Title: Shape Dynamics: Relativity Without Relativity

Date: May 09, 2012 04:00 PM

URL: http://pirsa.org/12050067

Abstract: I review the best-matching construction, and the striking properties of a Jacobi-type action first introduced by Baierelein, Sharp and Wheeler. The simplest theories compatible with such an action principle must have a universal light-cone and gauge symmetry. I also describe the implementation of three-dimensional conformal symmetries on the basis of the BSW action, which gives a first-principles derivation of York's solution of the initial value problem in General Relativity.

THE JACOBI PRINCIPLE

For a holonomic conservative system the action

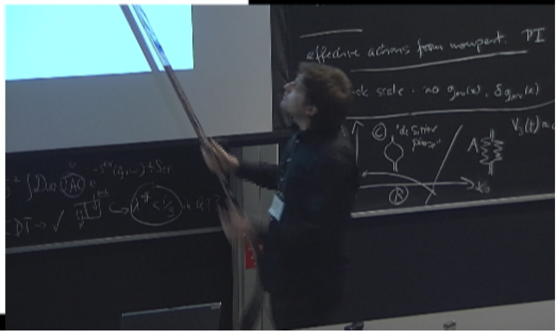
$$S_J = \int \sqrt{2(E_{\mathrm{tot}} - U)}\,ds\,, \qquad ds^2 = m^{IJ}\,\mathrm{d}\mathbf{q}_I \cdot \mathrm{d}\mathbf{q}_J$$

where $U = U(q)$ is the potential, and m^{ij} is the mass tensor and E_{tot} is a constant, has a minimum on the actual trajectory $q(t)$

The dynamical problem of finding the actual trajectory is reduced to a geometrical problem: finding the geodesics of the metric

$$g_{ab}^{IJ} = 2(E_{\mathrm{tot}} - U)\, m^{IJ}\, \delta_{ab}$$

S_J is reparametrization-invariant: there is no notion of time



An Jacobi-type action for geometrodynamics

$$S = \int d\tau \int d^3x \sqrt{g}\sqrt{U}\sqrt{T},$$

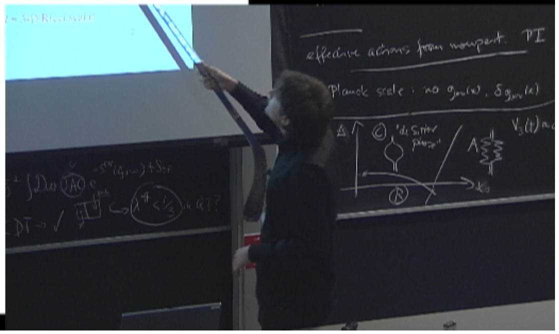
T = "kinetic energy": $\quad T = G^{abcd}\frac{\mathrm{d}g_{ab}}{\mathrm{d}\tau}\frac{\mathrm{d}g_{cd}}{\mathrm{d}\tau}$

g_{ab} = 3-D positive-definite metric

G^{abcd} = DeWitt "supermetric": $\quad G^{abcd} = g^{ac}g^{bd} - g^{ab}g^{cd}$

The momenta are underdetermined by the speeds $\dot{g}_{ij} \Rightarrow$ primary constraints. The theory is consistent if the constraints propagate, *i.e.* close a first-class system. This is what the RWR approach rests on.

Let's try the simplest choice: $U = R$ = 3-D Ricci scalar



An Jacobi-type action for geometrodynamics

$$S = \int d\tau \int d^3x \sqrt{g}\,\sqrt{U}\,\sqrt{T}\,,$$

T = "kinetic energy": $\quad T = G^{abcd}\,\frac{\mathrm{d}g_{ab}}{\mathrm{d}\tau}\,\frac{\mathrm{d}g_{cd}}{\mathrm{d}\tau}$

g_{ab} = 3-D positive-definite metric

G^{abcd} = DeWitt "supermetric": $\quad G^{abcd} = g^{ac}\,g^{bd} - g^{ab}\,g^{cd}$

The momenta are underdetermined by the speeds $\dot{g}_{ij} \Rightarrow$ primary constraints. The theory is consistent if the constraints propagate, *i.e.* close a first-class system. This is what the RWR approach rests on.

Let's try the simplest choice: $U = R$ = 3-D Ricci scalar

2

Momenta, primary constraints and their propagation

$$p^{ab} = \frac{\delta \mathscr{L}}{\delta \dot{g}_{ab}} = \sqrt{g}\sqrt{\frac{R}{T}} G^{abcd}\,\dot{g}_{cd}\,, \qquad p^{ab} \text{ not independent}$$

$$\boxed{\mathscr{H} = \frac{1}{\sqrt{g}}(G^{-1})_{abcd}\, p^{ab}\, p^{cd} - \sqrt{g}R} = 0 \qquad (G^{-1})_{abcd} = g_{ab}g_{cd} - \frac{1}{2} g_{ac}\, g_{bd}$$

the constraint $\mathscr{H}$ has to propagate:

$$\frac{\mathrm{d}(\sqrt{g}\mathscr{H})}{\mathrm{d}\tau} = 2\sqrt{g}N^{-1}\left(N^2\, p^{ab}{}_{;b}\right)_{;a}, \qquad N = \sqrt{T/4R}$$

this requires the introduction of a secondary constraint:

$$\boxed{\mathscr{H}^i = p^{ij}{}_{;j}} \approx 0$$

3

We have to check that the new constraint propagates as well:

$$\frac{\mathrm{d}\mathscr{H}^i}{\mathrm{d}\tau} = \frac{1}{2}\sqrt{g}N^{-1}(N\mathscr{H})^{;i} - \frac{2N}{\sqrt{g}}\left(p^{ij} - \frac{1}{2}g^{ij}p\right)\mathscr{H}_j \approx 0$$

constraint algebra closes.

$\mathscr{H}^i$ generates infinitesimal diffeomorphisms:

$$P_\xi = \int d^3y\, \xi_i(y)\, \mathscr{H}^i(y) \qquad \{P_\xi, g_{ab}(x)\} = \pounds_\xi g_{ab}(x),$$

$$\pounds_\xi g_{ab} = \nabla_a \xi_b + \nabla_b \xi_a \quad \text{Lie derivative}$$

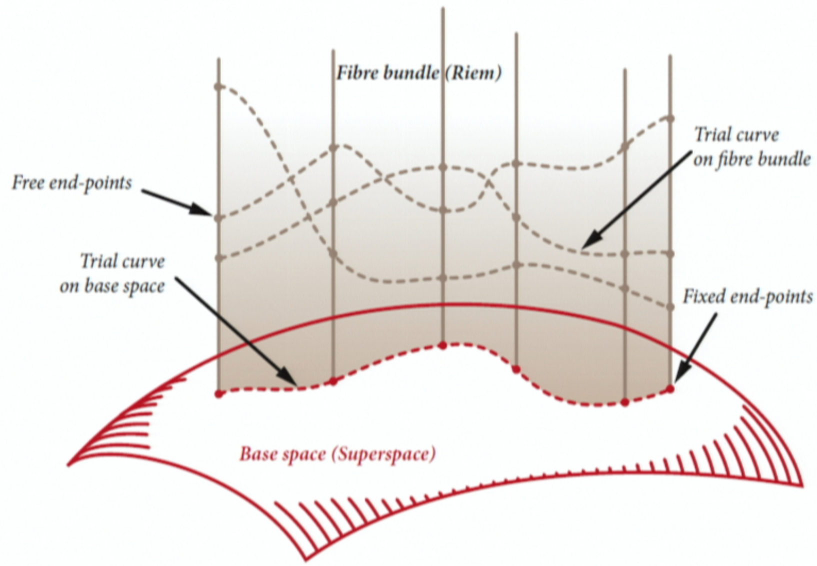
Our action:

$$S = \int d\tau \int d^3x \sqrt{g}\sqrt{R}\sqrt{T}$$

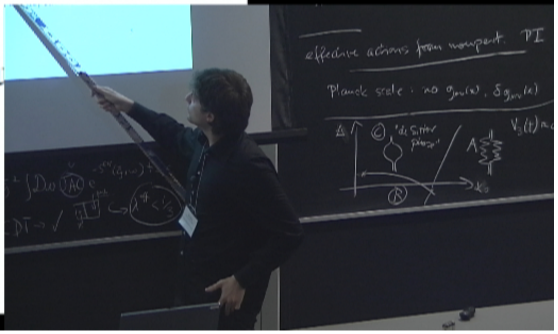
is invariant under *time-independent* diffeomorphisms $\xi_i = \xi_i(x)$

4

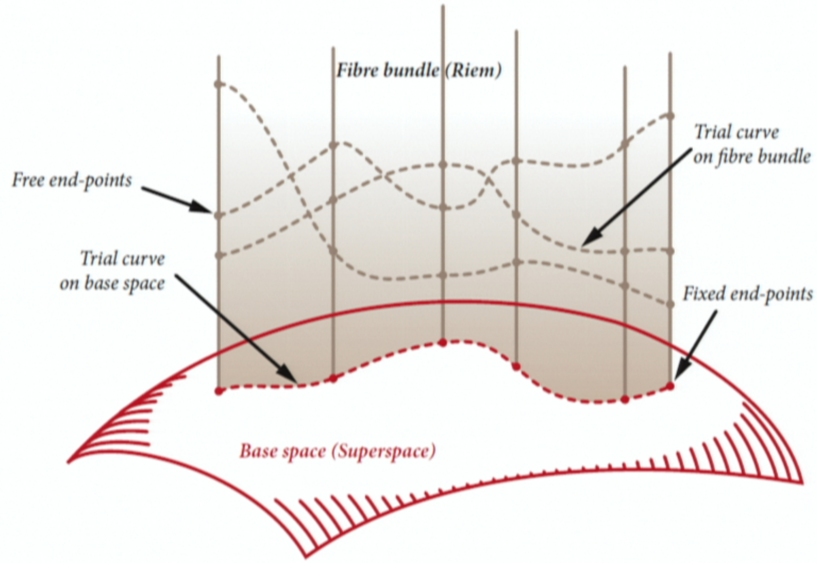
Machian free-end-point variation



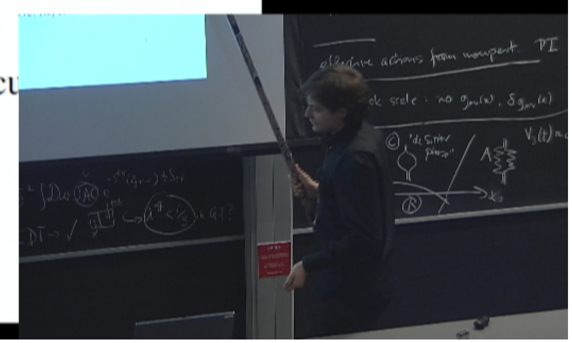
1. Take a trial curve in Superspace = Riem/3-Diffeos
2. Lift it into the fibre bundle through diffeos: you get a sheet in Riem
3. Minimize the action on that sheet *leaving the end-points free*
4. This is the Best-Matched action on the trial curve. Repeat on all trial cu *with fixed end-points* to find the extremal one.



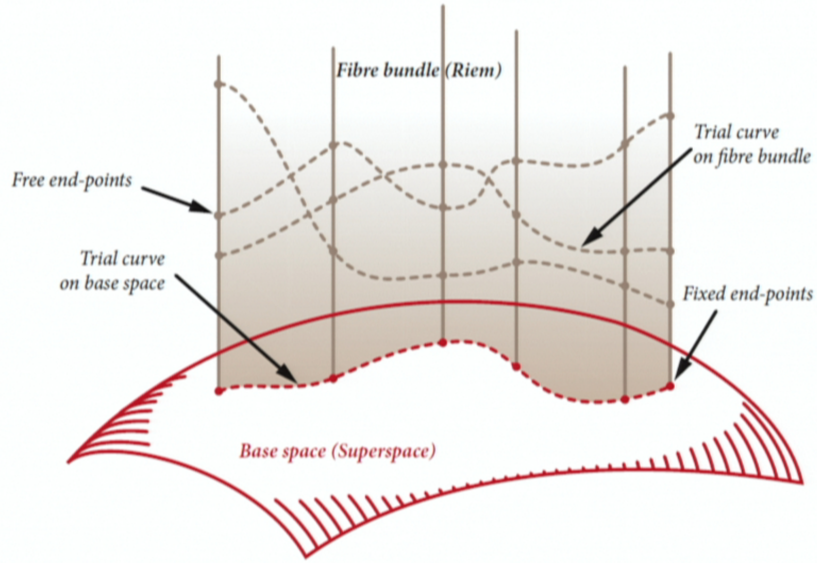
Machian free-end-point variation



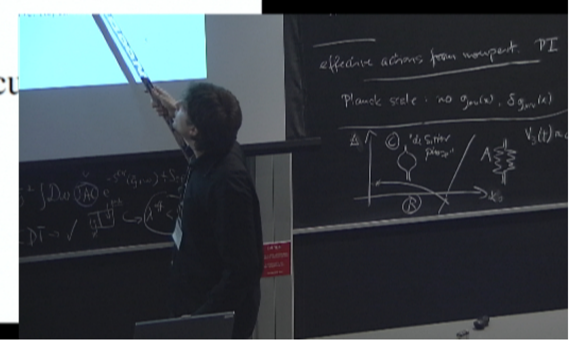
1. Take a trial curve in Superspace = Riem/3-Diffeos
2. Lift it into the fibre bundle through diffeos: you get a sheet in Riem
3. Minimize the action on that sheet *leaving the end-points free*
4. This is the Best-Matched action on the trial curve. Repeat on all trial cu
with fixed end-points to find the extremal one.



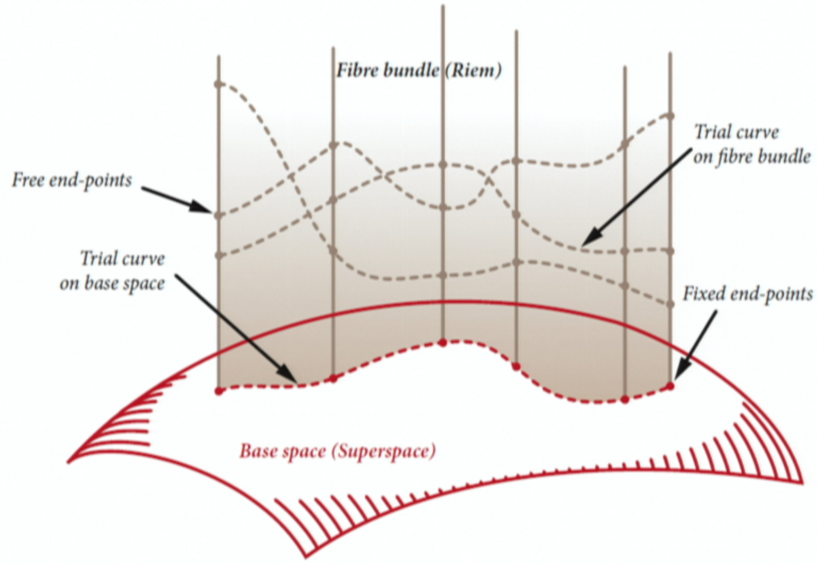
MACHIAN FREE-END-POINT VARIATION



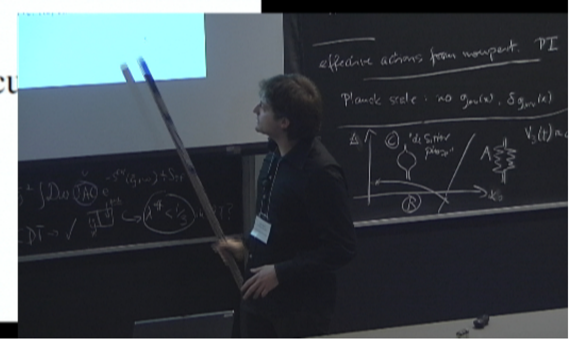
1. Take a trial curve in Superspace = Riem/3-Diffeos
2. Lift it into the fibre bundle through diffeos: you get a sheet in Riem
3. Minimize the action on that sheet *leaving the end-points free*
4. This is the Best-Matched action on the trial curve. Repeat on all trial cu
with fixed end-points to find the extremal one.



MACHIAN FREE-END-POINT VARIATION



1. Take a trial curve in Superspace = Riem/3-Diffeos
2. Lift it into the fibre bundle through diffeos: you get a sheet in Riem
3. Minimize the action on that sheet *leaving the end-points free*
4. This is the Best-Matched action on the trial curve. Repeat on all trial cu *with fixed end-points* to find the extremal one.



MACHIAN FREE-END-POINT VARIATION

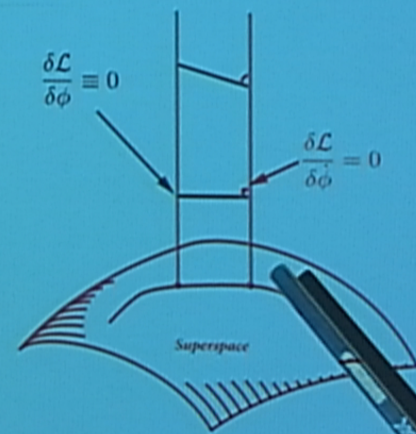
Take a reference metric $g_{ab}(x,\tau)$at each point of the curve τ.
Lift the metric all over the sheet through a field $\xi_i(x,\tau)$:

$$\frac{dg_{ab}}{d\tau} \to \frac{dg_{ab}}{d\tau} - \pounds_\xi g_{ab}, \qquad \mathscr{L} \to \sqrt{g}\sqrt{R}\sqrt{G^{abcd}\frac{d}{d\tau}\left(g_{ab}-\pounds_\xi g_{ab}\right)\frac{d}{d\tau}\left(g_{cd}-\pounds_\xi g_{cd}\right)}$$

then minimize without keeping the end point fixed.

$$\frac{\delta \mathscr{L}}{\delta \dot{\xi}_i} = 0$$ selects horizontal section

$$\frac{\delta \mathscr{L}}{\delta \xi_i} = 0$$ selects best-matched section



6

MACHIAN FREE-END-POINT VARIATION

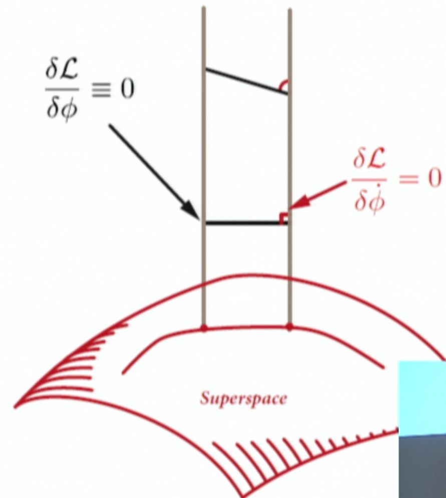
Take a reference metric $g_{ab}(x,\tau)$at each point of the curve τ.
Lift the metric all over the sheet through a field $\xi_i(x,\tau)$:

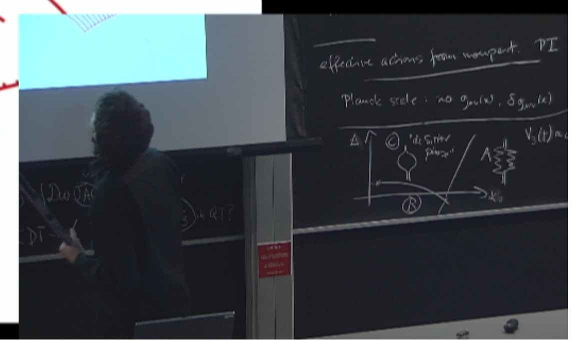
$$\frac{\mathrm{d}g_{ab}}{\mathrm{d}\tau} \to \frac{\mathrm{d}g_{ab}}{\mathrm{d}\tau} - \pounds_\xi g_{ab}, \qquad \mathscr{L} \to \sqrt{g}\sqrt{R}\sqrt{G^{abcd}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{ab}-\pounds_\xi g_{ab}\right)\frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{cd}-\pounds_\xi g_{cd}\right)}$$

then minimize without keeping the end point fixed.

$$\frac{\delta\mathscr{L}}{\delta\dot{\xi}_i}=0$$ selects horizontal section

$$\frac{\delta\mathscr{L}}{\delta\xi_i}=0$$ selects best-matched section





MACHIAN FREE-END-POINT VARIATION

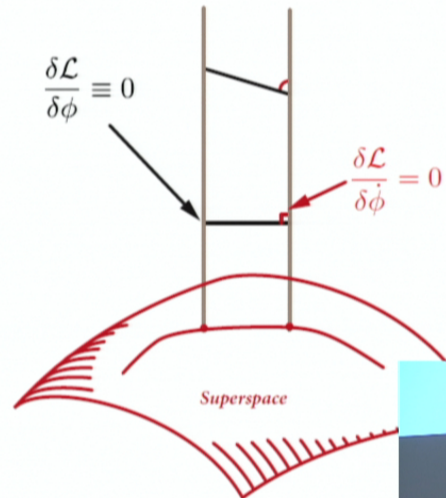
Take a reference metric $g_{ab}(x,\tau)$at each point of the curve τ.
Lift the metric all over the sheet through a field $\xi_i(x,\tau)$:

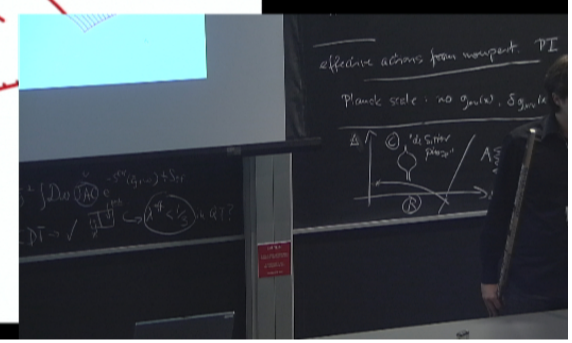
$$\frac{\mathrm{d}g_{ab}}{\mathrm{d}\tau} \to \frac{\mathrm{d}g_{ab}}{\mathrm{d}\tau} - \pounds_\xi g_{ab}\,, \qquad \mathscr{L} \to \sqrt{g}\sqrt{R}\sqrt{G^{abcd}\frac{\mathrm{d}}{\mathrm{d}\tau}\Big(g_{ab}-\pounds_\xi g_{ab}\Big)\frac{\mathrm{d}}{\mathrm{d}\tau}\Big(g_{cd}-\pounds_\xi g_{cd}\Big)}$$

then minimize without keeping the end point fixed.

$$\frac{\delta\mathscr{L}}{\delta\dot{\xi}_i} = 0 \quad \text{selects horizontal section}$$

$$\frac{\delta\mathscr{L}}{\delta\xi_i} = 0 \quad \text{selects best-matched section}$$





Machian free-end-point variation

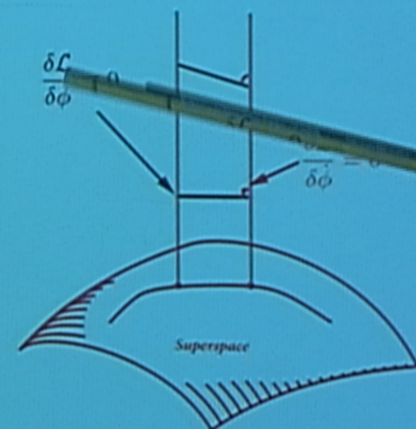
Take a reference metric $g_{ab}(x,\tau)$at each point of the curve τ.
Lift the metric all over the sheet through a field $\xi_i(x,\tau)$:

$$\frac{\mathrm{d}g_{ab}}{\mathrm{d}\tau} \to \frac{\mathrm{d}g_{ab}}{\mathrm{d}\tau} - \pounds_\xi g_{ab}, \qquad \mathscr{L} \to \sqrt{g}\sqrt{R}\sqrt{G^{abcd}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{ab} - \pounds_\xi g_{ab}\right)\frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{cd} - \pounds_\xi g_{cd}\right)}$$

then minimize without keeping the end point fixed.

$$\frac{\delta \mathscr{L}}{\delta \dot{\xi}_i} = 0 \quad \text{selects horizontal section}$$

$$\frac{\delta \mathscr{L}}{\delta \xi_i} = 0 \quad \text{selects best-matched section}$$



6

The Baierlein–Sharp–Wheeler action

What we found is the "BSW action":

In 1962 Baierlein, Sharp and Wheeler found a way to rewrite the Arnowitt–Deser–Misner action of General Relativity as a Jacobi-type action, which is explicitly reparametrization-invariant

$$S_{BSW} = \int d\tau \int d^3x \sqrt{g}\sqrt{R}\sqrt{k^{ij}k_{ij} - \mathrm{tr}\, k^2}$$

$$k_{ij} = \frac{\mathrm{d}g_{ij}}{\mathrm{d}\tau} - N_{(i;j)}$$

[Baierlein, Sharp, Wheeler, Phys. Rev. **126**, (1962)]

8

The Baierlein–Sharp–Wheeler action

What we found is the "BSW action":

In 1962 Baierlein, Sharp and Wheeler found a way to rewrite the Arnowitt–Deser–Misner action of General Relativity as a Jacobi-type action, which is explicitly reparametrization-invariant

$$S_{BSW} = \int d\tau \int d^3x \sqrt{g}\sqrt{R}\sqrt{k^{ij}k_{ij} - \mathrm{tr}\, k^2}$$

$$k_{ij} = \frac{\mathrm{d}g_{ij}}{\mathrm{d}\tau} - N_{(i;j)}$$

[Baierlein, Sharp, Wheeler, Phys. Rev. **126**, (1962)]

8

RIGIDITY OF BSW

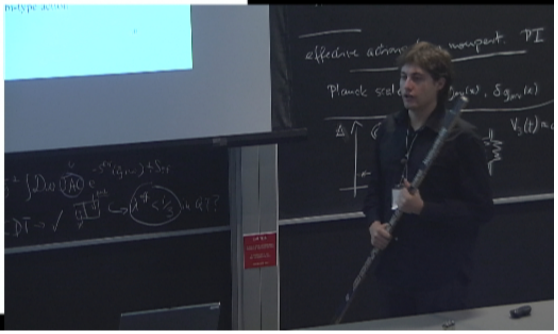
Barbour, Foster and Ó Murchadha [Class.Quant.Grav. **19** (2002) 3217] tested the consistency of different choices of the potential:

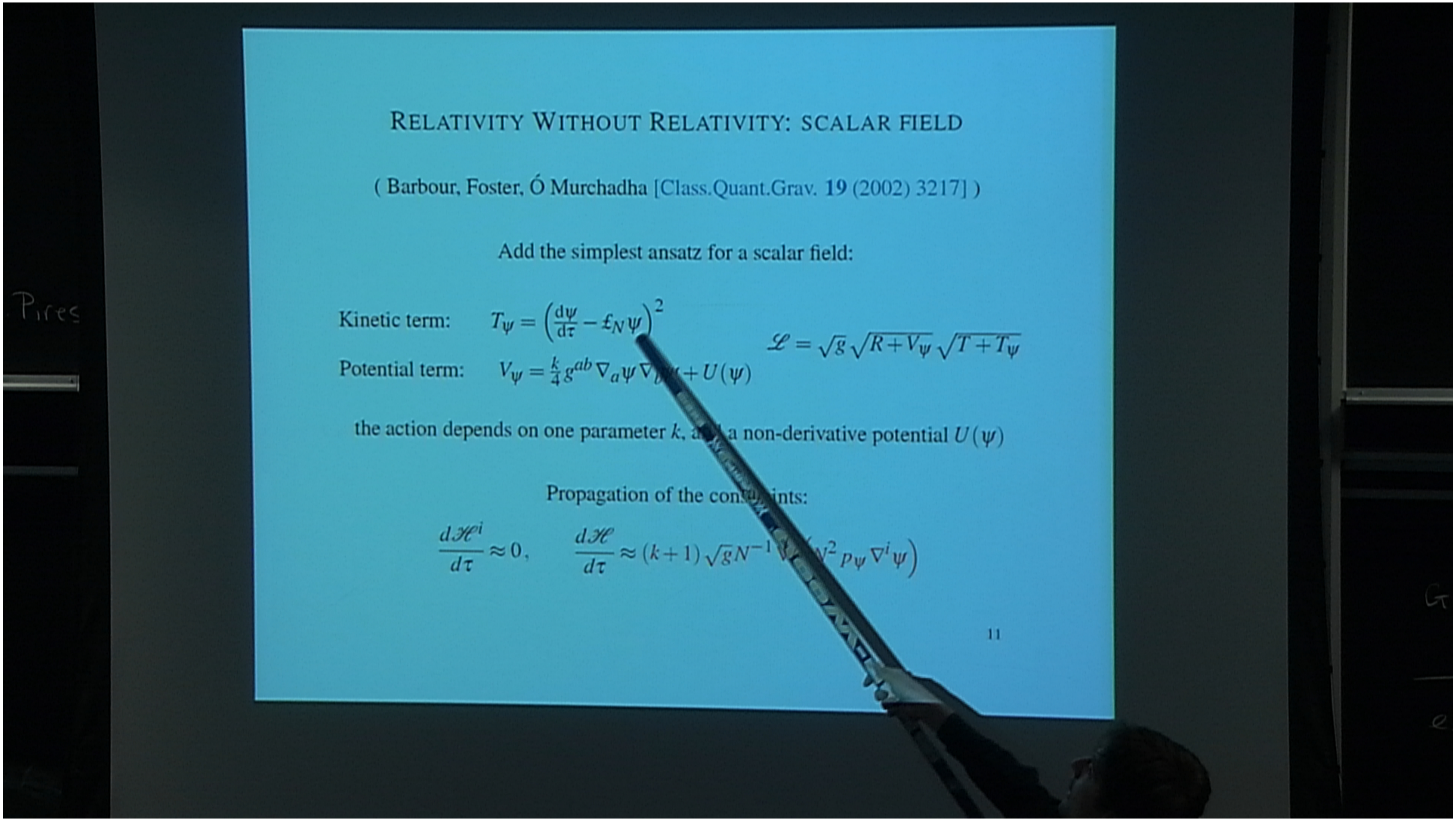
$$R - 2\Lambda, \quad R^{\alpha}, \quad R^{ab}R_{ab}, \quad \nabla^2 R, \quad \ldots$$

and linear combinations thereof. The propagation of the quadratic constraint always led to a proliferation of secondary constraints that trivialize the theory, *apart from the case $R - 2\Lambda$ of a cosmological constant.*

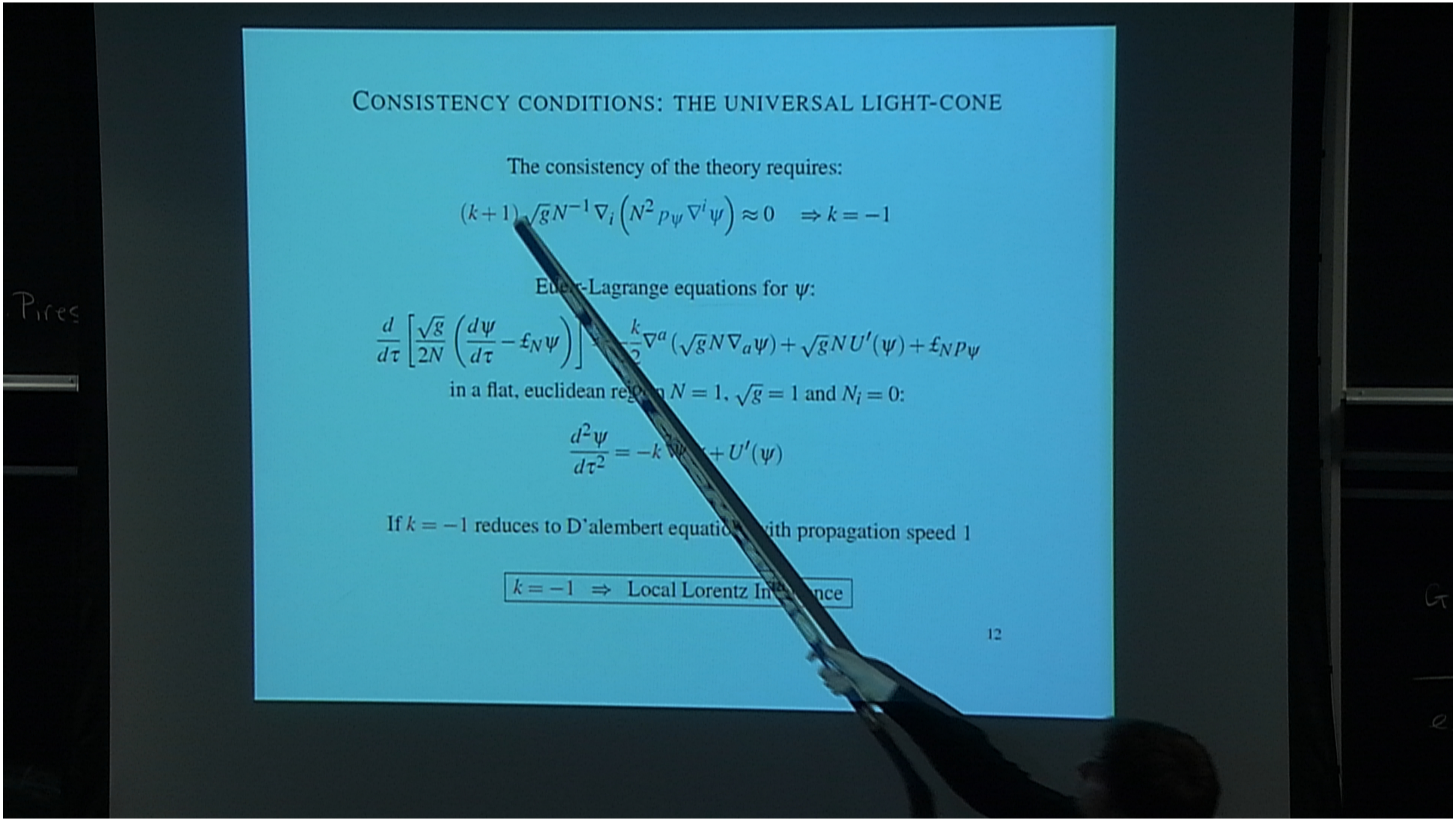
(Niall Ó Murchadha input)

BSW is the simplest consistent matter-free theory on superspace based on a Jacobi-type action

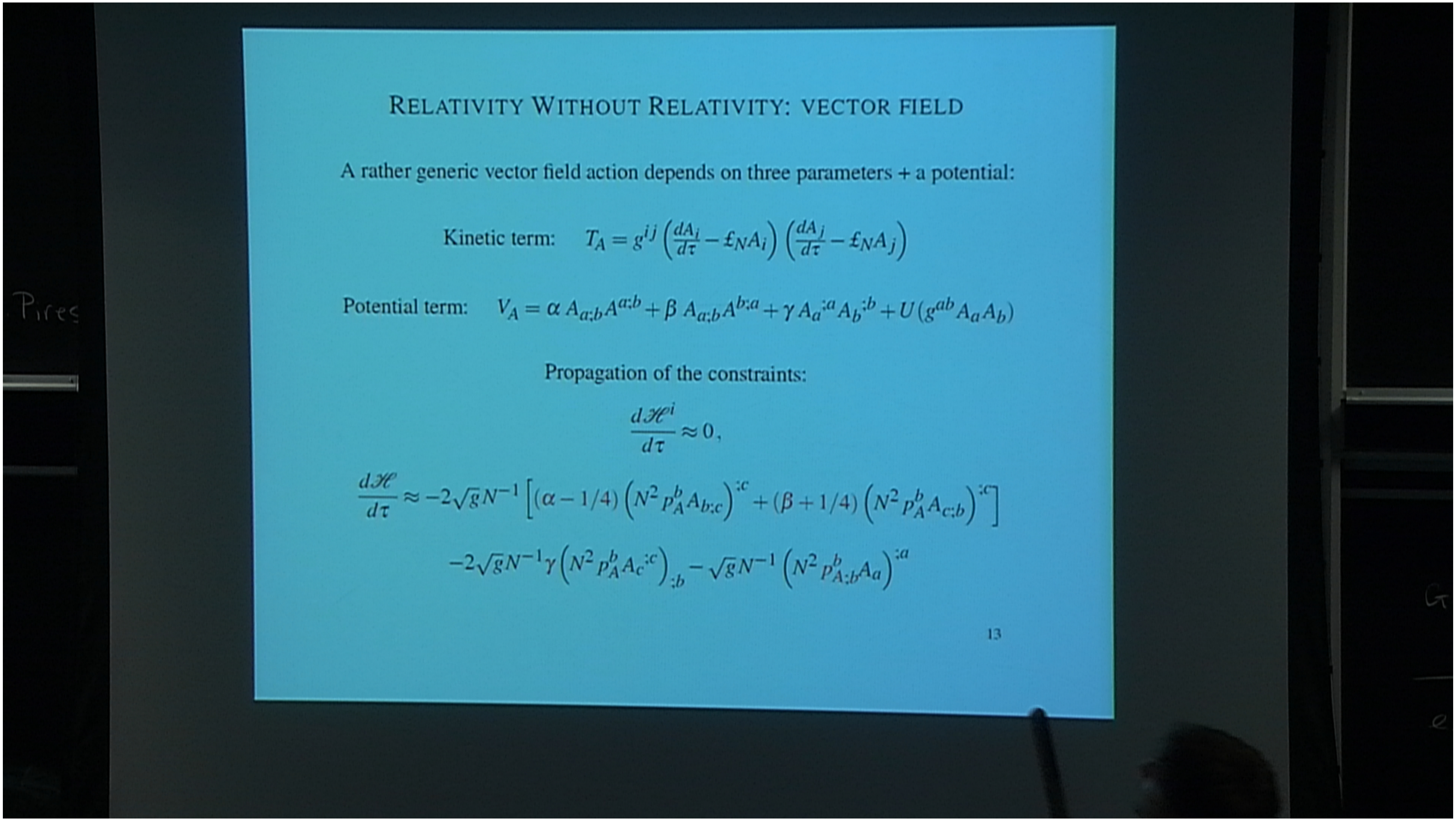




RELATIVITY WITHOUT RELATIVITY: SCALAR FIELD
(Barbour, Foster, Ó Murchadha [Class.Quant.Grav. 19 (2002) 3217])
Add the simplest ansatz for a scalar field:
Kinetic term: $T_\psi = \left(\frac{d\psi}{d\tau} - \pounds_N \psi\right)^2$
Potential term: $V_\psi = \frac{k}{4} g^{ab} \nabla_a \psi \nabla$
$+ U(\psi)$
$\mathscr{L} = \sqrt{g}\sqrt{R + V_\psi}\sqrt{T + T_\psi}$
the action depends on one parameter k,
a non-derivative potential $U(\psi)$
Propagation of the con
ints:
$\frac{d\mathscr{H}^i}{d\tau} \approx 0,$
$\frac{d\mathscr{H}}{d\tau} \approx (k+1)\sqrt{g}N^{-1}$
$p_\psi \nabla^i \psi)$
11



CONSISTENCY CONDITIONS: THE UNIVERSAL LIGHT-CONE
The consistency of the theory requires:
$(k+1)\sqrt{g}N^{-1}\nabla_i\left(N^2 p_\psi \nabla^i \psi\right) \approx 0 \quad \Rightarrow k=-1$
Lagrange equations for ψ:
$\frac{d}{d\tau}\left[\frac{\sqrt{g}}{2N}\left(\frac{d\psi}{d\tau} - \pounds_N \psi\right)\right]$
$\frac{k}{2}\nabla^a\left(\sqrt{g}N\nabla_a\psi\right) + \sqrt{g}NU'(\psi) + \pounds_N p_\psi$
in a flat, euclidean re
$N=1$, $\sqrt{g}=1$ and $N_i=0$:
$\frac{d^2\psi}{d\tau^2} = -k$
$+U'(\psi)$
If $k=-1$ reduces to D'alembert equati
ith propagation speed 1
$k=-1 \Rightarrow$ Local Lorentz In
nce
12
Pires


RELATIVITY WITHOUT RELATIVITY: VECTOR FIELD
A rather generic vector field action depends on three parameters + a potential:
Kinetic term: $T_A = g^{ij}\left(\frac{dA_i}{d\tau} - \pounds_N A_i\right)\left(\frac{dA_j}{d\tau} - \pounds_N A_j\right)$
Potential term: $V_A = \alpha\, A_{a;b}A^{a;b} + \beta\, A_{a;b}A^{b;a} + \gamma\, A_a{}^{;a} A_b{}^{;b} + U(g^{ab} A_a A_b)$
Propagation of the constraints:
$\frac{d\mathscr{H}^i}{d\tau} \approx 0\,,$
$\frac{d\mathscr{H}}{d\tau} \approx -2\sqrt{g}N^{-1}\left[(\alpha - 1/4)\left(N^2 p_A^b A_{b;c}\right)^{;c} + (\beta + 1/4)\left(N^2 p_A^b A_{c;b}\right)^{;c}\right]$
$-2\sqrt{g}N^{-1}\gamma\left(N^2 p_A^b A_c{}^{;c}\right)_{;b} - \sqrt{g}N^{-1}\left(N^2 p^b_{A;b} A_a\right)^{;a}$
13

RELATIVITY WITHOUT RELATIVITY: VECTOR FIELD

A rather generic vector field action depends on three parameters + a potential:

Kinetic term: $$T_A = g^{ij}\left(\frac{dA_i}{d\tau} - \pounds_N A_i\right)\left(\frac{dA_j}{d\tau} - \pounds_N A_j\right)$$

Potential term: $$V_A = \alpha\, A_{a;b}A^{a;b} + \beta\, A_{a;b}A^{b;a} + \gamma\, A_a{}^{;a} A_b{}^{;b} + U(g^{ab}A_a A_b)$$

Propagation of the constraints:

$$\frac{d\mathscr{H}^i}{d\tau} \approx 0\,,$$

$$\frac{d\mathscr{H}}{d\tau} \approx -2\sqrt{g}N^{-1}\left[(\alpha - 1/4)\left(N^2 p_A^b A_{b;c}\right)^{;c} + (\beta + 1/4)\left(N^2 p_A^b A_{c;b}\right)^{;c}\right]$$

$$-2\sqrt{g}N^{-1}\gamma\left(N^2 p_A^b A_c{}^{;c}\right)_{;b} - \sqrt{g}N^{-1}\left(N^2 p^b_{A;b} A_a\right)^{;a}$$

13

RELATIVITY WITHOUT RELATIVITY: VECTOR FIELD

A rather generic vector field action depends on three parameters + a potential:

Kinetic term: $T_A = g^{ij}\left(\frac{dA_i}{d\tau} - \pounds_N A_i\right)\left(\frac{dA_j}{d\tau} - \pounds_N A_j\right)$

Potential term: $V_A = \alpha\, A_{a;b}A^{a;b} + \beta\, A_{a;b}A^{b;a} + \gamma\, A_a{}^{;a}A_b{}^{;b} + U(g^{ab}A_aA_b)$

Propagation of the constraints:

$$\frac{d\mathscr{H}^i}{d\tau} \approx 0,$$

$$\frac{d\mathscr{H}}{d\tau} \approx -2\sqrt{g}N^{-1}\left[(\alpha - 1/4)\left(N^2 p_A^b A_{b;c}\right)^{;c} + (\beta + 1/4)\left(N^2 p_A^b A_{c;b}\right)^{;c}\right]$$

$$-2\sqrt{g}N^{-1}\gamma\left(N^2 p_A^b A_c{}^{;c}\right)_{;b} - \sqrt{g}N^{-1}\left(N^2 p^b_{A;b}A_a\right)^{;a}$$

13

Best-Matching gauge transformations

Equivariant action.

Add a Lagrange multiplier A_0 to the kinetic term
(Best-Matching wrt gauge transformations)

$$T_A = g^{ij}\left(\frac{dA_i}{d\tau} - \nabla_i A_0 - \pounds_N A_i\right)\left(\frac{dA_j}{d\tau} - \nabla_i A_0 - \pounds_N A_j\right), \qquad \frac{\delta \mathscr{L}}{\delta A_0} = \mathscr{G}$$

then, in a Euclidean background, the Euler–Lagrange equations read:

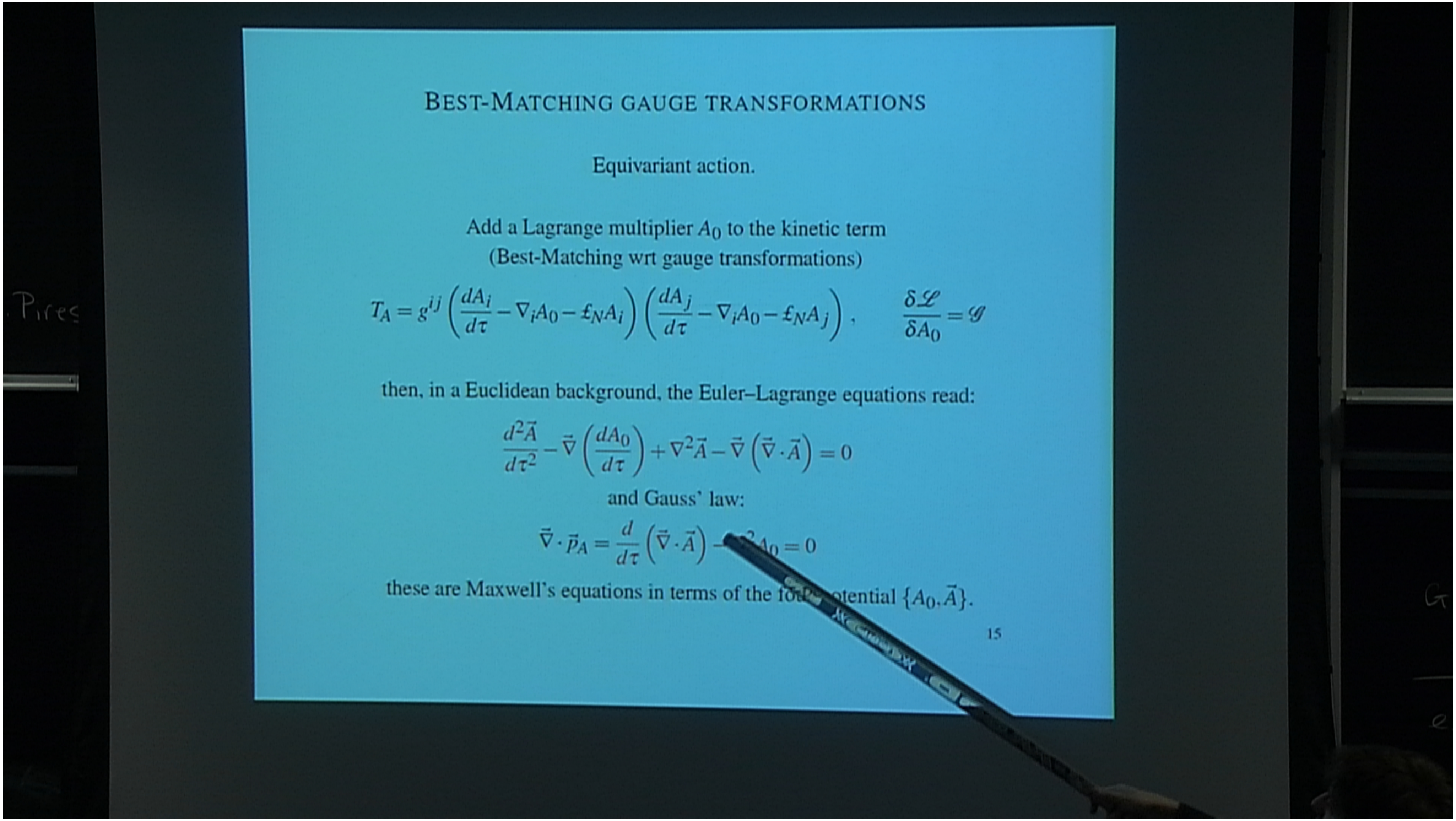
$$\frac{d^2\vec{A}}{d\tau^2} - \vec{\nabla}\left(\frac{dA_0}{d\tau}\right) + \nabla^2\vec{A} - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right) = 0$$

and Gauss' law:

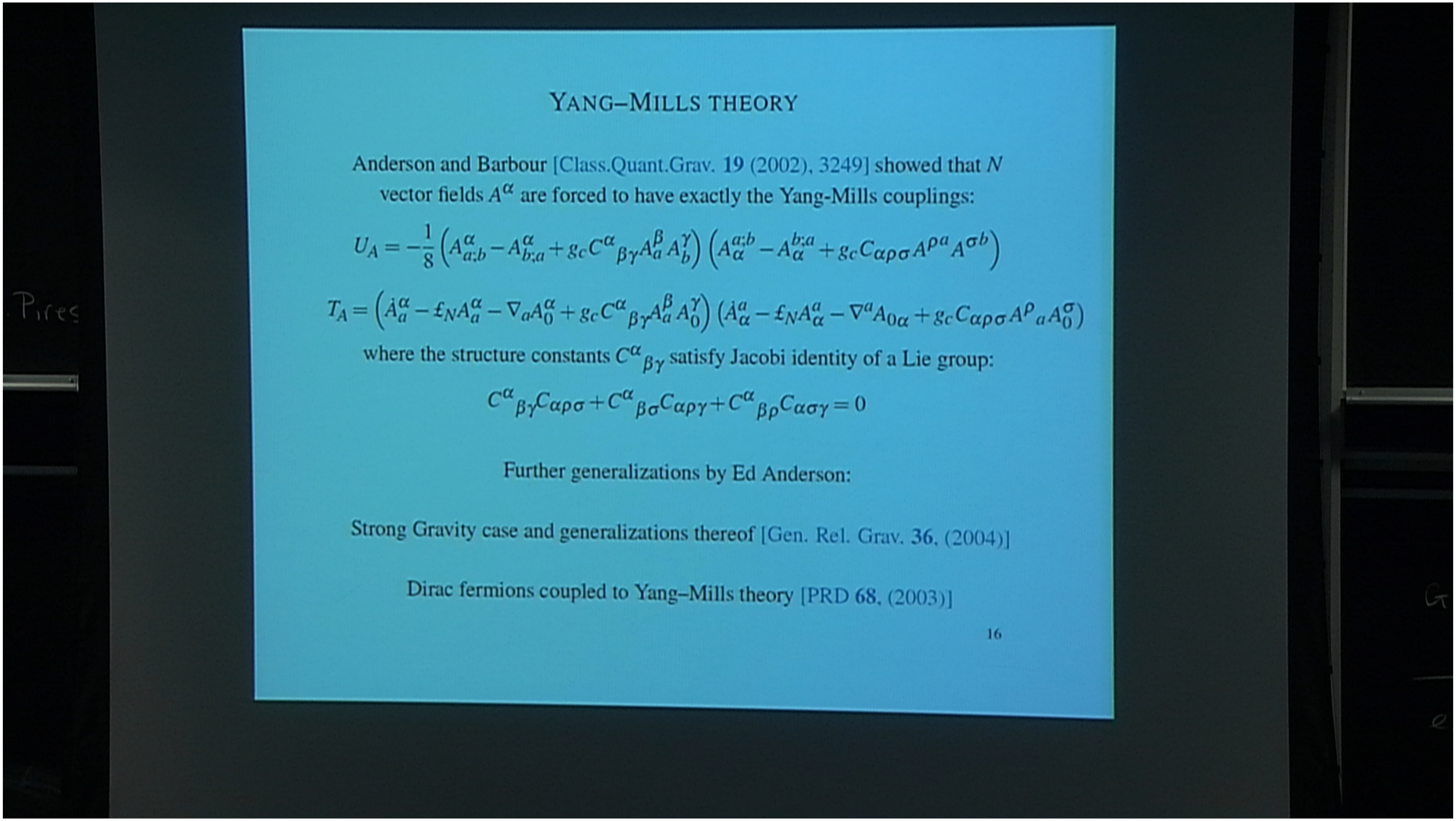
$$\vec{\nabla}\cdot\vec{p}_A = \frac{d}{d\tau}\left(\vec{\nabla}\cdot\vec{A}\right) - \nabla^2 A_0 = 0$$

these are Maxwell's equations in terms of the four-potential $\{A_0, \vec{A}\}$.

15



BEST-MATCHING GAUGE TRANSFORMATIONS
Equivariant action.
Add a Lagrange multiplier A_0 to the kinetic term
(Best-Matching wrt gauge transformations)
$T_A = g^{ij}\left(\frac{dA_i}{d\tau} - \nabla_i A_0 - \pounds_N A_i\right)\left(\frac{dA_j}{d\tau} - \nabla_i A_0 - \pounds_N A_j\right)$, $\frac{\delta \mathscr{L}}{\delta A_0} = \mathscr{G}$
then, in a Euclidean background, the Euler–Lagrange equations read:
$\frac{d^2\vec{A}}{d\tau^2} - \vec{\nabla}\left(\frac{dA_0}{d\tau}\right) + \nabla^2\vec{A} - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right) = 0$
and Gauss' law:
$\vec{\nabla}\cdot\vec{p}_A = \frac{d}{d\tau}\left(\vec{\nabla}\cdot\vec{A}\right)$
these are Maxwell's equations in terms of the
otential $\{A_0, \vec{A}\}$.
15


YANG–MILLS THEORY
Anderson and Barbour [Class.Quant.Grav. 19 (2002), 3249] showed that N vector fields A^α are forced to have exactly the Yang-Mills couplings:
$U_A = -\frac{1}{8}\left(A^\alpha_{a;b} - A^\alpha_{b;a} + g_c C^\alpha{}_{\beta\gamma}A^\beta_a A^\gamma_b\right)\left(A^{a;b}_\alpha - A^{b;a}_\alpha + g_c C_{\alpha\rho\sigma}A^{\rho a}A^{\sigma b}\right)$
$T_A = \left(\dot{A}^\alpha_a - £_N A^\alpha_a - \nabla_a A^\alpha_0 + g_c C^\alpha{}_{\beta\gamma}A^\beta_a A^\gamma_0\right)\left(\dot{A}^a_\alpha - £_N A^a_\alpha - \nabla^a A_{0\alpha} + g_c C_{\alpha\rho\sigma}A^\rho{}_a A^\sigma_0\right)$
where the structure constants $C^\alpha{}_{\beta\gamma}$ satisfy Jacobi identity of a Lie group:
$C^\alpha{}_{\beta\gamma}C_{\alpha\rho\sigma} + C^\alpha{}_{\beta\sigma}C_{\alpha\rho\gamma} + C^\alpha{}_{\beta\rho}C_{\alpha\sigma\gamma} = 0$
Further generalizations by Ed Anderson:
Strong Gravity case and generalizations thereof [Gen. Rel. Grav. 36, (2004)]
Dirac fermions coupled to Yang–Mills theory [PRD 68, (2003)]
16

Yang–Mills theory

Anderson and Barbour [Class.Quant.Grav. **19** (2002), 3249] showed that N vector fields A^α are forced to have exactly the Yang-Mills couplings:

$$U_A = -\frac{1}{8}\left(A^\alpha_{a;b} - A^\alpha_{b;a} + g_c C^\alpha{}_{\beta\gamma} A^\beta_a A^\gamma_b\right)\left(A^{a;b}_\alpha - A^{b;a}_\alpha + g_c C_{\alpha\rho\sigma} A^{\rho a} A^{\sigma b}\right)$$

$$T_A = \left(\dot{A}^\alpha_a - \pounds_N A^\alpha_a - \nabla_a A^\alpha_0 + g_c C^\alpha{}_{\beta\gamma} A^\beta_a A^\gamma_0\right)\left(\dot{A}^a_\alpha - \pounds_N A^a_\alpha - \nabla^a A_{0\alpha} + g_c C_{\alpha\rho\sigma} A^\rho{}_a A^\sigma_0\right)$$

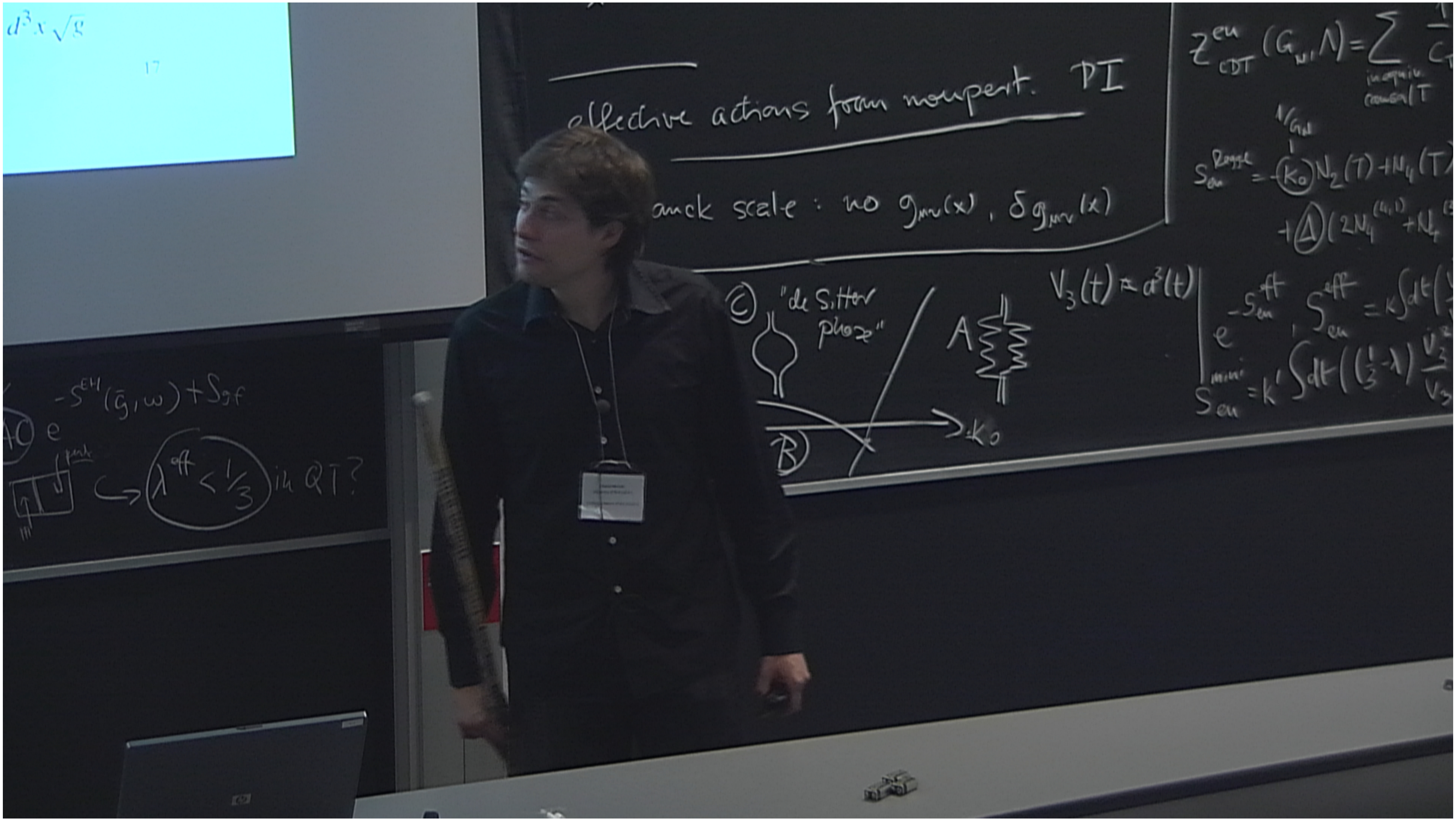
where the structure constants $C^\alpha{}_{\beta\gamma}$ satisfy Jacobi identity of a Lie group:

$$C^\alpha{}_{\beta\gamma} C_{\alpha\rho\sigma} + C^\alpha{}_{\beta\sigma} C_{\alpha\rho\gamma} + C^\alpha{}_{\beta\rho} C_{\alpha\sigma\gamma} = 0$$

Further generalizations by Ed Anderson:

Strong Gravity case and generalizations thereof [Gen. Rel. Grav. **36**, (2004)]

Dirac fermions coupled to Yang–Mills theory [PRD **68**, (2003)]


effective actions from nonpert. PI
"de Sitter phase"

An ambiguity in the supermetric

Nothing in the structure of configuration space fixes λ:

$$\boxed{G_\lambda^{abcd} = g^{ac}g^{bd} - \lambda\, g^{ab}g^{cd}}$$

What happens if we relax from the DeWitt value $\lambda = 1$?

$$\frac{d\mathscr{H}}{d\tau} \approx 2\left(\frac{\lambda - 1}{3\lambda - 1}\right)\sqrt{g}N^{-1}\left(N^2\, p_{;j}\right)^{;j} \quad \text{(even with bosonic matter)}$$

Two possibilities: either $\lambda = 1$ (DeWitt value), or we add a new constraint:

$$\boxed{\mathscr{D} = p - Y\sqrt{g}} \approx 0$$

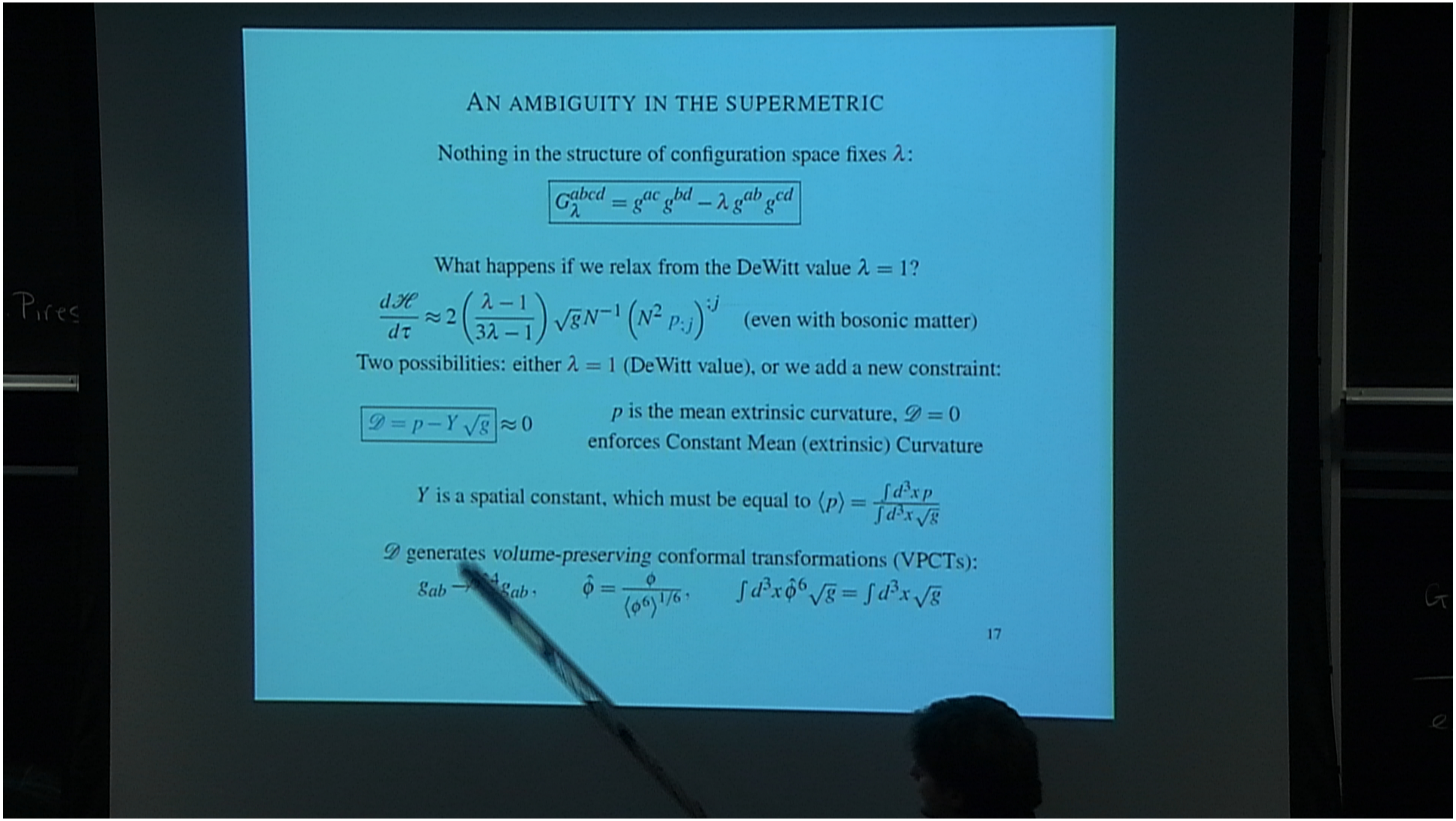
p is the mean extrinsic curvature, $\mathscr{D} = 0$ enforces Constant Mean (extrinsic) Curvature

Y is a spatial constant, which must be equal to $\langle p \rangle = \frac{\int d^3x\, p}{\int d^3x \sqrt{g}}$

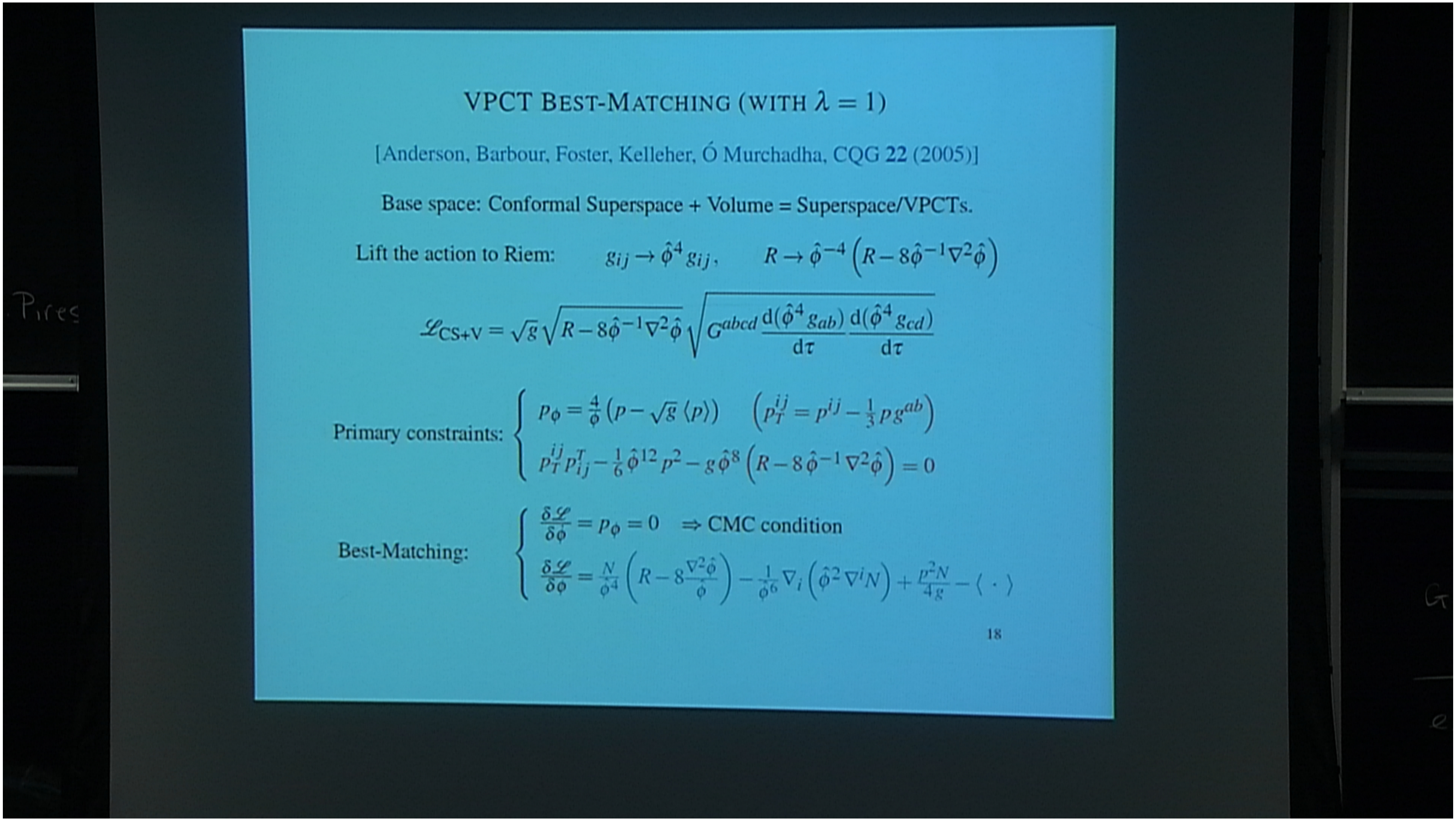
$\mathscr{D}$ generates *volume-preserving* conformal transformations (VPCTs):

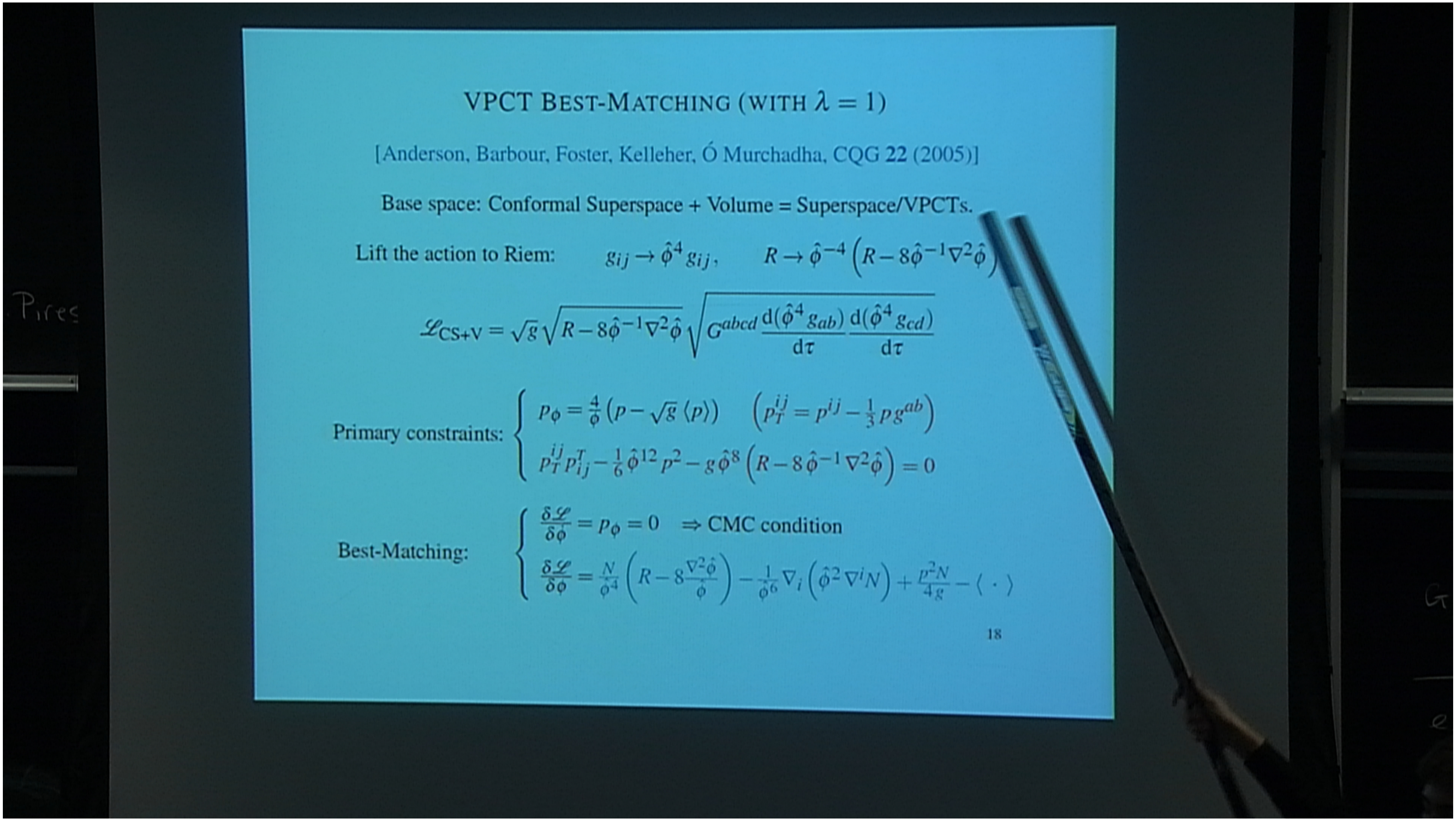
$$g_{ab} \to \hat{\phi}^4 g_{ab}, \qquad \hat{\phi} = \frac{\phi}{\langle \phi^6 \rangle^{1/6}}, \qquad \int d^3x\, \hat{\phi}^6 \sqrt{g} = \int d^3x \sqrt{g}$$

17



AN AMBIGUITY IN THE SUPERMETRIC
Nothing in the structure of configuration space fixes λ:
$G_\lambda^{abcd} = g^{ac}g^{bd} - \lambda g^{ab}g^{cd}$
What happens if we relax from the DeWitt value $\lambda = 1$?
$\frac{d\mathcal{H}}{d\tau} \approx 2\left(\frac{\lambda-1}{3\lambda-1}\right)\sqrt{g}N^{-1}\left(N^2 p_{;j}\right)^{;j}$ (even with bosonic matter)
Two possibilities: either $\lambda = 1$ (DeWitt value), or we add a new constraint:
$\mathscr{D} = p - Y\sqrt{g} \approx 0$
p is the mean extrinsic curvature, $\mathscr{D} = 0$ enforces Constant Mean (extrinsic) Curvature
Y is a spatial constant, which must be equal to $\langle p \rangle = \frac{\int d^3x\, p}{\int d^3x \sqrt{g}}$
$\mathscr{D}$ generates *volume-preserving* conformal transformations (VPCTs):
$g_{ab} \to$ … g_{ab}, $\hat{\phi} = \frac{\phi}{\langle \phi^6 \rangle^{1/6}}$, $\int d^3x\, \hat{\phi}^6 \sqrt{g} = \int d^3x \sqrt{g}$
17


VPCT BEST-MATCHING (WITH $\lambda = 1$)
[Anderson, Barbour, Foster, Kelleher, Ó Murchadha, CQG **22** (2005)]
Base space: Conformal Superspace + Volume = Superspace/VPCTs.
Lift the action to Riem: $g_{ij} \to \hat{\phi}^4 g_{ij}, \quad R \to \hat{\phi}^{-4}\left(R - 8\hat{\phi}^{-1}\nabla^2\hat{\phi}\right)$
$\mathscr{L}_{\text{CS+V}} = \sqrt{g}\sqrt{R - 8\hat{\phi}^{-1}\nabla^2\hat{\phi}}\sqrt{G^{abcd}\frac{\mathrm{d}(\hat{\phi}^4 g_{ab})}{\mathrm{d}\tau}\frac{\mathrm{d}(\hat{\phi}^4 g_{cd})}{\mathrm{d}\tau}}$
Primary constraints:
$p_\phi = \frac{4}{\phi}\left(p - \sqrt{g}\,\langle p\rangle\right) \quad \left(p_T^{ij} = p^{ij} - \frac{1}{3}\,p\,g^{ab}\right)$
$p_T^{ij} p^T_{ij} - \frac{1}{6}\hat{\phi}^{12} p^2 - g\,\hat{\phi}^8\left(R - 8\,\hat{\phi}^{-1}\,\nabla^2\hat{\phi}\right) = 0$
Best-Matching:
$\frac{\delta\mathscr{L}}{\delta\phi} = p_\phi = 0 \quad \Rightarrow$ CMC condition
$\frac{\delta\mathscr{L}}{\delta\phi} = \frac{N}{\hat{\phi}^4}\left(R - 8\frac{\nabla^2\hat{\phi}}{\hat{\phi}}\right) - \frac{1}{\hat{\phi}^6}\nabla_i\left(\hat{\phi}^2\,\nabla^i N\right) + \frac{p^2 N}{4g} - \langle\,\cdot\,\rangle$
18


VPCT BEST-MATCHING (WITH $\lambda = 1$)
[Anderson, Barbour, Foster, Kelleher, Ó Murchadha, CQG 22 (2005)]
Base space: Conformal Superspace + Volume = Superspace/VPCTs.
Lift the action to Riem: $g_{ij} \to \hat{\phi}^4 g_{ij}$, $R \to \hat{\phi}^{-4}\left(R - 8\hat{\phi}^{-1}\nabla^2\hat{\phi}\right)$
$\mathscr{L}_{\text{CS+V}} = \sqrt{g}\sqrt{R - 8\hat{\phi}^{-1}\nabla^2\hat{\phi}}\sqrt{G^{abcd}\frac{\mathrm{d}(\hat{\phi}^4 g_{ab})}{\mathrm{d}\tau}\frac{\mathrm{d}(\hat{\phi}^4 g_{cd})}{\mathrm{d}\tau}}$
Primary constraints:
$p_\phi = \frac{4}{\phi}\left(p - \sqrt{g}\,\langle p\rangle\right)$ $\left(p_T^{ij} = p^{ij} - \frac{1}{3}p\,g^{ab}\right)$
$p_T^{ij}p^T_{ij} - \frac{1}{6}\hat{\phi}^{12}p^2 - g\,\hat{\phi}^8\left(R - 8\,\hat{\phi}^{-1}\,\nabla^2\hat{\phi}\right) = 0$
Best-Matching:
$\frac{\delta\mathscr{L}}{\delta\phi} = p_\phi = 0$ ⇒ CMC condition
$\frac{\delta\mathscr{L}}{\delta\phi} = \frac{N}{\hat{\phi}^4}\left(R - 8\frac{\nabla^2\hat{\phi}}{\hat{\phi}}\right) - \frac{1}{\hat{\phi}^6}\nabla_i\left(\hat{\phi}^2\nabla^i N\right) + \frac{p^2 N}{4g} - \langle\,\cdot\,\rangle$
18

AN AMBIGUITY IN THE SUPERMETRIC

Nothing in the structure of configuration space fixes λ:

$$\boxed{G_{\lambda}^{abcd} = g^{ac}g^{bd} - \lambda\, g^{ab}g^{cd}}$$

What happens if we relax from the DeWitt value $\lambda = 1$?

$$\frac{d\mathscr{H}}{d\tau} \approx 2\left(\frac{\lambda-1}{3\lambda-1}\right)\sqrt{g}N^{-1}\left(N^2\, p_{;j}\right)^{;j} \quad \text{(even with bosonic matter)}$$

Two possibilities: either $\lambda = 1$ (DeWitt value), or we add a new constraint:

$$\boxed{\mathscr{D} = p - Y\sqrt{g}} \approx 0$$

p is the mean extrinsic curvature, $\mathscr{D} = 0$ enforces Constant Mean (extrinsic) Curvature

Y is a spatial constant, which must be equal to $\langle p \rangle = \frac{\int d^3x\, p}{\int d^3x \sqrt{g}}$

$\mathscr{D}$ generates *volume-preserving* conformal transformations (VPCTs):

$$g_{ab} \to \hat{\phi}^4 g_{ab}, \qquad \hat{\phi} = \frac{\phi}{\langle \phi^6 \rangle^{1/6}}, \qquad \int d^3x\, \hat{\phi}^6 \sqrt{g} = \int d^3x \sqrt{g}$$

17

An ambiguity in the supermetric

Nothing in the structure of configuration space fixes λ:

$$\boxed{G_\lambda^{abcd} = g^{ac}\,g^{bd} - \lambda\, g^{ab}\, g^{cd}}$$

What happens if we relax from the DeWitt value $\lambda = 1$?

$$\frac{d\mathscr{H}}{d\tau} \approx 2\left(\frac{\lambda - 1}{3\lambda - 1}\right)\sqrt{g}\,N^{-1}\left(N^2\, p_{;j}\right)^{;j} \qquad \text{(even with bosonic matter)}$$

Two possibilities: either $\lambda = 1$ (DeWitt value), or we add a new constraint:

$$\boxed{\mathscr{D} = p - Y\sqrt{g}} \approx 0$$

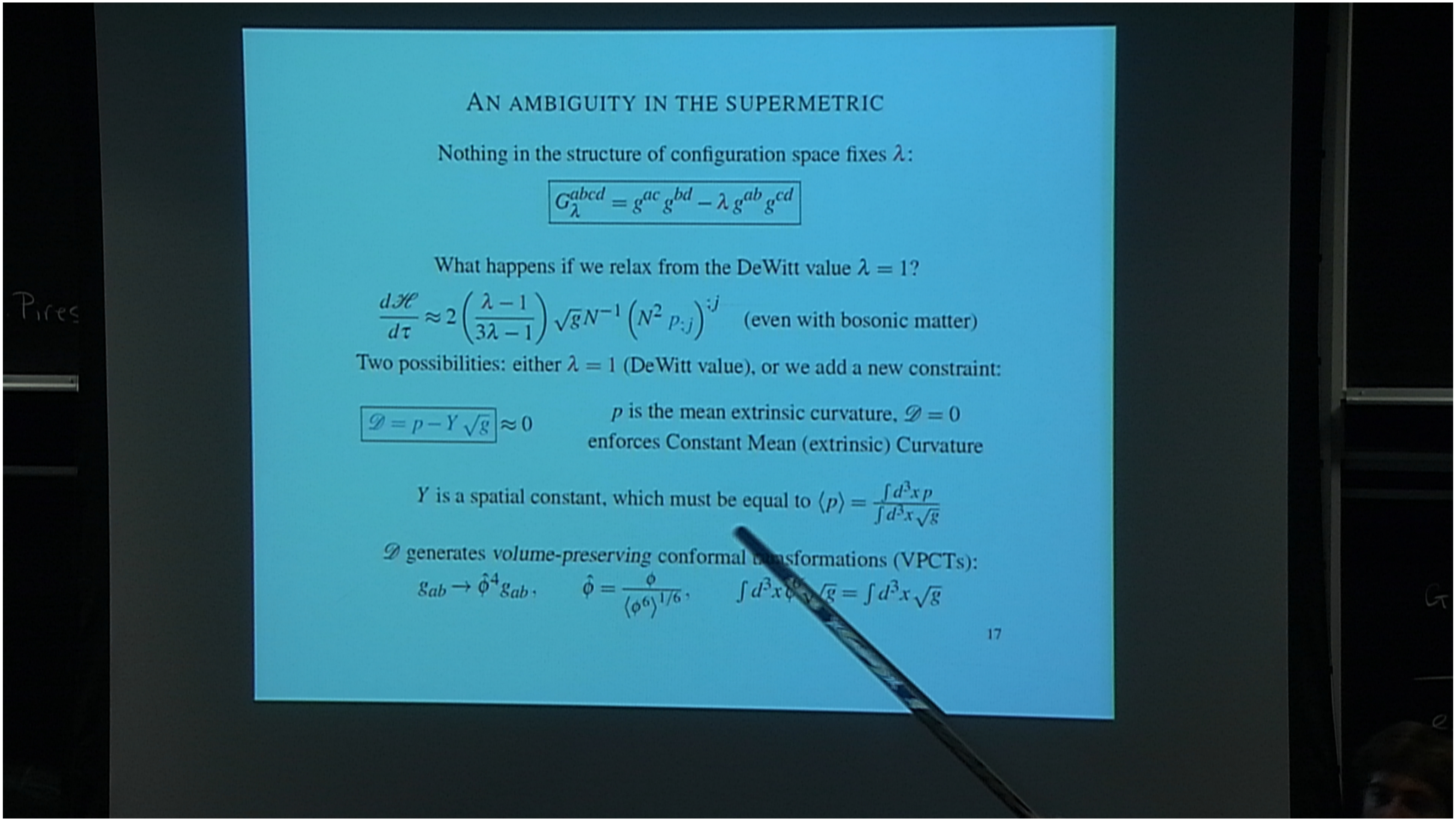
p is the mean extrinsic curvature, $\mathscr{D} = 0$ enforces Constant Mean (extrinsic) Curvature

Y is a spatial constant, which must be equal to $\langle p \rangle = \frac{\int d^3x\, p}{\int d^3x \sqrt{g}}$

$\mathscr{D}$ generates *volume-preserving* conformal transformations (VPCTs):

$$g_{ab} \to \hat{\phi}^4 g_{ab}\,, \qquad \hat{\phi} = \frac{\phi}{\langle \phi^6 \rangle^{1/6}}\,, \qquad \int d^3x\,\hat{\phi}^6\sqrt{g} = \int d^3x\sqrt{g}$$

17


AN AMBIGUITY IN THE SUPERMETRIC
Nothing in the structure of configuration space fixes λ:
$G_\lambda^{abcd} = g^{ac} g^{bd} - \lambda g^{ab} g^{cd}$
What happens if we relax from the DeWitt value $\lambda = 1$?
$\frac{d\mathscr{H}}{d\tau} \approx 2 \left(\frac{\lambda - 1}{3\lambda - 1} \right) \sqrt{g} N^{-1} \left(N^2 p_{;j} \right)^{;j}$ (even with bosonic matter)
Two possibilities: either $\lambda = 1$ (DeWitt value), or we add a new constraint:
$\mathscr{D} = p - Y \sqrt{g} \approx 0$
p is the mean extrinsic curvature, $\mathscr{D} = 0$ enforces Constant Mean (extrinsic) Curvature
Y is a spatial constant, which must be equal to $\langle p \rangle = \frac{\int d^3x\, p}{\int d^3x \sqrt{g}}$
$\mathscr{D}$ generates *volume-preserving* conformal sformations (VPCTs):
$g_{ab} \to \hat{\phi}^4 g_{ab}$, $\hat{\phi} = \frac{\phi}{\langle \phi^6 \rangle^{1/6}}$, $\int d^3x \sqrt{g} = \int d^3x \sqrt{g}$
17

An ambiguity in the supermetric

Nothing in the structure of configuration space fixes λ:

$$\boxed{G_\lambda^{abcd} = g^{ac}\,g^{bd} - \lambda\, g^{ab}\, g^{cd}}$$

What happens if we relax from the DeWitt value $\lambda = 1$?

$$\frac{d\mathscr{H}}{d\tau} \approx 2\left(\frac{\lambda-1}{3\lambda-1}\right)\sqrt{g}\,N^{-1}\left(N^2\,p_{;j}\right)^{;j} \quad \text{(even with bosonic matter)}$$

Two possibilities: either $\lambda = 1$ (DeWitt value), or we add a new constraint:

$$\boxed{\mathscr{D} = p - Y\sqrt{g}} \approx 0$$

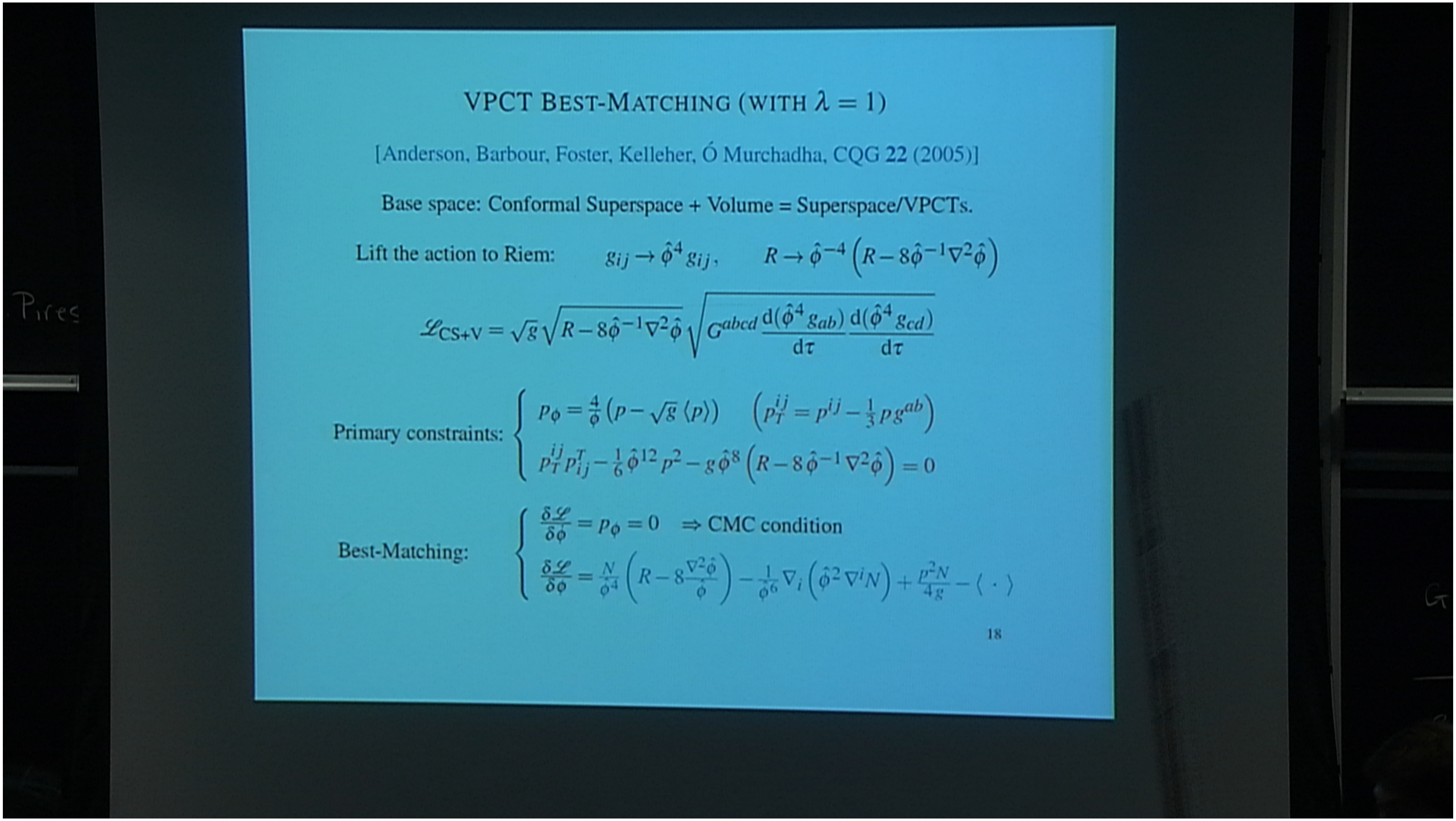
p is the mean extrinsic curvature, $\mathscr{D} = 0$ enforces Constant Mean (extrinsic) Curvature

Y is a spatial constant, which must be equal to $\langle p \rangle = \frac{\int d^3x\, p}{\int d^3x \sqrt{g}}$

$\mathscr{D}$ generates *volume-preserving* conformal transformations (VPCTs):

$$g_{ab} \to \hat{\phi}^4 g_{ab}\,, \qquad \hat{\phi} = \frac{\phi}{\langle \phi^6 \rangle^{1/6}}\,, \qquad \int d^3x\, \hat{\phi}^6 \sqrt{g} = \int d^3x \sqrt{g}$$

17


VPCT BEST-MATCHING (WITH $\lambda = 1$)
[Anderson, Barbour, Foster, Kelleher, Ó Murchadha, CQG 22 (2005)]
Base space: Conformal Superspace + Volume = Superspace/VPCTs.
Lift the action to Riem: $g_{ij} \to \hat\phi^4 g_{ij}$, $R \to \hat\phi^{-4}\left(R - 8\hat\phi^{-1}\nabla^2\hat\phi\right)$
$\mathscr{L}_{\text{CS+V}} = \sqrt{g}\sqrt{R - 8\hat\phi^{-1}\nabla^2\hat\phi}\sqrt{G^{abcd}\frac{\mathrm{d}(\hat\phi^4 g_{ab})}{\mathrm{d}\tau}\frac{\mathrm{d}(\hat\phi^4 g_{cd})}{\mathrm{d}\tau}}$
Primary constraints:
$p_\phi = \frac{4}{\phi}\left(p - \sqrt{g}\,\langle p\rangle\right)$ $\left(p_T^{ij} = p^{ij} - \frac{1}{3} p g^{ab}\right)$
$p_T^{ij} p^T_{ij} - \frac{1}{6}\hat\phi^{12} p^2 - g\hat\phi^8\left(R - 8\hat\phi^{-1}\nabla^2\hat\phi\right) = 0$
Best-Matching:
$\frac{\delta\mathscr{L}}{\delta\phi} = p_\phi = 0$ $\Rightarrow$ CMC condition
$\frac{\delta\mathscr{L}}{\delta\phi} = \frac{N}{\hat\phi^4}\left(R - 8\frac{\nabla^2\hat\phi}{\hat\phi}\right) - \frac{1}{\hat\phi^6}\nabla_i\left(\hat\phi^2\nabla^i N\right) + \frac{p^2 N}{4g} - \langle\,\cdot\,\rangle$
18

VPCT Best-Matching (with $\lambda = 1$)

[Anderson, Barbour, Foster, Kelleher, Ó Murchadha, CQG **22** (2005)]

Base space: Conformal Superspace + Volume = Superspace/VPCTs.

Lift the action to Riem: $g_{ij} \to \hat{\phi}^4 g_{ij}, \qquad R \to \hat{\phi}^{-4}\left(R - 8\hat{\phi}^{-1}\nabla^2\hat{\phi}\right)$

$$\mathscr{L}_{\text{CS+V}} = \sqrt{g}\sqrt{R - 8\hat{\phi}^{-1}\nabla^2\hat{\phi}}\sqrt{G^{abcd}\frac{\mathrm{d}(\hat{\phi}^4 g_{ab})}{\mathrm{d}\tau}\frac{\mathrm{d}(\hat{\phi}^4 g_{cd})}{\mathrm{d}\tau}}$$

Primary constraints:
$$\begin{cases} p_\phi = \frac{4}{\phi}\left(p - \sqrt{g}\,\langle p\rangle\right) \qquad \left(p_T^{ij} = p^{ij} - \frac{1}{3}p\,g^{ab}\right) \\ p_T^{ij}p^T_{ij} - \frac{1}{6}\hat{\phi}^{12}p^2 - g\,\hat{\phi}^8\left(R - 8\,\hat{\phi}^{-1}\nabla^2\hat{\phi}\right) = 0 \end{cases}$$

Best-Matching:
$$\begin{cases} \frac{\delta\mathscr{L}}{\delta\phi} = p_\phi = 0 \quad \Rightarrow \text{CMC condition} \\ \frac{\delta\mathscr{L}}{\delta\phi} = \frac{N}{\hat{\phi}^4}\left(R - 8\frac{\nabla^2\hat{\phi}}{\hat{\phi}}\right) - \frac{1}{\hat{\phi}}\nabla_i\left(\hat{\phi}^2\nabla^i N\right) + \frac{p^2 N}{4g} - \langle\,\cdot\,\rangle \end{cases}$$

18

THE LICHNEROWICZ–YORK EQUATION

J. York solved the initial-value problem of GR [PRL **26-28** (1971-2)]

If $p = Y\sqrt{g}$ Momentum and Hamiltonian constraints decouple:

$$\nabla_i p^{ij}_{TT} = 0, \qquad p^{ij}_{TT} p^{TT}_{ij} - \frac{1}{6} Y^2 - g R = 0$$

the latter can be transformed into an elliptic equation for a conformal factor:

$$g_{ij} = \phi^4 g^{\text{reference}}_{ij}, \qquad p^{ij}_{TT} p^{TT}_{ij} - \frac{1}{6}\phi^{12} Y^2 - g\,\phi^8 \left(R - 8\,\phi^{-1}\nabla^2\phi\right) = 0$$

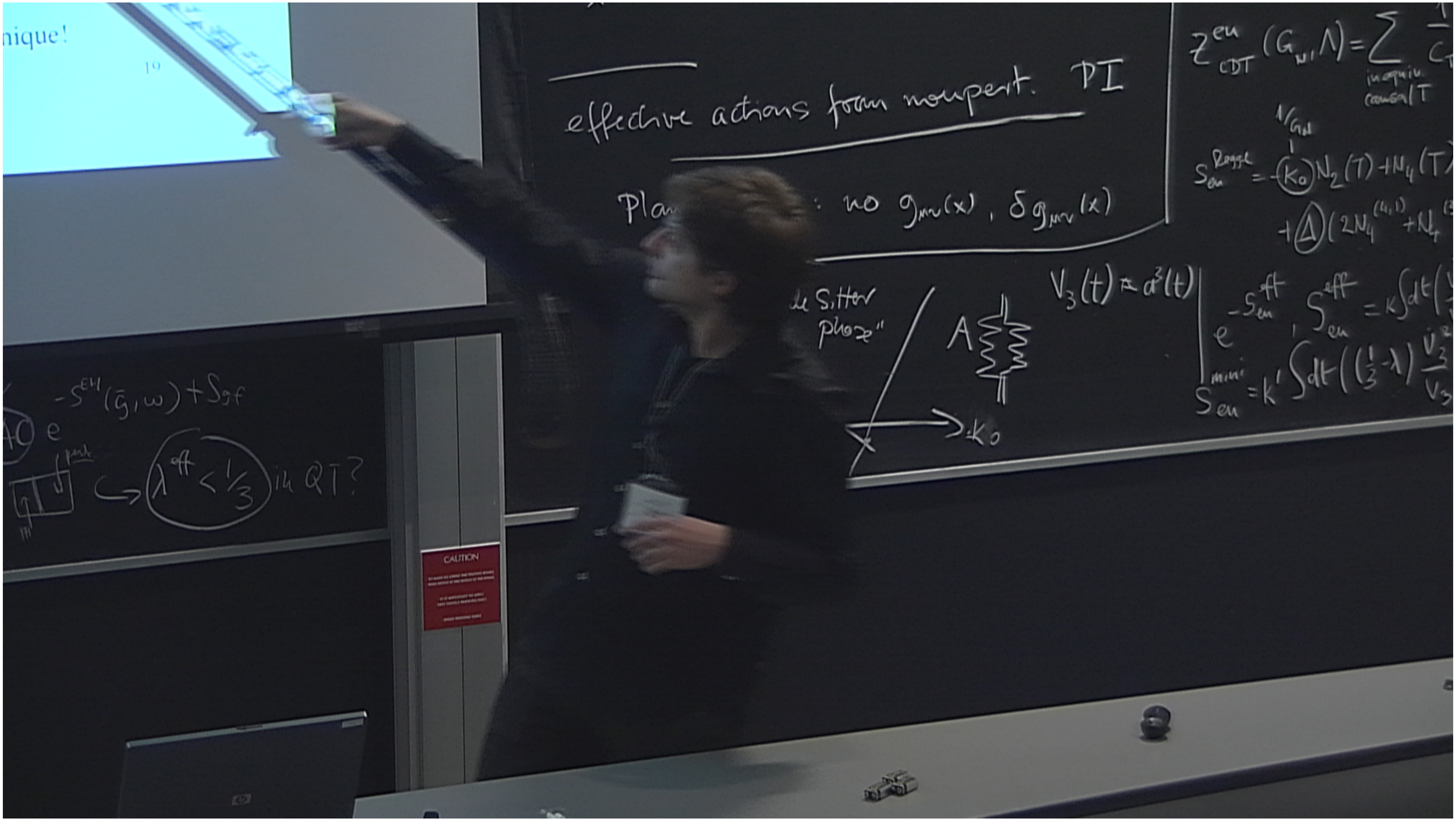
the independent initial data are a *conformal metric* $g^{\text{reference}}_{ij}$,
a TT-momentum p^{ij}_{TT} and the constant Y, the *York time*

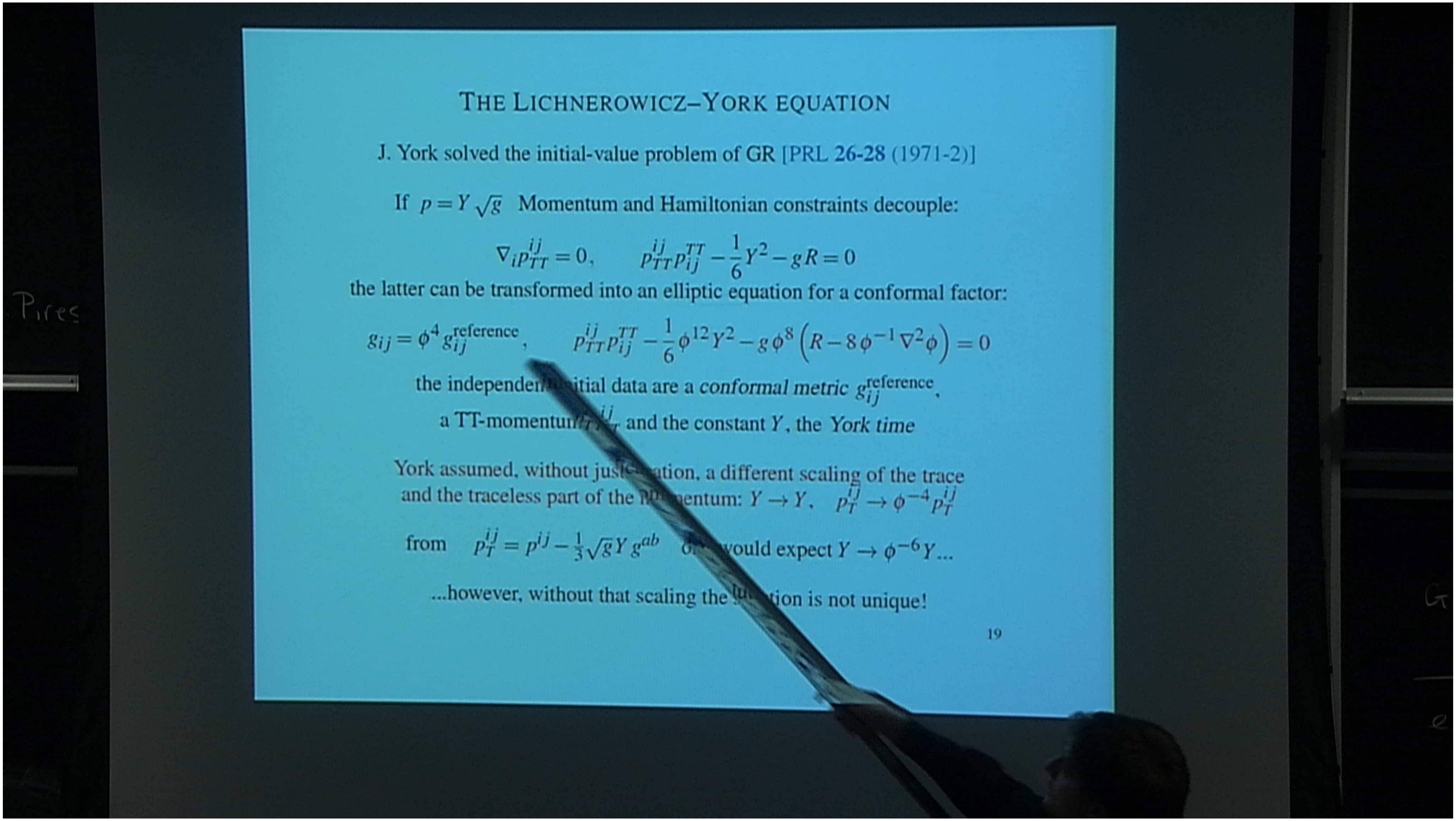
York assumed, without justification, a different scaling of the trace
and the traceless part of the momentum: $Y \to Y$, $p^{ij}_T \to \phi^{-4} p^{ij}_T$

from $p^{ij}_T = p^{ij} - \frac{1}{3}\sqrt{g}\, Y g^{ab}$ one would expect $Y \to \phi^{-6} Y$...

...however, without that scaling the solution is not unique!

19


nique!
effective actions from nonpert. PI
no gμν(x), δgμν(x)
phase"
CAUTION


THE LICHNEROWICZ–YORK EQUATION
J. York solved the initial-value problem of GR [PRL 26-28 (1971-2)]
If $p = Y\sqrt{g}$ Momentum and Hamiltonian constraints decouple:
$\nabla_i p^{ij}_{TT} = 0, \qquad p^{ij}_{TT} p^{TT}_{ij} - \frac{1}{6} Y^2 - g R = 0$
the latter can be transformed into an elliptic equation for a conformal factor:
$g_{ij} = \phi^4 g^{\text{reference}}_{ij}, \qquad p^{ij}_{TT} p^{TT}_{ij} - \frac{1}{6} \phi^{12} Y^2 - g \phi^8 \left(R - 8 \phi^{-1} \nabla^2 \phi \right) = 0$
the independen itial data are a conformal metric $g^{\text{reference}}_{ij}$,
a TT-momentu and the constant Y, the York time
York assumed, without jus ation, a different scaling of the trace
and the traceless part of the entum: $Y \to Y, \quad p^{ij}_T \to \phi^{-4} p^{ij}_T$
from $p^{ij}_T = p^{ij} - \frac{1}{3}\sqrt{g}\, Y g^{ab}$ ould expect $Y \to \phi^{-6} Y$...
...however, without that scaling the tion is not unique!
19

THE LICHNEROWICZ–YORK EQUATION

J. York solved the initial-value problem of GR [PRL **26-28** (1971-2)]

If $p = Y\sqrt{g}$ Momentum and Hamiltonian constraints decouple:

$$\nabla_i p^{ij}_{TT} = 0, \qquad p^{ij}_{TT} p^{TT}_{ij} - \frac{1}{6} Y^2 - g R = 0$$

the latter can be transformed into an elliptic equation for a conformal factor:

$$g_{ij} = \phi^4 g^{\text{reference}}_{ij}, \qquad p^{ij}_{TT} p^{TT}_{ij} - \frac{1}{6}\phi^{12} Y^2 - g\,\phi^8 \left(R - 8\,\phi^{-1}\nabla^2 \phi \right) = 0$$

the independent initial data are a *conformal metric* $g^{\text{reference}}_{ij}$,
a TT-momentum p^{ij}_{TT} and the constant Y, the *York time*

York assumed, without justification, a different scaling of the trace
and the traceless part of the momentum: $Y \to Y$, $p^{ij}_{T} \to \phi^{-4} p^{ij}_{T}$

from $p^{ij}_{T} = p^{ij} - \frac{1}{3}\sqrt{g}\,Y\,g^{ab}$ one would expect $Y \to \phi^{-6} Y$...

...however, without that scaling the solution is not unique!

19

The Lichnerowicz–York equation

J. York solved the initial-value problem of GR [PRL **26-28** (1971-2)]

If $p = Y\sqrt{g}$ Momentum and Hamiltonian constraints decouple:

$$\nabla_i p^{ij}_{TT} = 0, \qquad p^{ij}_{TT} p^{TT}_{ij} - \frac{1}{6} Y^2 - g R = 0$$

the latter can be transformed into an elliptic equation for a conformal factor:

$$g_{ij} = \phi^4 g^{\text{reference}}_{ij}, \qquad p^{ij}_{TT} p^{TT}_{ij} - \frac{1}{6}\phi^{12} Y^2 - g\,\phi^8 \left(R - 8\,\phi^{-1}\nabla^2\phi\right) = 0$$

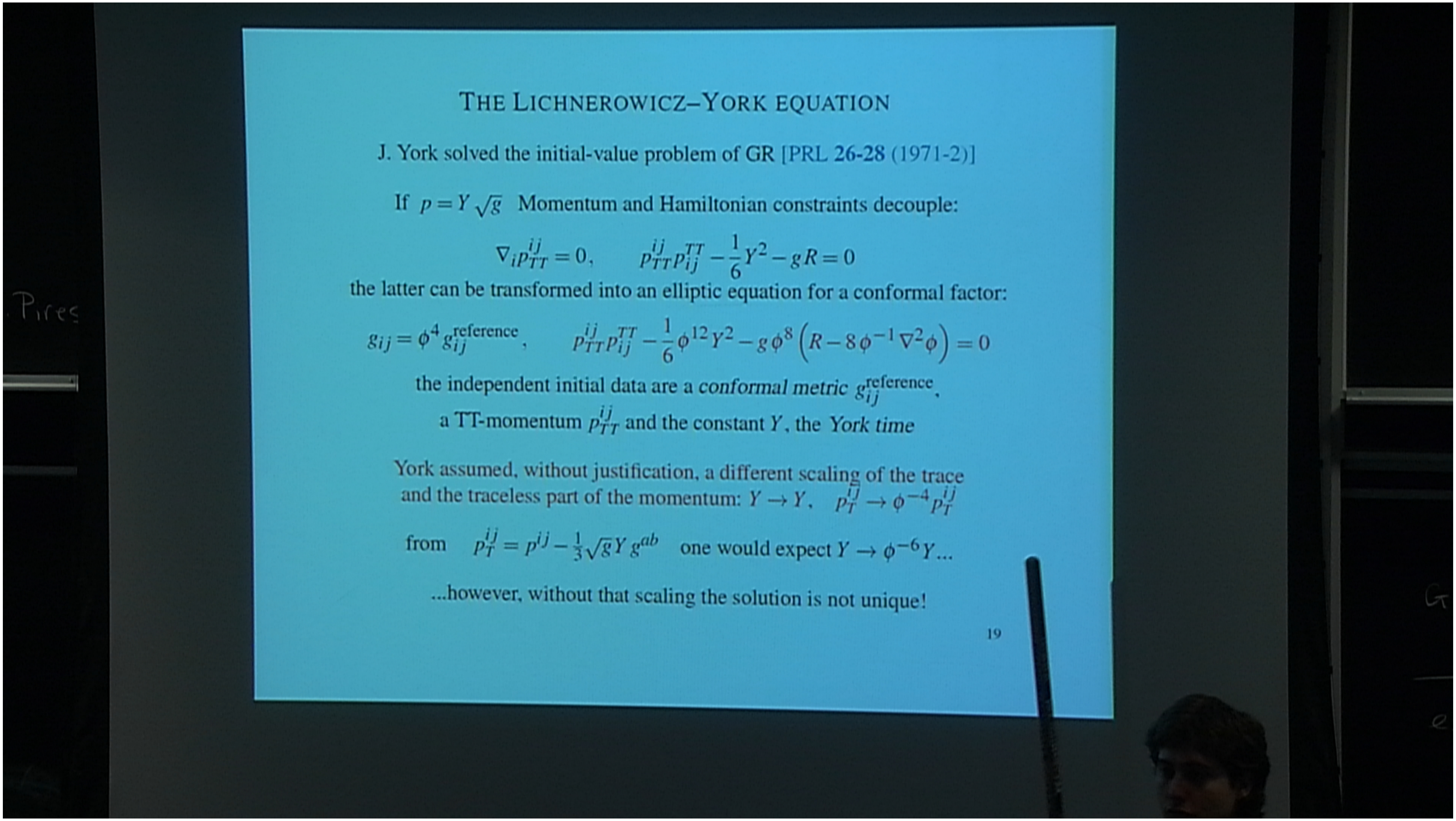
the independent initial data are a *conformal metric* $g^{\text{reference}}_{ij}$,
a TT-momentum p^{ij}_{TT} and the constant Y, the *York time*

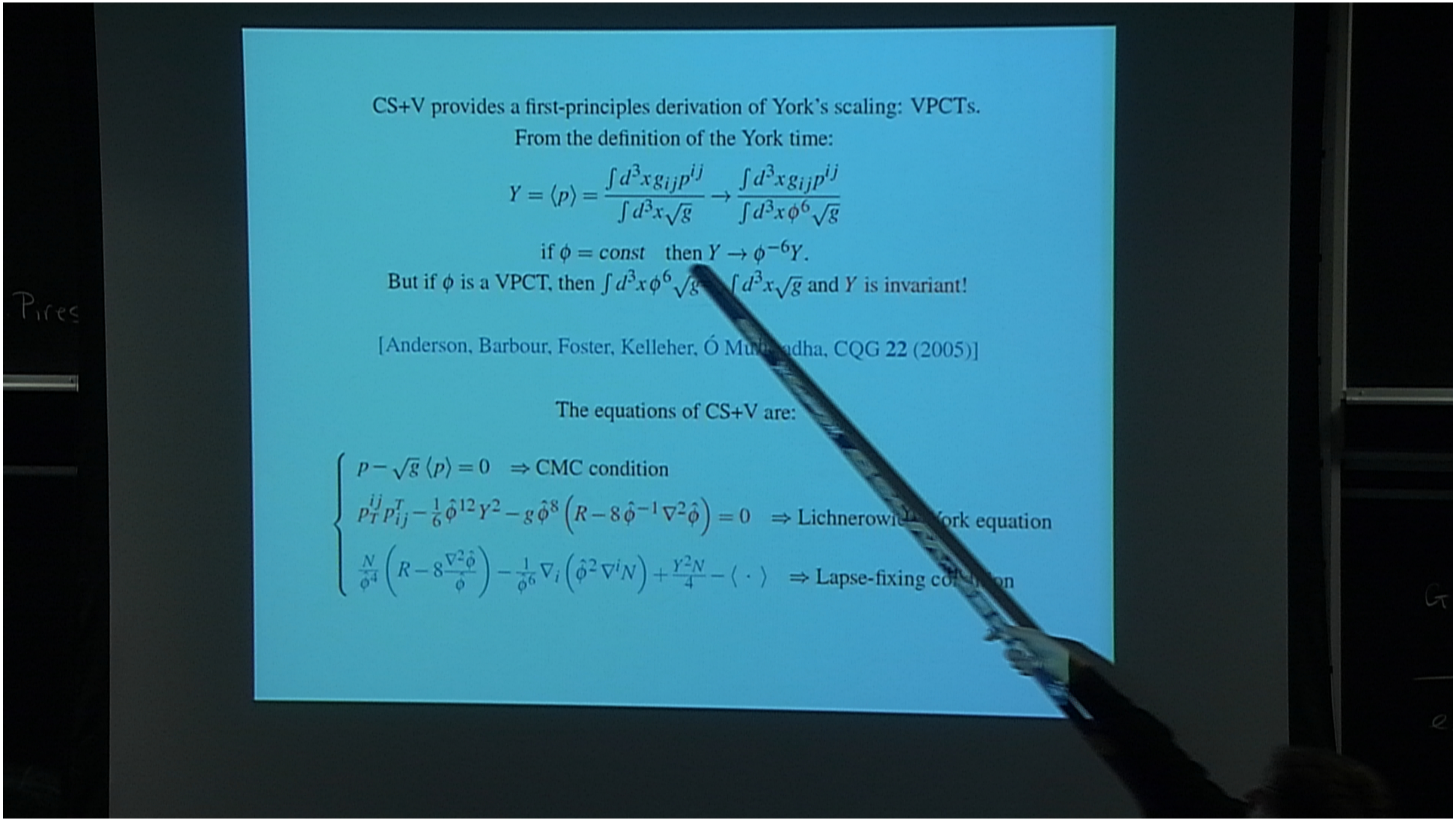
York assumed, without justification, a different scaling of the trace and the traceless part of the momentum: $Y \to Y$, $p^{ij}_T \to \phi^{-4} p^{ij}_T$

from $p^{ij}_T = p^{ij} - \frac{1}{3}\sqrt{g}\,Y g^{ab}$ one would expect $Y \to \phi^{-6} Y$...

...however, without that scaling the solution is not unique!

19


THE LICHNEROWICZ–YORK EQUATION
J. York solved the initial-value problem of GR [PRL 26-28 (1971-2)]
If $p = Y\sqrt{g}$ Momentum and Hamiltonian constraints decouple:
$\nabla_i p^{ij}_{TT} = 0, \qquad p^{ij}_{TT} p^{TT}_{ij} - \frac{1}{6} Y^2 - g R = 0$
the latter can be transformed into an elliptic equation for a conformal factor:
$g_{ij} = \phi^4 g^{\text{reference}}_{ij}, \qquad p^{ij}_{TT} p^{TT}_{ij} - \frac{1}{6}\phi^{12} Y^2 - g\,\phi^8 \left(R - 8\phi^{-1}\nabla^2\phi\right) = 0$
the independent initial data are a conformal metric $g^{\text{reference}}_{ij}$,
a TT-momentum p^{ij}_{TT} and the constant Y, the York time
York assumed, without justification, a different scaling of the trace
and the traceless part of the momentum: $Y \to Y, \quad p^{ij}_T \to \phi^{-4} p^{ij}_T$
from $p^{ij}_T = p^{ij} - \frac{1}{3}\sqrt{g}\, Y\, g^{ab}$ one would expect $Y \to \phi^{-6} Y$...
...however, without that scaling the solution is not unique!
19


CS+V provides a first-principles derivation of York's scaling: VPCTs.
From the definition of the York time:
$Y = \langle p \rangle = \frac{\int d^3x g_{ij} p^{ij}}{\int d^3x \sqrt{g}} \to \frac{\int d^3x g_{ij} p^{ij}}{\int d^3x \phi^6 \sqrt{g}}$
if $\phi = const$ then $Y \to \phi^{-6} Y$.
But if ϕ is a VPCT, then $\int d^3x \phi^6 \sqrt{g}$ $\int d^3x \sqrt{g}$ and Y is invariant!
[Anderson, Barbour, Foster, Kelleher, Ó Mu adha, CQG 22 (2005)]
The equations of CS+V are:
$p - \sqrt{g}\,\langle p \rangle = 0$ ⇒ CMC condition
$p_T^{ij} p^T_{ij} - \frac{1}{6}\hat{\phi}^{12} Y^2 - g\hat{\phi}^8 \left(R - 8\hat{\phi}^{-1}\nabla^2\hat{\phi}\right) = 0$ ⇒ Lichnerowi ork equation
$\frac{N}{\hat{\phi}^4}\left(R - 8\frac{\nabla^2\hat{\phi}}{\hat{\phi}}\right) - \frac{1}{\hat{\phi}^6}\nabla_i\left(\hat{\phi}^2\nabla^i N\right) + \frac{Y^2 N}{4} - \langle \cdot \rangle$ ⇒ Lapse-fixing co on
Pires

SHAPE DYNAMICS ($\lambda = 1$)

The system of constraints of CS+V is second-class:

$$\frac{\mathrm{d}\mathscr{H}}{\mathrm{d}\tau} = \frac{N}{\hat{\phi}^4}\left(R - 8\frac{\nabla^2\hat{\phi}}{\hat{\phi}}\right) - \frac{1}{\hat{\phi}^6}\nabla_i\left(\hat{\phi}^2\nabla^i N\right) + \frac{Y^2 N}{4} - \langle \cdot \rangle$$

the rhs cannot be included as a new constraint. It is instead a *gauge-fixing condition*, that ensures that the Lapse N propagates the CMC condition. One can then perform a Dirac procedure, ending up with a first-class theory which is a gauge-fixed version of GR.

The Hamiltonian constraint gets all gauge-fixed by the LFE, apart from a single *global* mode which is still first-class:

$\mathscr{H}_{\text{global}} = \int d^3x\, \tilde{N}(x)\,\mathscr{H}(x)$ where $\tilde{N}(x)$ is the (unique!) solution of the LFE.

In [Gomes, Gryb, Koslowski, CQG 28, (2011)] this was the starting point for the definition of a new theory with VPCT gauge invariance, which in a particular gauge coincides with CMC GR.
(check out the talks by Gomes, Koslowski and Gryb later...)

Shape Dynamics ($\lambda = 1$)

The system of constraints of CS+V is second-class:

$$\frac{\mathrm{d}\mathscr{H}}{\mathrm{d}\tau} = \frac{N}{\hat{\phi}^4}\left(R - 8\frac{\nabla^2\hat{\phi}}{\hat{\phi}}\right) - \frac{1}{\hat{\phi}^6}\nabla_i\left(\hat{\phi}^2\nabla^i N\right) + \frac{Y^2 N}{4} - \langle \cdot \rangle$$

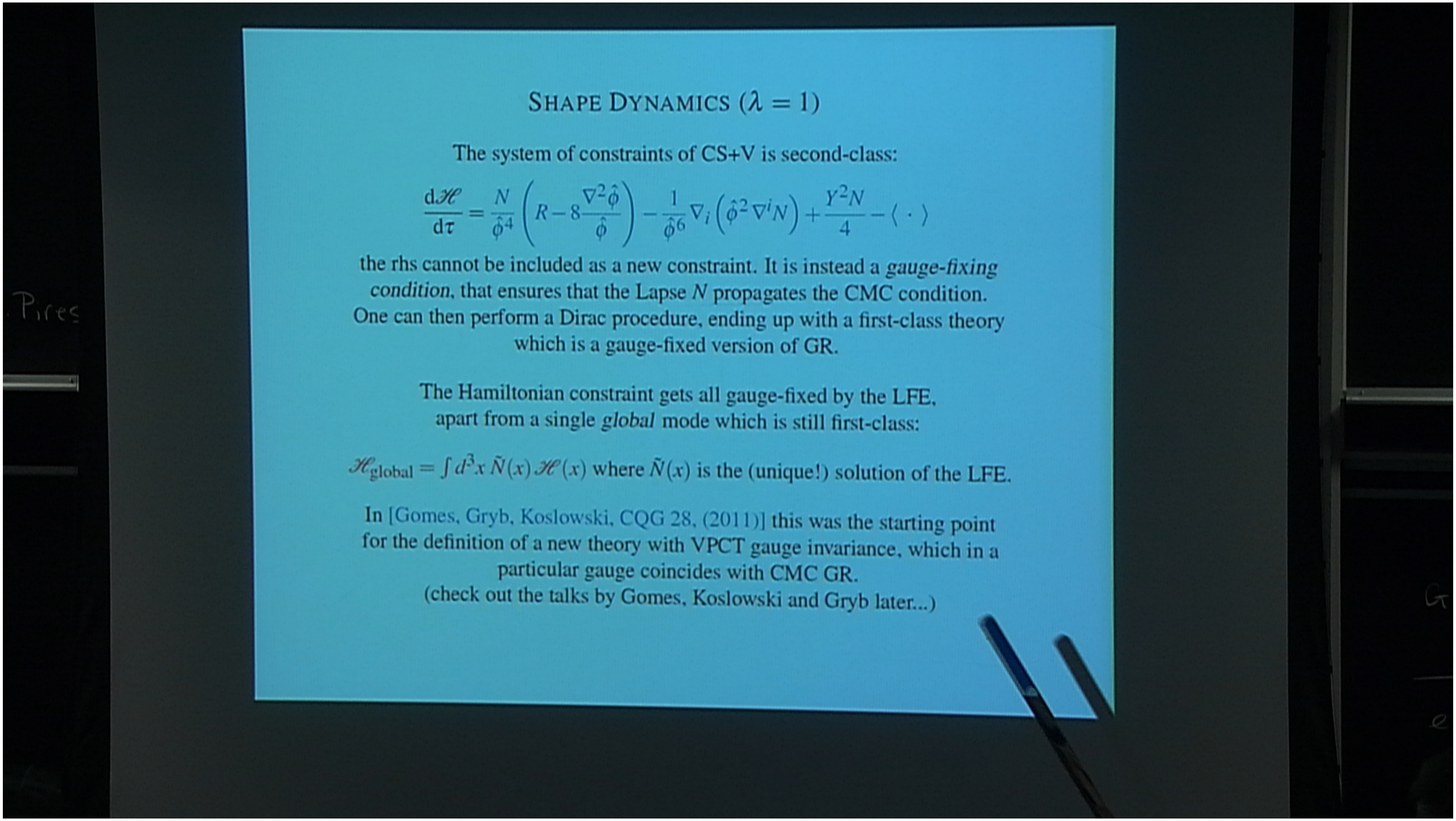
the rhs cannot be included as a new constraint. It is instead a *gauge-fixing condition*, that ensures that the Lapse N propagates the CMC condition. One can then perform a Dirac procedure, ending up with a first-class theory which is a gauge-fixed version of GR.

The Hamiltonian constraint gets all gauge-fixed by the LFE, apart from a single *global* mode which is still first-class:

$\mathscr{H}_{\text{global}} = \int d^3x\, \tilde{N}(x)\, \mathscr{H}(x)$ where $\tilde{N}(x)$ is the (unique!) solution of the LFE.

In [Gomes, Gryb, Koslowski, CQG 28, (2011)] this was the starting point for the definition of a new theory with VPCT gauge invariance, which in a particular gauge coincides with CMC GR.
(check out the talks by Gomes, Koslowski and Gryb later...)

21


SHAPE DYNAMICS ($\lambda = 1$)
The system of constraints of CS+V is second-class:
$\frac{d\mathscr{H}}{d\tau} = \frac{N}{\hat{\phi}^4}\left(R - 8\frac{\nabla^2\hat{\phi}}{\hat{\phi}}\right) - \frac{1}{\hat{\phi}^6}\nabla_i\left(\hat{\phi}^2\nabla^i N\right) + \frac{Y^2 N}{4} - \langle \cdot \rangle$
the rhs cannot be included as a new constraint. It is instead a gauge-fixing condition, that ensures that the Lapse N propagates the CMC condition. One can then perform a Dirac procedure, ending up with a first-class theory which is a gauge-fixed version of GR.
The Hamiltonian constraint gets all gauge-fixed by the LFE, apart from a single global mode which is still first-class:
$\mathscr{H}_{\text{global}} = \int d^3x\, \tilde{N}(x)\mathscr{H}(x)$ where $\tilde{N}(x)$ is the (unique!) solution of the LFE.
In [Gomes, Gryb, Koslowski, CQG 28, (2011)] this was the starting point for the definition of a new theory with VPCT gauge invariance, which in a particular gauge coincides with CMC GR.
(check out the talks by Gomes, Koslowski and Gryb later...)

Shape Dynamics ($\lambda = 1$)

The system of constraints of CS+V is second-class:

$$\frac{\mathrm{d}\mathscr{H}}{\mathrm{d}\tau} = \frac{N}{\hat{\phi}^4}\left(R - 8\frac{\nabla^2\hat{\phi}}{\hat{\phi}}\right) - \frac{1}{\hat{\phi}^6}\nabla_i\left(\hat{\phi}^2\nabla^i N\right) + \frac{Y^2 N}{4} - \langle\,\cdot\,\rangle$$

the rhs cannot be included as a new constraint. It is instead a *gauge-fixing condition*, that ensures that the Lapse N propagates the CMC condition. One can then perform a Dirac procedure, ending up with a first-class theory which is a gauge-fixed version of GR.

The Hamiltonian constraint gets all gauge-fixed by the LFE, apart from a single *global* mode which is still first-class:

$\mathscr{H}_{\text{global}} = \int d^3x\, \tilde{N}(x)\,\mathscr{H}(x)$ where $\tilde{N}(x)$ is the (unique!) solution of the LFE.

In [Gomes, Gryb, Koslowski, CQG 28, (2011)] this was the starting point for the definition of a new theory with VPCT gauge invariance, which in a particular gauge coincides with CMC GR.
(check out the talks by Gomes, Koslowski and Gryb later...)

...AND WHAT ABOUT λ?

All the steps of CS+V and Gomes, Gryb and Koslowski's version of SD go through straightforwardly also if $1/3 < \lambda \leq 1$:

$$\begin{cases} p - \sqrt{g}\,\langle p \rangle = 0 & \Rightarrow \text{CMC condition} \\ p^{ij}_T p^T_{ij} - \frac{1}{9\lambda - 3}\hat{\phi}^{12}\gamma^2 - g\,\hat{\phi}^8\left(R - 8\,\hat{\phi}^{-1}\nabla^2\hat{\phi}\right) = 0 & \Rightarrow \text{Lichnerowicz–York equation} \\ \frac{N}{\hat{\phi}^4}\left(R - 8\frac{\nabla^2\hat{\phi}}{\hat{\phi}}\right) - \frac{1}{\hat{\phi}^6}\nabla_i\left(\hat{\phi}^2\nabla^i N\right) + \frac{\gamma^2 N}{2(3\lambda - 1)} - \langle\,\cdot\,\rangle & \Rightarrow \text{Lapse-fixing condition} \end{cases}$$

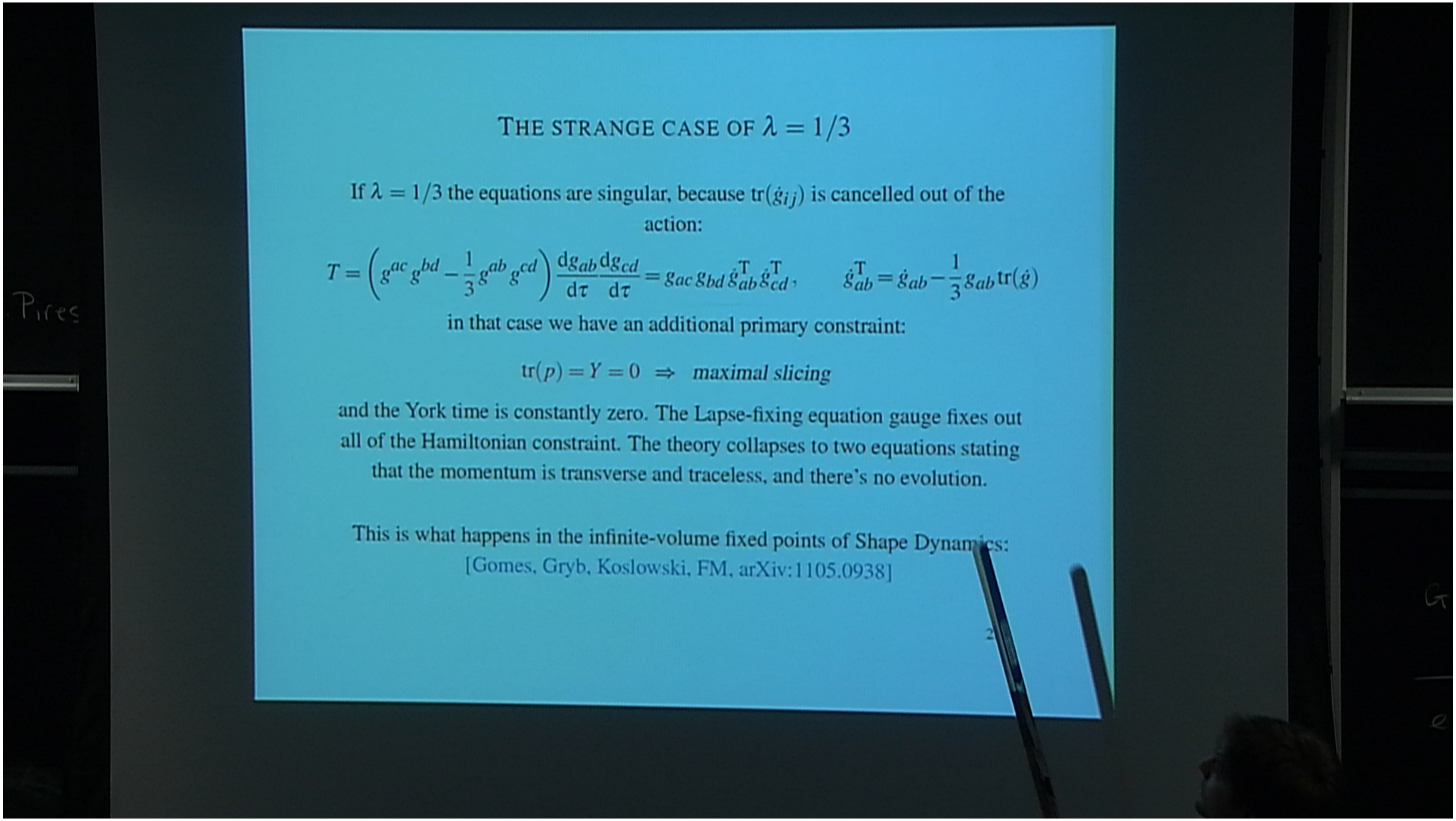
$\lambda \neq 1$ doesn't affect the propagation speed of gravitational waves:

$$\dot{p}^{ij} = Rg^{ij} - R^{ij} - \frac{1}{2}\left(g^{ik}g^{jl} - \frac{\lambda}{3\lambda - 1}g^{ij}g^{kl}\right)\dot{g}_{ij}\dot{g}_{kl}, \quad (N \sim 1, \sqrt{g} \sim 1, N_i = 0)$$

expanding $g_{ij} = \delta_{ij} + h_{ij}$ we see that λ enters at $\mathcal{O}(h^2)$.

look at cosmological data to constrain λ... [N. Ashfordi *et al.*, PRD **75**, (2007)]

22


THE STRANGE CASE OF $\lambda = 1/3$
If $\lambda = 1/3$ the equations are singular, because $\mathrm{tr}(\dot{g}_{ij})$ is cancelled out of the action:
$T = \left(g^{ac} g^{bd} - \frac{1}{3} g^{ab} g^{cd}\right) \frac{\mathrm{d}g_{ab}}{\mathrm{d}\tau} \frac{\mathrm{d}g_{cd}}{\mathrm{d}\tau} = g_{ac} g_{bd} \dot{g}^{\mathrm{T}}_{ab} \dot{g}^{\mathrm{T}}_{cd}, \qquad \dot{g}^{\mathrm{T}}_{ab} = \dot{g}_{ab} - \frac{1}{3} g_{ab} \mathrm{tr}(\dot{g})$
in that case we have an additional primary constraint:
$\mathrm{tr}(p) = Y = 0 \Rightarrow$ *maximal slicing*
and the York time is constantly zero. The Lapse-fixing equation gauge fixes out all of the Hamiltonian constraint. The theory collapses to two equations stating that the momentum is transverse and traceless, and there's no evolution.
This is what happens in the infinite-volume fixed points of Shape Dynamics:
[Gomes, Gryb, Koslowski, FM, arXiv:1105.0938]