Title: Pitfalls of Path Integrals: Amplitudes for Spacetime Regions and the Quantum Zeno Effect

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Abstract:

Pitfalls of Path Integrals: Amplitudes for Spacetime Regions and the Quantum Zeno Effect

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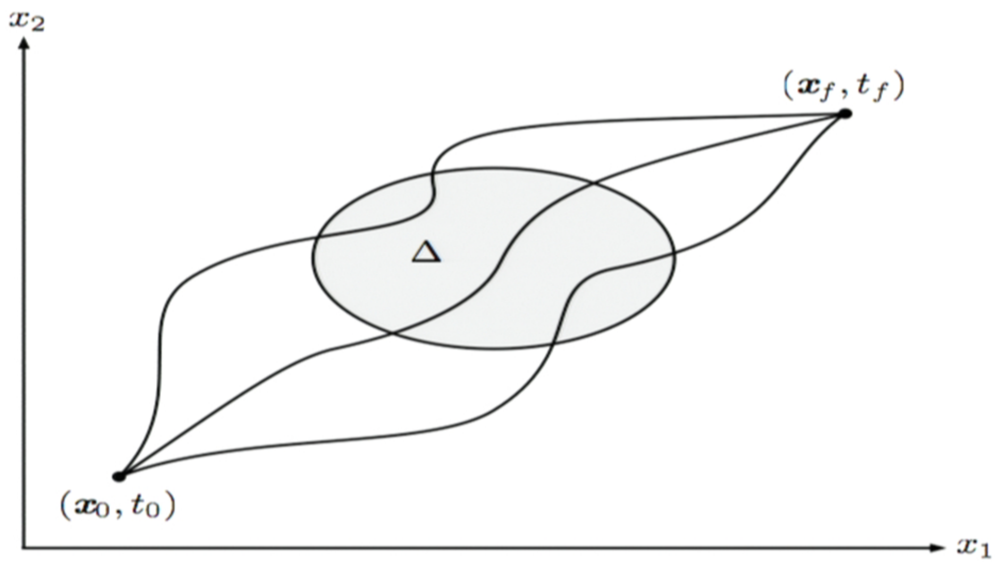
JJ Halliwell & JMY arXiv:1205.3773

Perimeter Institute Quantum Foundations Seminar

Introduction: What is amplitude for entering Δ?



"Obvious" answer is:

$$g_\Delta(\mathbf{x}_f, t_f|\mathbf{x}_0, t_0) = \int_\Delta D\mathbf{x}(t) \exp\left(i \int_{t_0}^{t_f} dt \left[\frac{1}{2} m\dot{\mathbf{x}}^2 - U(\mathbf{x}) \right] \right)$$

Standard Picture

Therefore;

$$g_\triangle(\mathbf{x}_f, t_f|\mathbf{x}_0, t_0) = \int_\triangle D\mathbf{x}(t) \exp\left(i \int_{t_0}^{t_f} dt \left[\frac{1}{2}m\dot{\mathbf{x}}^2 - U(\mathbf{x})\right]\right)$$

$$g_r(\mathbf{x}_f, t_f|\mathbf{x}_0, t_0) = \int_r D\mathbf{x}(t) \exp\left(i \int_{t_0}^{t_f} dt \left[\frac{1}{2}m\dot{\mathbf{x}}^2 - U(\mathbf{x})\right]\right)$$

and

$$g(\mathbf{x}_f, t_f|\mathbf{x}_0, t_0) = g_\triangle(\mathbf{x}_f, t_f|\mathbf{x}_0, t_0) + g_r(\mathbf{x}_f, t_f|\mathbf{x}_0, t_0)$$

where g is the usual free propagator.

Standard Picture

In operator form, $g_r(\mathbf{x}_f, t_f|\mathbf{x}_0, t_0) = \langle \mathbf{x}_f | \hat{g}_r(t_f, t_0) | \mathbf{x}_0 \rangle$ etc and,

$$
\begin{aligned}
p_\Delta &= \langle \psi | \hat{g}_\Delta(t_f, t_0)^\dagger \hat{g}_\Delta(t_f, t_0) | \psi \rangle \\
p_r &= \langle \psi | \hat{g}_r(t_f, t_0)^\dagger \hat{g}_r(t_f, t_0) | \psi \rangle
\end{aligned}
$$

For these to be sensible probabilities we need,

$$p_\Delta + p_r = 1$$

which requires,

$$\mathrm{Re} \langle \psi | \hat{g}_r(t_f, t_0)^\dagger \hat{g}_\Delta(t_f, t_0) | \psi \rangle = 0$$

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$$\begin{aligned} p_\Delta &= \langle \psi | \hat{g}_\Delta(t_f, t_0)^\dagger \hat{g}_\Delta(t_f, t_0) | \psi \rangle \\ p_r &= \langle \psi | \hat{g}_r(t_f, t_0)^\dagger \hat{g}_r(t_f, t_0) | \psi \rangle \end{aligned}$$

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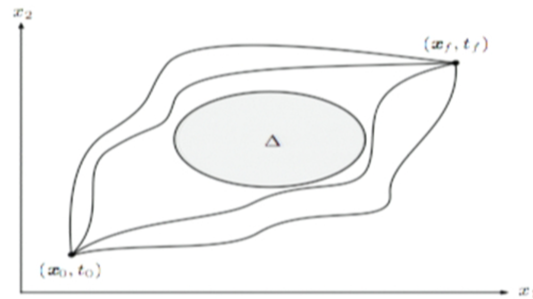
Standard Picture

- These constructions used in quantum cosmology and quantum gravity, and to address issues concerning time in QT such as arrival, dwell and tunneling times.
- Feynman's original paper was entitled "Space-time approach to non-relativistic quantum mechanics," suggesting spacetime features of path integral should be made use of.

The Problem: First Look

Focus on g_r,

$$g_r(\mathbf{x}_f, t_f|\mathbf{x}_0, t_0) = \int_r D\mathbf{x}(t) \exp\left(i \int_{t_0}^{t_f} dt \left[\frac{1}{2}m\dot{\mathbf{x}}^2 - U(\mathbf{x})\right]\right)$$



How do we define this?

Time slicing, $t_f - t_0 = n\epsilon$, consider propagation between slices $t_k = t_0 + k\epsilon$

$$g_r(\mathbf{x}_f, t_f|\mathbf{x}_0, t_0) := \lim_{\epsilon\to 0, n\to\infty} \int_r d^d x_1 \ldots \int_r d^d x_{n-1} \prod_{k=1}^{n} \left(\frac{m}{2\pi i\epsilon}\right)^{d/2} \exp\left(iS(\mathbf{x}_k, t_k|\mathbf{x}_{k-1}, t_{k-1})\right)$$

Where $\mathbf{x}_n = \mathbf{x}_f$.

- Key problem, this describes unitary propagation in Hilbert space of states with support outside Δ.
- Therefore $p_r = 1$ for any initial state. So either $p_r + p_\Delta \neq 1$ or $p_\Delta = 0$ for any initial state, which is physically nonsensical.
- Differently put, restriction on paths in $\hat{g}_r$ effectively sets reflecting boundary conditions on the boundary of Δ, whereas to obtain the intuitive result one needs the incoming state to be absorbed.

A Brief Note On Generality

- Discussion so far is NRQM in d-dims.
- Δ could consist of multiple regions and could vary in time.
- Assume Δ is large and smooth (compared to $1/p$).
- Can also consider RQM and path integrals in curved space-time.

An important application of these ideas is quantum cosmology.

$$\hat{g}_r(t_f, t_0) = \lim_{\epsilon \to 0, n \to \infty} \bar{P} e^{-iH\epsilon} \bar{P} \ldots e^{-ih\epsilon} \bar{P}$$

Mathematically equivalent to continuous measurement - Quantum Zeno Effect

Indeed limit may be computed explicitly,

$$\hat{g}_r(t_f, t_0) = \bar{P} \exp\left(-i\bar{P}H\bar{P}(t_f - t_0)\right)$$

Unitary on space of states with support outside Δ, so state never leaves Δ.

This is the origin of the counter-intuitive properties of these path integral expressions

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Proposed Solution

Since the root of the problem is the Zeno effect solution is to soften the "monitoring" of the system to minimise reflection

Two ways:

- Decline to take the limit $\epsilon \longrightarrow 0$
- Replace $\bar{P}$ by a POVM

Note that we are not defining these amplitudes in terms of measurements.

Using the relationship between the PI and the operator expression to come up with an alternative PI expression not suffering from Zeno.

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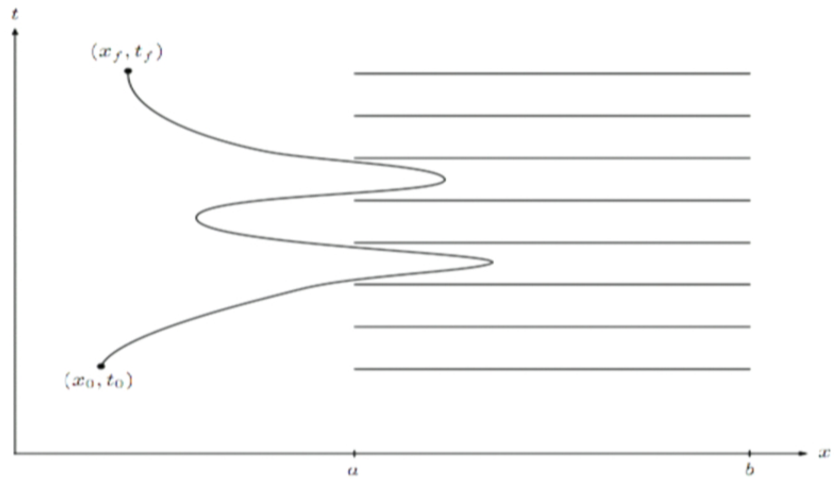
First approach, define

$$\hat{g}_r^{\epsilon}(t_f, t_0) = \bar{P} e^{-iH\epsilon} \bar{P} \dots e^{-iH\epsilon} \bar{P}$$

Again $n+1$ projectors, n unitary evolutions and $n\epsilon = t_f - t_0$



- New coarse-graining parameter ϵ.
- Tends to g_r for $\epsilon \to 0$.
- Interesting regime where ϵ small enough to monitor the paths well, but large enough to avoid reflection.

Leads to second approach.

Useful approximate alternative to strings of projectors.

Proven that in 1D and for $P = \theta(\hat{x})$

$$\bar{P}e^{-iH\epsilon}\bar{P}\dots e^{-iH\epsilon}\bar{P} \approx \exp\left(-i(H - iV_0P)(t_f - t_0)\right)$$

which is propagation with a complex potential $U(x) = -iV_0P$.

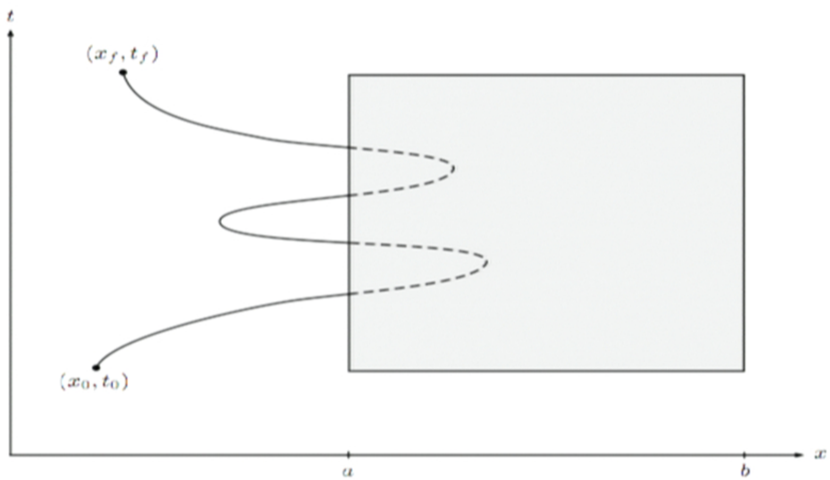
Equivalence for $V_0 \sim 1/\epsilon$ and for states with $\epsilon << 1/\Delta H$

General arguments for equivalence not tied to this particular case so expect it to hold generally

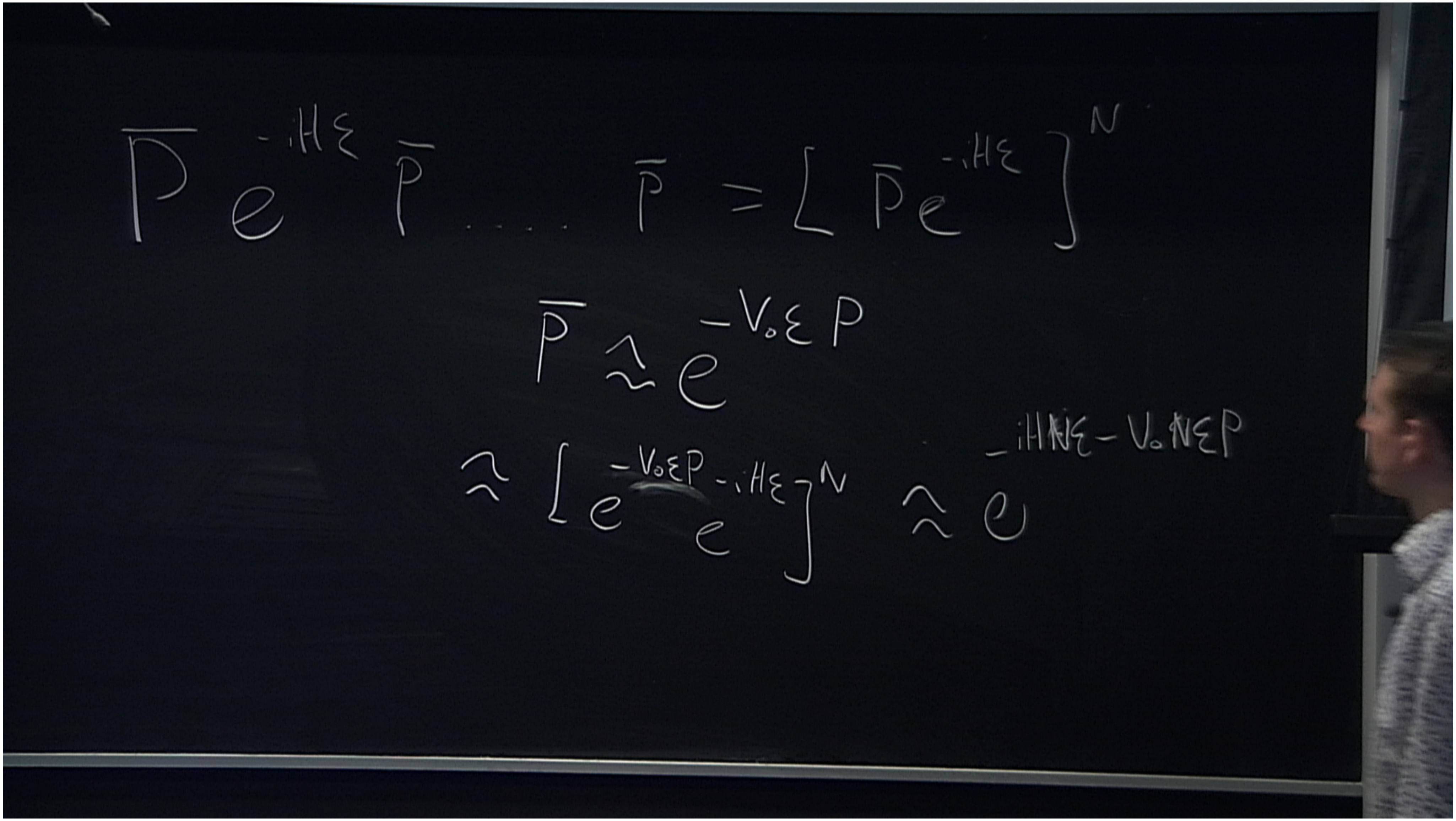
JJ Halliwell & JMY, J.Phys.A 43, 445303 (2010).

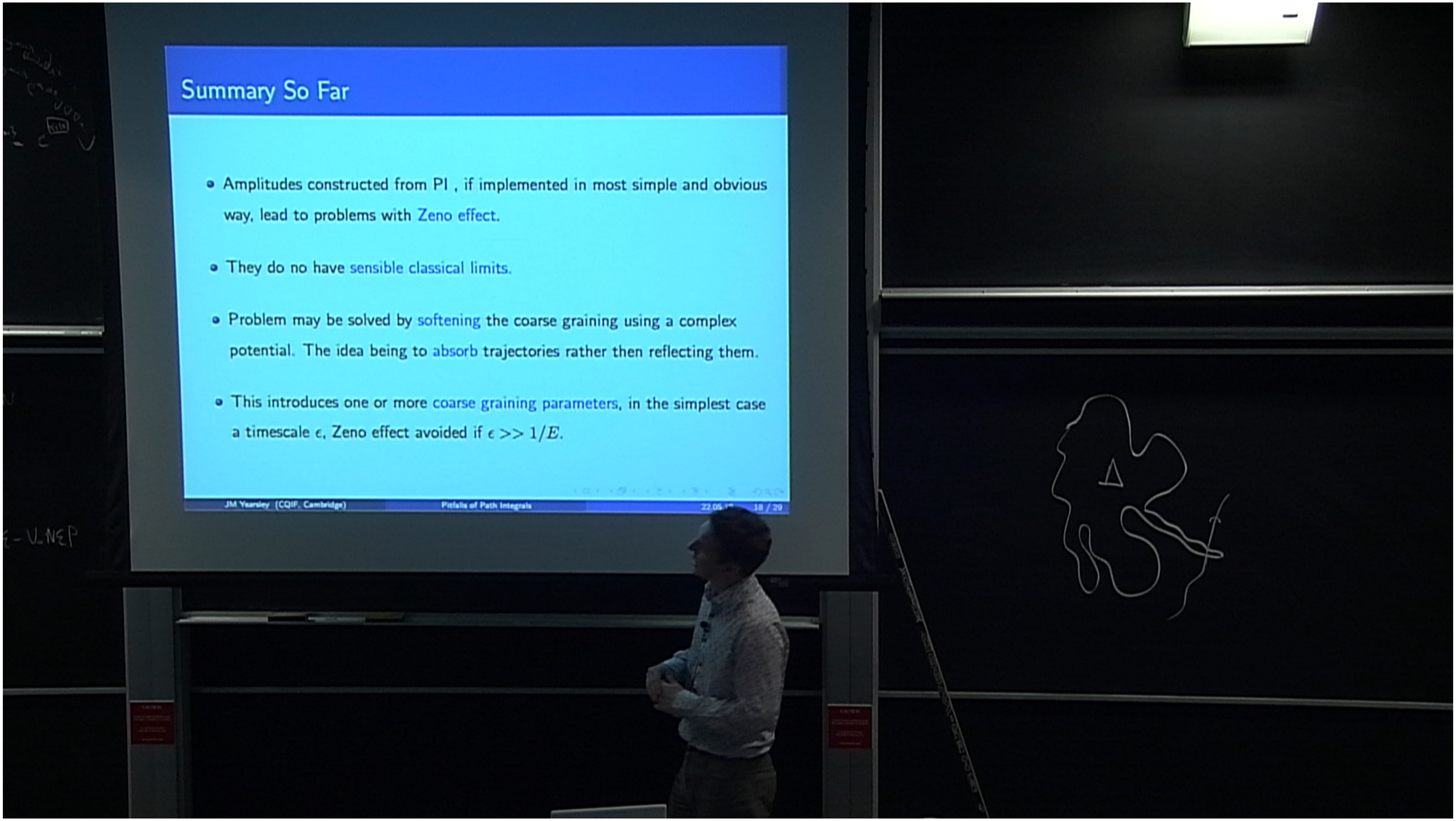
$$\hat{g}_r^V(t_f, t_0) := \exp\left(-i(H - iV_0 P)(t_f - t_0)\right)$$
$$g_r^V(\mathbf{x}_f, t_f|\mathbf{x}_0, t_0) = \int D\mathbf{x}(t) \exp\left(i\int_{t_0}^{t_f} dt \left[\frac{1}{2}m\dot{\mathbf{x}}^2 - U(\mathbf{x}) + iV_0 f_\Delta(\mathbf{x})\right]\right)$$

Equivalent to replacing $\bar{P}$ with POVM $\exp(-V_0 \epsilon f_\Delta(\mathbf{x}))$



- $f_\Delta(\mathbf{x})$ is a window function onto Δ.
- Tends to g_r for $V_0 \to \infty$.
- Interesting regime where V_0 large enough to suppress paths entering Δ, but small enough to avoid reflection.


$\bar{P} e^{-iH\varepsilon} \bar{P} \cdots \bar{P} = [\bar{P} e^{-iH\varepsilon}]^N$
$\bar{P} \approx e^{-V_0 \varepsilon P}$
$\approx [e^{-V_0 \varepsilon P} e^{-iH\varepsilon}]^N \approx e^{-iHN\varepsilon - V_0 N \varepsilon P}$


Summary So Far
Amplitudes constructed from PI , if implemented in most simple and obvious way, lead to problems with Zeno effect.
They do no have sensible classical limits.
Problem may be solved by softening the coarse graining using a complex potential. The idea being to absorb trajectories rather then reflecting them.
This introduces one or more coarse graining parameters, in the simplest case a timescale ϵ, Zeno effect avoided if $\epsilon >> 1/E$.
JM Yearsley (CQIF, Cambridge)
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Simple Example I

Consider some simple 1D examples to illustrate the ideas

Consider a free particle in 1D consisting of mainly positive momenta and initially localised in $x < 0$ so $P\,|\psi\rangle = |\psi\rangle$ where $P = \theta(\hat{x})$

Take Δ to be $x > 0$.

What is probability that particle either enters or does not enter Δ during $[0, \tau]$ and ends at a point $x_f < 0$ at time τ?

This is related to arrival time problem.

Other approaches to this problem give $\text{Prob}(0, \tau) \sim \int_0^\tau dt\, \langle J(t) \rangle$

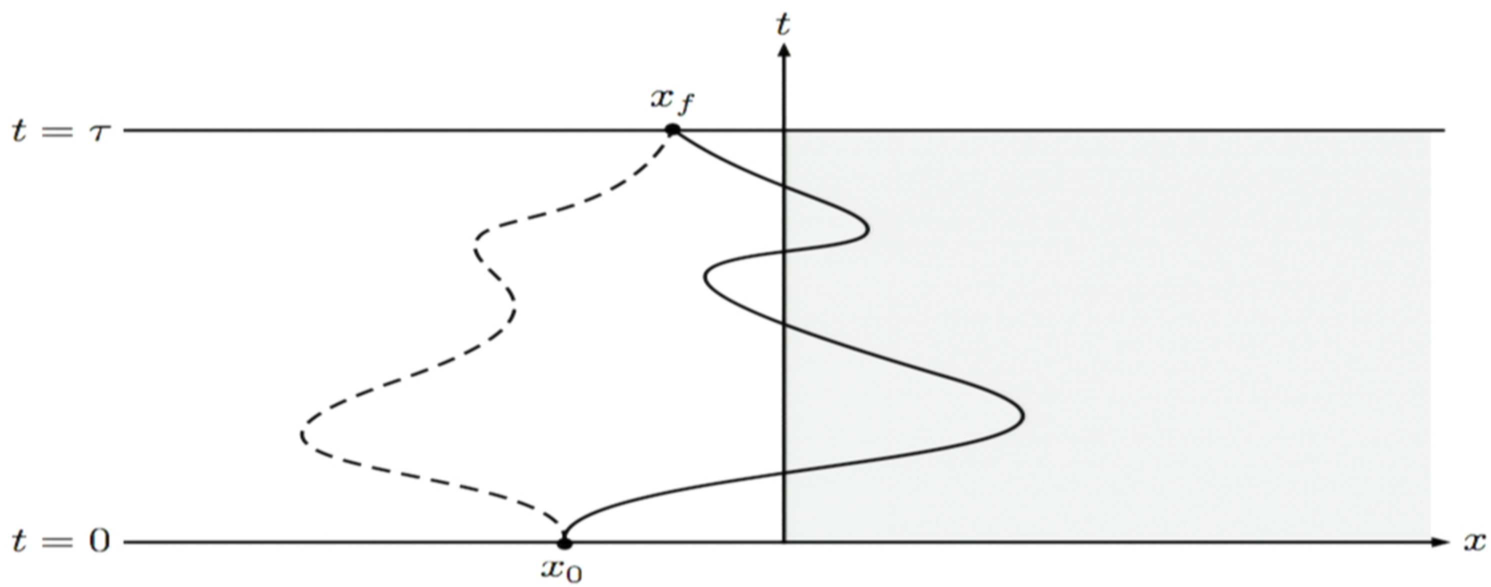

Figure: Amplitude for entering or not entering $x > 0$ during $[0, \tau]$ and ending at $x_f < 0$ is given by sum over paths that respectively enter (solid line) or do not enter (dashed line) $x > 0$

Amplitude for not entering $x > 0$ given by restricted propagator, which in this case can be constructed by method of images

$$g_r(x,t|x_0,0) = \theta(-x)\theta(-x_0)[g(x,\tau|x_0,0) - g(x,\tau|-x_0,0)]$$

Here g is the usual free particle propagator

$$g(x,\tau|x_0,0) = \left(\frac{m}{2\pi i\tau}\right)^{1/2} \exp\left(\frac{im(x-x_0)^2}{2\tau}\right)$$

Can also be written as

$$\hat{g}_r(\tau,0) = \bar{P}(1-R)e^{-iH\tau}\bar{P}$$

where $\bar{P} = \theta(-\hat{x})$ and $R = \int dx\, |-x\rangle\langle x|$

This gives Zeno effect and $p_r = 1$ for any ψ.

Nevertheless let's explore $g_\triangle$, given by

$$\hat{g}_\triangle(\tau, 0) = \bar{P}(\hat{g}(\tau, 0) - \hat{g}_r(\tau, 0))\bar{P} = \bar{P} R e^{-iH\tau} \bar{P}$$

We then have

$$p_\triangle(\tau, 0) = \langle\psi| \, \hat{g}_\triangle(\tau, 0)^\dagger \hat{g}_\triangle(\tau, 0) \, |\psi\rangle = \langle\psi| \, \bar{P} P(\tau) \bar{P} \, |\psi\rangle$$

Now note that

$$P(\tau) = P + \int_0^\tau dt \hat{J}(t), \quad \hat{J} = \frac{1}{2m}(\hat{p}\delta(\hat{x}) + \delta(\hat{x})\hat{p})$$

So finally,

$$p_\triangle(0, \tau) = \int_0^\tau dt \, \langle\psi| \, \hat{J}(t) \, |\psi\rangle$$

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$$p_\Delta(0, \tau) = \int_0^\tau dt \langle\psi| \hat{J}(t) |\psi\rangle$$

Slight modification, paths now end at any x_f at $t_f > \tau$

Propagator for not entering is similar,

$$\hat{g}_r(t_f, 0) = e^{-iH(t_f - \tau)} \bar{P}(1 - R) e^{-iH\tau} \bar{P}$$

Still have $p_r = 1$ for any ψ

However

$$\begin{aligned} \hat{g}_\Delta(t_f, 0) &= (\hat{g}(t_f, 0) - \hat{g}_r(t_f, 0))\bar{P} \\ &= e^{-iH(t_f - \tau)} \left(\bar{P} R e^{-iH\tau} \bar{P} + P e^{-iH\tau} \bar{P} \right) \end{aligned}$$

These terms make identical contributions to the probability, so

$$p_\Delta(t_f, 0) = 2 \int_0^\tau dt \, \langle\psi| \hat{J}(t) |\psi\rangle$$

The underlying problem is the reflection produced by the restricted propagator

Solution to this is to soften the coarse graining by using a complex potential

For $V_0 << E$ there is negligible reflection, so part of state that crosses during $[0, \tau]$ is absorbed and remainder unchanged.

$$\hat{g}_r^V(0,\tau)\,|\psi\rangle \approx \bar{P}(\tau)\,|\psi\rangle$$

For states which are single wavepackets reasonably well peaked in position and momentum $p_r + p_\Delta = 1$ and

$$p_\Delta(\tau, 0) \approx \int_0^\tau dt\, \langle\psi|\, \hat{J}(t)\,|\psi\rangle$$

JJ Halliwell & JMY, Phys.Rev.A 79, 062101 (2009).

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Simple Example II

General initial state, what is probability of either crossing or not crossing origin in either direction during $[0, \tau]$?

$$\hat{g}_r(\tau, 0) = \bar{P}(1 - R)e^{-iH\tau}\bar{P} + P(1 - R)e^{-iH\tau}P$$

Sum rules satisfied only for states antisymmetric about origin, and $p_\Delta = 0$.

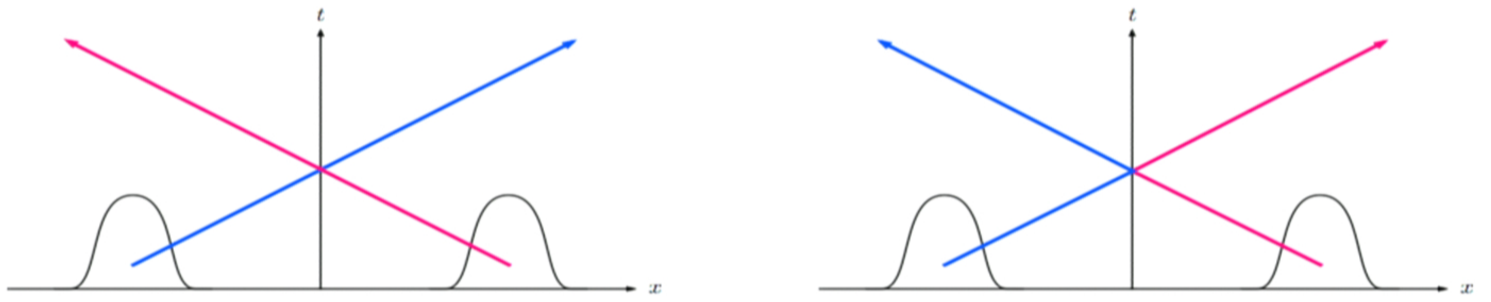
Modified analysis with complex potential straightforward. For superpositions of incoming wavepackets sum rules approximately satisfied if $V_0 << E$ and probabilities are expected semiclassical ones.

N Yamada & S Takagi, Prog.Theor.Phys. 85, 985 (1991); 86, 599 (1991); 87, 77 (1992).
JMY, arXiv:1110.5790.

What is interesting is that there are two regimes where the sum rules satisfied.

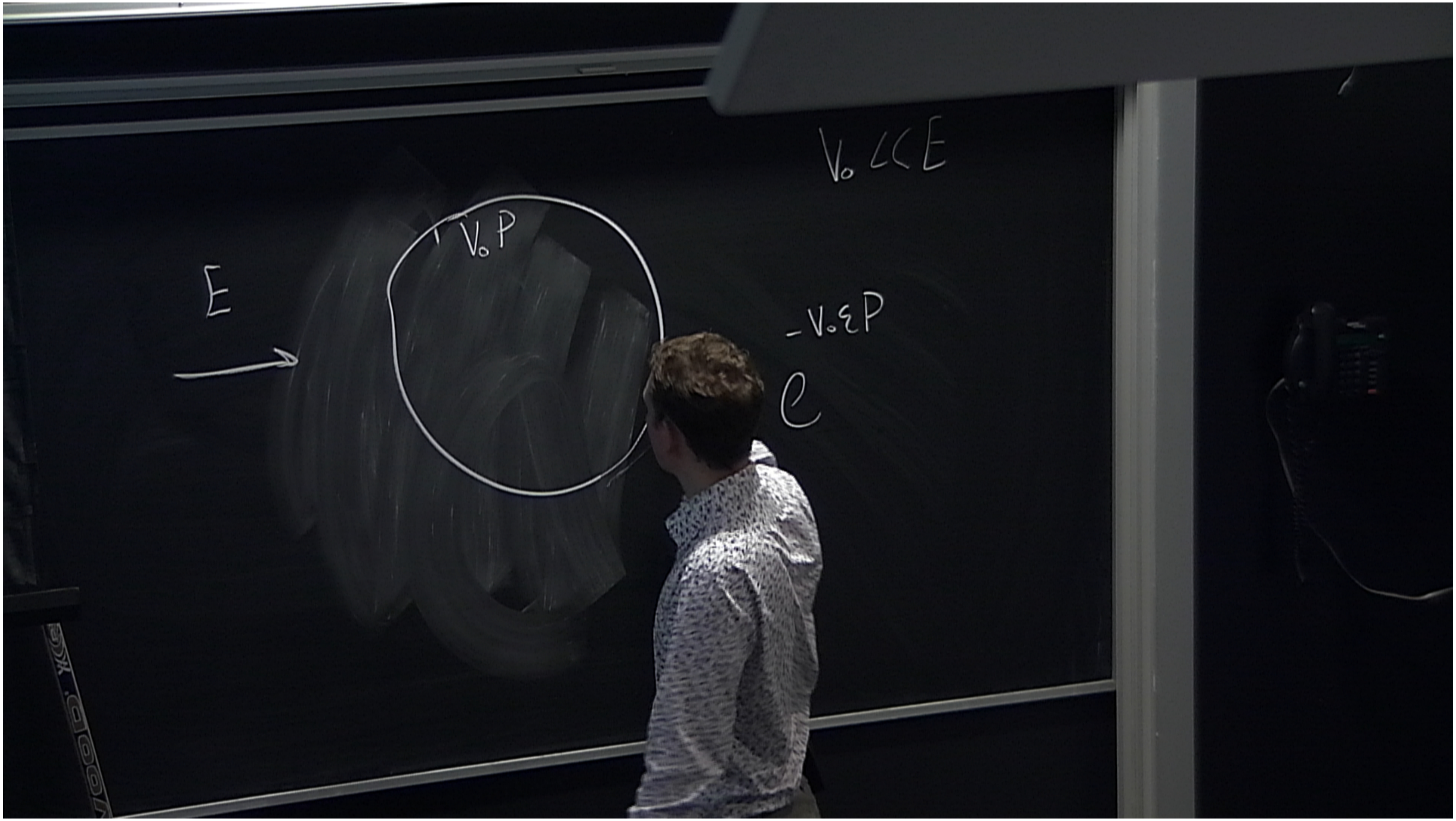
- Approximately, for small V_0 where we recover the semiclassical results.
- Exactly, for $V_0 \to \infty$ for antisymmetric states.

But the probabilities in each case are very different



Summary

- Amplitudes constructed from path integrals can, if implemented in the simplest and most obvious way, lead to problems to to the Zeno effect.
- They do not have a sensible classical limit or, in DH approach, sum rules not satisfied except for special initial states.
- Although these problems have been observed in specific examples have argued this is a very general issue.
- Outlined a solution to the problem, based on a softening of the coarse graining.
- This softening introduces one or more coarse graining parameters, in the simplest case a timescale ϵ. Zeno problem then avoided if $\epsilon << 1/E$.


$V_0 \ll E$
$V_0 P$
E
$-V_0 \varepsilon P$
e