Title: Pentahedral Volume, Chaos, and Quantum Gravity

Date: May 30, 2012 04:00 PM

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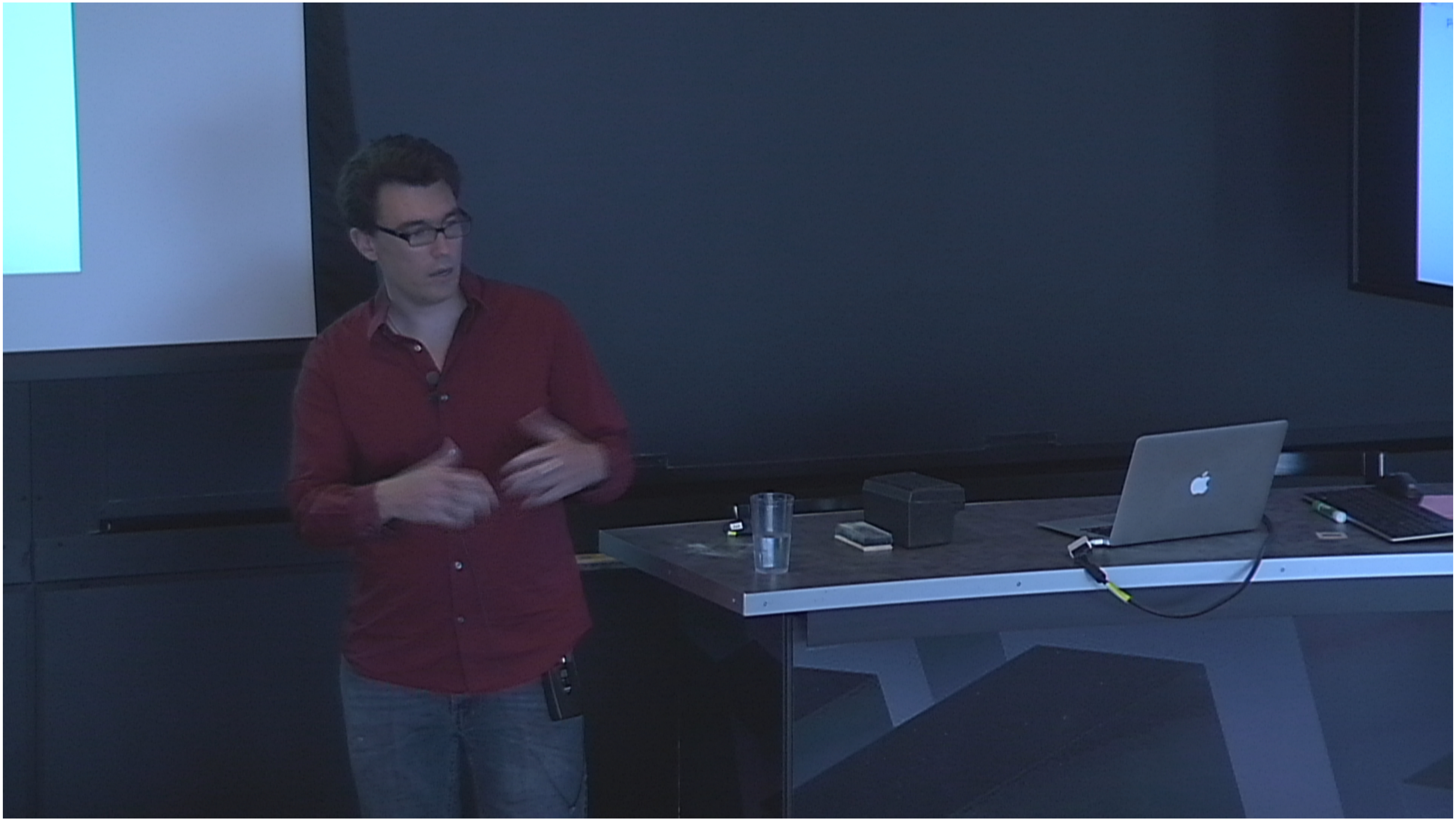
Abstract: The space of convex polyhedra can be given a dynamical structure. Exploiting this dynamics we have performed a Bohr-Sommerfeld quantization of the volume of a tetrahedral grain of space, which is in excellent agreement with loop gravity. Here we present investigations of the volume of a 5-faced convex polyhedron. We give for the first time a constructive method for finding these polyhedra given their face areas and normals to the faces and find an explicit formula for the volume. This results
in new information about cylindrical consistency in loop gravity and a couple of surprises about polyhedra. In particular, we are interested in discovering whether the evolution generated by this volume is chaotic or integrable as this will impact the interpretation of the spin network basis in loop gravity.



Pentahedral Volume, Chaos, and Quantum Gravity

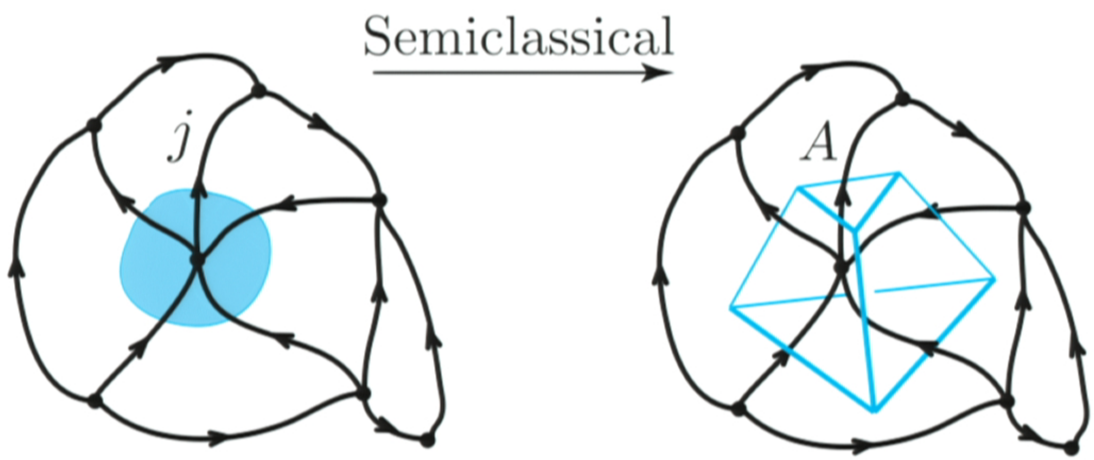
Hal Haggard

May 30, 2012

Volume

Polyhedral Volume (Bianchi, Doná and Speziale):

$$\hat{V}_{\text{Pol}} = \text{The volume of a quantum polyhedron}$$



Outline

1. Pentahedral Volume

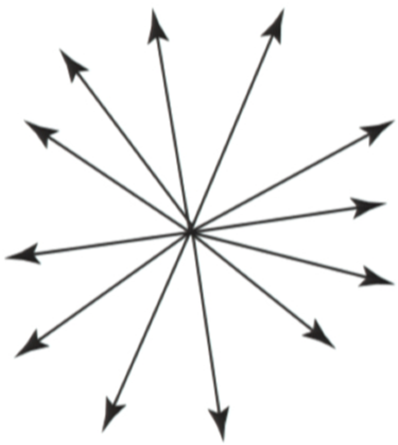
2. Chaos & Quantization

3. Volume Dynamics and Quantum Gravity

Minkowski's theorem: polyhedra

The area vectors of a convex polyhedron determine its shape:

$$\vec{A}_1 + \cdots + \vec{A}_n = 0.$$



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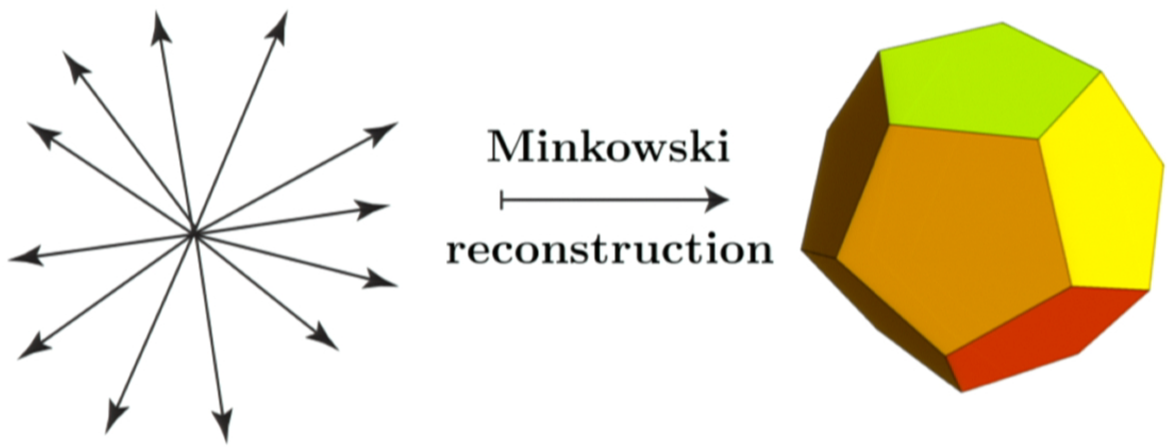
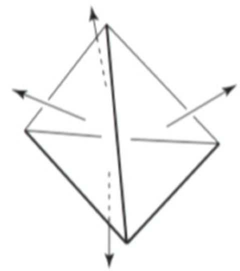


Only an existence and uniqueness theorem.

Minkowski's theorem: a tetrahdedron

Interpret the area vectors of tetrahedron as angular momenta:

$$\vec{A}_1 + \vec{A}_2 + \vec{A}_3 + \vec{A}_4 = 0 \quad \Longleftrightarrow$$



For fixed areas $A_1, \ldots, A_4$ each area vector lives in S^2.

Symplectic reduction of $(S^2)^4$ gives rise to the Poisson brackets:

$$\{f, g\} = \sum_{l=1}^{4} \vec{A}_l \cdot \left(\frac{\partial f}{\partial \vec{A}_l} \times \frac{\partial g}{\partial \vec{A}_l} \right)$$

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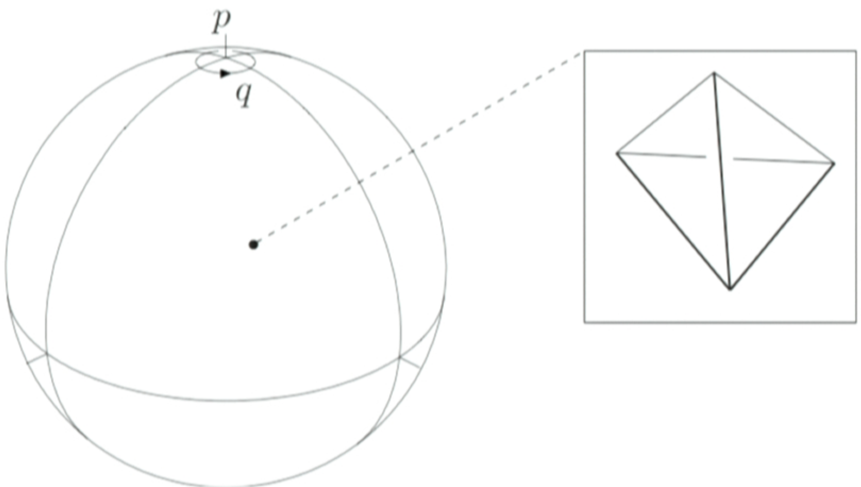

$p = |\vec{A}_1 + \vec{A}_2|$ $\qquad$ $q =$ Angle of rotation generated by p:

$$\{q, p\} = 1$$

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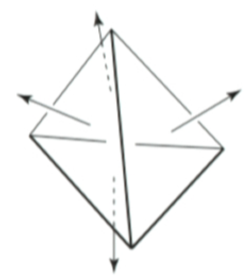

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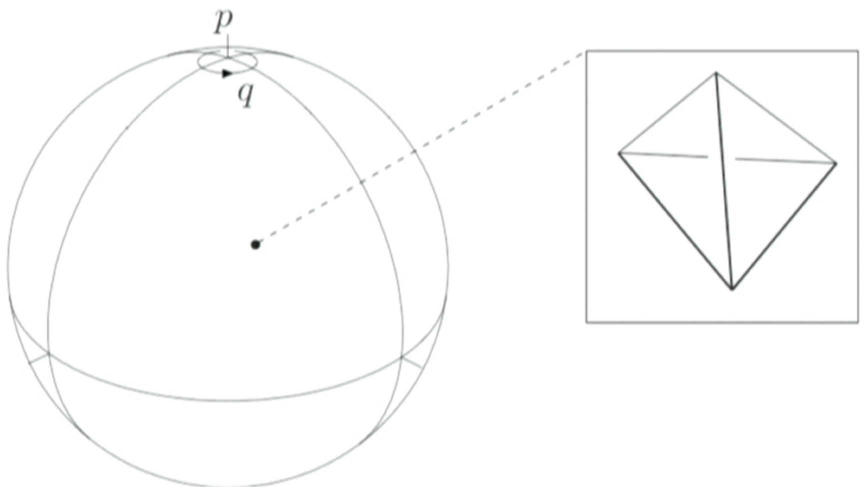

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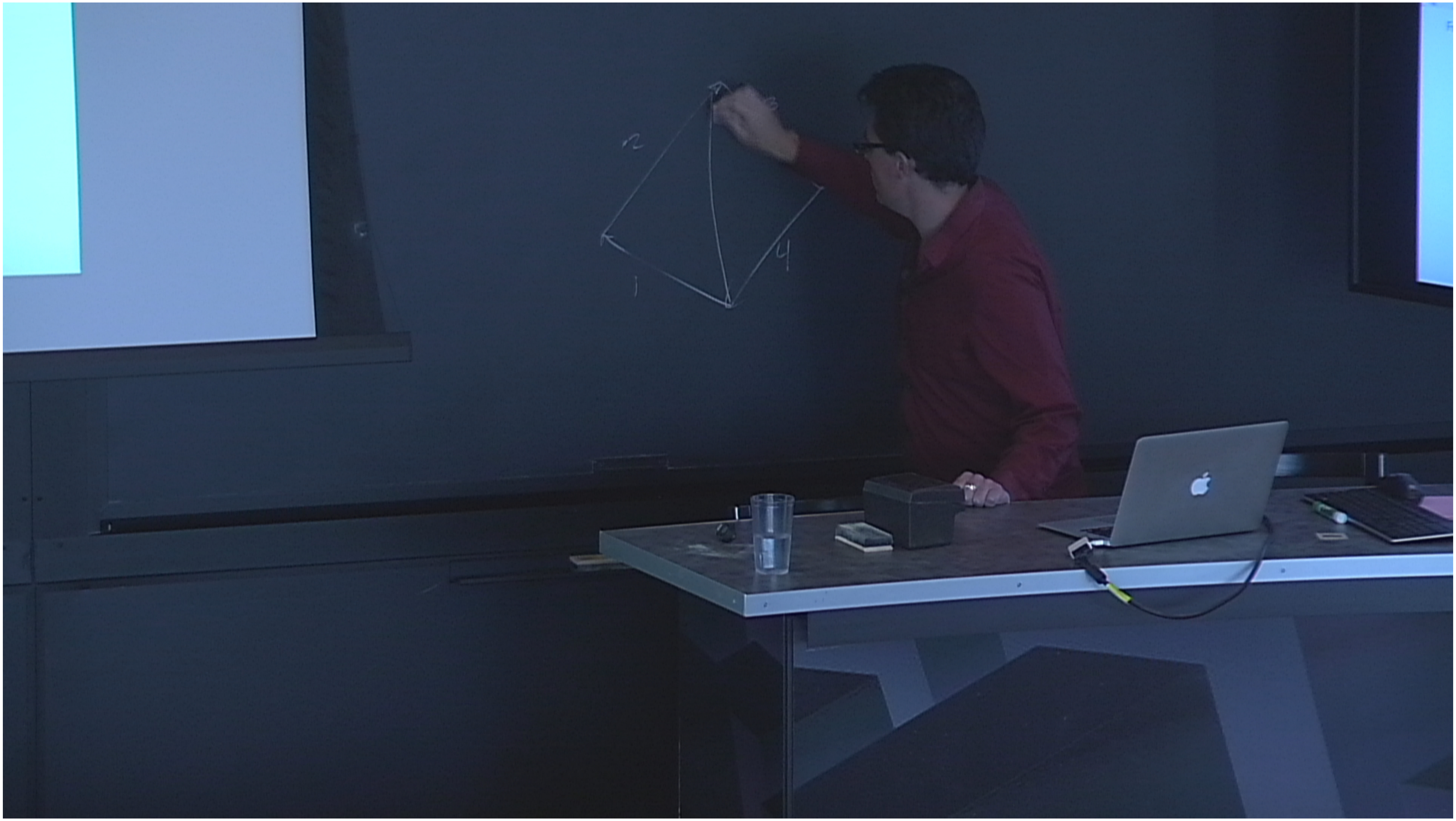
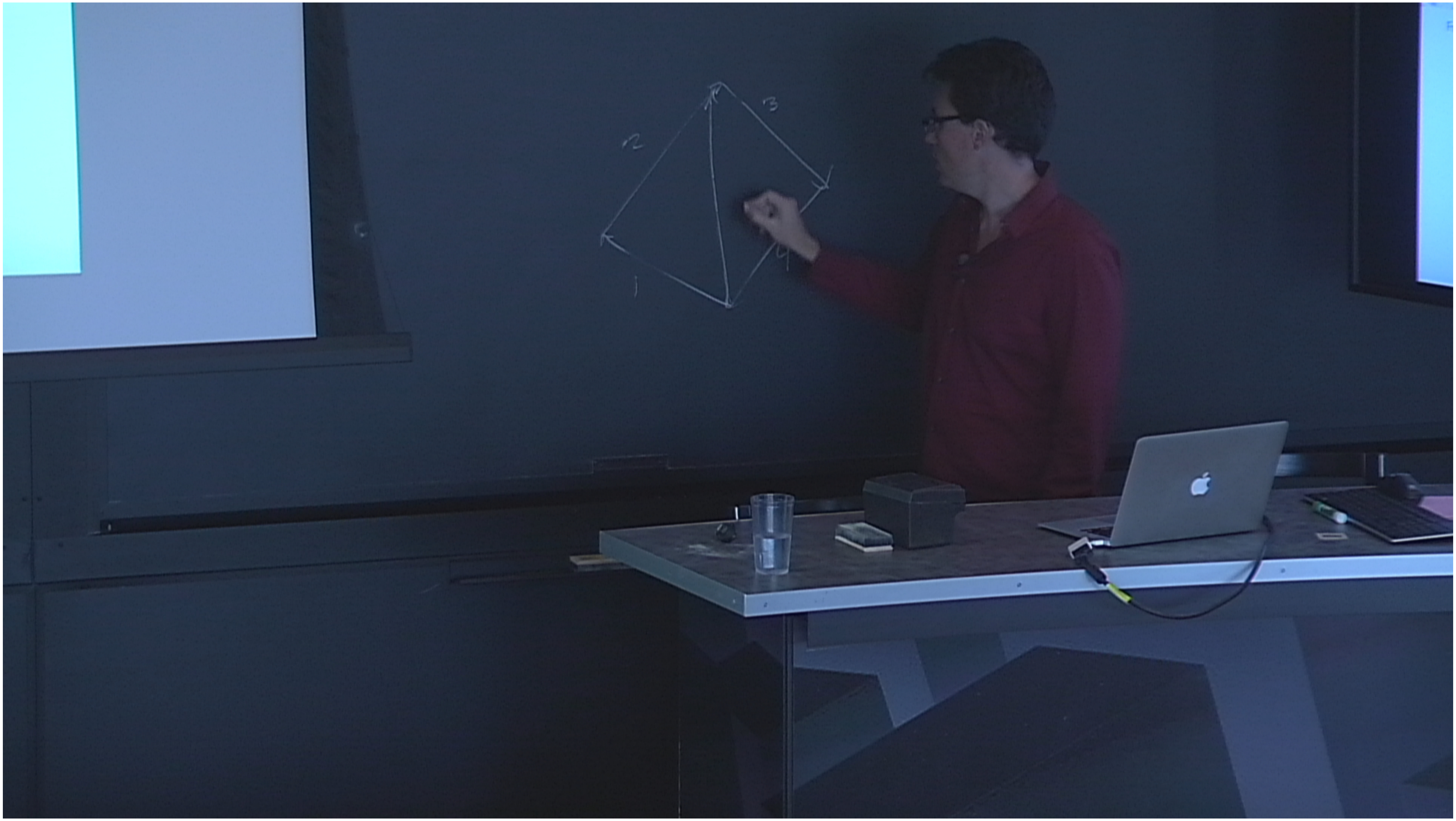
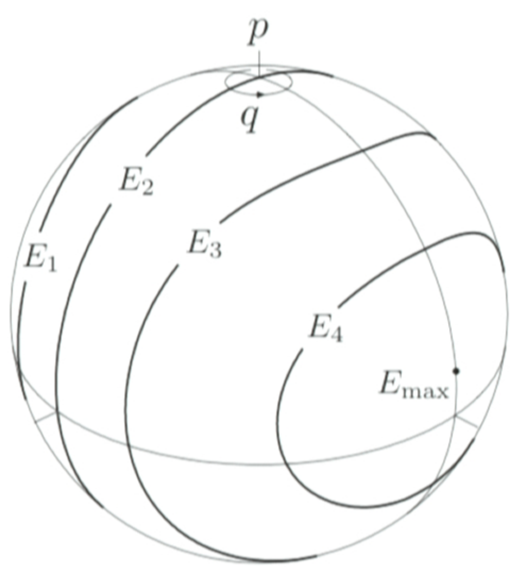

2
1
4



Dynamics

Take as Hamiltonian the Volume:

$$H = V^2 = \frac{2}{9}\vec{A}_1 \cdot (\vec{A}_2 \times \vec{A}_3)$$



Bohr-Sommerfeld quantization

Require Bohr-Sommerfeld quantization condition,

$$S = \oint_\gamma p dq = (n + \frac{1}{2})2\pi\hbar.$$

Area of orbits given in terms of complete elliptic integrals,

$$S(E) = \left(\sum_{i=1}^{4} a_i K(m) + \sum_{i=1}^{4} b_i \Pi(\alpha_i^2, m) \right) E$$

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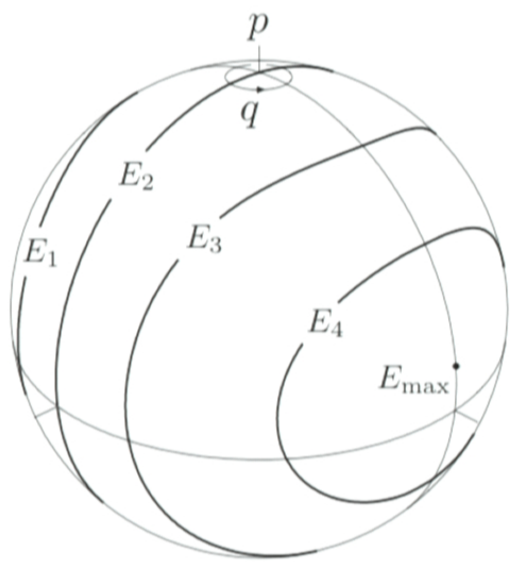

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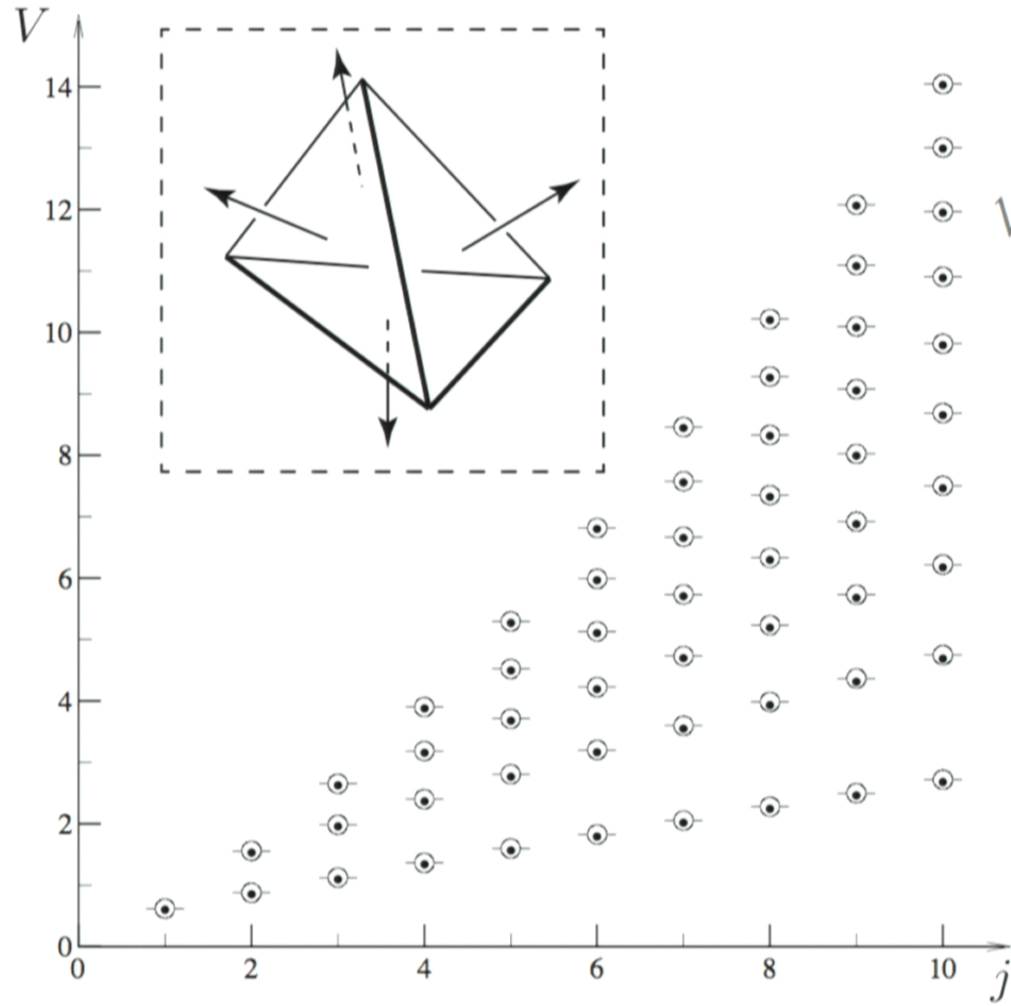

$$V_{\text{Tet}} = \frac{\sqrt{2}}{3} \times \sqrt{|\vec{A}_1 \cdot (\vec{A}_2 \times \vec{A}_3)|}$$

$$A_1 = j + 1/2$$
$$A_2 = j + 1/2$$
$$A_3 = j + 1/2$$
$$A_4 = j + 3/2$$

○ = Numerical
● = Bohr-Som

[PRL **107**, 011301]

Table			
$j_1\ j_2\ j_3\ j_4$	Loop gravity	Bohr-Sommerfeld	Accuracy
6 6 6 7	1.828	1.795	1.8%
	3.204	3.162	1.3%
	4.225	4.190	0.8%
	5.133	5.105	0.5%
	5.989	5.967	0.4%
	6.817	6.799	0.3%
$\frac{11}{2}\ \frac{13}{2}\ \frac{13}{2}\ \frac{13}{2}$	1.828	1.795	1.8%
	3.204	3.162	1.3%
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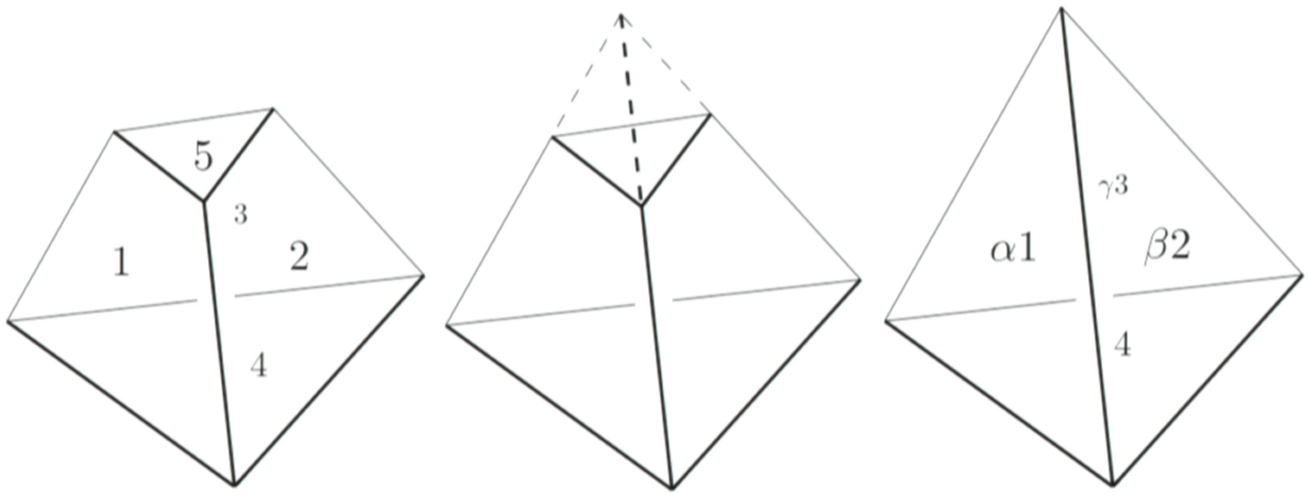
Volume of a pentahedron

A pentahedron can be completed to a tetrahedron



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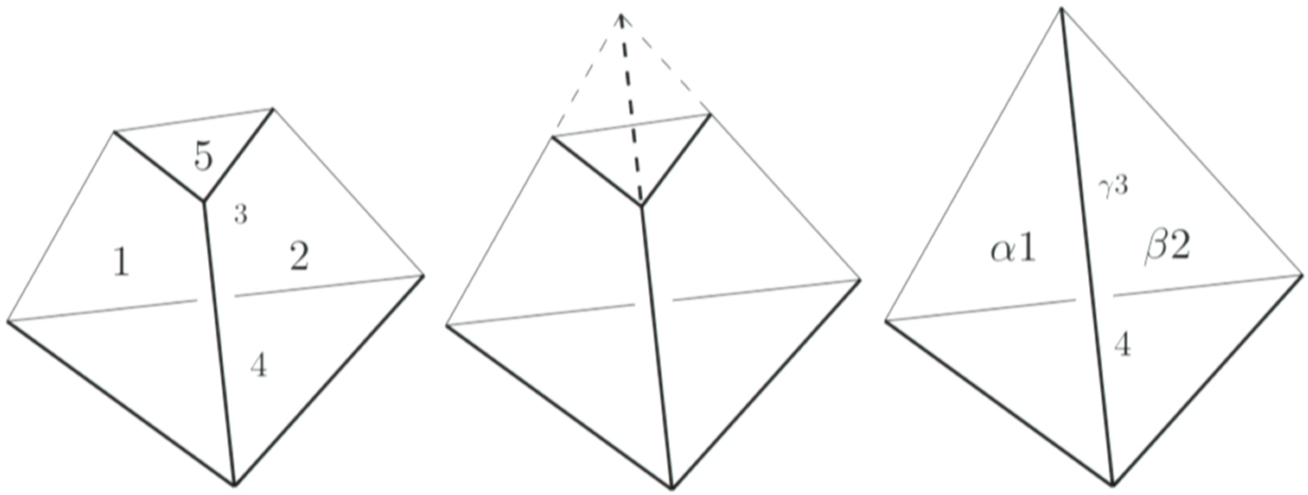

$\alpha, \beta, \gamma > 1$ found from,

$$\alpha \vec{A}_1 + \beta \vec{A}_2 + \gamma \vec{A}_3 + \vec{A}_4 = 0$$

e.g. $\implies \alpha = -\vec{A}_4 \cdot (\vec{A}_2 \times \vec{A}_3) / \vec{A}_1 \cdot (\vec{A}_2 \times \vec{A}_3)$

Volume of a pentahedron

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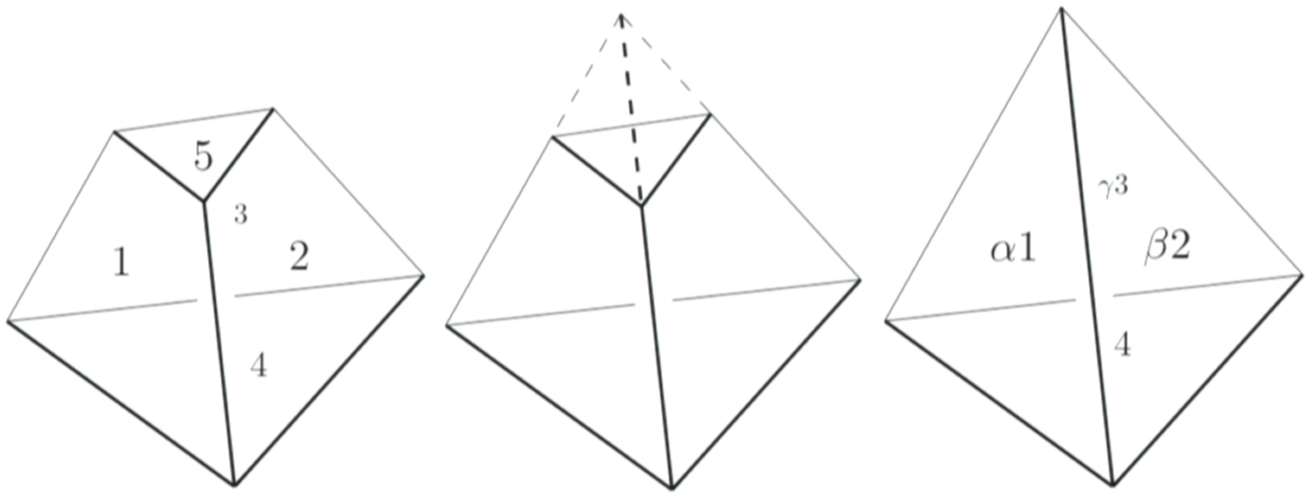
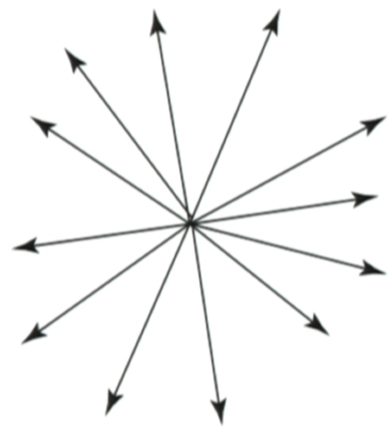


The volume of the prism is then,

$$V = \frac{\sqrt{2}}{3}\left(\sqrt{\alpha\beta\gamma} - \sqrt{(\alpha-1)(\beta-1)(\gamma-1)}\right)\sqrt{\vec{A}_1 \cdot (\vec{A}_2 \times \vec{A}_3)}$$

Adjacency and reconstruction

What's most difficult about Minkowski reconstruction? Adjacency!



Remarkable side effect of introducing α, β and γ: they completely solve the adjacency problem!

Determining the adjacency

Let $W_{ijk} = \vec{A}_i \cdot (\vec{A}_j \times \vec{A}_k)$. Different closures imply,

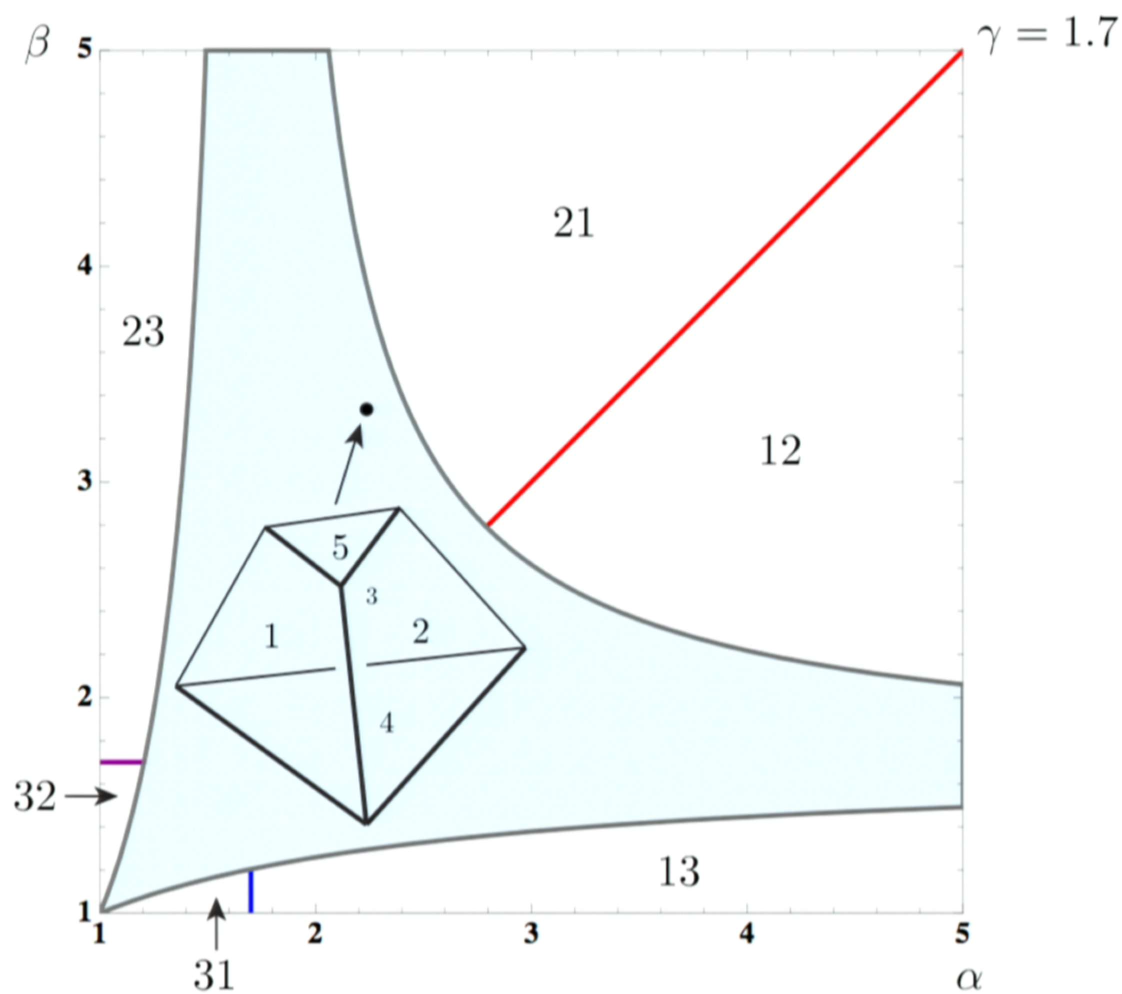
- $\alpha_1 \vec{A}_1 + \beta_1 \vec{A}_2 + \gamma_1 \vec{A}_3 + \vec{A}_4 = 0,$

$$\alpha \equiv \alpha_1 = -\frac{W_{234}}{W_{123}} \quad \beta \equiv \beta_1 = \frac{W_{134}}{W_{123}} \quad \gamma \equiv \gamma_1 = -\frac{W_{124}}{W_{123}}$$

- $\alpha_2 \vec{A}_1 + \beta_2 \vec{A}_2 + \vec{A}_3 + \gamma_2 \vec{A}_4 = 0,$

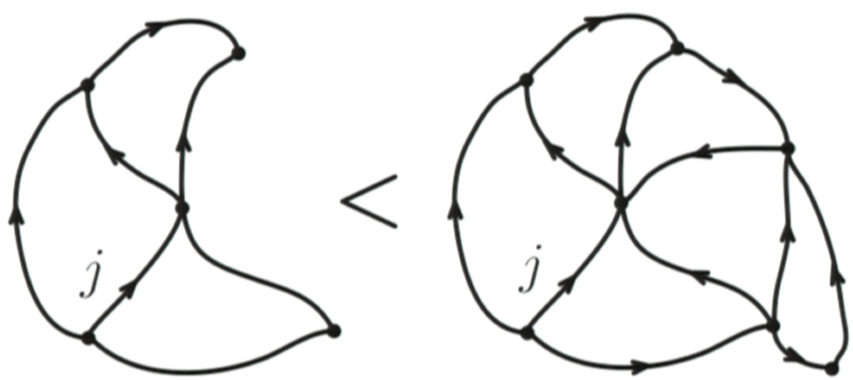
$$\alpha_2 = \frac{W_{234}}{W_{124}} = \frac{\alpha}{\gamma} \quad \beta_2 = -\frac{W_{134}}{W_{124}} = \frac{\beta}{\gamma} \quad \gamma_2 = -\frac{W_{123}}{W_{124}} = \frac{1}{\gamma}$$

They are mutually incompatible!


β
α
γ = 1.7
21
23
12
13
32
31
5
1
2
3
4
1
2
3
4
5
1
2
3
4
5

Cylindrical consistency

Smaller graphs and the associated observables can be consistently included into larger ones



Cylindrical consistency is non-trivially implemented for the polyhedral volume

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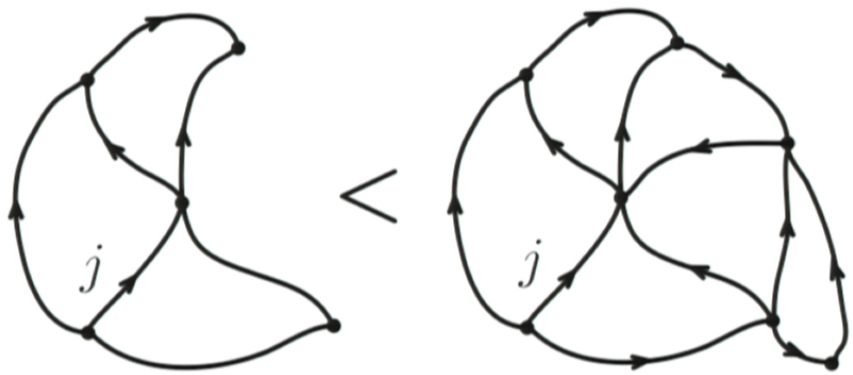

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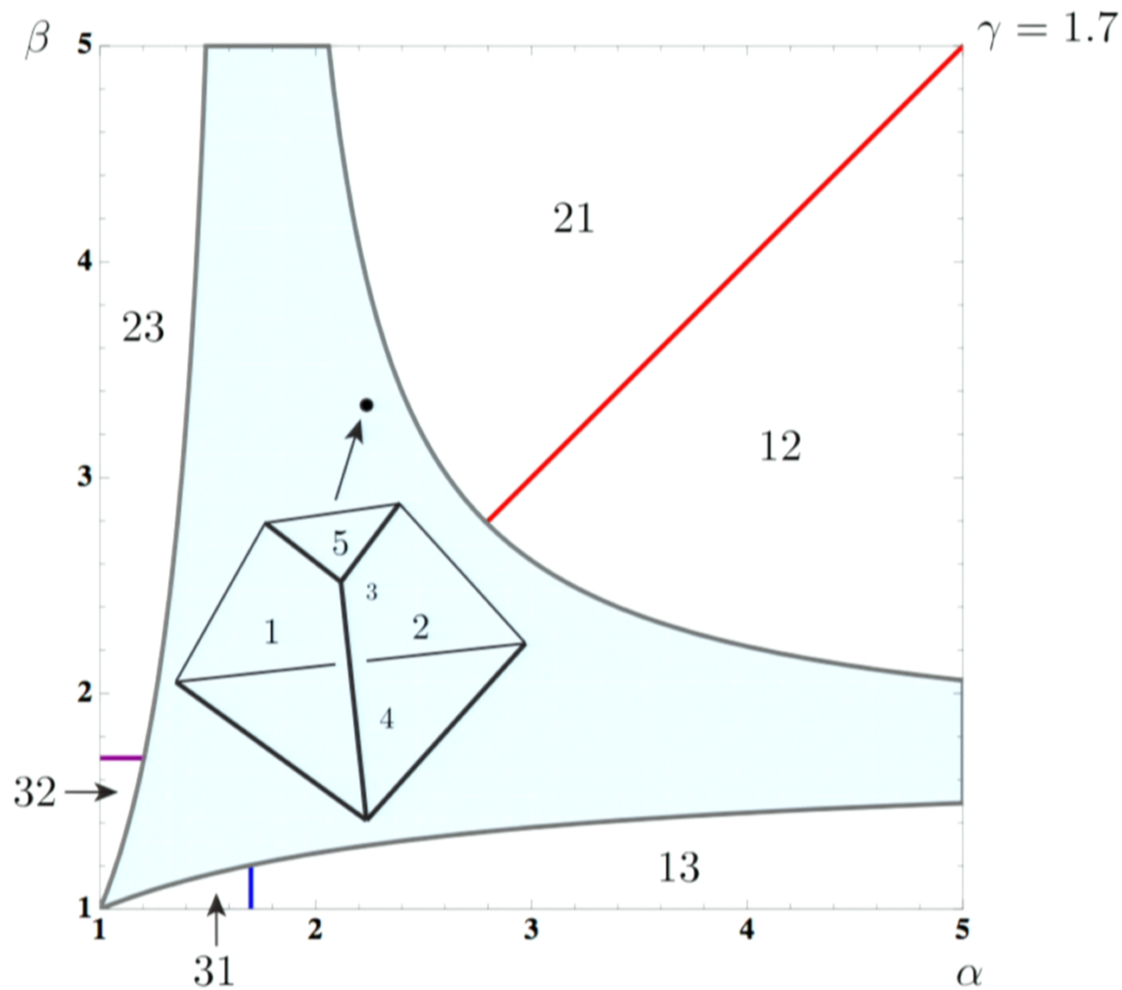
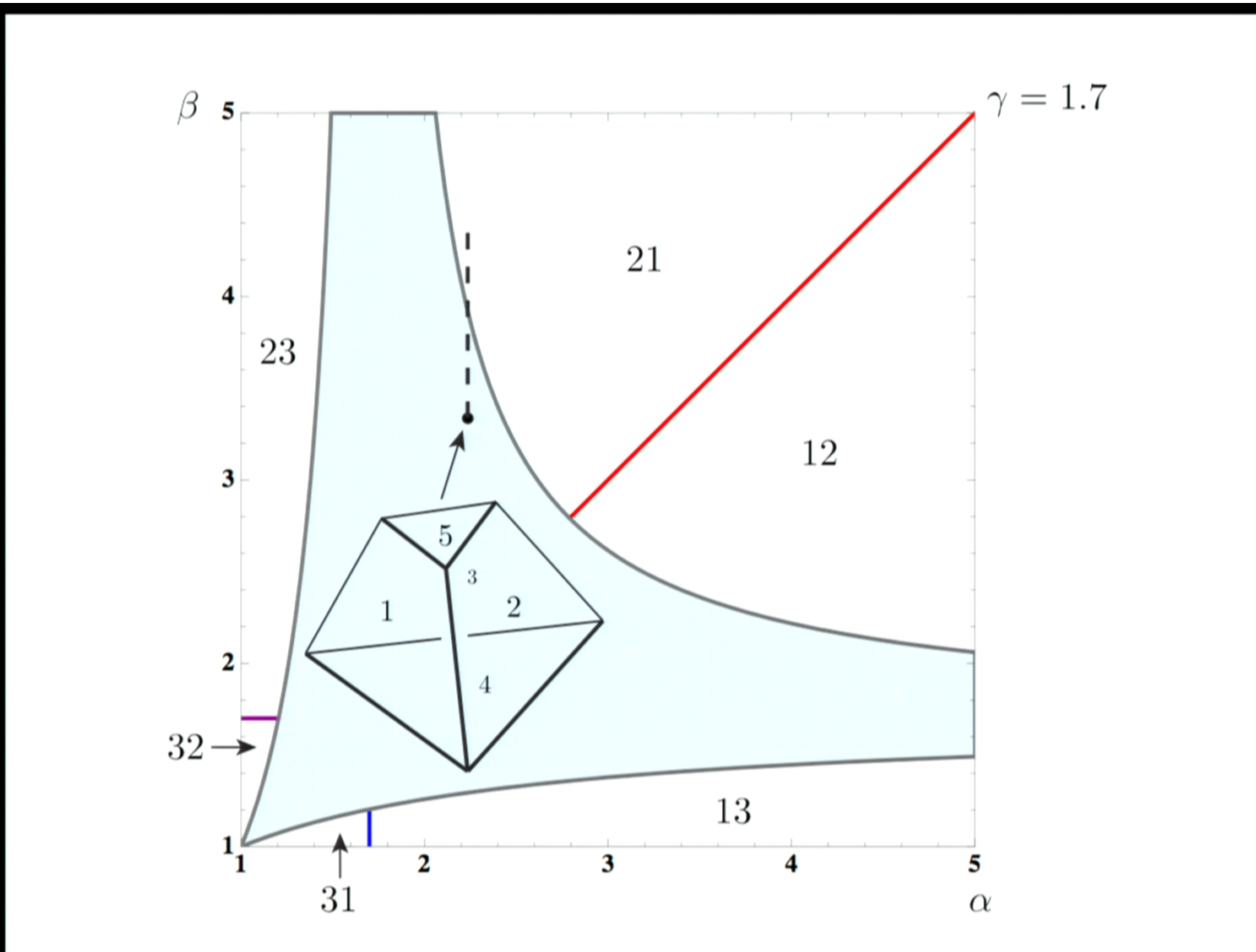


β
5
4
3
2
1
1
2
3
4
5
α
γ = 1.7
21
23
12
13
32
31
5
3
1
2
4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β
α
21
23
12
13
32
31
5
3
1
2
4

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1. Pentahedral Volume
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EBK quantization

Sommerfeld and Epstein extended Bohr's condition, $L = n\hbar$, as we have seen

$$S = \int_0^T p \frac{dq}{dt} dt = nh$$

and applied it to bounded, separable systems with d degrees of freedom,

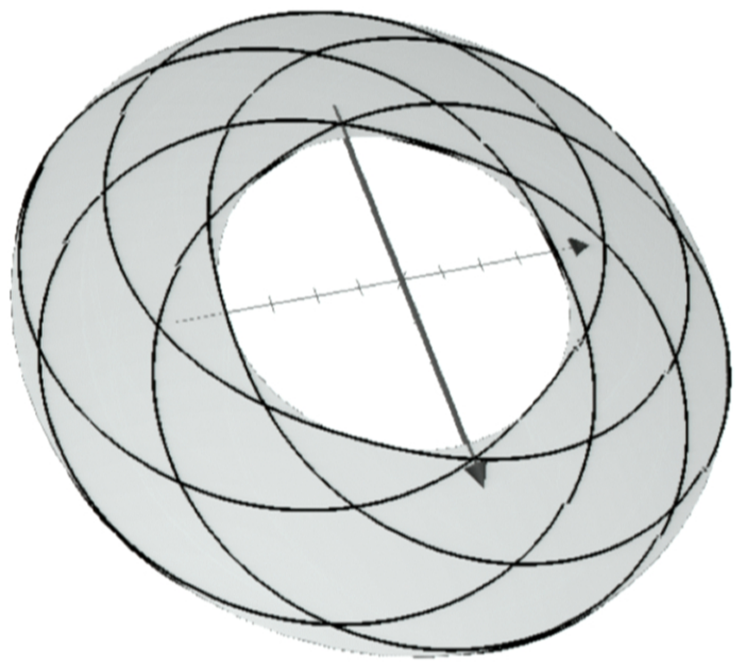
$$\int_0^{T_i} p_i \frac{dq_i}{dt} dt = n_i h, \qquad i = 1, \ldots, d$$

Here the T_i are the periods of each of the coordinates.

Einstein(!) was not satisfied. These conditions are not invariant under phase space changes of coordinates.

EBK quantization II

Motivating example: central force problems



In configuration space trajectories cross

EBK quantization II

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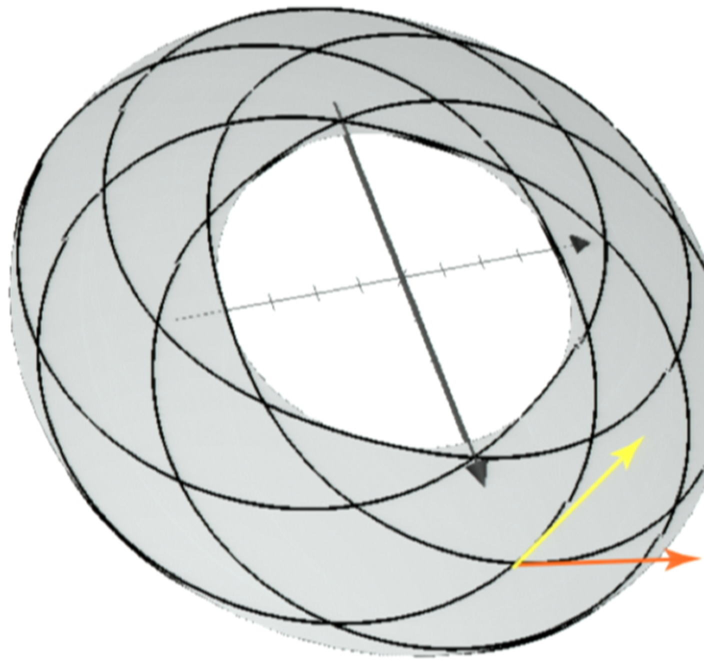

Momenta are distinct at such a crossing

EBK quantization II

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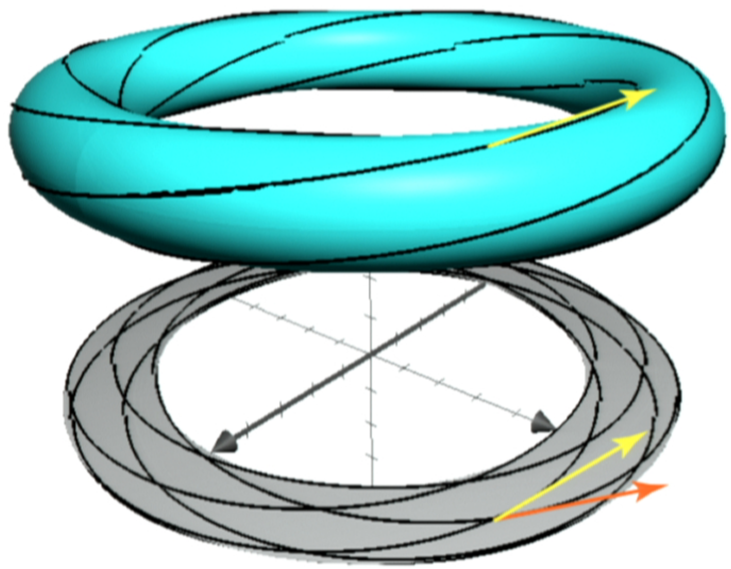


In phase space the distinct momenta lift to the two sheets of a torus

EBK quantization III

Following Poincaré, Einstein suggested that we use the invariant

$$\sum_{i=1}^{d} p_i dq_i$$

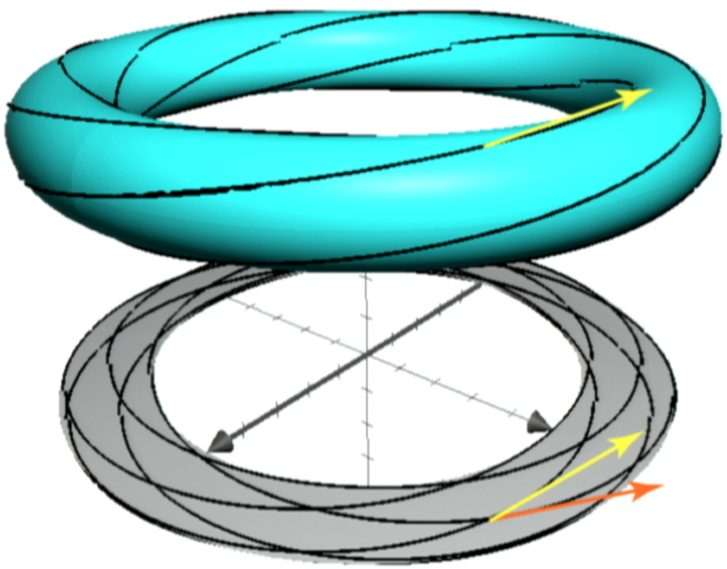
to perform the quantization.

The topology of the torus remains under coordinate changes, and so the quantization condition should be,

$$S_i = \oint_{C_i} \vec{p} \cdot d\vec{q} = n_i h.$$

EBK quantization II

Motivating example: central force problems



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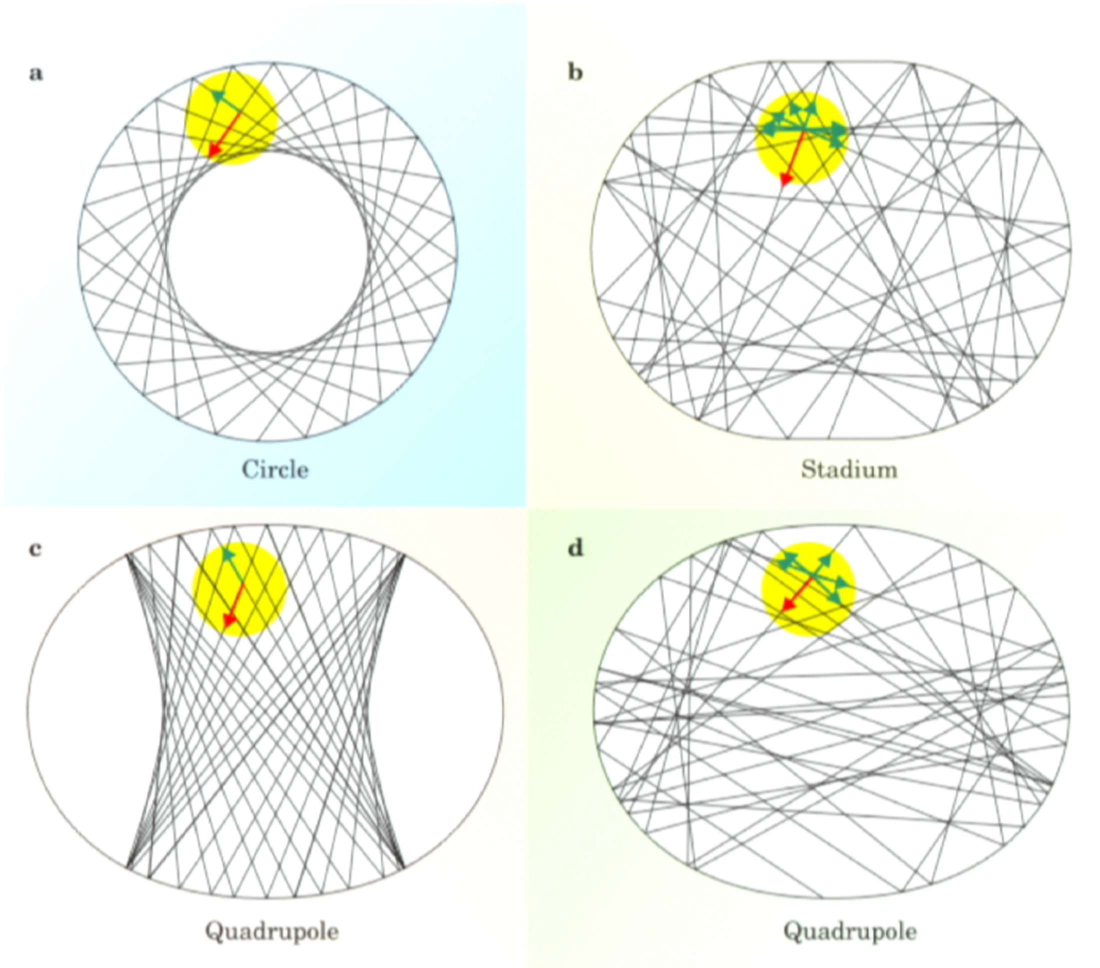
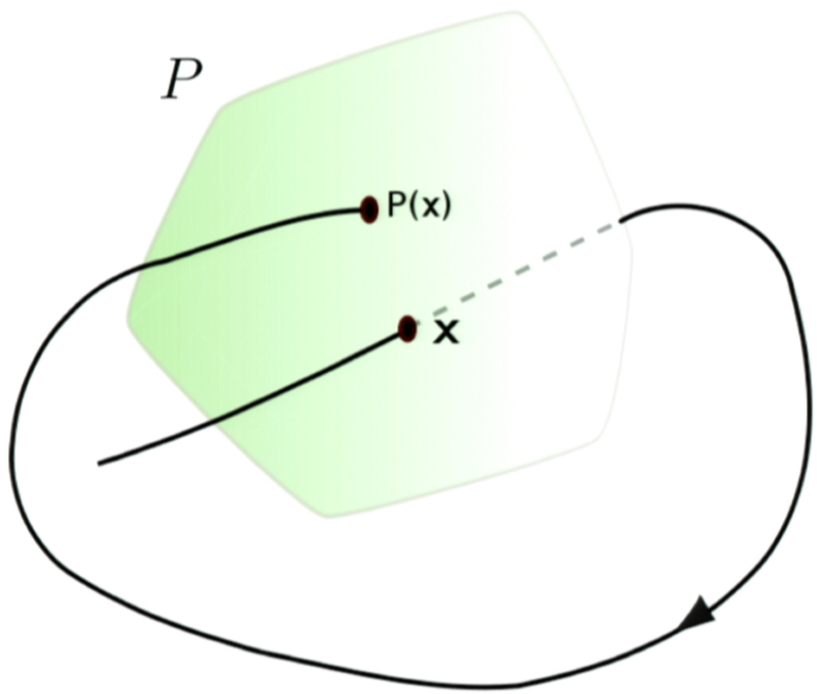
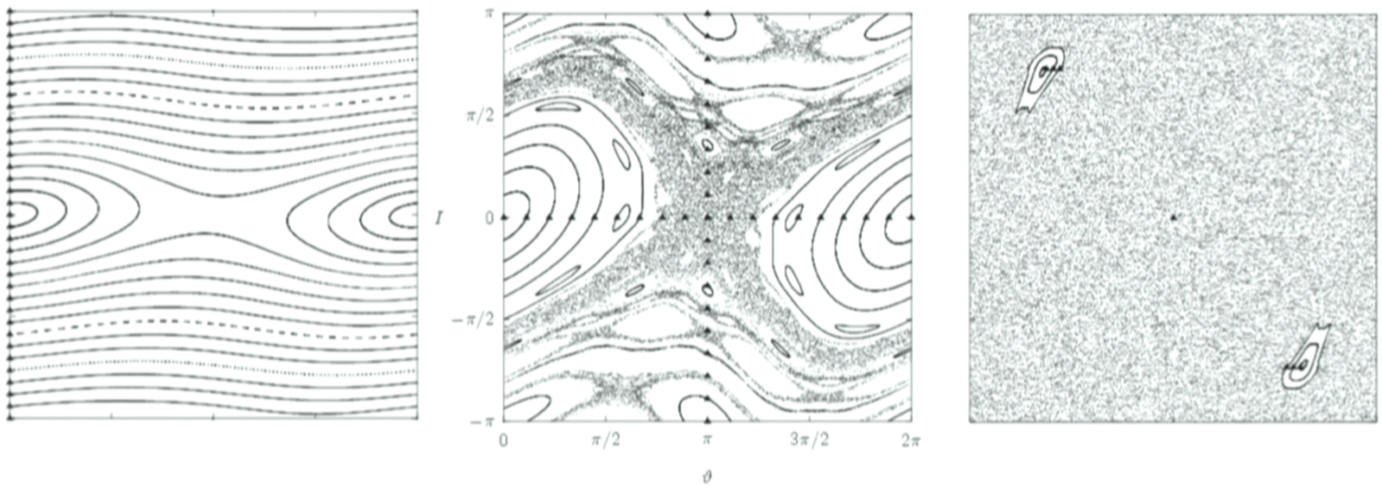

a
Circle
b
Stadium
c
Quadrupole
d
Quadrupole

Surface of section

Visualizing dynamics with a surface of section



KAM: Weak perturbation of an integrable system → Break up of those tori foliated by trajectories with rational frequency ratios



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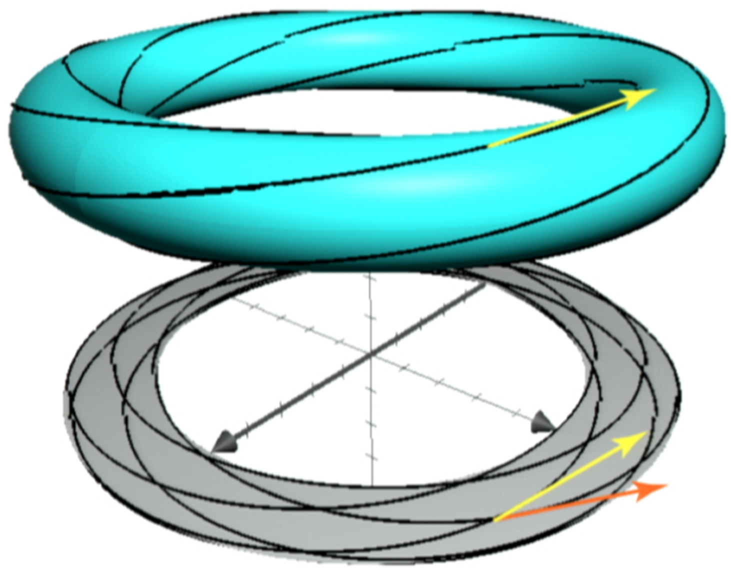

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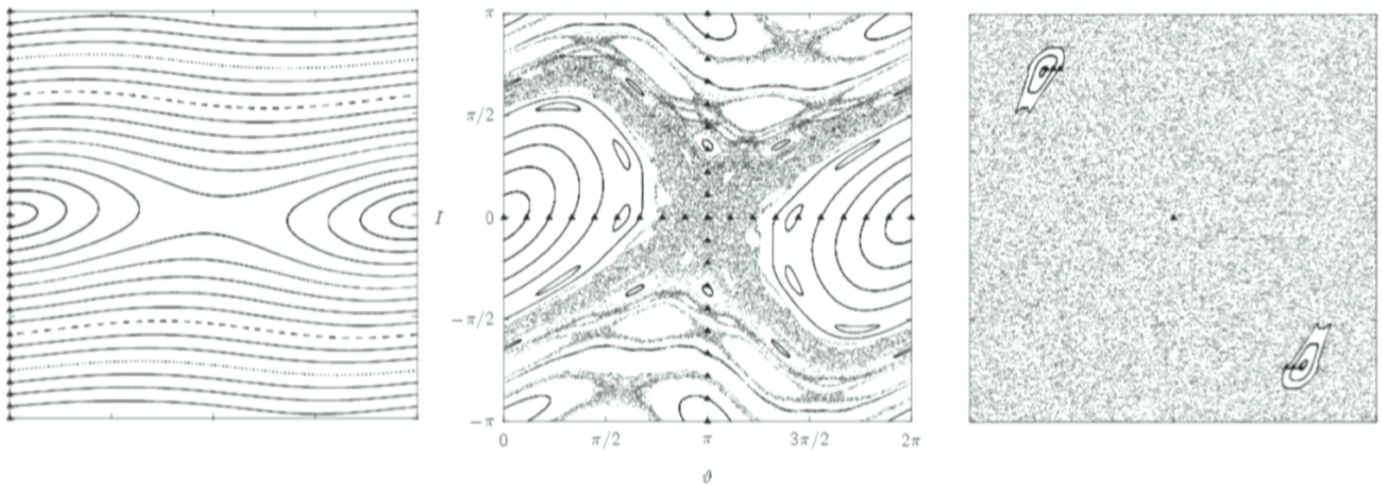

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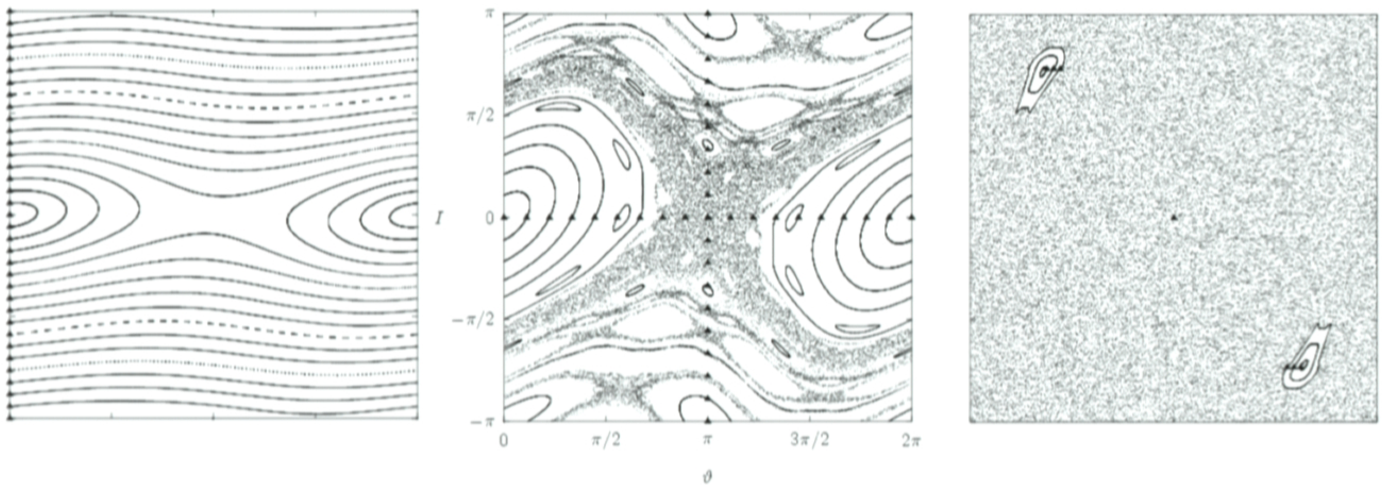

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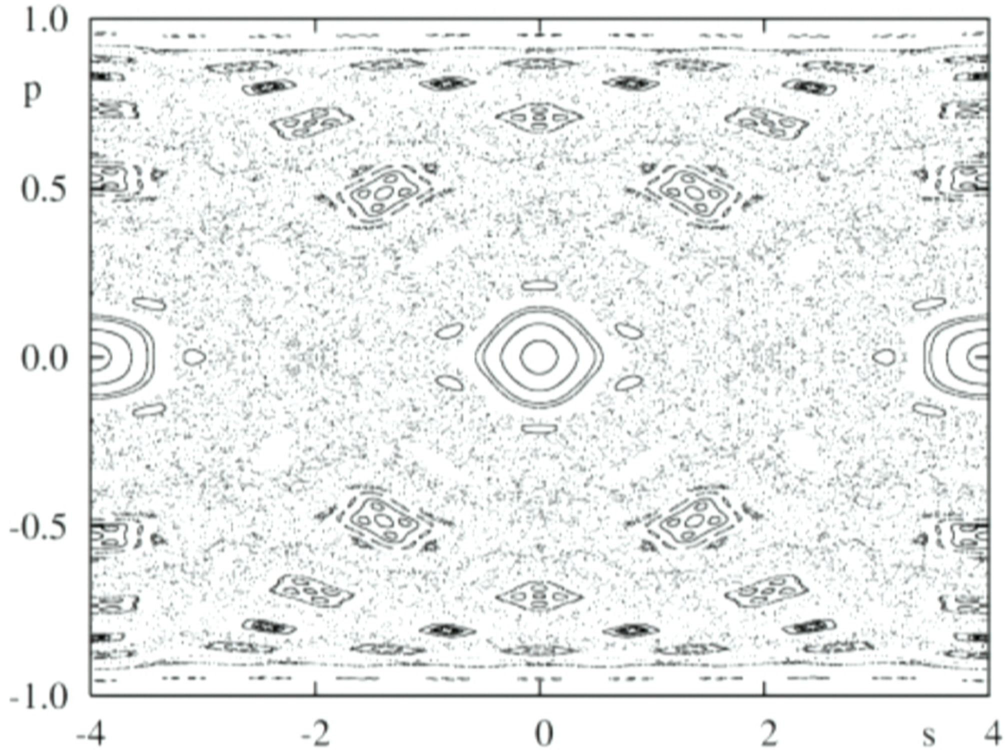

Toroidal Islands and island chains are left within a sea of chaos

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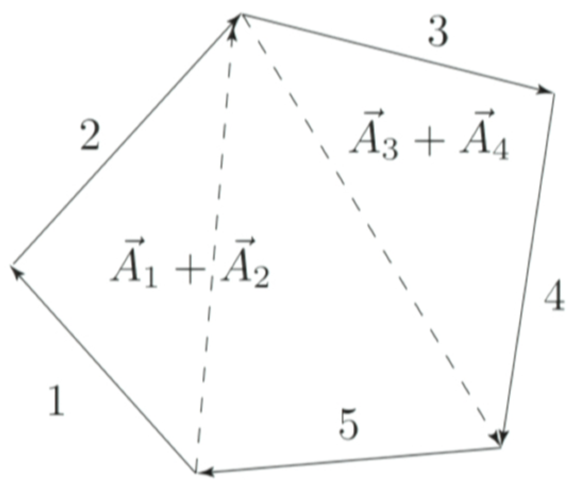
Phase space of the pentahedron I

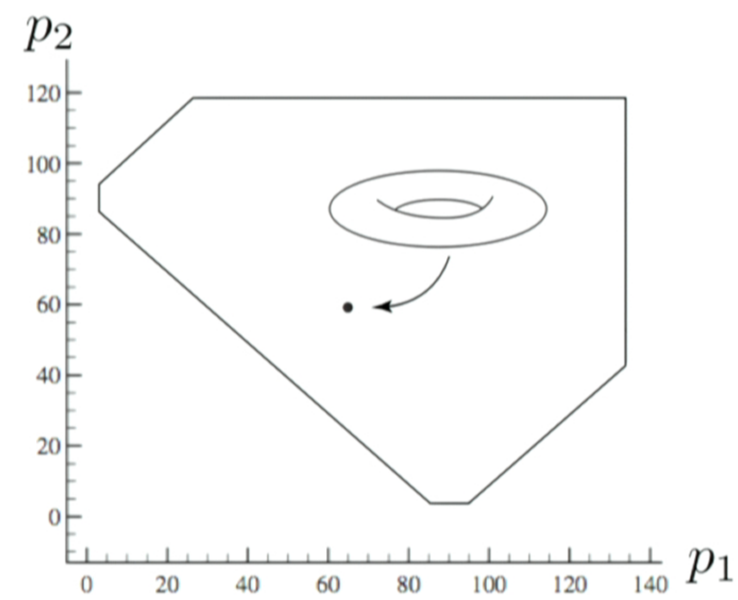
The pentahedron has two fundamental degrees of freedom,



The angles generated by $p_1 = |\vec{A}_1 + \vec{A}_2|$ and $p_2 = |\vec{A}_3 + \vec{A}_4|$.

Phase space of the pentahedron II

For fixed p_1 and p_2 these angles sweep out a torus.



The phase space consists of tori over a convex region of the $p_1 p_2$-plane.

Volume is nonlinear

The volume is a very nonlinear function of any of the variables we have considered:

$$V = \frac{\sqrt{2}}{3}\left(\sqrt{\alpha\beta\gamma} - \sqrt{(\alpha-1)(\beta-1)(\gamma-1)}\right)\sqrt{|\vec{A}_1 \cdot (\vec{A}_2 \times \vec{A}_3)|}$$

Recall,

$$\alpha = -\frac{\vec{A}_4 \cdot (\vec{A}_2 \times \vec{A}_3)}{\vec{A}_1 \cdot (\vec{A}_2 \times \vec{A}_3)}, \qquad \text{similarly for } \beta,\ \gamma$$

Forced to integrate it numerically.

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Numerical integration

Fortunately, the angular momenta can be lifted into the phase space of a collection of harmonic oscillators. This allows the use of a geometric (i.e. symplectic) integrator.

Explicit Euler: $u_{n+1} = u_n + h \cdot a(u_n)$

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Symplectic Euler:

$$u_{n+1} = u_n + h \cdot a(u_n, v_{n+1})$$
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Implementation: Symplectic integrator preserves face areas to machine precision and volume varies in 14th digit

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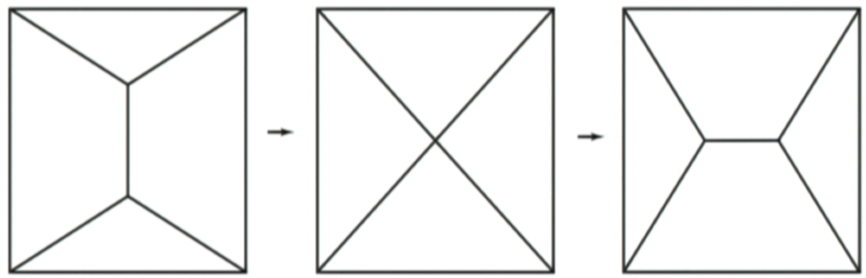
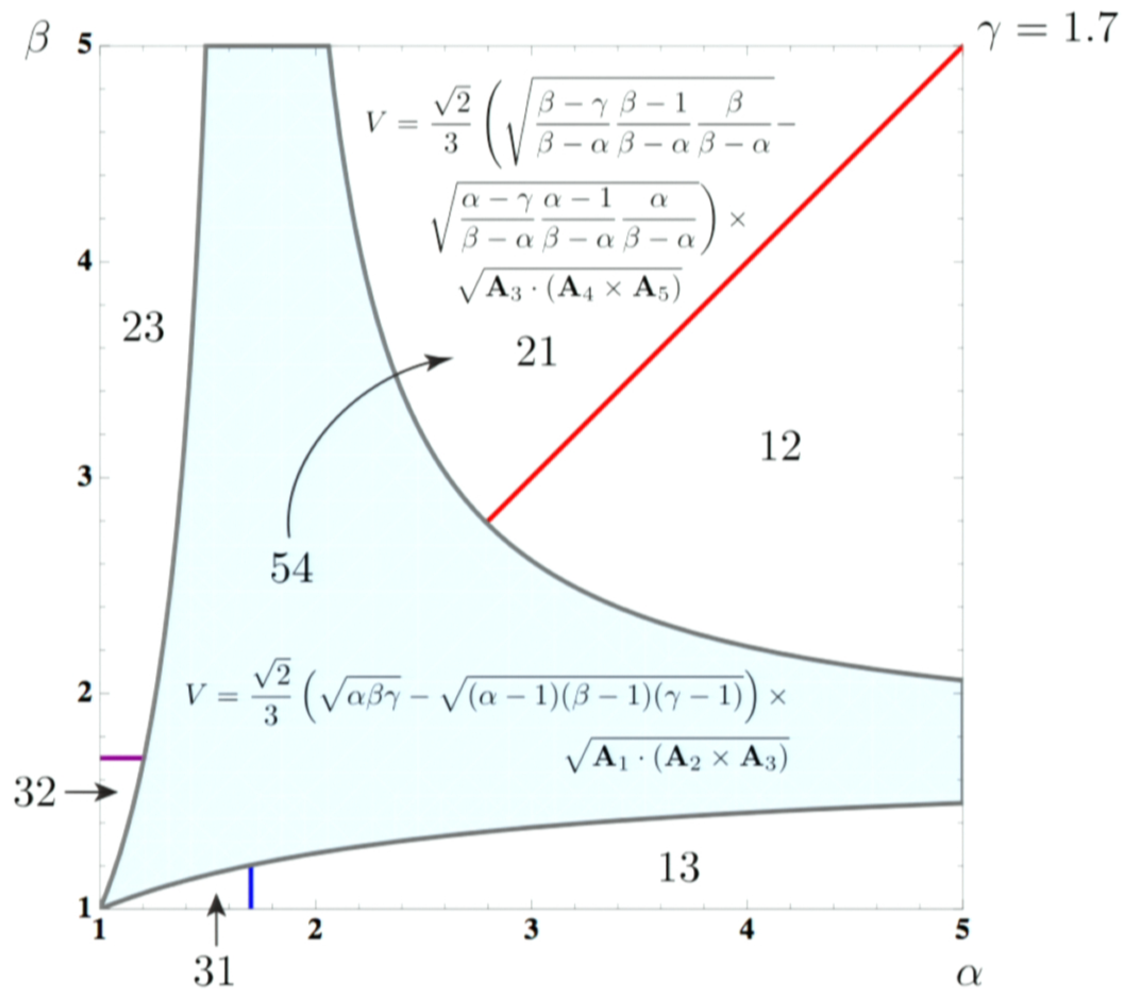
Volume dynamics: first results

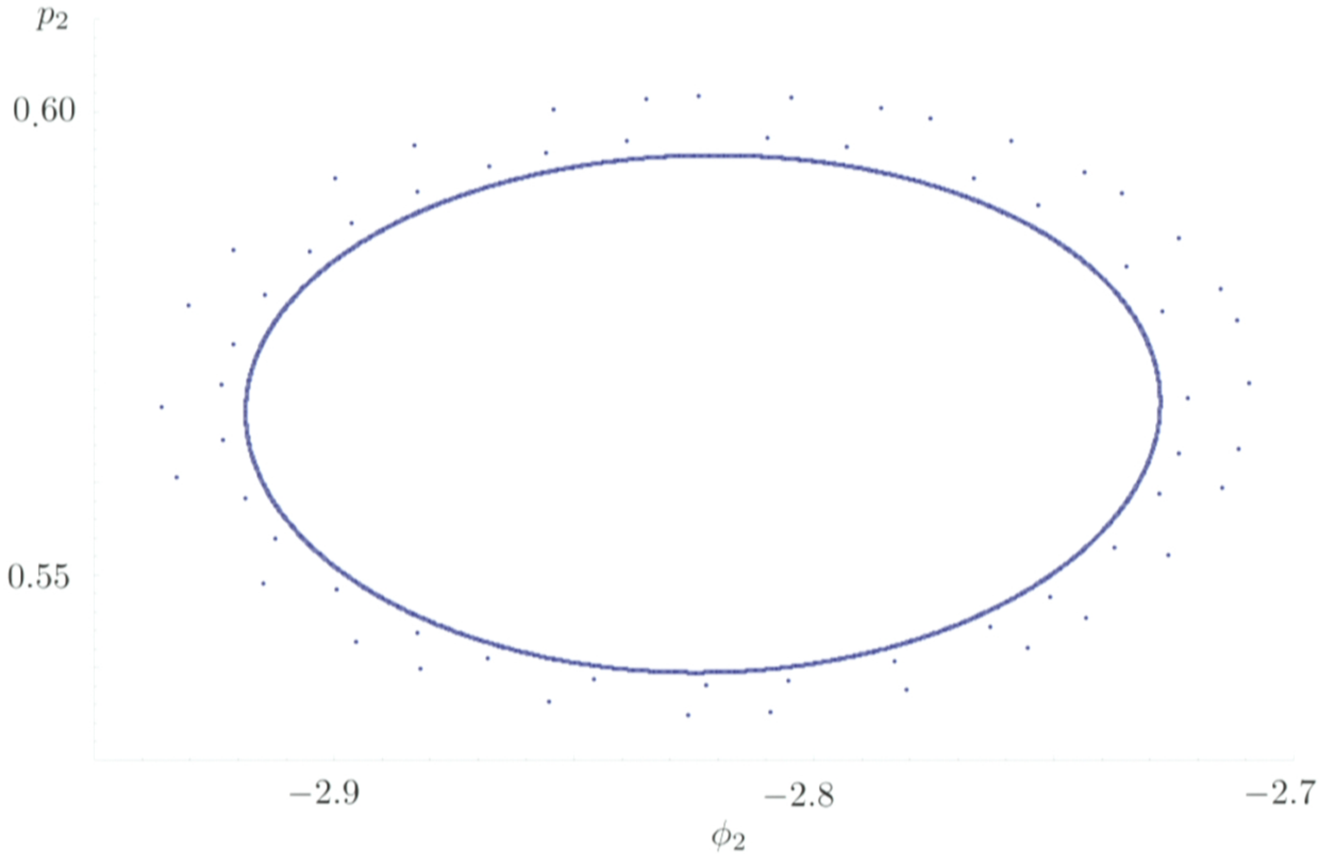
A Schlegel diagram projects a 3D polyhedron into one of its faces (left panel):



A Schlegel move merges two vertices of the diagram and and splits them apart in a different manner. This is precisely how the volume dynamics changes adjacency.


$\gamma = 1.7$
β
α
1
2
3
4
5
$V = \frac{\sqrt{2}}{3}\left(\sqrt{\frac{\beta-\gamma}{\beta-\alpha}\frac{\beta-1}{\beta-\alpha}\frac{\beta}{\beta-\alpha}} - \sqrt{\frac{\alpha-\gamma}{\beta-\alpha}\frac{\alpha-1}{\beta-\alpha}\frac{\alpha}{\beta-\alpha}}\right) \times \sqrt{\mathbf{A}_3 \cdot (\mathbf{A}_4 \times \mathbf{A}_5)}$
23
21
12
54
$V = \frac{\sqrt{2}}{3}\left(\sqrt{\alpha\beta\gamma} - \sqrt{(\alpha-1)(\beta-1)(\gamma-1)}\right) \times \sqrt{\mathbf{A}_1 \cdot (\mathbf{A}_2 \times \mathbf{A}_3)}$
32
13
31

Poincaré section of pentahedral volume dynamics



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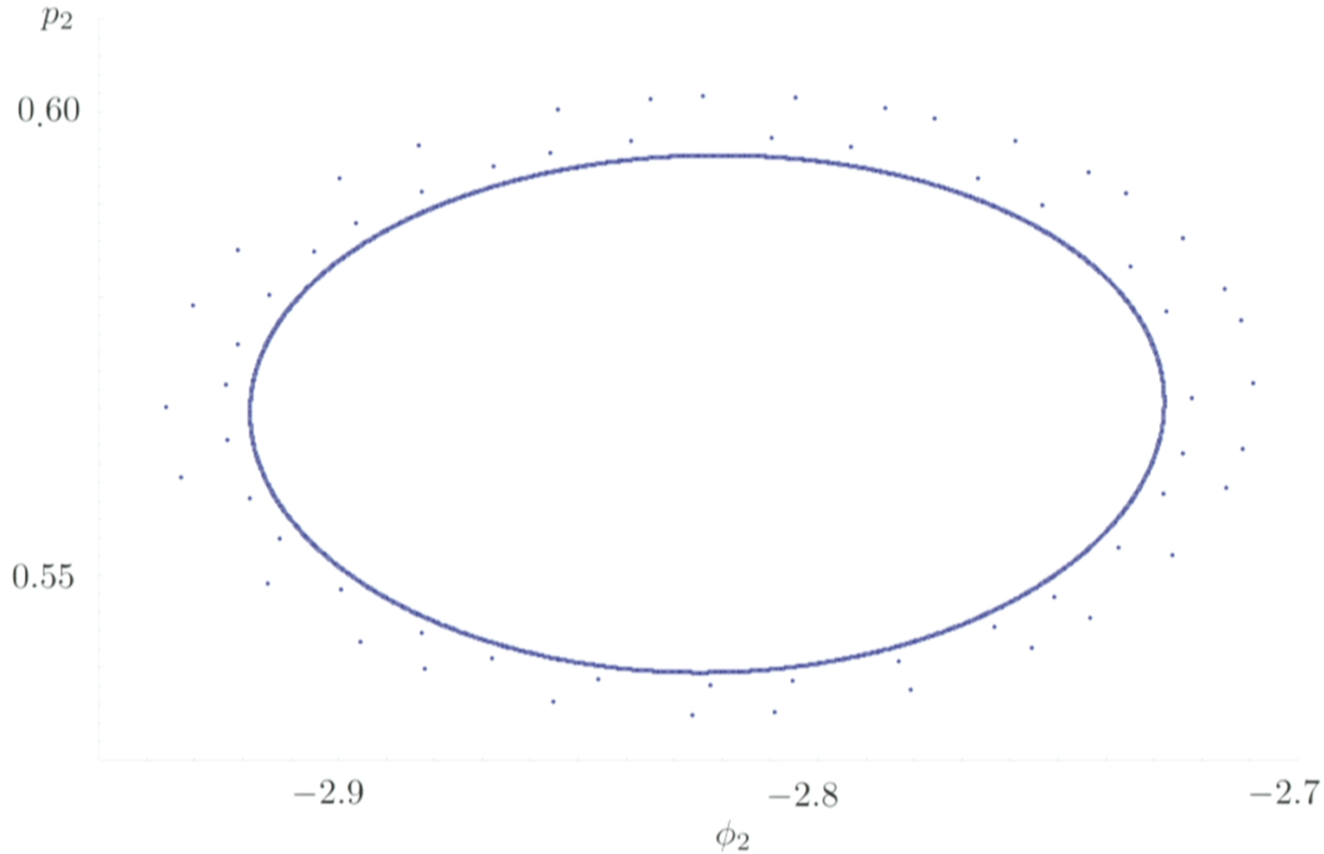
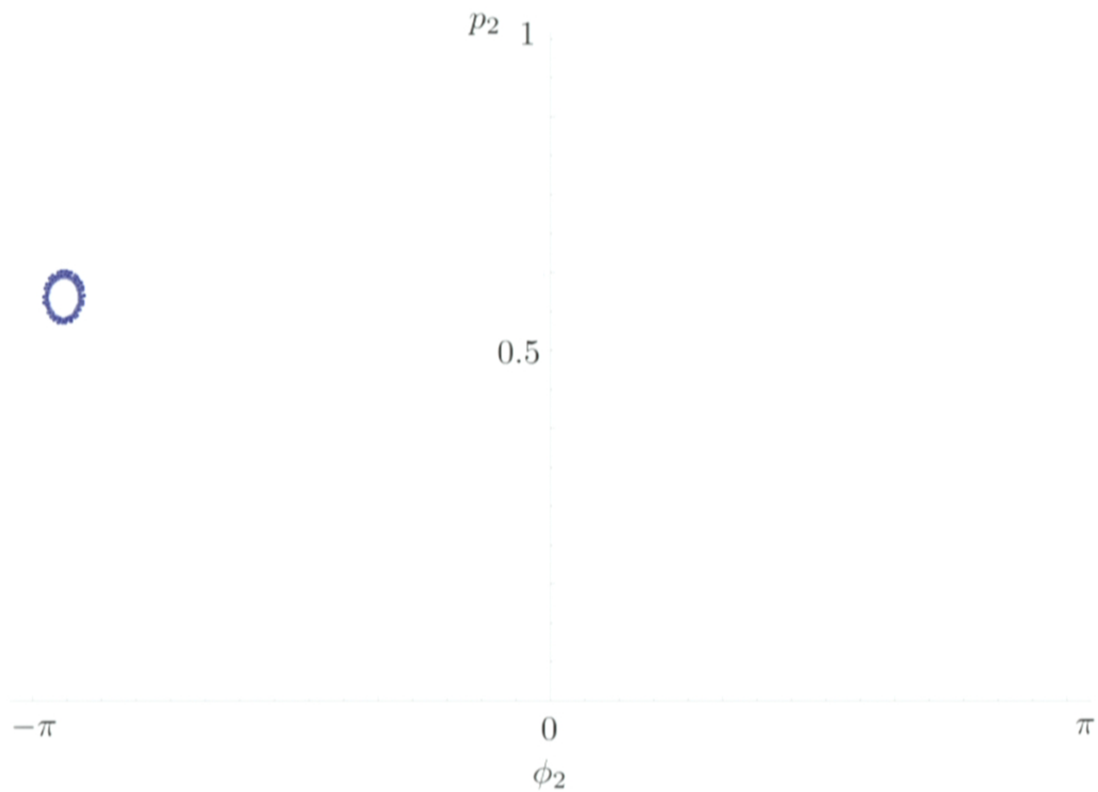


Guess: Analogy with billiards systems suggests that the dynamics will be mixed, containing chaos



Conclusions

- Minkowski reconstruction for 5 vectors solved
- There is cylindrical consistency in the polyhedral picture and it is non-trivial
- The classical polyhedral volume is only twice continuously differentiable
- Can explore the classical dynamics of the volume operator in the case of a polyhedron with 5 faces
- Does this dynamics exhibit chaos?