

Title: 2+1 gravity as a conformal gauge theory and some frontiers for Shape Dynamics

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Abstract: I will start by showing that gravity, with positive cosmological constant in 2+1 dimensions, can be formulated as a theory of dynamic conformal spatial geometry. Exploiting the isomorphism between the isometry group of de Sitter space in $D+1$ dimensions and the conformal group in D dimensions, I will reinterpret the Chern--Simons formulation of 2+1 gravity as a gauge theory of a conformal connection. In Cartan's generalization of geometry, this connection represents an evolving spatial geometry locally modeled off the conformal sphere. After a suitable phase space reduction, we obtain shape dynamics. This remodeling explains, in 2+1 dimensions, the remarkable success of the York procedure for solving the initial value problem of general relativity and the uniqueness of the shape dynamics Hamiltonian. I will finish by speculating about possible connections between this work and the general shape dynamics program with holographic renormalization, AdS/CFT, and Horava gravity.

2+1 gravity as a conformal gauge theory and some frontiers for Shape Dynamics

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Collaborators

- 2 + 1: F. Mercati. (paper to appear soon)
- Large Volume: H. Gomes, T. Koslowski, F. Mercati. (arxiv:1105.0938)
- General: J. Barbour, L. Smolin.



A well-known isomorphism

Shape Dynamics:

Spacetime symmetry is *traded* for spatial scale invariance.

\therefore many fingered *time* + global *scale* \Leftrightarrow many fingered *scale* + global *time*

Relation between $g_{ab} \rightarrow e^\phi g_{ab}$ and $Conf(D)$?

Consider:

Isomorphism:

- Isometry group of $dS^{D,1} = SO(D+1, 1) = Conf(D, 0)$.
- Isometry group of $AdS^{D,1} = SO(D, 2) = Conf(D-1, 1)$.

Can this be used to get SD?

2 + 1: Yes!

- Isomorphism \rightarrow spacetime geometry \sim spatial “conformal” geometry.
- Key: relation between different *Cartan geometries*.
- Dynamic “conformal” geometry \Rightarrow SD constraints.

Higher dimensions? Gauge/gravity duality?



HJ equation in large volume limit

The large volume expansion (Gomes, Koslowski, Mercati, sg '11)

The SD Ham can be expanded in terms of $1/V^{1/D}$:

$$H_{\text{sd}} = 2\Lambda - \frac{D}{4(D-1)} \langle \pi \rangle^2 - R_{\text{Yamabe}} \left(\frac{V}{V_0} \right)^{-2/D} + \langle \pi^{\text{TT}} \cdot \pi^{\text{TT}} \rangle \left(\frac{V}{V_0} \right)^{-2} + \dots$$

for trajectories that reach $V \gg V_0$.

⇒ diff-invariant def'n of dS boundary + boundary is **conformal**.

HJ eq'n (in 3 + 1, asymptotically homogeneous)

The HJ functional → solved for *all* constraints: (Yamabe trick)

$$S = \pm \left(\sqrt{\frac{16}{3}} \Lambda V - \sqrt{\frac{3}{\Lambda}} R_{\text{Yamabe}} V^{1/3} + \dots \right).$$

Structure of holographic RG counter-terms (Skenderis '00) **but** in V-expansion.

HJ = RG flow equation?



AdS vs dS

Can translate results to AdS.

How?

- Use isomorphism $\text{AdS}^{(D,1)} \cong \text{Conf}(D-1, 1)$.
- Go to Euclidean \Rightarrow radial hypersurfaces = Cauchy surfaces.
- Use **radial** Ham constraint (minus sign between V and T).

Arguments *should* go through for $2+1$.

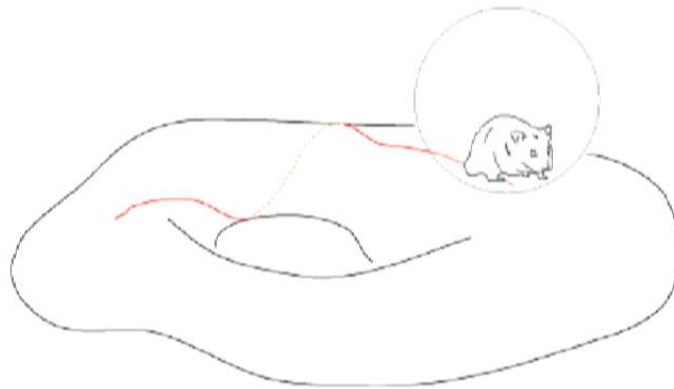
Boundaries:

- dS \rightarrow spacelike infinite past.
- AdS \rightarrow “timelike” spatial infinity.



Cartan geometry

Hamster ball: (source: D. Wise, gr-qc/0611154)



Roll homogeneous space over manifold.
⇒ point of contact stabilizes symmetries of hamster ball.

Cartan geometry

Generalized hamster “ball”! (local best matching)



Cartan geometry II

Ingredients

- Gauge group G (eg., $SO(3)$)
- Subgroup $H \subset G$ (eg., $SO(2)$)
 \Rightarrow Homogeneous space $Y = G/H$ (eg., S^2).
- Manifold \mathcal{M} st $\dim(\mathcal{M}) = \dim(Y)$.

Geometry: given by 'rolling' Y over \mathcal{M} .

\Rightarrow \mathfrak{g} -valued connection, A , on a principle right H -bundle over \mathcal{M} .

\Rightarrow metric: projection of A on $\mathfrak{g}/\mathfrak{h}$. (non-degenerate)

Riemannian geometry

$G = ISO(p, q)$; $H = SO(p, q)$; $Y = R^{p,q}$.

$$A = e^a P_a + \omega^a J_a$$

$$F(A) = T^a P_a + R^a J_a$$

e^a - coframe field; ω^a - spin connection; T^a - Torsion; R^a - Curvature.



2 + 1 Chern–Simons gravity

Define a Cartan geometry

- $G = ISO(2, 1), SO(2, 2), SO(3, 1), \quad k = 0, -1, +1$
- $H = SO(2, 1), \therefore \dim(\mathcal{M}) = 2 + 1.$

$$\Rightarrow A = e^\alpha P_\alpha + \omega^\alpha J_\alpha \qquad F = T^\alpha P_\alpha + \Omega^\alpha J_\alpha$$

$$\Omega^\alpha = R^\alpha + \frac{k}{l^2} \epsilon^{\alpha\beta\gamma} (e_\beta \wedge e_\gamma)$$

Action (Witten '89)

Palatini action \rightarrow Chern–Simons for A :

$$S_{\text{Palatini}}(e, \omega) = S_{\text{CS}}(A) = \frac{k'}{4\pi} \int \text{Tr} \left(A \wedge dA + \frac{2}{3} A^3 \right)$$

EOMs: Vanishing curvature

$$T^\alpha = 0 \quad (\text{Torsionless}) \qquad R^\alpha = \frac{k}{l^2} \epsilon^{\alpha\beta\gamma} (e_\beta \wedge e_\gamma) \quad (\text{EEs})$$



2 + 1 split and ADM

Decompose action:

$$S_{CS} = \frac{k'}{4\pi} \int dt \int d^2x [2\dot{e}^\alpha \wedge \omega_\alpha + e_0^\alpha \Omega_\alpha + \omega_0^\alpha T_\alpha] \quad (1)$$

Read off:

- Symplectic structure: $\{e_\mu^\alpha(x), \omega_\nu^\beta(y)\} \propto \epsilon_{\mu\nu} \eta^{\alpha\beta} \delta(x, y)$
- Lagrange multipliers: $A_0^\alpha = (e_0^\alpha, \omega_0^\alpha)$.
- Constraints: $T^\alpha = 0$ and $\Omega^\alpha = 0$
- Symmetries:
 - $\Omega \rightarrow$ spacetime translations (diffeos on-shell).
 - $T \rightarrow SO(2, 1)$ rotations.

ADM

Phase space reduction:

- Gauge fix $T^a = 0$ with $e_i^0 = 0$.
- $\omega_i^0 \rightarrow$ metric compatible $SO(2)$ connection.
- Variables left: e_i^a, ω_i^a (spatial)
- Constraints left: Ham + 2-diffeos (Ω^α), $SO(2)$ Gauss constraint (T^0).

Isomorphism

Note:

$$A_0 = \text{Lagrange multiplier}$$

New decomposition:

$$\begin{aligned} A_i &= e_i^\alpha P_\alpha + \omega_i^\alpha J_\alpha \\ &= E_i^a p_a + B_i^a k_a + \omega_i j + \phi_i D \end{aligned}$$

Use isomorphism

$$M = \begin{pmatrix} 0 & j & -\frac{1}{\sqrt{2}}(p_1 + k_1) & \frac{1}{\sqrt{2}}(p_1 - k_1) \\ -j & 0 & -\frac{1}{\sqrt{2}}(p_2 + k_2) & \frac{1}{\sqrt{2}}(p_1 - k_1) \\ \frac{1}{\sqrt{2}}(p_1 + k_1) & \frac{1}{\sqrt{2}}(p_2 + k_2) & 0 & D \\ -\frac{1}{\sqrt{2}}(p_1 - k_1) & \frac{1}{\sqrt{2}}(p_2 - k_2) & -D & 0 \end{pmatrix}.$$

Then

$$\begin{aligned} E_i^a &= \frac{1}{\sqrt{2}} (\epsilon^{ab} \omega_i^b + e_i^a) & \phi_i &= e_i^3 \\ B_i^a &= \frac{1}{\sqrt{2}} (\epsilon^{ab} \omega_i^b - e_i^a) & \omega_i &= \omega_i^3. \end{aligned}$$



Shape dynamics

$$F = T^a p_a + S^a k_a + \Omega j + \Theta D$$

New structure

- Symplectic structure:

$$\{E_i^a(x), B_j^b(y)\} \propto \frac{1}{2} \epsilon^{ab} \epsilon_{ij} \delta(x, y), \{\phi_i(x), \omega_j(y)\} \propto \epsilon_{ij} \delta(x, y).$$

- Symmetries:
 - $T \rightarrow$ special conformal transformations.
 - $S \rightarrow$ translations (2 diffs on-shell).
 - $\Omega \rightarrow SO(2)$ rotations.
 - $\Theta \rightarrow$ dilatations.

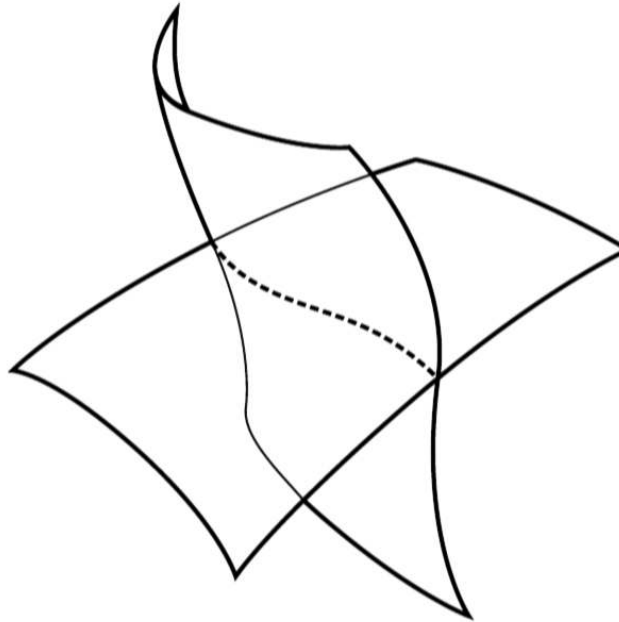
SD

Gauge fixing (inspired by Witten '89)

- Gauge fix T^a with $\phi_i = 0$.
- $\omega \rightarrow SO(2)$ spin connection.
- Variables left: E_i^a, B_i^a (dynamic spatial conformal geometry)
- Constraints left: S, Ω, Θ (SD!!)

Picture on ADM phase space

2 intersecting surfaces (full symmetry trading)



Outlook

- Extend to AdS.
- Extend to $\Lambda = 0$. ($2 + 1$: same trace)
- Extend to $D + 1$. (Stabilizer, action, etc...)
- gauge/gravity: pullback into bulk, a nice set of variables?



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Extensions: Construction principle for SD?

2 Requirements:

The SD Hamiltonian is constructed from:

- ① Local scale invariance.
- ② Dynamical equivalence to GR (or locality).

But:

- Observables are non-local → locality in bulk?
- SD should stand *on its own*.

Idea

Local description at conformal fixed point (boundary).

⇒ Time evolution given by **holographic** RG flow.

Interesting observation: (particle) shape space is holographic.

Can we use ideas from entropic gravity or Jacobson's argument?







