Title: Scale Without Conformal Invariance in Relativistic Quantum Field Theory

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Abstract: In 2-dim it is known that a unitary, well defined quantum field theory, if scale invariant must also be invariant under conformal transformations. Whether this is also true in dimensions higher than two has been an open question for decades. We have discovered renomalization group flows in 4-epsilon dimensions corresponding to scale but not conformal invariant theories. The flows correspond to limit cycles or ergodic behavior, neither of which had been reported in relativistic quantum field theories either.   There seems to be a deep connection between scale without conformal invariance and this type of renormalization group behavior. We will present these results and list some of open questions, including the possibility of such behavior in integral dimensions.

Scale Invariance without Conformal Invariance in Relativistic QFT

Benjamin Grinstein
with Jean-Francois Fortin and Andreas Stergiou

Conformal Nature of the Universe
Perimeter Institute, May 9-12, 2012

Caution
No GR here

Caution
QM here

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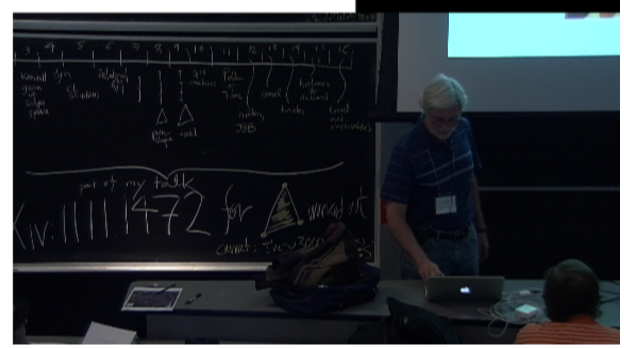


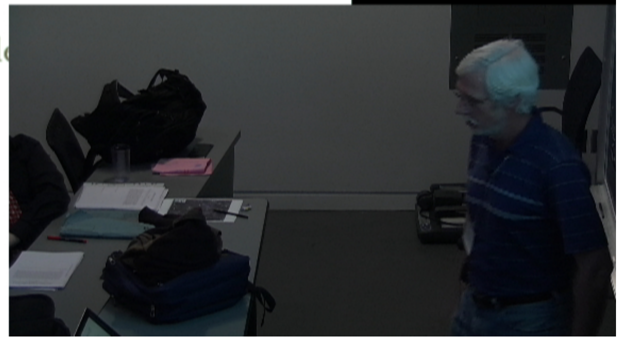
Two Unsolved Mysteries in QFT

- Are there QFT models in $D = 4$ with SCALE invariance but without being CONFORMAL (*i.e.,* invariant under special conformal transformations)?

  (In this talk QFT means Quantum Field Theories that are also Poincare invariant, ie, relativistic; one can ask similar questions about non-relativistic QFTs but won't here)

- Are there Renormalization Group (RG) flows in QFTs with limit cycles? Are there limit ergodic flows?

  (Known to exist in non-relativistic Quantum Mechanics: Effimov cycl



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Why care?

- Classification: "Phases" of QFT models. Or:
  What are the possible behaviors of QFT models at very long distances:
  - IR-Free:
    - With mass gap: exponentially decaying correlators (eg, confinement)
    - Without mass gap: trivial correlators (eg, coulomb phase)
  - IR-Interacting
    - Interacting CFTs: power-law correlators
    - Interacting SwC (scale without conformal): power-law ????
- Alternative classification: IR-limit of RG-flows (Wilson):
  - Strong (eg, QCD)
  - Fixed Point (ie, IR-CFT)
  - Limit Cycles
  - Limit ergodic flows
- New, unknown phenomena/behaviors?
- New, unknown applications? (e.g., "cylcunparticles" as we'll propose later)

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Scale without Conformal

- Condition for Scale Invariance?

$$\partial_\mu D^\mu = 0$$

where the dilatation (scale) current is given in terms of the improved energy-momentum tensor

$$D^\mu = x_\nu T^{\mu\nu}$$

so that

$$\partial_\mu D^\mu = T^\mu_\mu$$

- Condition for Conformal Invariance?

$$\partial_\mu K^{\mu\nu} = -x^\nu T^\mu_\mu = 0$$

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- Improvements? If

$$T^{\mu}_{\mu} = \partial_\mu \partial_\nu L^{\mu\nu}$$

one can improve $T^{\mu\nu}$ so that scale and conformal still conserved.

- But! What if the unbroken symmetry is a combination of two broken symmetries? This happens in other familiar contexts:
  - For spontaneously broken symmetries, as in the SM: SU(2)×U(1) → U(1)$_{EM}$
  - For anomalous currents, as in B and L in SM, but not $B-L$

- *Look for a conserved current of the form (Polchinski '87)*

$$\boxed{D^\mu = x_\nu T^{\mu\nu} - V^\mu}$$

where V^μ *(the "virial current") is a non-conserved current that does not depend explicitly on coordinates.*

(and which is not of the form $V^\mu = \partial_\nu L^{\mu\nu}$)

THEN: We can have

$$\partial_\mu D^\mu = T^\mu_\mu - \partial_\mu V^\mu = 0$$ scale invariance

while

$$T^\mu_\mu = \partial_\mu V^\mu \neq 0$$ no conformal symmetry

A scale transformation together with a U(1) rotation is still a symmetry.

Immediate implication: limit cycles or ergodic limit flows:

Let $$Q = \int d^3x \, V^0$$ be the generator of rotations

Rotation on fields:

$$\Phi_I \to (e^{itQ}\Phi)_I = (e^{itQ})_I^J \Phi_J$$

A scale transformation corresponds to RG-motion of coupling constants: if $\mathcal{L}_{\text{int}} = g_{IJ\ldots}\Phi_I\Phi_J\cdots$

$$g_{IJ\ldots}(t_0) \to g_{IJ\ldots}(t)$$

This can undo the rotation (so we have a symmetry) if

$$g_{IJ\ldots}(t) = \left[(e^{-itQ})_I^M (e^{-itQ})_J^N \cdots\right] g_{MN\ldots}(t_0)$$

For fixed t, this transformation is an element of the group of internal global transformations of the model (the "flavor" symmetry group of the kinetic terms)

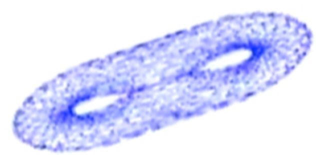
$$e^{-itQ} \in G_F$$

As a function of t: one parameter trajectory in compact space

→

- Trajectory closes

- Trajectory comes arbitrarily close to initial point (Poincare recurrence)



Outline

- Introduction
- Searching for SwC models (in $D = 4 - \varepsilon$)
- Some General Properties of SI in $D = 4$
  - Scheme Dependence
  - Stability Properties
  - Correlation Functions
  - Cyclunparticles
  - Perturbative Solutions in $D = 4$
  - A word about the a-theorem

Searching For SwC Models

(SwC = Scale without Conformal Symmetry)

Want: $D^{\mu} = x_{\nu} T^{\mu\nu} - V^{\mu}$

$$\partial_{\mu} D^{\mu} = T^{\mu}_{\mu} - \partial_{\mu} V^{\mu} = 0$$

$$T^{\mu}_{\mu} = \partial_{\mu} V^{\mu} \neq 0$$

$$V^{\mu} \neq \partial_{\nu} L^{\mu\nu}$$

Considerations:

- Interacting
- Renormalizable
- Perturbative
- Enough DOFs for nontrivial virial current
- $D = 4$ models: YM with scalars and spinors. Most interesting
- $D = 4 - \varepsilon$ models: scalars and spinors. More tractable (avoid complication of YM by going to $4 - \varepsilon$ to get asymptotic freedom and possibility of fixed points)

Need to be more explicit in order to write candidates for the virial current: vector operators of dimension-3 with non-vanishing divergence

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For now: focus on D = 4 − ε models $\mathcal{L} = \mathcal{L}_K + \mathcal{L}_{int}$

$$\mathcal{L}_K = \sum_{a=1}^{n_s} \frac{1}{2}\partial^\mu \phi_a \partial_\mu \phi_a + \sum_{k=1}^{n_f} \bar{\psi}_k i \bar{\sigma}^\mu \partial_\mu \psi_k$$

real scalars — Weyl spinors

with: $$-\mathcal{L}_{int} = \frac{1}{4!}\lambda_{abcd}\phi_a\phi_b\phi_c\phi_d + \frac{1}{2} y_{a|ij}\phi_a\psi_i\psi_j + \text{h.c.}$$

Only candidate:

$$V^\mu = Q_{ab}\phi_a\partial^\mu\phi_b + iP_{ij}\bar{\psi}_i\bar{\sigma}^\mu\psi_j$$

Note that:

- $Q_{ab} = -Q_{ba}$ (symmetric combination gives total divergence)
- $P^\dagger = -P$ (current is real)
- Q and P are generators of SO(n_s) and SU(n_f) symmetries of $\mathcal{L}_K$
- This U(1) subgroup should be broken by $\mathcal{L}_{int}$

Recall: $\partial_\mu D^\mu = T^\mu_\mu - \partial_\mu V^\mu = 0$ with $T^\mu_\mu = \partial_\mu V^\mu \neq 0$

Trace anomaly: $$T_\mu{}^\mu(x) = -\tfrac{1}{4!}\beta_{abcd}\phi_a\phi_b\phi_c\phi_d - \tfrac{1}{2}\beta_{a|ij}\phi_a\psi_i\psi_j + \text{h.c.}\,.$$

Div of virial: $$\partial_\mu V^\mu(x) = Q_{aa'}\partial^2\phi_a\phi_{a'} - P^*_{i'i}\bar\psi_i i\bar\sigma^\mu\partial_\mu\psi_{i'} + P_{ii'}\partial_\mu\bar\psi_i i\bar\sigma^\mu\psi_{i'}$$

Using Equations of Motion (EOM), eg

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obtain conditions:

$$\beta_{abcd} = -Q_{a'a}\lambda_{a'bcd} - Q_{b'b}\lambda_{ab'cd} - Q_{c'c}\lambda_{abc'd} - Q_{d'd}\lambda_{abcd'}$$

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These are not functional equations

Solution: specific values of coupling constants (and Q and P) that satisfy these equations

Precisely as in searching for conformal fixed points (with $Q = P = 0$)

We look for solutions using perturbation theory:

- beta-functions to fixed order in the loop expansion
- coupling constants on limit cycle (eg, on solution) remain small
- Q and P consistent with beta-function loop expansion

Polchinski ('87): scalars only, 1-loop solutions: if SI then CFT

$$\beta_{abcd} = Q_{ae}\lambda_{ebcd} + \text{permutations}$$

$$\Rightarrow \qquad \sum_{a,b,c,d} \beta^2_{abcd} = \sum_{a,b,c,d} \beta_{abcd}(Q_{ae}\lambda_{ebcd} + \text{permutations})$$

now show RHS vanishes identically (for any value of coupling constant λ)

$$\beta_{abcd} = -\epsilon\lambda_{abcd} + \frac{\#}{16\pi^2}\left(\lambda_{abgh}\lambda_{cdgh} + \text{permutations}\right)$$

"classical" term: $\beta_{abcd}(Q_{ae}\lambda_{ebcd}) \propto Q_{ae}\lambda_{ebcd}\lambda_{abcd} = 0$

1-loop term: $\beta_{abcd}(Q_{ae}\lambda_{ebcd}) \propto Q_{ae}\lambda_{ebcd}\lambda_{abgh}\lambda_{cdgh} = 0$

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ε-expansion

Recall: $\beta_\lambda \sim -\varepsilon\lambda + \frac{1}{16\pi^2}\left(\lambda^2 + (y^\dagger y)^2\right) + \cdots$ and $\beta_y \sim -\varepsilon y + \frac{1}{16\pi^2} y y^\dagger y + \cdots$

Expansion ("flavor" indices implicit):

$$y = \sum_{n\geq 1} y^{(n)} \varepsilon^{n-\frac{1}{2}} \qquad \lambda = \sum_{n\geq 1} \lambda^{(n)} \varepsilon^n$$

$$Q = \sum_{n\geq 2} Q^{(n)} \varepsilon^n \qquad P = \sum_{n\geq 2} P^{(n)} \varepsilon^n$$

Match powers of $\varepsilon^{\frac{1}{2}}$ on both sides of $\beta_\lambda = Q\lambda$ and $\beta_y = Qy + Py$

- Lowest order: non-linear. Many solutions. Discard "bad" ones
- Higher orders: linear

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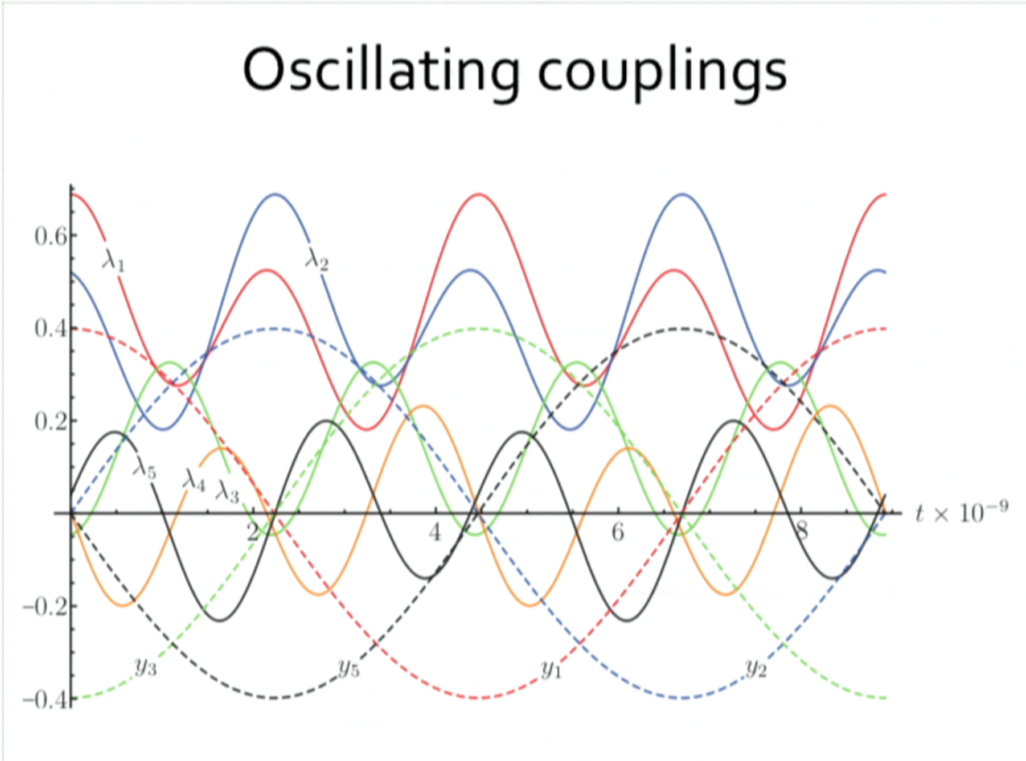
Summary of findings:

- No obstruction for pure scalar theory at 2-loops
- No obstruction to all orders in perturbation theory for any n_f if $n_s < 2$ (that is SI implies CI in these theories — to all orders in perturbation theory)
- Solutions with $P = 0$ but $Q \neq 0$ at 3-loops in:
  - $n_f = 1, n_s = 2$, with unbounded tree-level potential
  - $n_f = 2, n_s = 2$, with bounded tree-level potential



(too many) Comments:

- Tree level potential bounded vs unbounded: not really an issue for perturbative analysis:
  - Much like studying RG flows in ϕ^3 in D = 6
  - Vacuum stability determined by effective potential
- Solutions with $n_f = 1, 2$, $n_s = 2$, already found at 2-loops (used Jack & Osborn betas)
  - Numerically
  - Via ε-expansion
  - Perfect agreement between them
  - Checked by integrating RGE using both full nonlinear 2-loop beta functions as compared with analytic computation of cycle (in terms of Q and P computed either numerically or analytically via ε-expansion)
  - However $\beta = Q\lambda \sim \varepsilon^4$ which means $Q \sim \varepsilon^3$. Hence terms in beta function of order of 3-loops can contribute at same order in ε: result cannot be trusted
- At 3-loops SI solutions with $n_f = 1, 2$, $n_s = 2$, are still obtained
  - It is still true that $\beta = Q\lambda \sim \varepsilon^4$ but now next order (4-loops) can only enter at order ε^5
  - 3-loop beta functions generally unknown, but:
    - Only Yukawa's beta-function can modify Q at this order
    - About 250 3-loop 1PI graphs: only 12 can modify Q at this order (see next slide)
- Scheme dependence
  - Computations described above are all in MS-scheme
  - Result is scheme dependent in $D = 4 - \varepsilon$: "classical" term in beta-function $(-\varepsilon\lambda)$ is not covariant $\Rightarrow$ could have avoided computation of 3-loop corrections by scheme choice

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Diagrams that can contribute to Q in the $n_s = n_f = 2$ model

2-loops

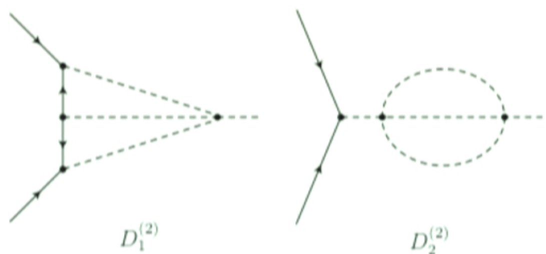

Fig. 1: Diagrams that contribute to q at two-loop order.

3-loops

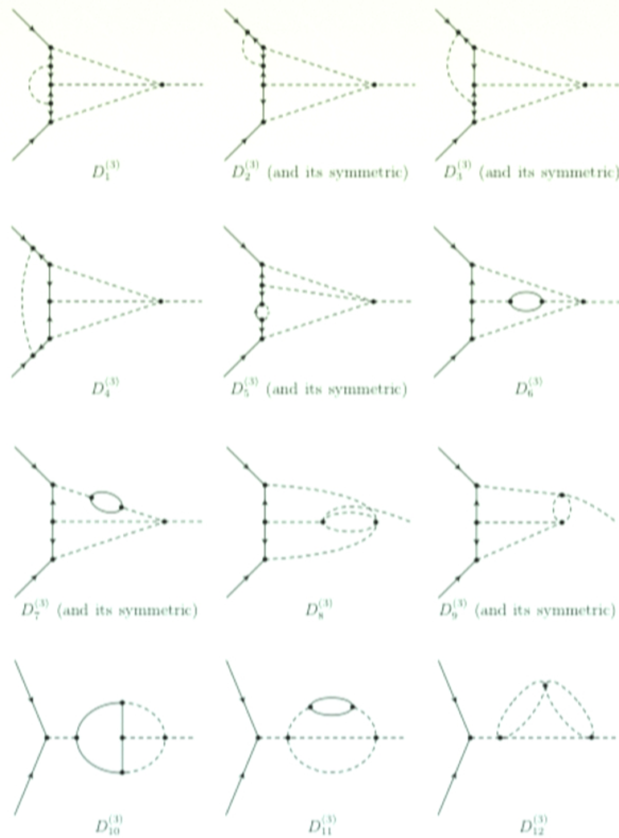

Fig. 2: Diagrams that contribute to q at three-loop order.

Some General Properties of SI Solutions

(in $D = 4$, but readily extended to other dimensions)

Scheme Dependence

Properties that have physical consequences must be independent of the scheme:

Single coupling fixed point

I. The existence of a conformal fixed point

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II. The anomalous dimension at a conformal fixed point, which determines the scaling behavior of Green functions

III. The first derivative of the beta function at a conformal fixed point, which determines the sign and rate of approach of the coupling to the conformal fixed point and thus modifies asymptotic formulae

IV. The first two coefficients in the loop expansion of the beta function, which govern the UV or IR asymptotics of the coupling

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Extension to Many couplings (FP or LC)

I. The existence of conformal fixed points and scale-invariant trajectories

II. The eigenvalues of $\gamma + Q$ at conformal fixed points and scale-invariant trajectories

III. The eigenvalues of $\partial\beta/\partial g + Q$ at conformal fixed points and scale-invariant trajectories

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IV. The first coefficient in the loop expansion of beta functions

V. The first coefficient in the anomalous dimension matrix.

To be sure, these only guaranteed under ***natural*** scheme changes

A scheme change

$$\tilde{\lambda}_{abcd} = \lambda_{abcd} + \eta_{abcd}(\lambda, y, g)$$
$$\tilde{y}_{a|ij} = y_{a|ij} + \xi_{a|ij}(\lambda, y, g)$$

is *natural* if all couplings transform covariantly with respect to G_F (the symmetry group of the kinetic terms)

that is, if

$$\lambda_{abcd} \to R_{aa'} R_{bb'} R_{cc'} R_{dd'} \lambda_{a'b'c'd'} \quad \Rightarrow \quad \tilde{\lambda}_{abcd} \to R_{aa'} R_{bb'} R_{cc'} R_{dd'} \tilde{\lambda}_{a'b'c'd'}$$

and $\quad y_{a|ij} \to R_{aa'} \hat{R}_{ii'} \hat{R}_{jj'} y_{a'|i'j'} \quad \Rightarrow \quad \tilde{y}_{a|ij} \to R_{aa'} \hat{R}_{ii'} \hat{R}_{jj'} \tilde{y}_{a'|i'j'}$

Stability Properties

$$\delta g(t) = [g(t) - g_*(t)]e^{-Qt}$$

Measures small deviations of flow from cycle (generic vector of couplings, matrix notation)

$$g_*(t) = g_*(0)e^{Qt}$$

Then $$\boxed{\delta g(t) = \delta g(0)e^{-St}}$$

where $$S = \left(\frac{\partial \beta}{\partial g}\bigg|_{g=g_*(0)} + Q \right)$$ is the "stability matrix" (scheme independent eigenvalues)

Limit cycle: there is always one vanishing eigenvalue

For example: in n_f, $n_s = 2$, 2 eigenvalues are 2.4, 1, 0.99, 0.74, 0.095, −0.19, 0 (in units of ε)

Correlation Functions

Determined from RGE. Less constrained than in CFTs (less symmetry)

By example here (rather than in generality).
Consider scalar and vector operators under $SO(n_s) \subset G_F$

scalar-scalar:

$$\langle \mathcal{O}(p)\mathcal{O}'(-p)\rangle = C(-p^2 - i\epsilon)^{\frac{1}{2}(\Delta+\Delta'-4)}$$

Dimensions constrained by unitarity: for (j_1, j_2) operator $\Delta \geq j_1 + j_2 + 1$
(for CFT, operators with $j_1 j_2 \neq 0$ have $\Delta \geq j_1 + j_2 + 2$)

scalar-vector:

$$\langle \mathcal{O}_a(p)\mathcal{O}'(-p)\rangle = (-p^2 - i\epsilon)^{\frac{1}{2}(\Delta'-4)} \left[(-p^2 - i\epsilon)^{\frac{1}{2}(\Delta+Q)}\right]_{ab} C_b$$

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vector-vector:

$$\langle \mathcal{O}_a(p)\mathcal{O}_b(-p)\rangle = (-p^2 - i\epsilon)^{-3}\left[(-p^2 - i\epsilon)^{\frac{1}{2}(\Delta+Q)}C(-p^2 - i\epsilon)^{\frac{1}{2}(\Delta-Q)}\right]_{ab}$$

with C an $n_s \times n_s$ matrix

and for (Lorentz) vectors:

$$\langle \mathcal{O}^\mu_a(p)\mathcal{O}^\nu_b(-p)\rangle = (-p^2 - i\epsilon)^{-3}\left[(-p^2 - i\epsilon)^{\frac{1}{2}(\Delta+Q)}(p^2 g^{\mu\nu}C_1 + p^\mu p^\nu C_2)(-p^2 - i\epsilon)^{\frac{1}{2}(\Delta-Q)}\right]_{ab}$$

$C_{1,2}$ are $n_s \times n_s$ matrices, relation between them not forced by symmetry (as opposed to CFT case) vector-vector:

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Cyclunparticles

As Georgi's unparticles for CFTs, use SM to probe SI sector:

- weakly couple SM to SI model (possibly strongly coupled)
- use irrelevant operators to retain IR behavior
- see fractional phase space, but also possibly oscillatory behavior
- see odd scaling *and* oscillations in interference term in scattering

Cyclunparticle phase space: discontinuity of correlation function across real axis

For example, if $\mathcal{L} \supset g_a \chi \mathcal{O}_a + \text{h.c.}$

$\chi \to \chi$ forward scattering amplitude: $\mathcal{M}^{\text{fwd}} = g_a g_b |\chi|^2 \left[(-p^2 - i\epsilon)^{\frac{1}{2}(\Delta+Q)-1} C(-p^2 - i\epsilon)^{\frac{1}{2}(\Delta-Q)-1}\right]_{ab}$

Take imaginary part:

$$F(p^2) = -g_a g_b \left[(p^2)^{\frac{1}{2}(\Delta+Q)-1} \left\{\cos\left[\left(\frac{\Delta+Q}{2}\right)\pi\right] C \sin\left[\left(\frac{\Delta-Q}{2}\right)\pi\right] + \sin\left[\left(\frac{\Delta+Q}{2}\right)\pi\right] C \cos\left[\left(\frac{\Delta-Q}{2}\right)\pi\right]\right\} (p^2)^{\frac{1}{2}(\Delta-Q)-1}\right]_{ab}$$

- Yang-Mills with Weyl spinors and scalars
- Couplings:
  - YM: g
  - Scalar: λ
  - Yukawa: y
- Arrange for perturbative Caswell-Banks-Zaks fixed point g_*
- No $-\varepsilon\lambda$ term in beta as in $D=4-\varepsilon$, but now g_* drives λ and y toward fixed points and cycles

- Particular example:
  - YM: SU(N)
  - Two real scalars, singlets under SU(N)
  - Two Weyl spinors in fundamental + two in anti-fundamental
  - The above produces at least as much complexity in flavor space as our $D=4-\varepsilon$, $n_s, n_f = 2,2$ model
  - Additional spinors in fundamental + anti-fundamental to achieve CBZ fixed point perturbatively
  - Have done $N = 2,3$ ($N = 2$ is questionable for perturbation theory)
  - Cycles found at 2-loops
  - Potentially undone by 3-loops (just as in)
  - 3-loop calculation in progress

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One word on a-theorem

H. Osborn, "Weyl consistency conditions and a local renormalization group equation for general renormalizable field theories", Nucl.Phys. B363, 486 (1991).

A perturbative argument for the a-theorem was given by Osborn: in general, for the classically scale invariant $D = 4$ theory with dimensionless couplings g_I the Weyl anomaly coefficient a must satisfy

$$\frac{\partial a}{\partial g_I} = -(G_{IJ} + \cdots)B^J$$

where ellipsis = anti-symmetric in $I \leftrightarrow J$, G_{IJ} is a positive definite metric and $B^J = \beta^J - (Qg)^J$

Hence

$$\boxed{\frac{da}{dt} = -(G_{IJ} + \cdots)B^J \beta^I}$$

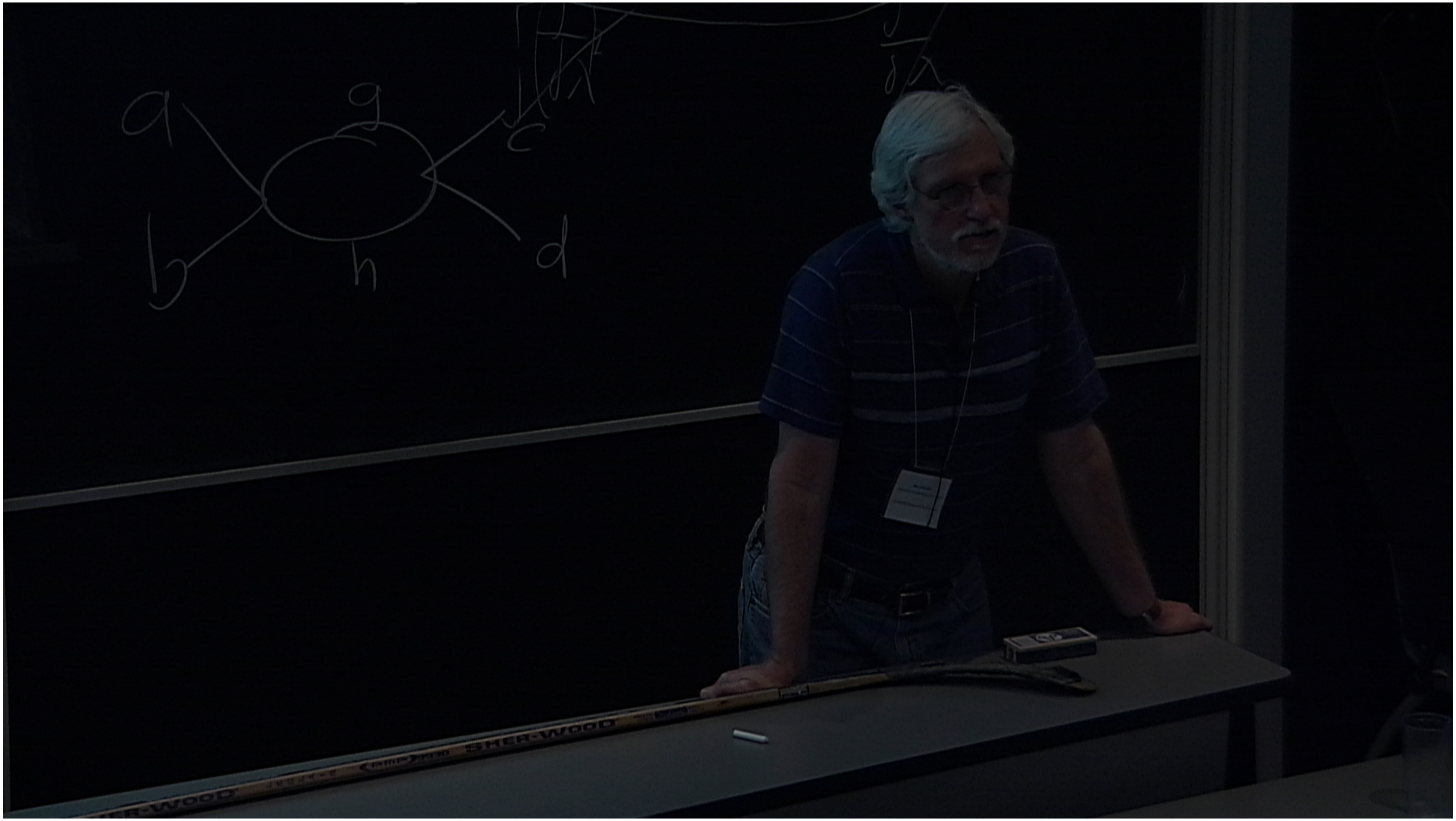
Note that $\frac{da}{dt} = 0 \leftrightarrow (\beta^I = 0 \mid B^I = 0)$ fixed points or cycles!

$$Q = 0 \quad \Rightarrow \quad \frac{da}{dt} = -G_{IJ}\beta^J\beta^I \leq 0 \quad \text{with} \quad \frac{da}{dt} = 0 \leftrightarrow \beta^I = 0$$

Obviously, lots of things left to do ...

- 3-loops
- Explore models in $D = 4$
- Supersymmetry?
- $D = 2 + \varepsilon$?
- $D = 6$? $D = 3$?
- Flows, globally (from where to where?)
- Relation to NR-QM cycles (Effimov)?
- Gravity duals? (Nakayama)
- Strong coupling
- ...

The End


a
g
c
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SHER-WOOD