Title: The Theory of Shape Dynamics

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Abstract:

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Intro to Shape Dynamics

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May 9, 2012

In collaboration with Tim Koslowski, Sean Gryb and Flavio Mercatti

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Notational warning!!

This will be a talk focused on 3+1 formulations of gravity.

3D vs 4D

Here g is a 3D Riemannian metric (the spatial metric) that evolves in time. If we ever need to use 4D Lorentzian metric, we'll use h.

- Over a closed (compact without boundary) 3-dimensional manifold Σ!
- We will avoid indices as much as we can, but they are there!
- When we talk about a conformal transformation, we mean Weyl transformations, as $g \mapsto \alpha g$.

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What is Shape Dynamics?

What it is

A Hamiltonian formulation of gravity with the following proeminent features:

- Possesses the same canonical variables as Hamiltonian GR: (g, π).
- Does *not* possess refoliation invariance (boosts).
- Trades that symmetry for foliation preserving conformal transformations (Weyl) + unique, non-local global Hamiltonian.

What it is not

- York's approach to the initial value problem (and its related constant-mean-curvature gauge for GR).
- Barbour et al's CS+V re-derivation of York.

Both very useful and necessary for Shape Dynamics, but neither has conformal symmetry manifest in the dynamics.

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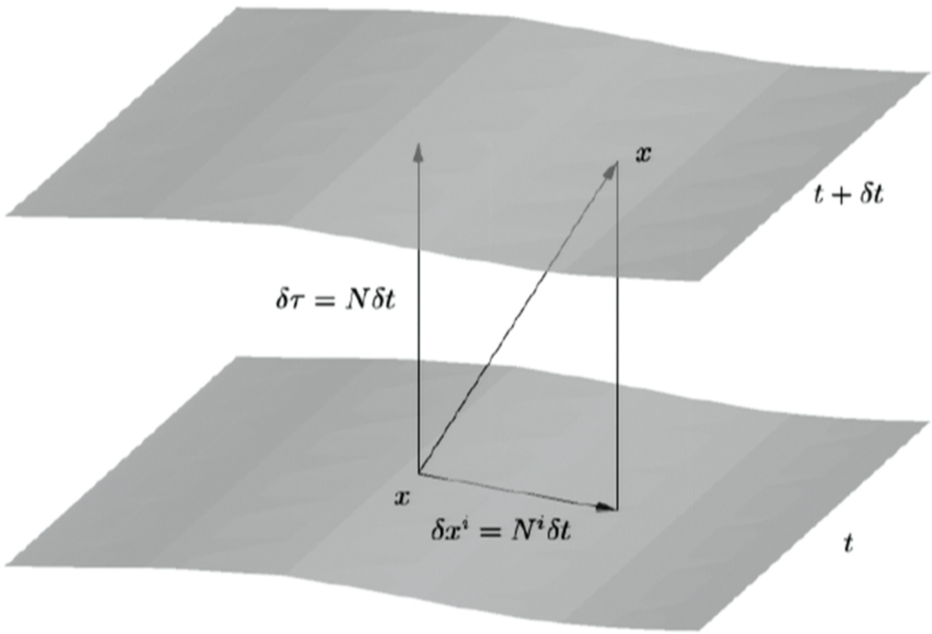
Hypersurface foliation

- Assume global hyperbolic: $M \simeq \Sigma \times \mathbb{R}$



Intermezzo: Dirac analysis

Constraints are a convenient way to encode over-parametrization of physical degrees of freedom.

Let $\phi_i(q,p) = 0, i \in I$ denote constraints. They define surfaces and flow in phase space, and can have different degrees of mutual "conservation":

- **Compatible.**
  These are called first class constraints. They arise when the dynamical flow generated by one constraint conserves the set: $\delta_{\phi_i}\phi_j = \{\phi_i, \phi_j\} = a^k \phi_k$
- **Impose further constraints.** This occurs when the flow is only conserved on some subsurface:
  $\delta_{\phi_i}\phi_j = f(p,q) \Rightarrow f = 0$ must now be added to the list of constraints.
- **Second class.** These arise when the two constraints are conjugate. $\delta_{\phi_i}\phi_j = 1$. Must either find a coordinate system where they don't appear, or project dynamics to constraint surface.

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Canonical framework: ADM

- Use Gauss-Codazzi relations + Einstein equations: ${}^4R \mapsto (R, K)$
- Over-parametrization: relations that extrinsic curvature have to satisfy.
- Legendre: $(g, \dot{g}) \mapsto (g, \pi)$
  - Constraints: ensure relations hold.

  $H_a(x) = {\pi_a}^b{}_{;b} = 0$

  $S(x) = \mathrm{tr}(\pi \cdot \pi) - \frac{1}{2}(\mathrm{tr}\pi)^2 - R = 0$

$S(x)$ and $H_a(x)$ are *first class*. They generate compatible symmetries (on the constraint surface).

Total Hamiltonian: $H_{\text{ADM}} = \int_\Sigma d^3x\, (N(x)S(x) + \xi^a(x)H_a(x))$
Is "pure constraint" . This is the ADM system.

Momentum and scalar constraints

- $\mathcal{H}_a(x)$ generates 3-diffeomorphisms.

  True (infinite-dimensional) Lie algebra.
- $S(x)$ generates time refoliation (and thus evolution).

  In contrast to the action of 3-diffeomorphisms:

  Enormous difficulty in giving meaning to GR's physical degrees of freedom!

  Not subalgebra (commutation relations involve 3-diffeomorphisms), and entire constraint algebra is "soft" (structure functions).

Introduces many, many difficulties in quantization.

- $S(x)$ quadratic in momenta: $\frac{\delta}{\delta g(x)} \frac{\delta}{\delta g(x)}$: ill-defined.
- Constraints imposed at the quantum level: $\hat{S}\Psi[g] = 0$.

  Klein-Gordon type equation. Inner product?

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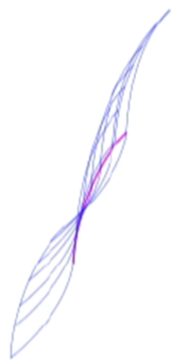
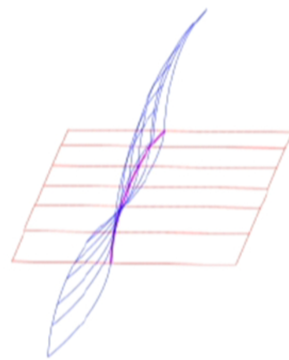
Outline

1. Hamiltonian GR
2. Shape Dynamics
3. Matter and large volume expansion
4. Outlook

Motivation: "pure constraint" systems such as GR may have observables coinciding with that of systems with different symmetries.





ADM Gravity

$$S(N) = \int N\left(\frac{G(\pi,\pi)}{\sqrt{|g|}} - (R - 2\Lambda)\sqrt{|g|}\right)$$
$$H(v) = \int \pi^{ab}\mathcal{L}_v g_{ab}$$

Shape Dynamics

$$H_{SD} = V - V_o$$
$$Q(\rho) = \int(\pi - \langle\pi\rangle\sqrt{|g|})$$
$$H(v) = \int \pi^{ab}\mathcal{L}_v g_{ab}$$

Refoliation invariance has famous gauge fixing exploring spatial conformal transformations ([York]).

Preliminary: intrinsic constant mean curvature gauge

What is the constant mean curvature condition?

- The trace of the extrinsic curvature of each leaf is spatially constant.

  Roughly, means that observers use the Hubble constant as a clock.

- Mathematically: set $\mathrm{tr}\pi = \frac{1}{V}\int \mathrm{tr}\pi =: \langle \mathrm{tr}\pi \rangle$.

  Note that $\mathrm{tr}\pi$ generates conformal transformations. I.e.

  - $\{\mathrm{tr}\pi(\epsilon), g\} = \epsilon g$
  - $\{\mathrm{tr}\pi(\epsilon), \pi\} = -\epsilon\pi$
  - $\mathrm{tr}\pi - \frac{1}{V}\int \mathrm{tr}\pi$ generates volume-preserving conformal transformations.

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Shape Dynamics: Main message (words)

- On a certain region in phase space, there exists a very special system dynamically equivalent to ADM.
  - Region is that of constant-mean curvature (CMC) foliable Einstein spacetimes (with closed Σ). (see Isenberg's talk for counter-examples)
- System is one that does not possess refoliation symmetry.
  - Instead it possesses local 3D scale invariance. Symmetry trading!
  - All constraints linear in momenta.
  - Individual sets of constraints form subalgebras. Easy to quotient. Physical degrees of freedom clear.
  - Exists in the original ADM phase space (g, π) with the canonical Poisson bracket.
  - Possesses one global Hamiltonian which depends only on (g, π) (no explicit "time" dependence).

Shape Dynamics: Main message

ADM ($\Sigma \times \mathbb{R}$)

Local 1st class constraints:

- 3-diffeomorphisms
- refoliations

Shape Dynamics

Local 1st class constraints

- 3-diffeomorphisms
- Conformal transformations

$$H_{\text{ADM}} = \int d^3x(N(x)S(x) + \xi^a(x)H_a(x))$$

$$H_{\text{dual}} = \mathcal{H}_{\text{gl}} + \int d^3x[\lambda(x)D(x) + \xi^a(x)H_a(x)]$$

- $H_a(x)$: momentum constraint (one per x).
- $S(x)$: Scalar constraint (one per x).
- $D(x) = 4(\pi - \langle\pi\rangle\sqrt{g})(x)$: conformal constraint (one per x).
- $\mathcal{H}_{\text{gl}}$: Global Shape Dynamics Hamiltonian.

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Shape Dynamics

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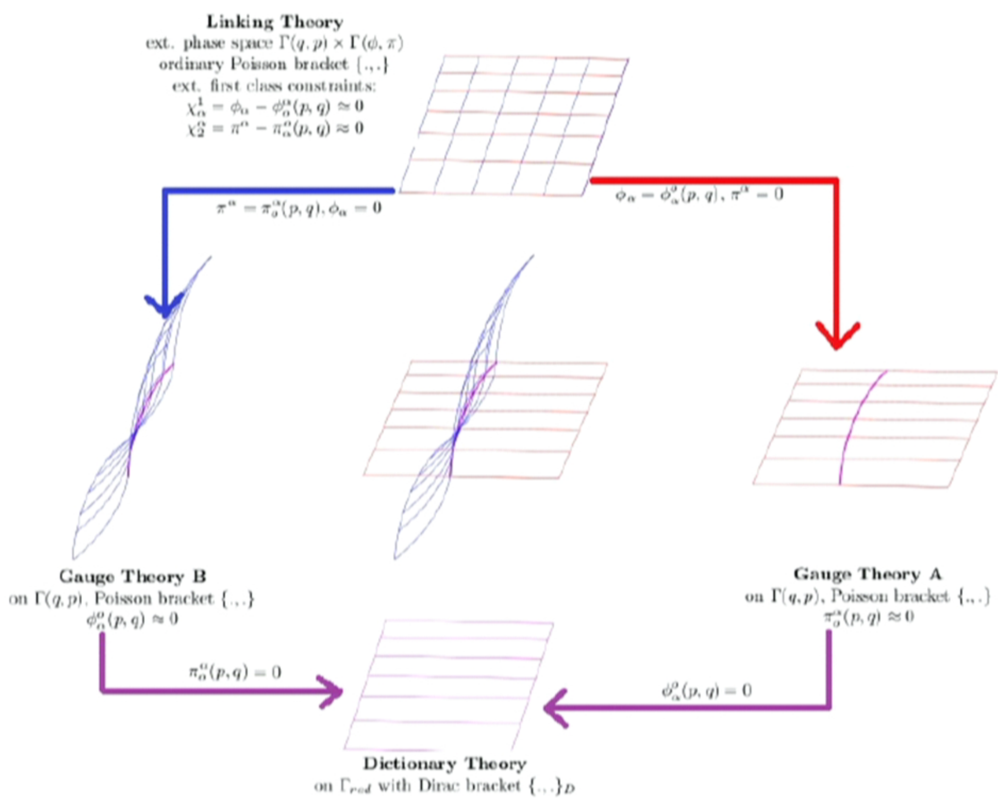
How is it constructed? Figure.



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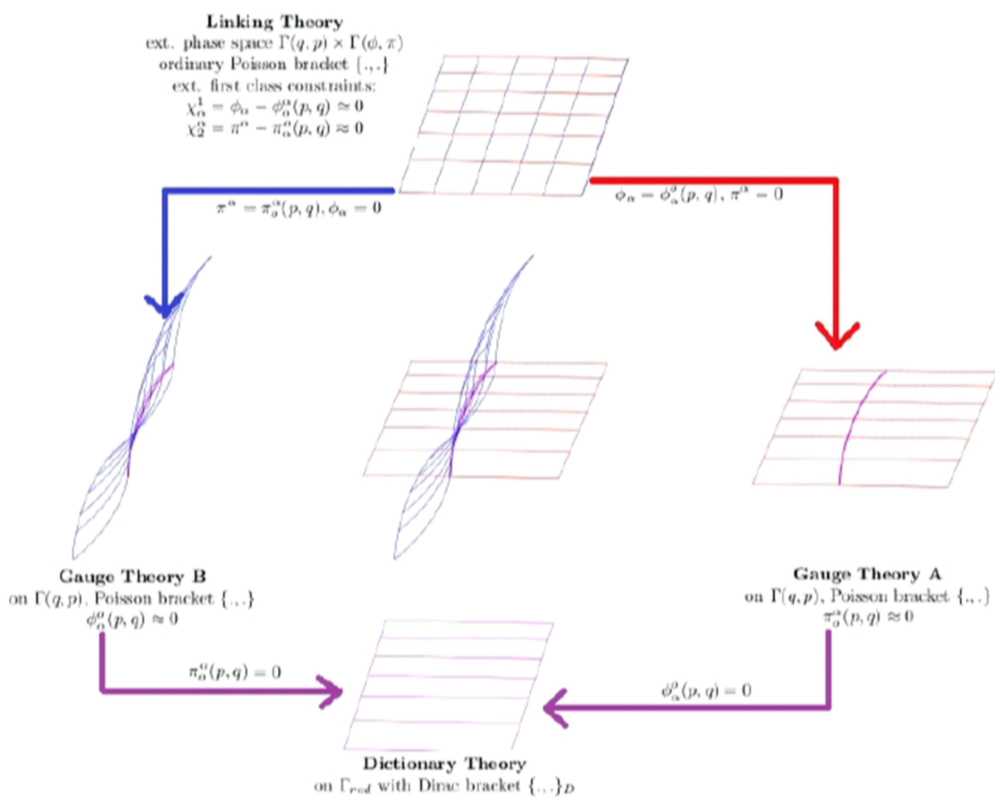

How is it constructed? Words.

1. Introduce artificial symmetry (Stuckelberg) and extend phase space (and functions)
2. Extra constraint arises (constrains functions on extended phase space to represent original functions: no extra physical degrees of freedom).

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3. Impose specific (natural) gauge fixings and separate first and second class parts of constraints.
4. Will get that $S(x)$ separates into 1 first class constraint (evolution) and the rest second class.
5. Solve 2nd class constraints *for extra Stuckelberg variables.*
6. Get back to *original phase space* g, π *with canonical Poisson bracket.*

   Leftover constraints first class, generating diffeomorphisms, local 3d conformal transformations, and global evolution.

$$\pi_\varphi = 0$$

$$C = \pi_\varphi - 4\left(\pi - \langle \pi \rangle \sqrt{g}\right)$$

$$\{C, g_{ab}\} =$$

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Take away message from SD.

ADM

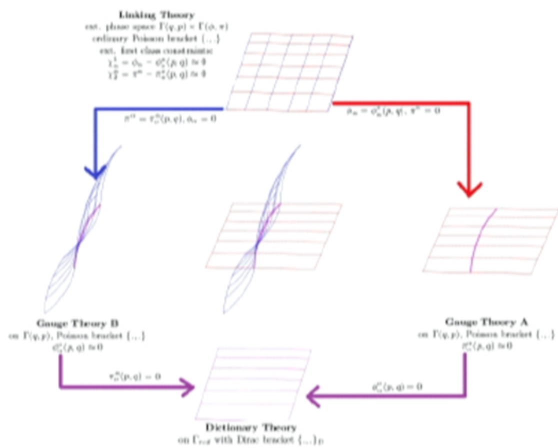
Local symmetries:

- 3-diffeomorphisms
- refoliations

Shape Dynamics

One Hamiltonian + local symmetries:

- 3-diffeomorphisms
- Conformal transformations



Shape Dynamics is to York CMC

as Electromagnetism is to
transverse gauge of vector potential.

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Coupling other fields.

Question: how should fields scale $\psi \to e^{n\hat{\phi}}\psi$?

Two problems:

1. Foliation depends on the field for scaling $n \neq 0$, not geometric (or worse, for YM depends on the gauge)

   Solution: only metric variables scale ("neutral coupling")

2. Uniqueness of global Hamiltonian: involves invertibility of elliptic 2nd order diff. op. Requires:

   $\frac{1}{2}\left(\frac{\delta H_m}{\delta g_{ab}} g_{ab} - \frac{1}{2} H_m\right) \leq \frac{1}{12}\langle\pi\rangle^2 + \sigma^2$

- Both issues solved with neutral coupling for Yang-Mills (and gauge invariance respected) and massless scalars.

  But invertibility (point 2) doesn't work always for massive scalars: bound on the field magnitude (e.g. bound on the cosmological constant).

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Tractability: Large-volume expansion.

- Global Hamiltonian is non-local. Solve order by order in a large volume expansion. First few terms:

$$\mathcal{H}_{gl} = 2(\Lambda - \frac{1}{12}\langle\pi\rangle^2) - \frac{R_o}{V^{2/3}} + \frac{1}{V^2}\langle\sigma^2\rangle + \mathcal{O}(V^{-8/3})$$

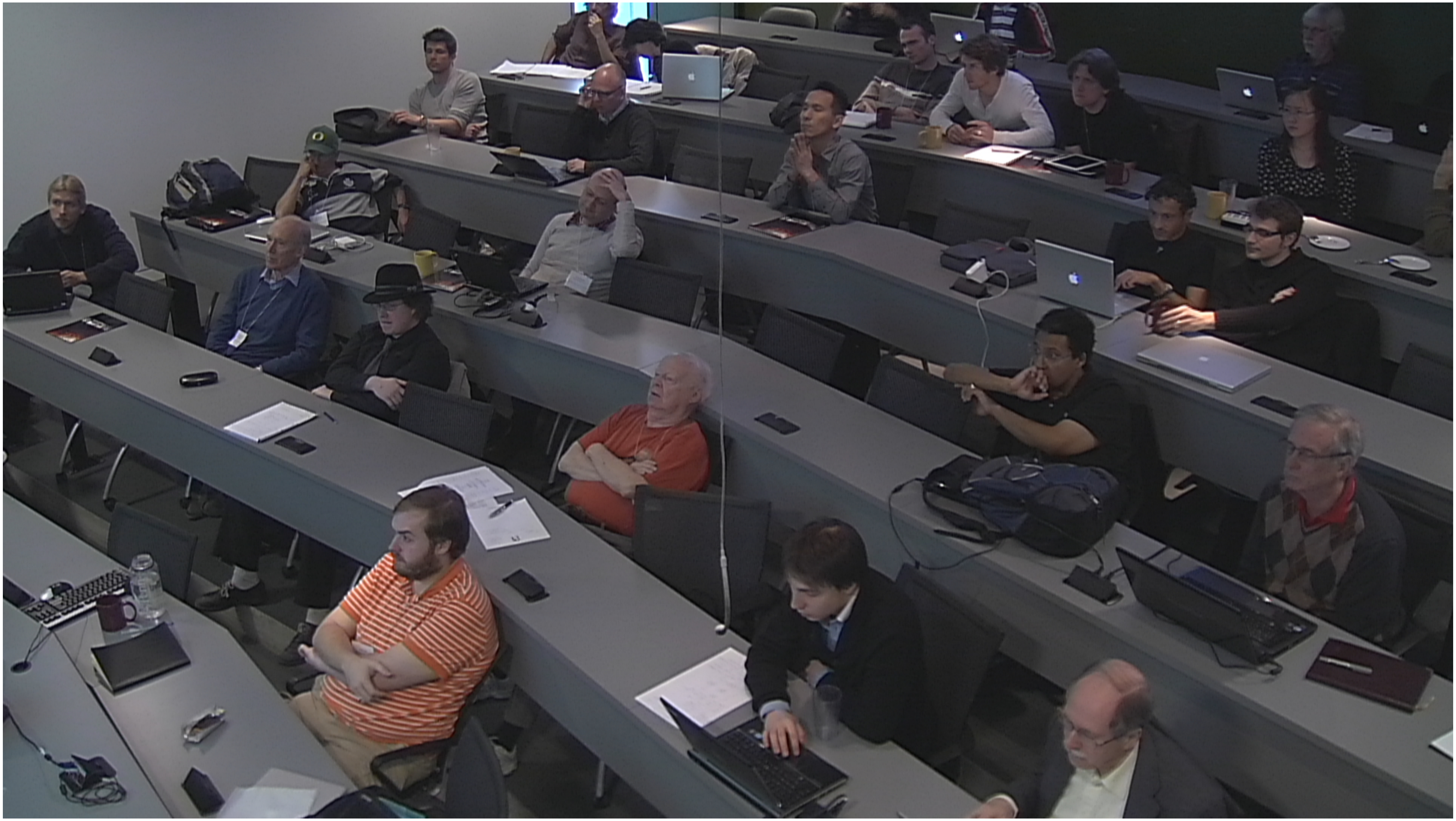
Here R_o is the unique constant scalar curvature in the conformal class of R (Yamabe gauge).

Global Hamiltonian can be seen as reparametrization constraint: for large volume reparam. invariance implies full conformal invariance.

- Also a Hamilton-Jacobi expansion for the on-shell action:

$\langle\pi\rangle \to \frac{\delta S}{\delta V}$, $\pi^{ab} \to \frac{\delta S}{\delta g_{ab}}$

$S = S_0 V + S_1 V^{1/3} + S_2 V^{-1/3} + \mathcal{O}(V^{-1})$
$=$
$\pm\left(\sqrt{\frac{16\Lambda}{3}}\, V - \sqrt{\frac{3}{\Lambda}}\, R_o\, V^{1/3} + \left(\frac{3}{\Lambda}\right)^{3/2}\left({R_o}^2 - \frac{8}{3}\langle R_o^{ab} R^o_{ab}\rangle\right) V^{-1/3} + \dots\right)$



$$\pi_\varphi = 0$$

$$C = \pi_\varphi - 4\left(\pi - \langle \pi \rangle \sqrt{g}\right)$$

$$\{C, g_{ab}\} =$$

$$\frac{\delta R_0}{\delta g_0} = R^0_{ab} - \frac{1}{2} R_0 g^0_{ab}$$

Possible advantages

Classically matches GR over (g, π) that satisfy $\text{tr}\pi = c$ (gauge choices in each) but

- Advantage over CMC gauge-fixed ADM in that variables *and* constraints on the dofs are "*local*".
- Different method to find solutions. Different symmetries. Different gauges.
  - Maybe find different solutions and go back to ADM gauge (and covariantize)?
  - First try: finding a solution for "KSdS" without imposing the ADM constraints.
- ADM cosmological perturbation theory complicated (because we can't separate evolution from constraints). Perturbations must satisfy all constraints at each level.
  - Here, introduce perturbations that only need to satisfy the local constraints, and use unperturbed global Hamiltonian to evolve?

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Issues and outlook

The elephant in the room: global Hamiltonian is non-local.

- We saw some ways around it: large-volume expansions. Other expansions?
- Theory is non-local because we include a volume-preserving condition on conformal transfs.
  - This is necessary to have a non-trivial leftover Hamiltonian in Shape Dynamics. I.e. to match ADM trajectories with Shape Dynamics trajectories (to just match Cauchy data for a conformal theory and ADM, no such problem arises).
- If we are interested in the pure quantum theory, so what if we don't match trajectories?
  - BRST: A modification of Shape Dynamics possesses full Weyl and special conformal symmetry (no diffeos) and serves as a complete gauge-fixing fermion for the BRST-extended ADM.
  - The gauge-fixed ADM BRST-extended Hamiltonian possesses a hidden symmetry: "symmetry doubling". (Koslowski's talk)