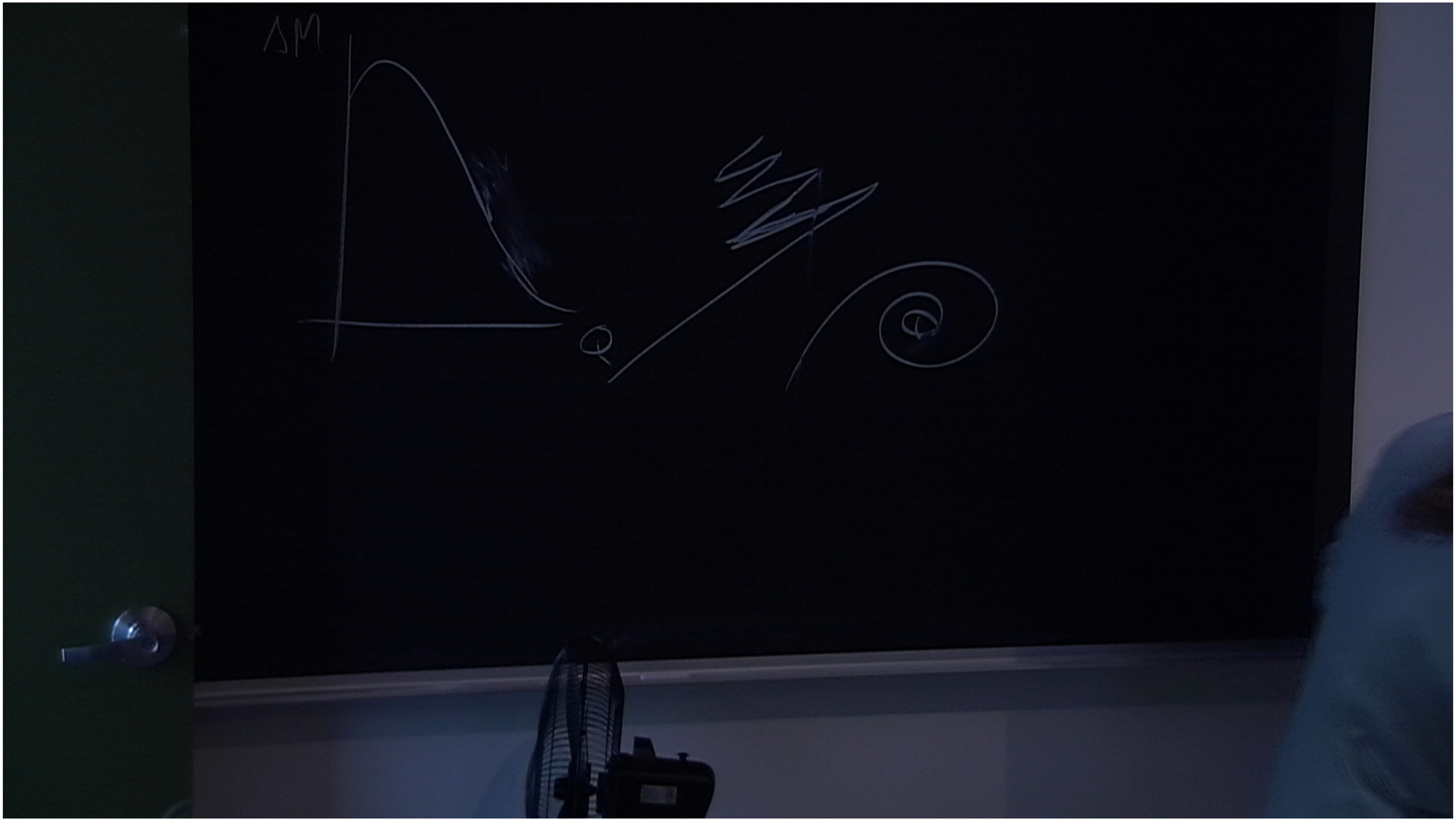
• Actually the story for the **boson stars** is slightly **more intricate:**

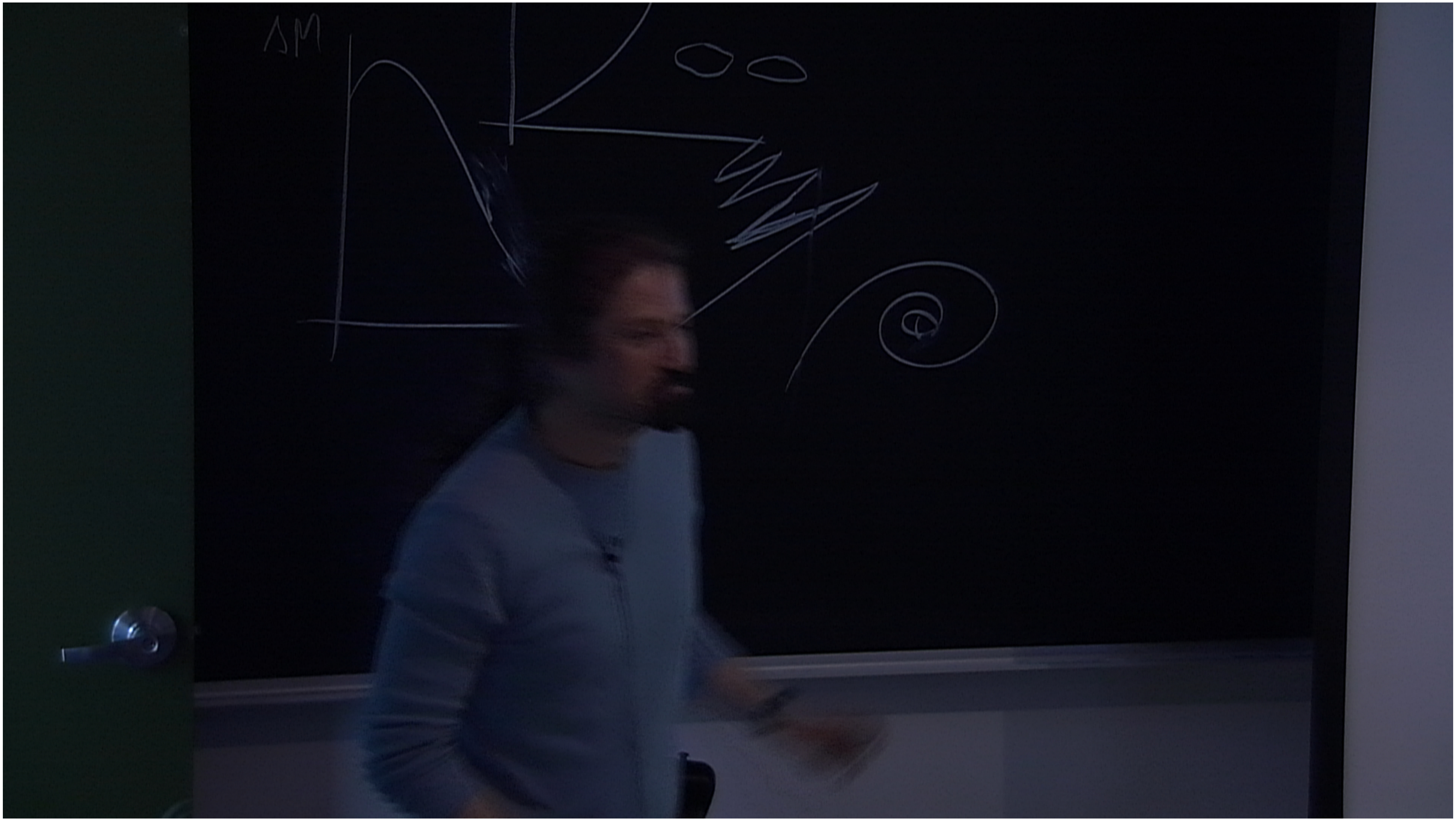
1112.3979, Gentle, Rangamani, Withers

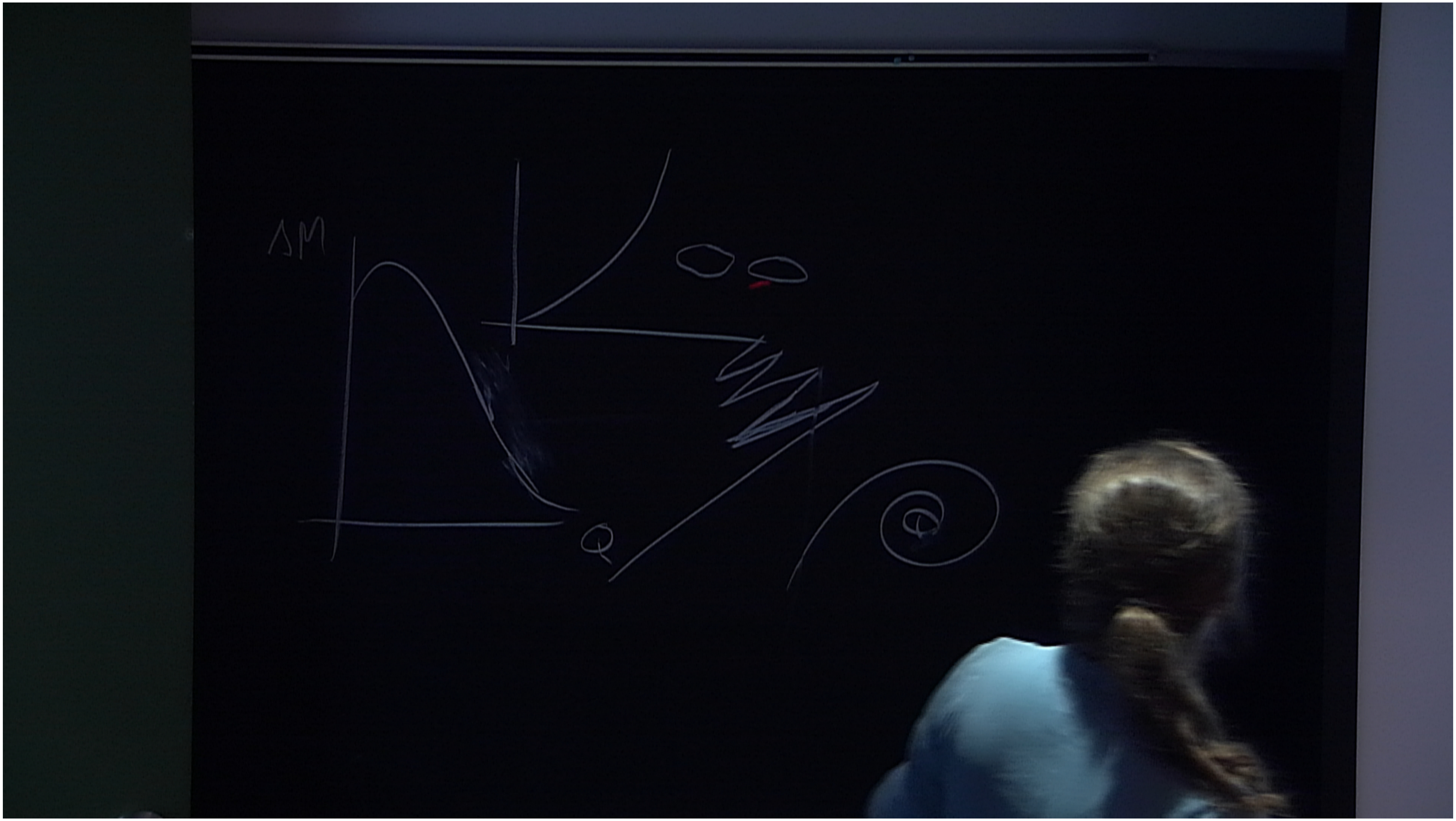
For $e^2 l^2 < 32/3$ there is not *one* but ***two*** **BS branches** that merge for $e^2 l^2 > 32/3$




ΔM


ΔM
Q

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What have we learned so far ?

- RN AdS is unstable **both** to **superrandiant** and **near-horizon** scalar condensation instabilities
- The **onset of superradiant / NH** scalar condensation instabilities is a **merger/bifurcation curve** to new family of **charged hairy BHs**
- RN-AdS not only static BH. Intricate BH / soliton phase diagram structure that depends on range of e.
- Phase space of static BHs of the Einstein-Maxwell theory, minimally coupled to a charged scalar field, in global AdS is now probably complete.

(If spatial horizon topology is R^3 (instead of S^3) hairy BHs describe holographic superconductor phase; Our BHs reduce to these in limit radius $S^3 \to \infty$, and are dual to superfluid phases of QFT on $R_t \times S^3$.

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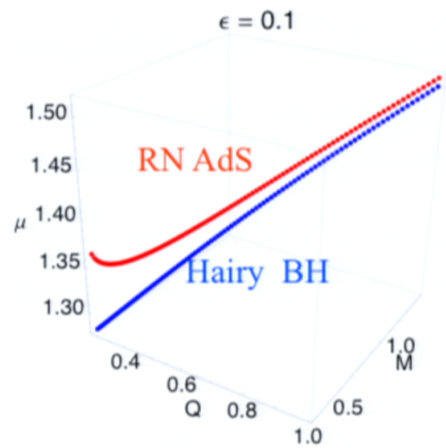
- However, **could the hairy BHs** be only a metastable state? Ie, shouldn't we expect hairy BHs to **be superradiant unstable?**

NO: Given $\{M, Q\}$, $\mu_{\text{hairy BH}} < \mu_{\text{RN AdS}}$ and such that **superradiant modes no longer fit inside AdS:**

$$\text{Re}\,\omega_{QN} > e\,\mu$$



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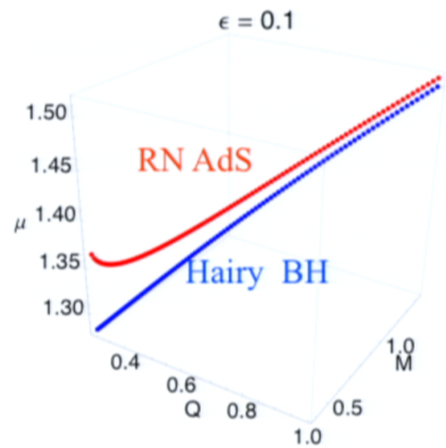

BHs with a single Killing field. Rotating Superradiance

OD, Gary Horowitz, Jorge Santos, 1105.4167

- **AdS-Einstein gravity** (in $d = 5$) minimally coupled to **2 complex massless scalar fields** Π^j :

$$S = \frac{1}{16\pi}\int_{\mathcal{M}} d^5x\,\sqrt{-g}\left[R + \frac{12}{\ell^2} - 2\left|\nabla\vec{\Pi}\right|^2\right] \qquad G_{ab} - 6\ell^{-2}g_{ab} = T_{ab}$$

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Most unstable mode is $m = 1$

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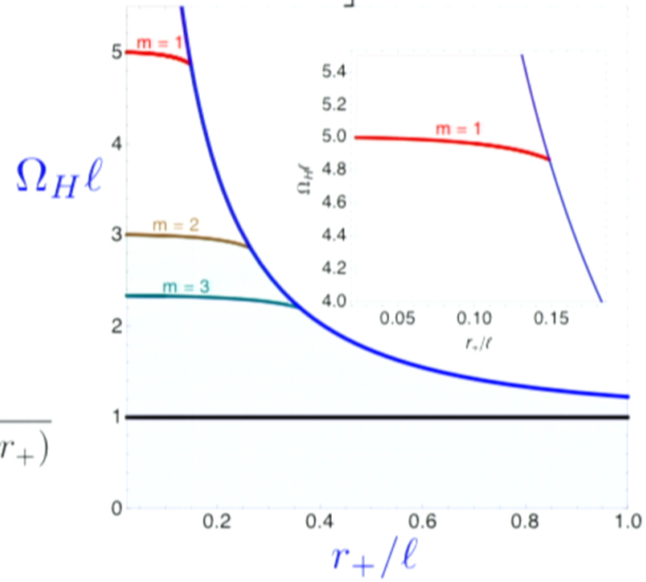

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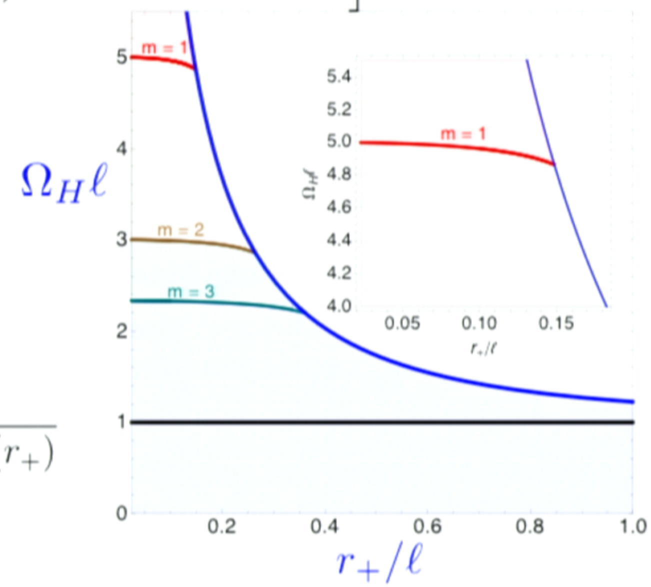

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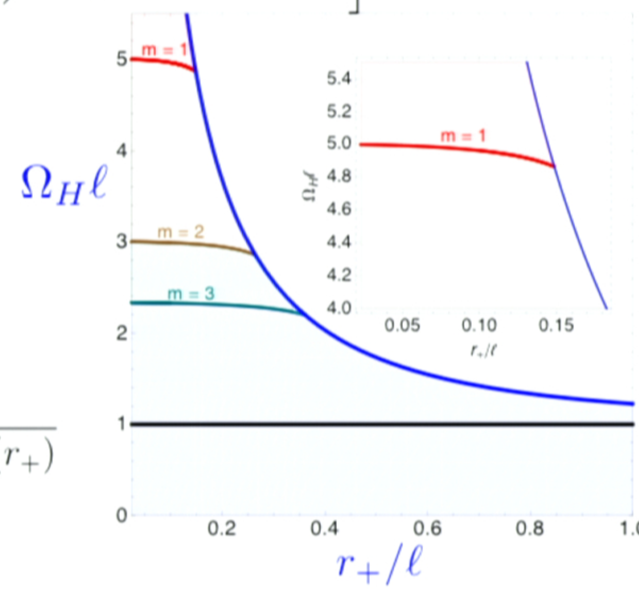

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➡ **Boundary Conditions:**

- **Asymptotic BC:**

BS & BH asymptote to global AdS w/ next-to-leading order terms fixing $\{M,J\}$. Π must be normalizable:

$$f\big|_{r\to\infty} = \frac{r^2}{\ell^2} + 1 + \frac{C_f\,\ell^2}{r^2} + \mathcal{O}\left(r^{-3}\right), \quad g\big|_{r\to\infty} = 1 - \frac{C_h\,\ell^4}{r^4} + \mathcal{O}\left(r^{-5}\right),$$

$$h\big|_{r\to\infty} = 1 + \frac{C_h\,\ell^4}{r^4} + \mathcal{O}\left(r^{-5}\right), \quad \Omega\big|_{r\to\infty} = \frac{C_\Omega\,\ell^4}{r^4} + \mathcal{O}\left(r^{-5}\right), \quad \Pi\big|_{r\to\infty} = \frac{\epsilon\,\ell^4}{r^4} + \mathcal{O}\left(r^{-5}\right).$$

ε: asymptotic amplitude of condensate Π

$$E = \frac{\pi\,\ell^2}{8}\left(4C_h - 3C_f\right), \qquad J = \frac{\pi\,\ell^3\,C_\Omega}{2}$$

- **Inner BC:**

BS: are smooth horizonless solutions. Functions must be regular at $r = 0$. Regularity of $\vec{\Pi}$ $\Rightarrow$ $\Pi|_{r\to 0} \sim r$:

$$f\big|_{r\to 0} = 1 + \mathcal{O}\left(r^2\right), \quad g\big|_{r\to 0} = \mathcal{O}(1), \quad h\big|_{r\to 0} = 1 + \mathcal{O}\left(r^2\right), \quad \Omega\big|_{r\to 0} = \mathcal{O}(1), \quad \Pi\big|_{r\to 0} = \mathcal{O}(r)$$

BH: inner bdry is Horizon at $r = r_+$ *defined* as location where $f(r_+) = 0$. Other functions are regular:

$$f\big|_{r\to r_+} = \mathcal{O}(r - r_+), \quad g\big|_{r\to r_+} = \mathcal{O}(1), \quad h\big|_{r\to r_+} = \mathcal{O}(1), \quad \Omega\big|_{r\to r_+} = \mathcal{O}(1), \quad \Pi\big|_{r\to r_+} = \mathcal{O}(1)$$

From the EOM evaluated at the horizon we further find that we must have: $\omega = \Omega_H \equiv \Omega(r_+)$

➡ **First law of thermodynamics:** **BS:** $dE = \omega\, dJ$ **BH:** $dE = \omega\, dJ + T_H\, dS$, with $\omega \equiv \Omega_H$

➡ **Properties of single KVF:** $K = \partial_t + \omega\partial_\psi$ with norm $|K|^2 = -f g + r^2 h(\omega - \Omega)^2$

- BCs $\Rightarrow$ $\boldsymbol{|K|_H = 0}$ $\rightarrow$ event horizon is **Killing horizon**. K is always timelike just outside H and in neighboorhood of $r = 0$.
- BCs $\Rightarrow$ $\boldsymbol{|K|_{r\to\infty} \rightarrow r^2(\omega^2 - 1/l^2)}$. KVF is asymp. timelike/null/spacelike depending on whether $\omega l < 1$, $\omega l = 1$ or $\omega l > 1$.
- Our solutions **all** have $\boldsymbol{\omega l > 1}$. So, not globally stationary: effective ergoregion at large r where Π is concentrated.

➡ **Perturbative construction of Boson stars** and hairy BHs:

- **Rotating Boson Stars:** smooth horizonless geometries with harmonic time dependence.
- **One-parameter** family of solutions: in the perturbative regime can be parametrized by ε ($\Pi|_{r\to\infty} \sim \varepsilon\, l^4/r^4$).
- Construct perturbatively BS fields through a **power expansion** in ε around **global AdS**:

$$F(r,\epsilon) = \sum_{j=0}^{n} F_{2j}(r)\,\epsilon^{2j}\,, \qquad \Pi(r,\epsilon) = \sum_{j=0}^{n} \Pi_{2j+1}(r)\,\epsilon^{2j+1}\,, \qquad \omega(\epsilon) = \sum_{j=0}^{n} \omega_{2j}\,\epsilon^{2j}$$

$$F = \{f, g, h, \Omega\}$$

Expand also ω: at linear order it is an AdS normal mode but receives corrections at higher order

- Leading order contribution in the expansion, $n = 0$, describes the linear perturbation problem: introduce non-trivial Π in global AdS, but this condensate does not back-react on g_{ab}.

BCs fix regular Π and quantize its ω (normal mode of AdS):

$$\Pi(r) = \frac{\epsilon \ell^4 r}{(r^2+\ell^2)^{5/2}}\, {}_2F_1\left[\frac{5-\omega\ell}{2}, \frac{5+\omega\ell}{2}, 3, \frac{\ell^2}{r^2+\ell^2}\right], \qquad \omega\ell = 5+2k\,, \quad (k=0,1,2,\cdots)$$

$k = 0$: $f_0 = 1 + \frac{r^2}{\ell^2}\,, \quad g_0 = 1\,, \quad h_0 = 1\,, \quad \Omega_0 = 0\,, \quad \Pi_1 = \frac{\ell^4 r}{(r^2+\ell^2)^{5/2}}\,, \quad \omega_0 = \frac{5}{\ell}$

- Go to higher order: back-reaction corrections in g_{ab} (odd n) and corrections in Π and its ω (even n):

$$\omega\ell = 5 - \frac{15}{28}\,\epsilon^2 - \frac{22456447}{35562240}\,\epsilon^4 + \mathcal{O}\left(\epsilon^6\right)$$

✔ **First law,** $\mathrm{d}E = \omega\, \mathrm{d}J$

$$E = \ell^2\,\frac{\pi}{4}\left[\frac{5}{6}\,\epsilon^2 + \frac{77951}{127008}\,\epsilon^4 + \mathcal{O}\left(\epsilon^6\right)\right], \qquad J = \ell^3\,\frac{\pi}{2}\left[\frac{1}{12}\,\epsilon^2 + \frac{83621}{1270080}\,\epsilon^4 + \mathcal{O}\left(\epsilon^6\right)\right]$$

➡ **Full non-linear construction of Boson stars and hairy BHs:**

- **Numerical method:**

  - standard **relaxation (Newton-Raphson)** method.
  - **spectral discretization** on a Chebyshev grid.
  - **residual gauge freedom**, that leaves the gravitational/scalar fields invariant:

$$\psi \to \psi + \lambda t\,, \qquad \Omega \to \Omega + \lambda\,, \qquad \omega \to \omega - \lambda$$

Choose λ to be such that $\Omega(\infty) = 0$ (the physical gauge) or $\omega = 0$.

Use this freedom to **optimize the numerical construction**:

convergence is better in different gauges in different regions of parameter space.

✓ Use **First Law** of thermodynamics to check numerics: maximal error of 0.005%

✓ Check with perturbative construction of BS and hairy BHs [also Stotyn, Park, McGrath, Mann,d>5]

- **Non-linear code** gives $\varepsilon \to 0$ **Hairy BHs** that **agree** w/ $m = 1$ **threshold instability curve** of **linear code.** **Onset** of **superradiant instability** signals, in phase diag., a **merger line** connecting **MP-AdS** with **hairy BHs**

- **Boson star:** In perturbative construction, ε appropriately parametrizes the solution.

  For **large** $\{E,J\}$, ε no longer defines the solution uniquely. Neither does $\{\omega, E, J\}$

  Use instead the **energy density** $\Pi'(0)^2$ at BS **center**.

➡ Full non-linear construction of Boson stars and hairy BHs:

- **Numerical method:**

  - standard **relaxation (Newton-Raphson)** method.
  - **spectral discretization** on a Chebyshev grid.
  - **residual gauge freedom**, that leaves the gravitational/scalar fields invariant:

$$\psi \to \psi + \lambda t\,, \qquad \Omega \to \Omega + \lambda\,, \qquad \omega \to \omega - \lambda$$

  Choose λ to be such that $\Omega(\infty) = 0$ (the physical gauge) or $\omega = 0$.

  Use this freedom to **optimize the numerical construction**:

  convergence is better in different gauges in different regions of parameter space.

  ✓ Use **First Law** of thermodynamics to check numerics: maximal error of 0.005%

  ✓ Check with perturbative construction of BS and hairy BHs [also Stotyn, Park, McGrath, Mann,d>5]

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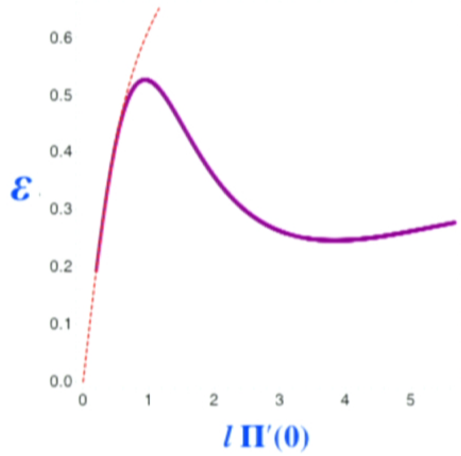
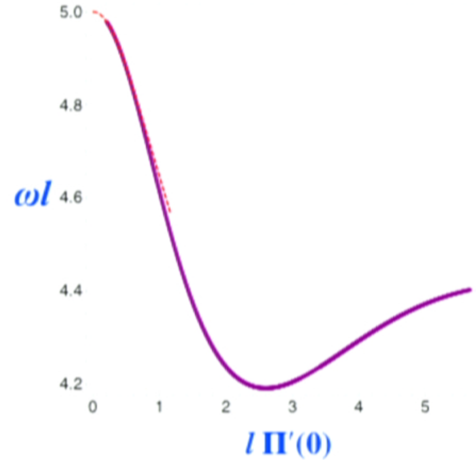




Similar oscillations for $\{E, J\}$ **vs** $\Pi'(0)$

- **Black hole:** In perturbative construction: $\{\varepsilon, r_+\}$ appropriately parametrizes the solution.

  For **large** $\{E, J\}$, $\{\varepsilon, r_+\}$ no longer defines uniquely solution. Neither does any pair of $\{\omega, E, J\}$

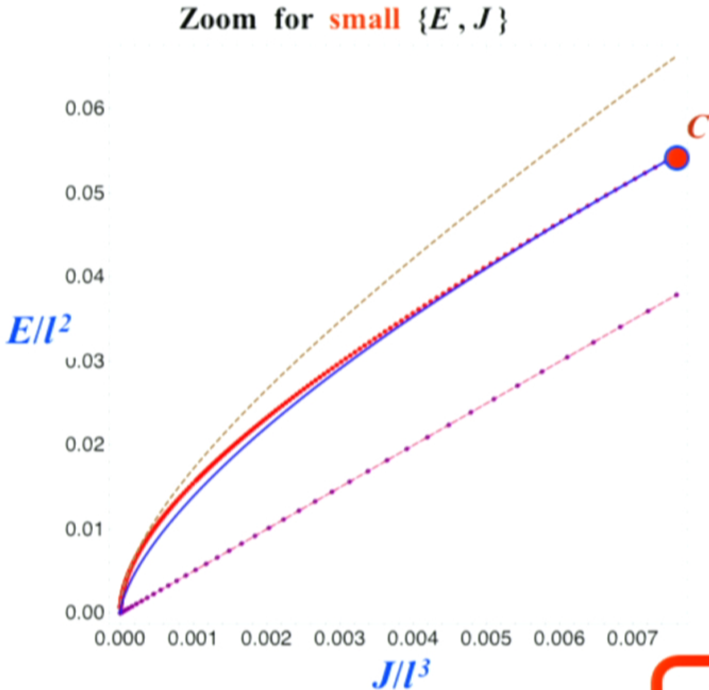
  BHs have squashed S^3 horizons: viewed as S^1 bundles over S^2.

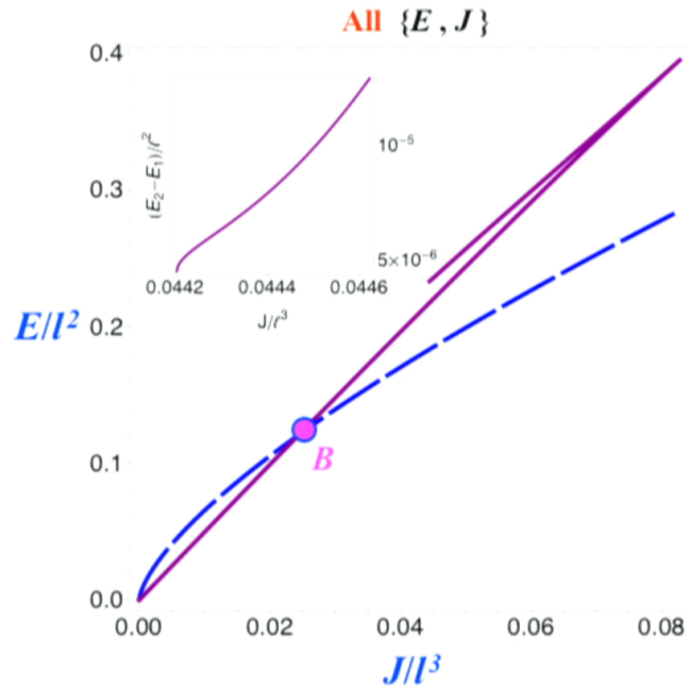
$$ds^2 = (\ldots) + r^2 \left[h \left(d\psi + \frac{\cos\theta}{2} d\phi - \Omega dt \right)^2 + \frac{1}{4} \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right]$$

Natural parametrization: size of $\{S^1, S^2\}$: $r_1 = r_+ \sqrt{h(r_+)}$ $\quad r_2 = r_+$

$$S = \frac{1}{4} A_3 \, r_1 \, r_2^2$$

➡ Full non-linear construction of Boson stars:



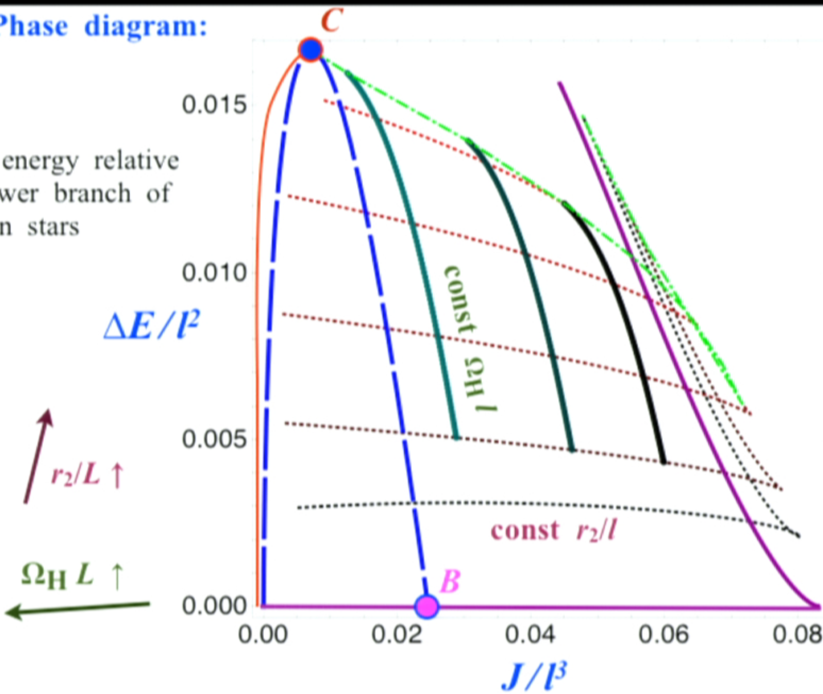


- **Perturbative estimate** for the **merger line.**
  Exact merger line using shooting linear code.
- **Extremal line** of **MP-AdS BHs.**
- **Perturbative estimate** for the **bosons stars.**
  Exact E **vs** J for the **bosons stars (Dots).**

- Damped oscillatory behavior (cusps, spirals):
  NOT captured by perturbative analysis.
  1st law ⇒ extrema of E, J are at same $\mathbf{\Pi'(0)}$
- Expect ∞ # of damped oscillations/cusps/spiral arms.
- BS is regular but $K|_{r=0}$ grows large along BS branch
- Some BS coexist with MP-AdS (purple is above blue).

➡ Phase diagram:

ΔE : energy relative to lower branch of boson stars



- Dashed blue: **extremal MP-AdS**
  (MP-AdS only exist above it **).**
- Solid purple curve: **boson stars.**
- Dotted red curves:
  hairy BH lines of **constant** r_2/l
  (r_1 ↗ **along const** r_2 **line)**
- Green "vertical" solid curves:
  hairy BH lines of **constant** $\Omega_H\, l$.
- Dotted-dashed green curve:
  singular extremal hairy BHs.
  $K|_H$ **finite** but **Tidal forces** $\rightarrow \infty$

- When **two** $r_2 =$ **const line cross** we have **non-uniqueness:** same E, J but different S.
- Close to the **merger**, the $S_{MP} < S_{\text{hairy BH}}$. $S_{MP} = S_{\text{hairy BH}}$ at merger → **2nd order phase transition**.
- However, for sufficiently large J, MP-AdS coexist with hairy BHs, and $S_{MP} > S_{\text{hairy BH}}$.
  Moreover, the transition is now **1st order,** because these solutions **never merge** for this range of J.
- In sum, in a **3d plot** of { S/l^3, $\Delta E/l^2$, J/l^3 }:
  $J < J_c$ **:** the hairy BH family is a 2d surface **bounded** by the **merger line** and the **boson star curve**
  $J > J_c$ **:** Surface continues but is now **bounded** by **extremal hairy BH curve** & **boson star line**.
  This 2d surface never intersects with itself and has a sequence of (regular) "cusp lines".

➡ Conclusion:

- BHs with a scalar field condensate & orbitating around horizon.
- ***First*** example of stationary **BH** with **single isometry**: stationary but not time symmetric nor axisymmetric
- Does not contradict **rigidity theorems**

➡ Stability? What is the endpoint of rotating superradiant instability?

- **Small** $\{E, J\}$ **BS** are deformations of AdS (linearly stable) and should be **linearly stable** (true for static BS)
- For **larger** $\{E, J\}$, we **expect BS** to become **unstable**. For static BS this occurs at maximum of E (1st cusp)
- Hairy BHs should be the **endpoint** of ***m*=1 superradiant instability**. ($S_{\text{hairy BH}} > S_{\text{MP}}$ for small $\{E, J\}$)
- **All** hairy BHs we find have $\Omega_H\, l > 1$ ⇒ **unstable** to superradiant ***m*>1 modes**.
- **Time evolution** will **never settle down?**

  That is, series of metastable configurations with **higher & higher *m*-structure?**
- What is the endpoint of the **competition** between **superradiant** and **turbulent instabilities ?**

  Time evolution of ***Superposition* of modes**:

  - **superradiance** cause **low** ω modes to **grow**; **high** ω are **absorbed**
  - **turbulent** instability will cause **higher** ω modes to be **created**