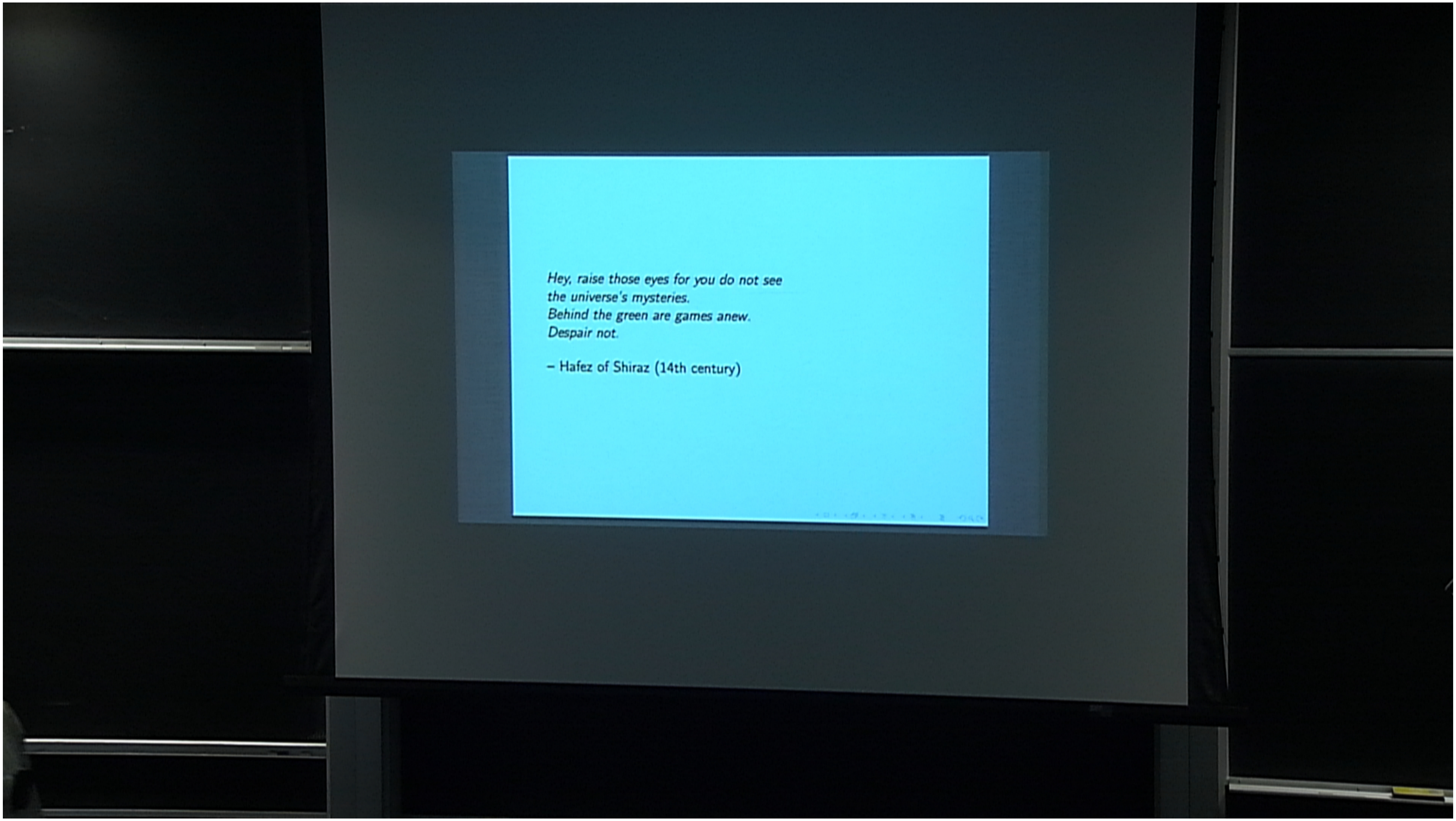
Title: Time and a Physical Hamiltonian for Quantum Gravity

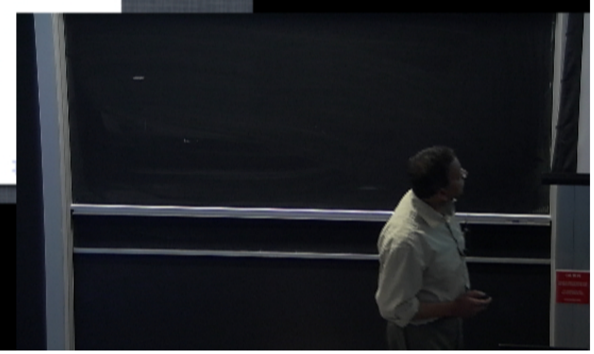
Date: May 12, 2012 09:50 AM

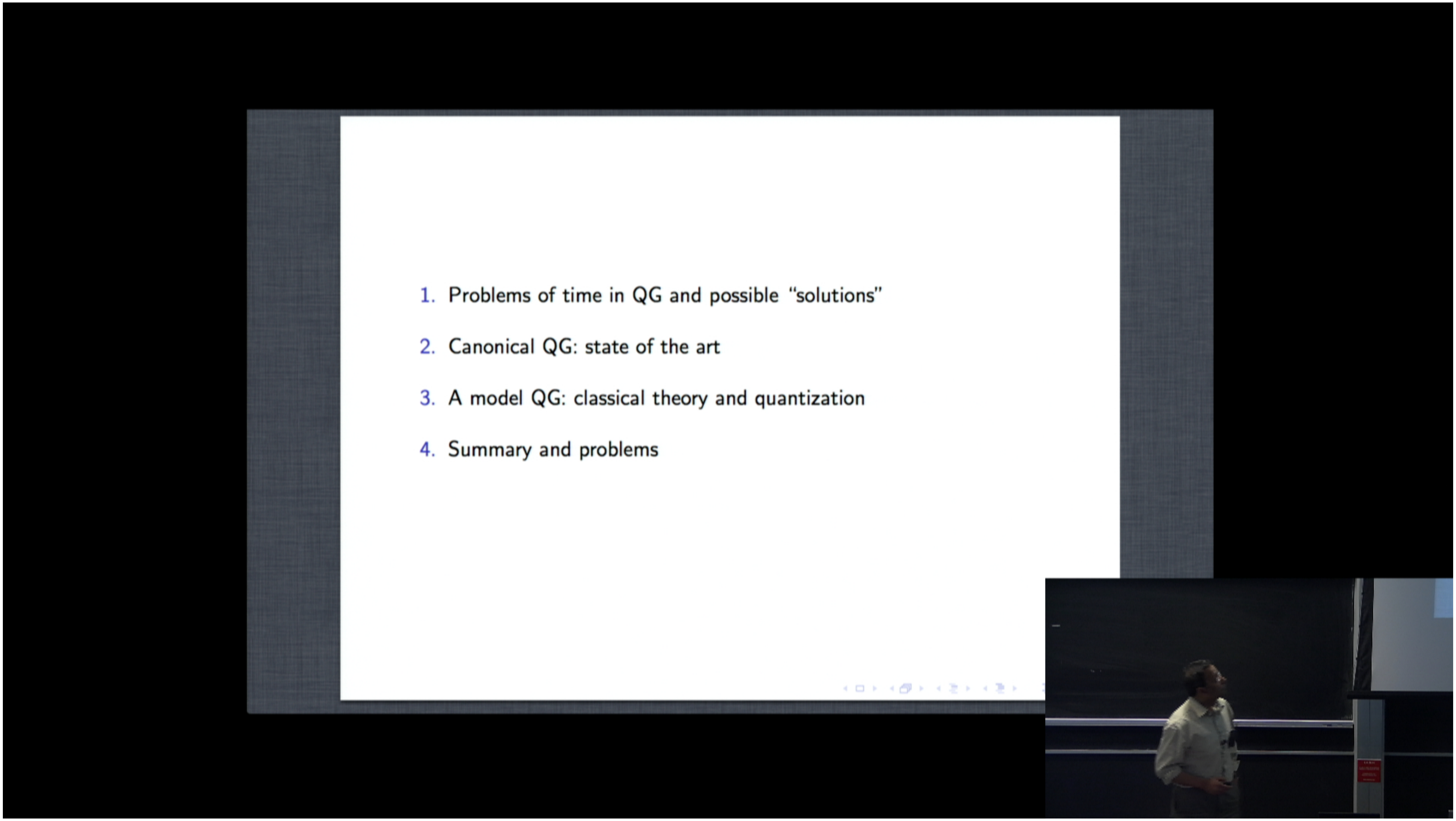
URL: http://pirsa.org/12050076

Abstract: I will describe an approach to the problem of time that uses dust as a time variable. The canonical theory is such that there is a true Hamiltonian with spatial diffeomorphisms as the only gauge symmetry. This feature, and the form of the Hamiltonian, suggest a model for non-perturbative quantum gravity that is computationally accessible using the formalism of loop quantum gravity.


Hey, raise those eyes for you do not see
the universe's mysteries.
Behind the green are games anew.
Despair not.
– Hafez of Shiraz (14th century)

1. Problems of time in QG and possible “solutions”
2. Canonical QG: state of the art
3. A model QG: classical theory and quantization
4. Summary and problems

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Problem of time in QG

3+1 ADM action

$$S = \int d^3x dt \left(\tilde{\pi}^{ab} \dot{q}_{ab} + P_\phi \dot{\phi} - N(\mathcal{H}_G + \mathcal{H}_M) - N^a (C_a^G + C_a^M) \right). \quad (1)$$

source of problem: Hamiltonian constraint

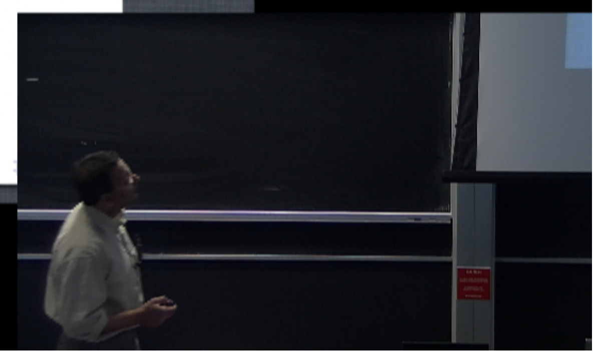
$$\mathcal{H} \equiv \mathcal{H}_G(q, \pi) + \mathcal{H}_M = 0. \quad (2)$$

"Dirac quantization"

$$\widehat{\mathcal{H}} \Psi[q, \phi] = 0. \quad (3)$$

Not a TDSE – evolution is gauge.

What is the Hilbert space in which $\Psi[q, \phi]$ lives?
How is $\widehat{\mathcal{H}}$ defined?
What is the physical Hilbert space?



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“Solutions” to the problem of time

Fix time gauge: (partially) reduced pause space quantization

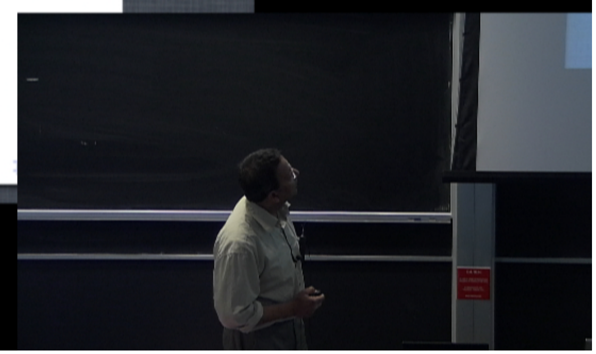
(eg. Horava-Lifshitz, Shape dynamics)

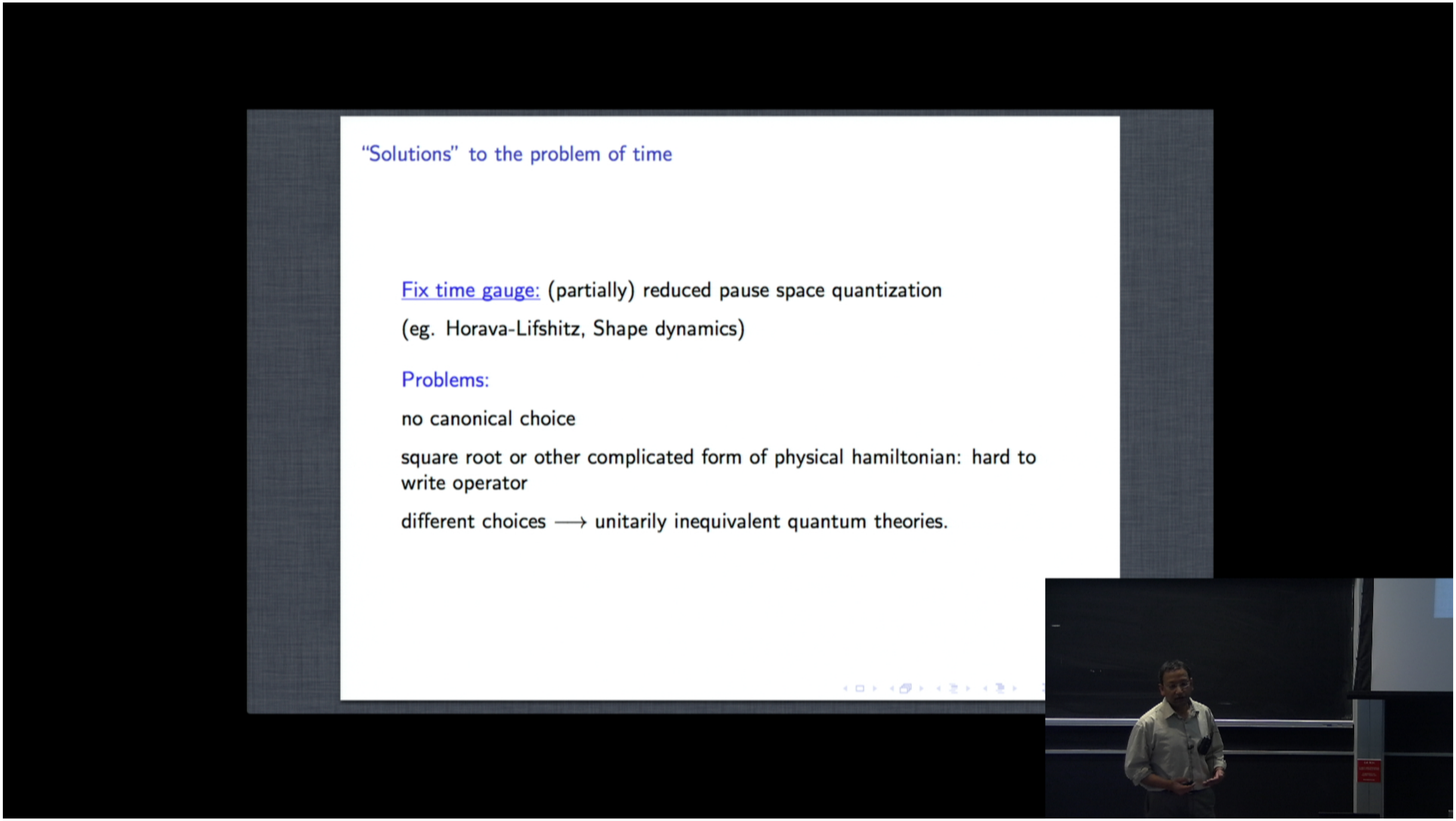
Problems:

no canonical choice

square root or other complicated form of physical hamiltonian: hard to write operator

different choices $\longrightarrow$ unitarily inequivalent quantum theories.

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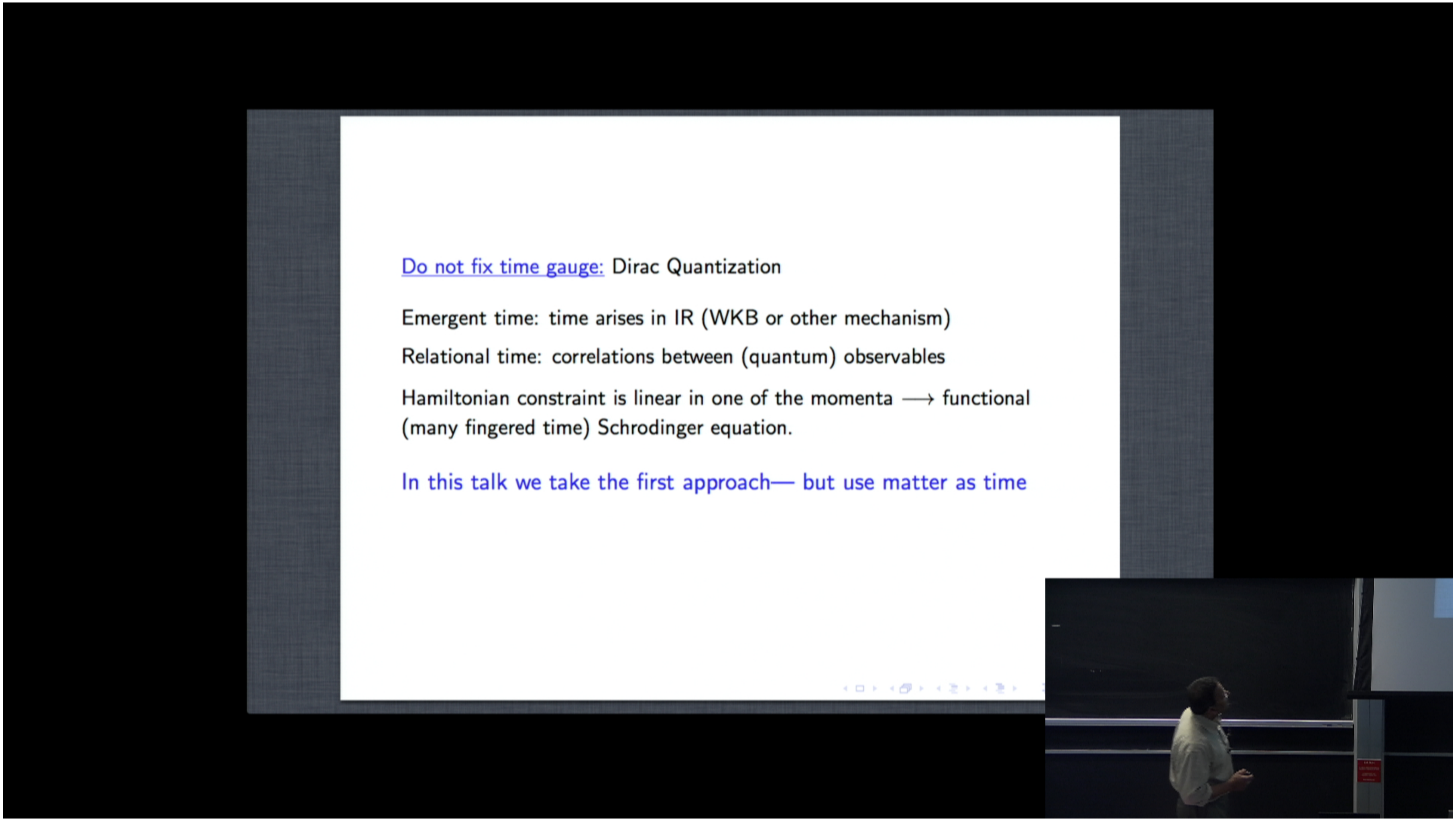
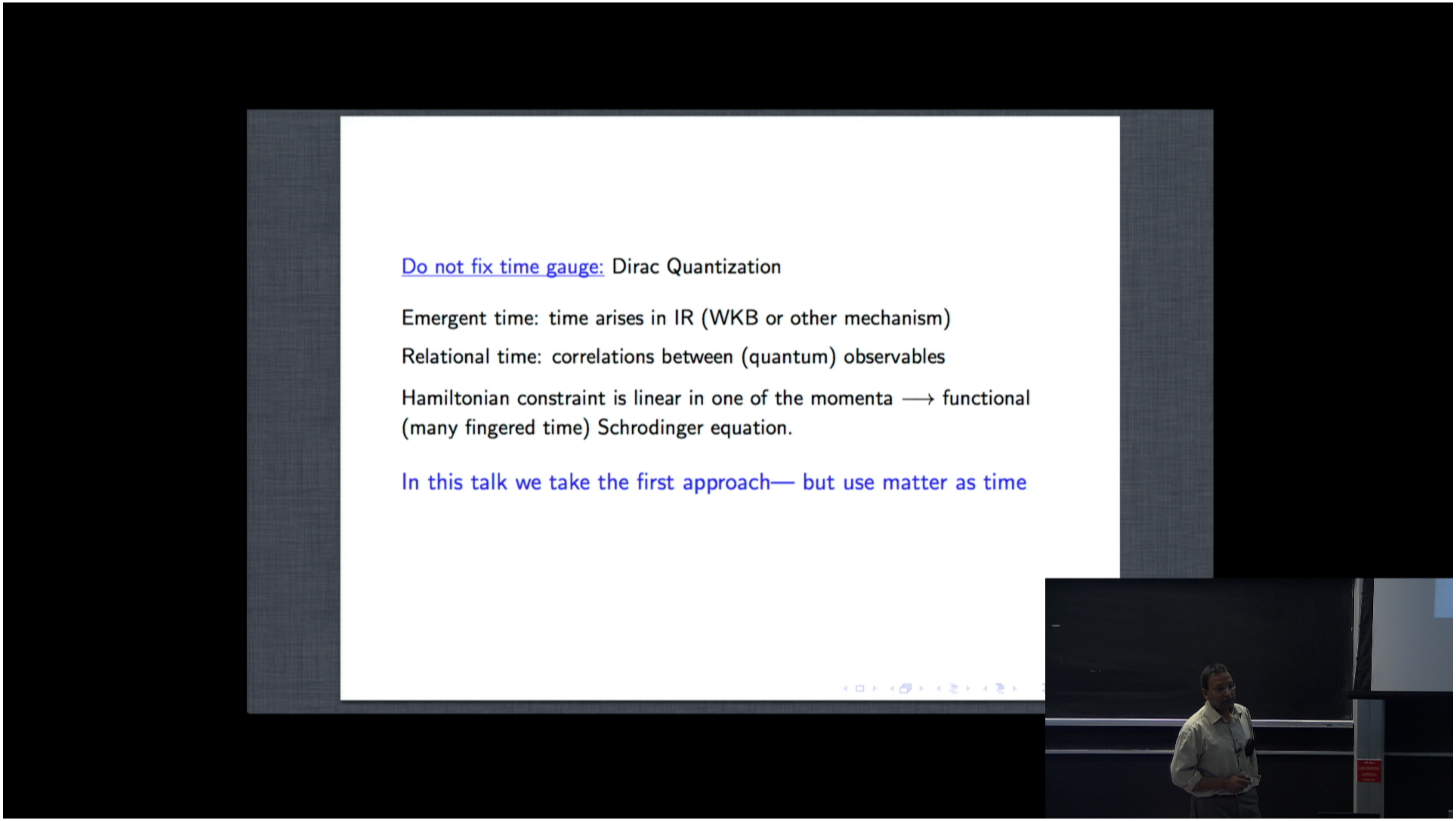

Do not fix time gauge: Dirac Quantization
Emergent time: time arises in IR (WKB or other mechanism)
Relational time: correlations between (quantum) observables
Hamiltonian constraint is linear in one of the momenta ⟶ functional (many fingered time) Schrodinger equation.
In this talk we take the first approach— but use matter as time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Non-perturbative QG: state of the Dirac quantization program

1. Define a suitable *kinematical Hilbert space* H_{kin}
2. Define constraints as operators on H_{kin}.
3. Check algebra of constraint operators
4. Find space of solution of the constraints: physical Hilbert space H_{phys}.
5. physical observables, semiclassical states $\cdots$.

State of the art in LQG: 1, 2 + some solutions of constraints that also satisfy

$$[H(N), H(M)]\psi = 0$$

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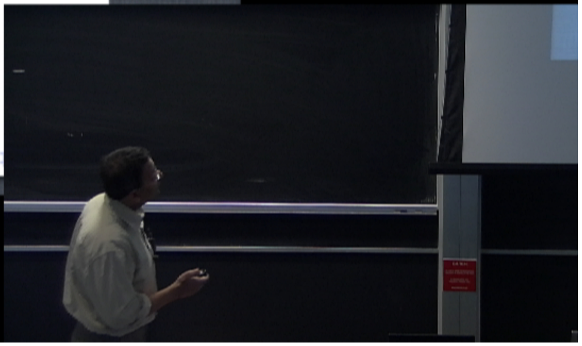

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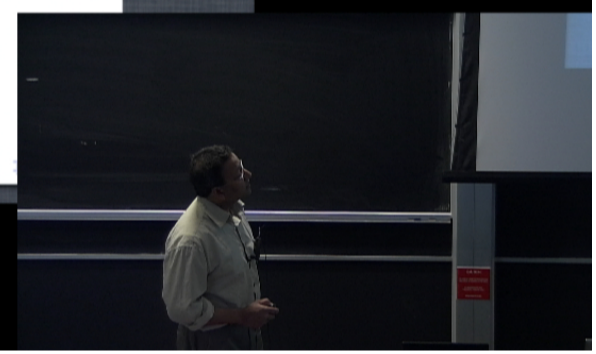

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The model

$$S = \frac{1}{4G}\int d^4x\sqrt{-g}R + S_{\rm M} - \frac{1}{2}\int d^4x\,\sqrt{-g}M(g^{ab}\partial_a T\partial_b T + 1),$$

Fields g, M, T. $S_{\rm M}$ is any matter action.

With $U_a = \partial_a T$ the dust stress-energy tensor is

$$T^{ab} = MU^aU^b + (M/2)g^{ab}\left(g_{cd}U^cU^d + 1\right)$$

Special case of Brown-Kuchar action which has 4 dust fields.

Considered much before: in spherical symmetry $\longrightarrow$ Tolman-Bondi-Lemaitre model.

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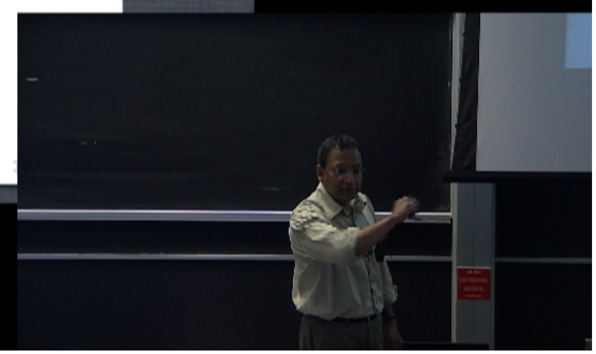

Hamiltonian theory of dust

Substitute ADM metric

$$ds^2 = -N^2dt^2 + q_{ab}(N^a dt + dx^a)(N^b dt + dx^b)$$

into dust action:

$$L_D = \frac{M\sqrt{q}}{2N}\left[(\dot{T} + N^a\partial_a T)^2 - N^2(q^{ab}\partial_a T\partial_b T + 1)\right].$$

$$p_T = \frac{\partial L_D}{\partial \dot{T}} = \sqrt{q}\frac{M}{N}(\dot{T} + N^a\partial_a T)$$

$$S_D = \int dt d^3x \left[p_T\dot{T} - N\mathcal{H}_D - N^a C_a^D\right].$$

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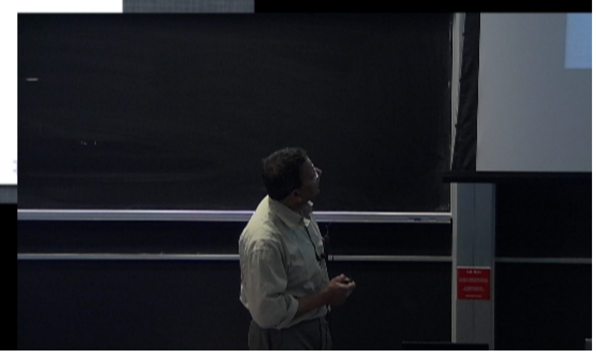

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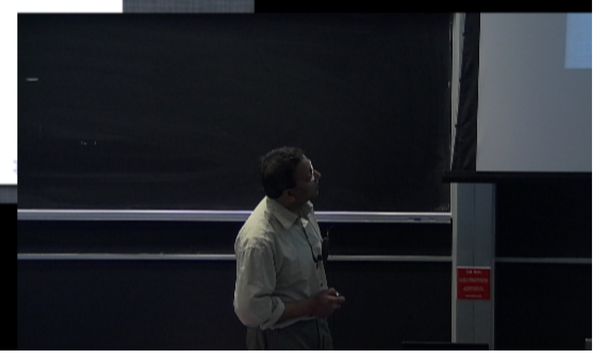

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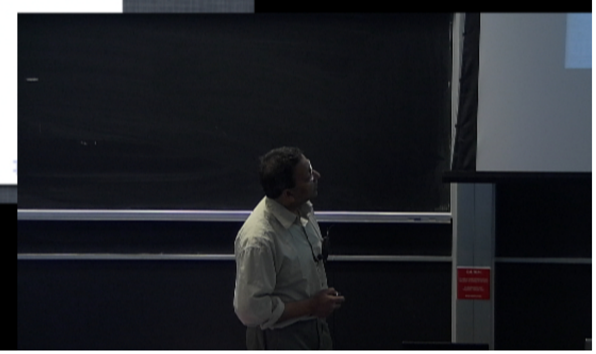

Equation of motion for M gives

$$M = [q]^{-1/2}\, p_T^2\, [p_T^2 + q^{ab} C_a^D C_b^D]^{-1/2}.$$

Plug back into dust action gives

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– a square root Hamiltonian.

Final canonical theory: $(q, \pi), (T, p_T), (\phi, P_\phi)$ subject to constraints

$$\mathcal{H}_G + \mathcal{H}_D + \mathcal{H}_M = 0$$
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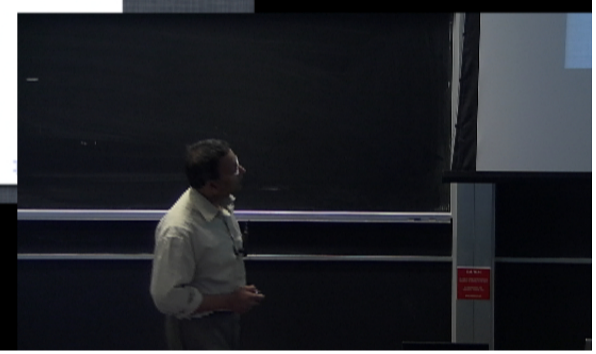

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Time gauge fixing

Dirac quantization is difficult.

But the form of the dust Hamiltonian suggest a natural time gauge:

$$T = t.$$

This is second class with $\mathcal{H}$.

Preservation of gauge under time evolution gives

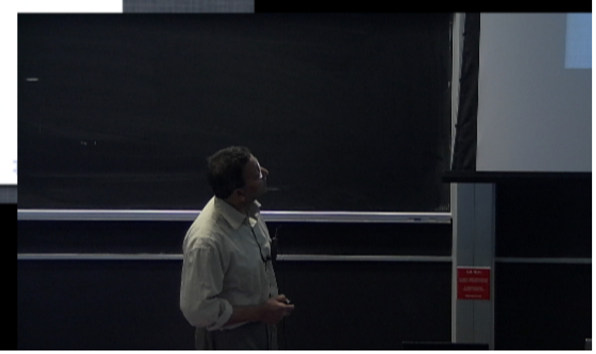
$$\dot{T} = \dot{t} = 1 = \{T, \int_\Sigma d^3x(N\mathcal{H} + N^a\mathcal{C}_a)\}|_{T=t}$$

This fixes $N = 1$ and leaves N^a arbitrary. Lapse and shift decouple!

$$-p_T = H_{phys} = \mathcal{H}_G(\pi, q) + \mathcal{H}_M(\phi, P_\phi)$$

– Physical hamiltonian is not a square root.

– It is independent of time.



Quantization

Basic idea

Use the Ashtekar-Barbero canonical variables for gravity: (A^i_a, E^a_i).

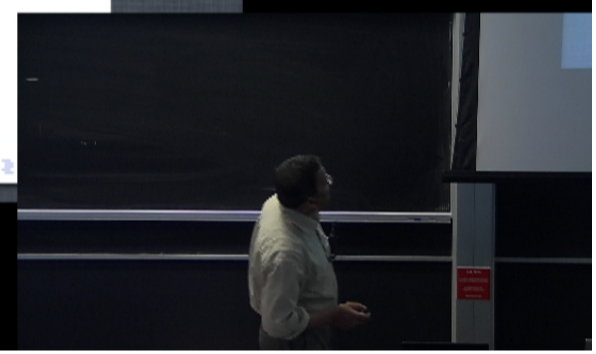
Use LQG H_{kin}: spin network basis.

H_{kin} carries a representation of the spatial diffeomorphism group. (cf. QFT Hilbert space carries a rep of Poincare group).

At least one construction of $\mathcal{H}_G$ available on H_{kin} (Thiemann)

But in this theory $\mathcal{H}_G$ is the physical Hamiltonian!

Page 15 of 24



H_{kin}

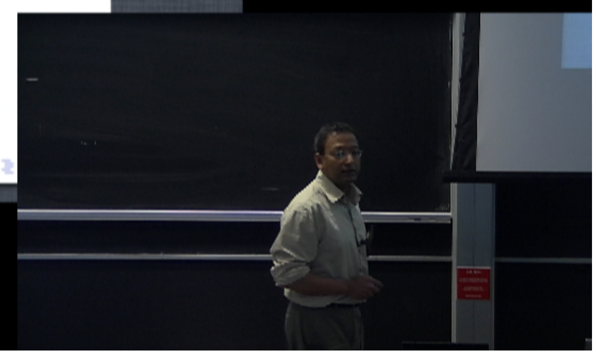
Connection triad formulation contains Gauss law constraint $D_a E^{ai} = 0$ in addition to $\mathcal{H} = 0$ and $\mathcal{C}_a = 0$.

Connection representation:

$$\psi[A] = f(U^{j_1}_{\gamma_1}(A), U^{j_2}_{\gamma_2}(A), \cdots U^{j_n}_{\gamma_n}(A))$$

γ_i: curves embedded in spatial slice; collection $\{\gamma_i\}$ is a graph Γ.

$U^{j_i}_{\gamma_i}$: holonomy of SU(2) connection in representation j_i.



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Spin network states:

Tie up holonomies at points of intersection of edges using *intertwiners* $\mathcal{I}$: eg. trivalent vertex

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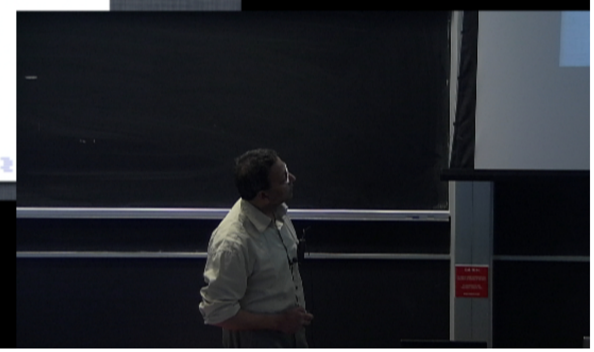
SU(2) Haar measure provides inner product.

$$|\Gamma; j_1, j_2 \cdots j_n; \mathcal{I}_1 \cdots \mathcal{I}_m\rangle$$



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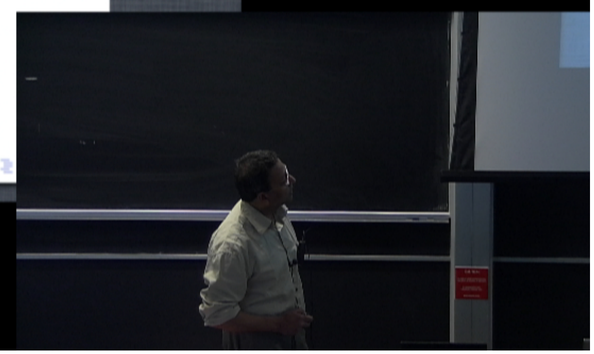

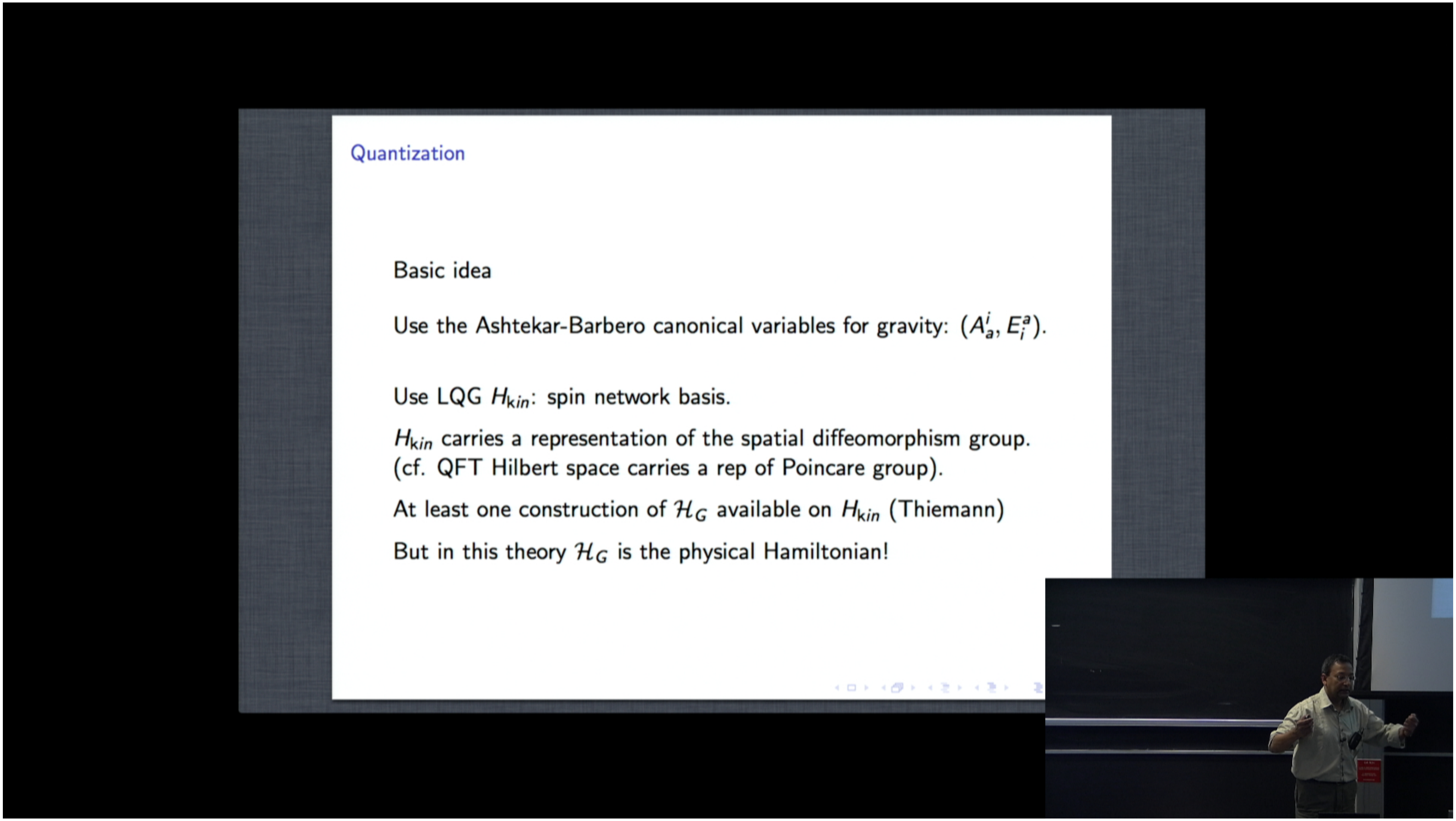
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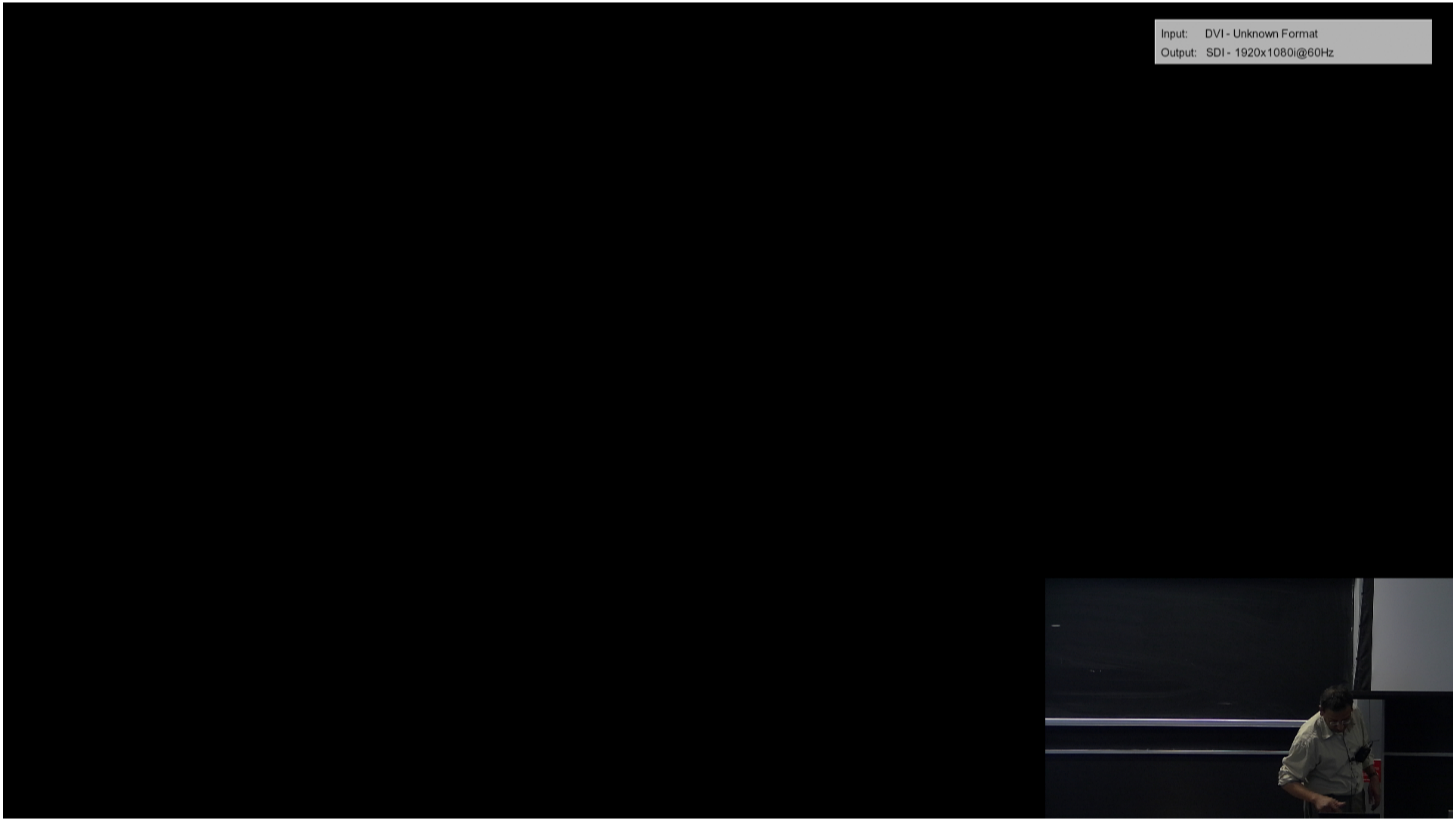
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Hamiltonian operator

Physical Hamiltonian density is

$$\mathcal{H}_G = \frac{\gamma^2}{2\sqrt{\det E}} E_i^a E_j^b \left(\epsilon^{ij}{}_k F_{ab}^k + 2(1-\gamma^2) K_{[a}^i K_{b]}^j \right)$$

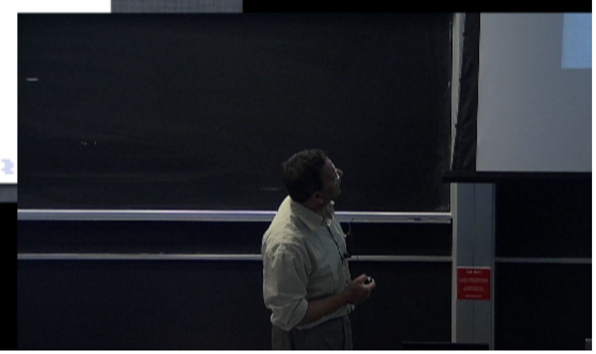
A possible operator (following Thiemann):

$$\hat{H}^G = \sum_{v \in V(\Gamma)} \hat{\mathcal{H}}_v^G$$

The sum is over vertices of a graph.

$\hat{\mathcal{H}}_v^G$ is composed of:
(i) the volume operator $\hat{V}(v)$
(ii) the combination $\hat{h}_e[\hat{h}_e^{-1}, \hat{V}]$ of holonomies and volume.
(iii) the holonomies $\hat{h}_\Box(v)$ along the minimal closed loops based at v .

(i) and (ii) are diagonal. (iii) changes the spin labels on the edges.



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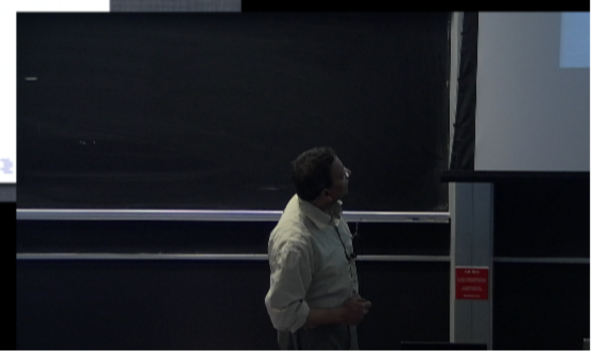
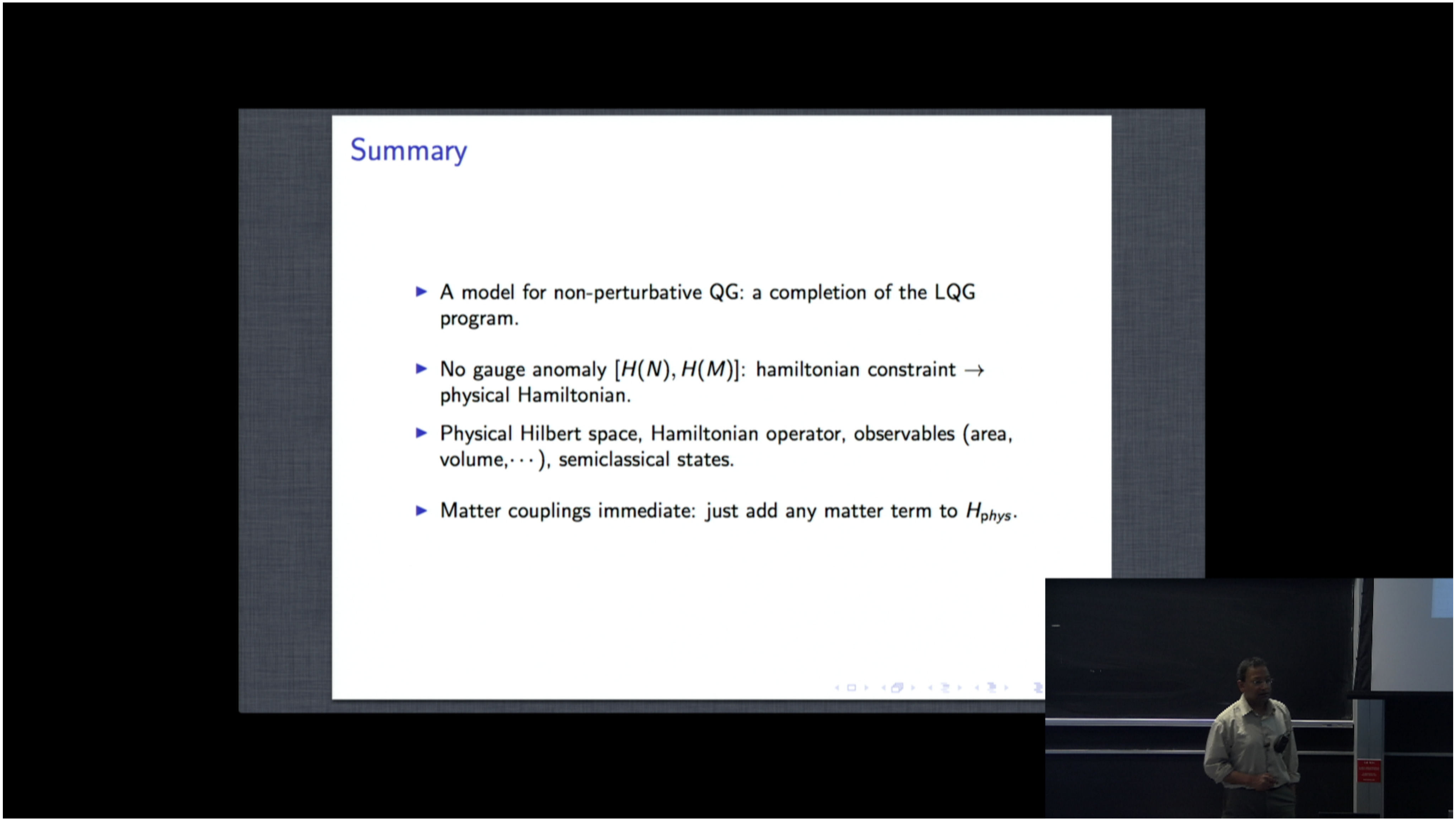



Summary
▶ A model for non-perturbative QG: a completion of the LQG program.
▶ No gauge anomaly $[H(N), H(M)]$: hamiltonian constraint $\rightarrow$ physical Hamiltonian.
▶ Physical Hilbert space, Hamiltonian operator, observables (area, volume,$\cdots$), semiclassical states.
▶ Matter couplings immediate: just add any matter term to H_{phys}.

Open problems

Model provides a test bed for physical/technical problems.

- Other ways to write the Hamiltonian operator. $\hat{H}$ on H_{diff}?
- What do reduced models look like? eg. Spherical symmetry + scalar field (work in progress)
- What is a black hole in quantum gravity?
- Emergence of semiclassical gravity. Hawking radiation?
- Cosmology with dust time: FRW reduction, fluctuations etc.
- Lorentz invariance?

Curiosities

1. Another use of spin network Hilbert space:
SU(2) theory (VH, K. Kuchar '91)

$$S = \frac{1}{L^2} \int Tr(e \wedge e \wedge F(A)) \left(+ \Lambda \int C.S.(A) \right)$$

e^i_a SU(2) dreibein in 4d. A^i_a connection.

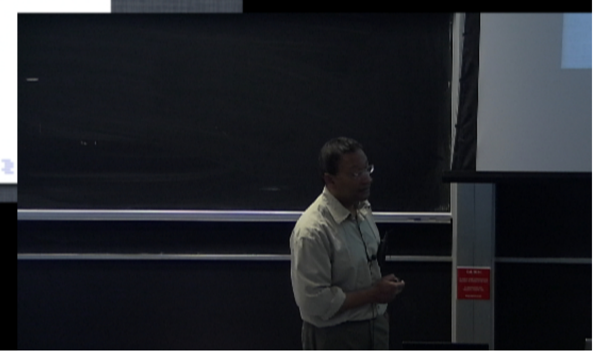
– power counting non-renormalizable.

– 3 local degrees of freedom.

– Hamiltonian constraint vanishes identically. Spatial diffeo + Gauss constraints only.

– Complete non-perturbative quantization: $H_{phys} = H_{diff}$ – Hilbert space of diffeomorphism invariant spin networks.

– What is the spinfoam for this model?



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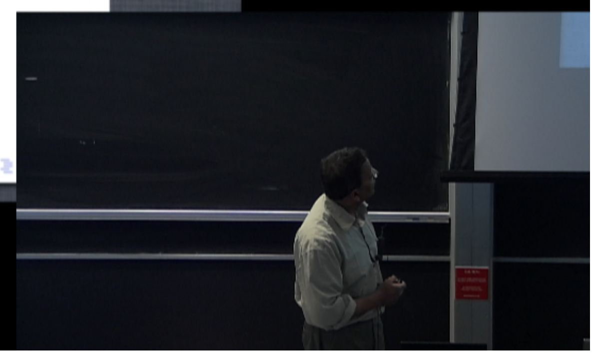


2. "Dusty string"

$$S = \int d^2x\sqrt{h}\left(G^{AB}h^{ab}\partial_a X_A \partial_b X_B + M(h^{ab}\partial_a T \partial_b T + 1)\right)$$

– not conformally invariant.

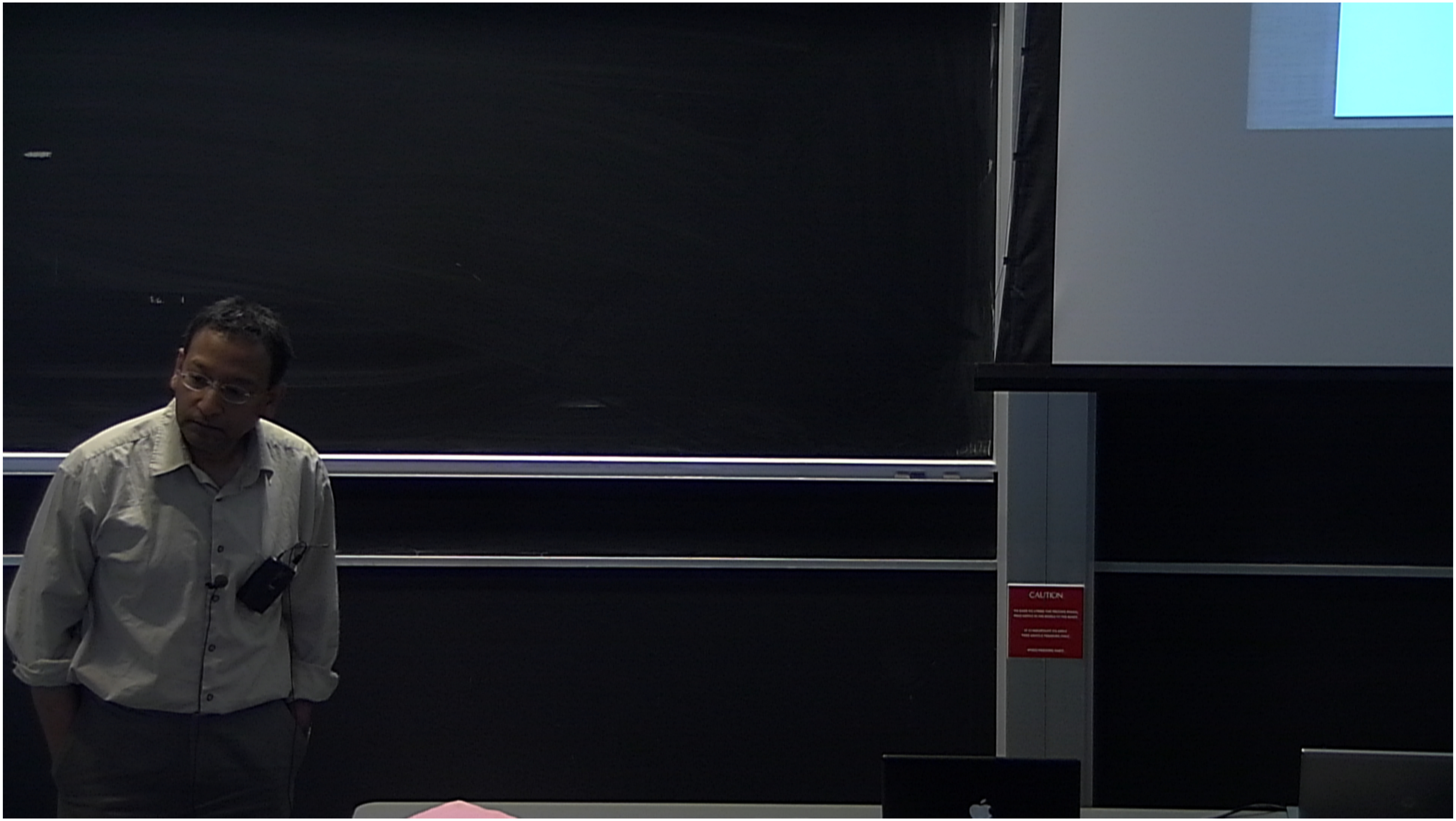
– $T = t$ gauge: physical Hamiltonian with Diff(S^1) constraint.

– manifest target space Poincare invariance.

– gravitons?

3. "Little λ": Can put this in by hand into the gravitational physical Hamiltonian in the dust gauge fixed action.

$$S^{GF} = \int d^3x dt \left[\pi^{ij}\dot{q}_{ij} - \frac{1}{\sqrt{q}}(\pi^{ij}\pi_{ij} - \lambda\pi^2) - \sqrt{q}V(q)\right]$$


CAUTION