Title: Conformal Gravity and AdS/CFT

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Conformal gravity and AdS/CFT

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Introduction

- In this talk I will discuss the connection between conformal gravity and the AdS/CFT correspondence
- In AdS/CFT correspondence one relates gravity in AdS to a non-gravitating CFT in one dimension less, so superficially there is no connection to conformal gravity: the bulk theory has gravity but it is not conformal while the boundary theory is conformal but it does not contain gravity.
- Nevertheless, we will see that AdS/CFT leads to systematic construction of conformal gravity coupled to conformal matter and leads to new results and intuition about these theories.

AdS/CFT: basics

To explain this connection we first need to recall some of the basics of AdS/CFT.

- Bulk fields correspondent to boundary operators. For example,
  - The bulk metric correspond to the boundary stress energy tensor T_{ij}.
  - A bulk scalar field of mass $m^2 = \Delta(\Delta - d)$ correspond to a boundary scalar operator of dimension Δ.
- AdS has a conformal boundary and we need to impose boundary conditions there. The fields parametrizing the boundary conditions are identified with sources for dual operators.
- The on-shell supergravity action, as a function of the fields parametrizing the boundary conditions, is identified with the generating functional of CFT correlators.

AdS/CFT: asymptotics

- In QFT the sources are unconstrained. This implies that the gravitational equations must admit a Dirichlet boundary problem specified by arbitrary functions.
- Such Dirichlet problem is not always possible. For example, asymptotically flat spacetimes do not admit such Dirichlet problem. It turns out however that the gravitational equations with a cosmological constant can be formulated in this way.

AdS and its conformal structure

To get some intuition let us first discuss Anti-de Sitter spacetime. The metric in global coordinates is given by

$$ds^2 = \frac{1}{\cos\theta^2}\left(-dt^2 + d\theta^2 + \sin^2\theta d\Omega_{d-1}^2\right)$$

where $0 \leq \theta < \pi/2$.

- The conformal boundary of AdS_{d+1} is at $\theta = \pi/2$.
- The bulk metric divergences there: there is a second order pole. So there is no well-defined boundary metric.
- There is however a well-defined conformal structure, i.e. a metric up to a Weyl transformation.

The boundary conformal structure

- To obtain a boundary metric we use a *defining function*, i.e. a function $r(x)$ which is positive in the interior but has a single zero at the boundary. We then define

$$g_{(0)} = \lim_{\theta \to \pi/2} r^2 G$$

This limit exits because the second order pole in G is canceled by the zero of r^2.

- However, any other $r'(x) = r(x)e^{\sigma(x)}$ is as good, so what is well-defined here is the conformal class

$$g_{(0)} \sim e^{2\sigma(x)} g_{(0)}$$

- For AdS we may pick $r = \cos\theta$, and this leads to the representative of the conformal class

$$ds_0^2 = -dt^2 + d\Omega_{d-1}^2$$

This metric is conformally flat and any other conformally flat metric (for example, the flat metric) is as good.

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The Dirichlet problem for AdS gravity

- Most of the GR literature ([Abbot, Deser],[Henneaux, Teitelboim], [Ashtekar, Magnon] ...) from the 80's discussed Asymptotically AdS spacetimes, i.e. fixed the boundary metric to be $ds_0^2 = -dt^2 + d\Omega_{d-1}^2$.
- For AdS/CFT we need to generalize the Dirichlet problem in two ways:

  - ➢ We would like to keep fixed a a conformal class.
  - ➢ We would like this to be a general conformal class.

- We need a general unconstrained $[g_{(0)}(x)]$ since it will serve as a source for T_{ij} and QFT sources are always unconstrained (we need to be able to functionally differentiate w.r.t. them).
- Spacetimes of this type were considered in the mathematics literature [Fefferman-Graham (1985)]. We will call them Asymptotically locally AdS.

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Asymptotically locally AdS spacetimes

Let's now consider a representative $g_{(0)}$ of $[g_{(0)}]$ and try to obtain a solution of Einstein's equations. It turns out that this can be done in complete generality near the conformal boundary [Fefferman-Graham (1985)]. Here we will focus on odd dimensional spacetimes because these yield the connection with conformal gravity.

- Near conformal infinity one can always choose coordinates such that the metric is

$$ds^2 = \frac{dr^2}{4\rho^2} + \frac{1}{\rho} g_{ij}(x,\rho) dx^i dx^j$$

where $\rho = 0$ is the position of the conformal boundary.

- Einstein equations, solved by expanding in ρ, become algebraic equations that can be readily solved.

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Structure of Asymptotically locally AdS spacetimes

- The solution is

$$g_{ij}(x,\rho) = \mathbf{g_{(0)}}(\mathbf{x}) + \rho g_{(2)ij} + \cdots + \rho^{d/2}(g_{(d)ij} + \log\rho\,\mathbf{h_{(d)ij}}) + \cdots$$

- The blue coefficient $g_{(2)ij}, \ldots, \mathbf{h_{(d)ij}}$ are uniquely and locally determined in terms of $\mathbf{g_{(0)}}(\mathbf{x})$.
- $\mathbf{h_{(d)ij}}$ is called the *obstruction tensor* in the mathematics literature and it will play an important role in our story.
- Only the trace and divergence of $g_{(d)ij}$ is determined by asymptotics. This coefficient is the most important coefficient for AdS/CFT: it gives the stress energy tensor of the dual theory [de Haro, Solodukhin, KS (2000)] but it will not play an important role today.

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Regularizing the on-shell action

- Now that we obtained the general asymptotic solution, the next step in AdS/CFT is to compute the on-shell gravitational action. However, the on-shell action is infinite because of the infinite volume of spacetime,

$$S = \int d^{d+1}\sqrt{G}(R+\Lambda) \sim \int \sqrt{G} \to \infty$$

- To regulate this infinity we cut-off the spacetime at $\rho = \epsilon$, with ϵ small. The result is [Henningson, KS (1998)]

$$S_{reg}[g_{(0)};\epsilon] = \sum_{n=1}^{d/2} \frac{a_n[g_{(0)}]}{\epsilon^n} - \mathbf{a}_{(\mathbf{d})}[\mathbf{g}_{(\mathbf{0})}] \log\epsilon + \mathit{finite}$$

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Holographic renormalization

- One can now renormalize the on-shell action by adding a set of boundary covariant counterterms [Henningson, KS (1998)] [de Haro, Solodukhin, KS (2000)]

$$S_{ren}[g_{(0)}] = \lim_{\epsilon \to 0} (S_{reg}[g_{(0)}; \epsilon] + S_{ct}[g_{(0)}; \epsilon])$$

- The counterterm action is given by

$$S_{ct}[g_{(0)}; \epsilon] = \int_{\rho=\epsilon} d^d x \sqrt{\gamma} \left(2(1-d) + \frac{1}{d-2} R + \cdots - \mathbf{a}_{(\mathbf{d})}[\gamma] \log \epsilon \right)$$

where γ is the induced metric at $\rho = \epsilon$.

Antoine Van Proeyen
is online

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The holographic Weyl anomaly [Henningson, KS (1998)]

- The renormalized on-shell action is now finite, but it turns out it depends on the specific representative of the conformal structure we started from

$$S_{ren}[e^{2\sigma(x)}g_{(0)}] = S_{ren}[g_{(0)}] + \mathcal{A}[g_{(0)}, \sigma]$$

- Considering infinitesimal σ one finds

$$\mathcal{A}[g_{(0)}, \sigma] = \int d^d x \sqrt{g_{(0)}}\sigma(x)\mathbf{a}_{(\mathbf{d})}[\mathbf{g}_{(\mathbf{0})}]$$

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Properties of the conformal anomaly

The holographic conformal anomaly is now

$$A[g_{(0)}] = \int d^d x \sqrt{g_{(0)}} \mathbf{a}_{(\mathbf{d})}[\mathbf{g}_{(\mathbf{0})}]$$

It has the following properties:

- It is conformally invariant,

$$A[e^{2\sigma(x)} g_{(0)}] = A[g_{(0)}]$$

  This means that the fact of whether there is a conformal anomaly or not depends only on the conformal class of $[g_{(0)}]$.
- The obstruction tensor is given by [de Haro, Solodukhin, KS (2000)]

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The variational problem [Papadimitriou, KS (2005)]

The counterterms that we just derived requiring finiteness of the on-shell action play another role, perhaps more fundamental.

- In even dimensions, a well-posed variational problem where a conformal class is kept fixed , requires additional boundary terms and these are precisely the boundary counterterms.
- In odd dimensions, one must specify a representative $g_{(0)}$ of the conformal class $[g_{(0)}]$ for the variational problem to be well-posed. In this case the boundary counterterms ensure that the dependence of the theory on the specific $g_{(0)}$ is governed only by the anomaly A, which itself depends only on the conformal class $[g_{(0)}]$.

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Conformall gravity from AdS

We have thus just obtained an action for conformal theory of gravity!

- Its Lagrangian,

$$L = a_{(d)}$$

  can be read-off from the logarithmic divergence of the on-shell action of AdS gravity in one dimension higher.

- Its field equations

$$h_{(d)ij} = 0$$

  can be read-off from the logarithmic term in the asymptotic expansion of the bulk metric.

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Holographic conformal gravity in $d = 4$

- In this case we get [Henningson, KS (1998)]

$$L = a_{(4)} \sim R_{ij}R^{ij} - \frac{1}{3}R^2$$

- This can be expressed also as

$$a_{(4)} \sim E_4 + W_{ijkl}W^{ijkl}$$

where E_4 is the Euler density and W_{ijkl} is the Weyl tensor.

- This is also exactly equal to the conformal anomaly of $N = 4$ SYM, providing a highly non-trivial check of the AdS/CFT correspondence.

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Einstein gravity from holographic conformal gravity

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Holographic proof [KS, unpublished (2006)]

- One can prove by straightforward computation that the $(d+1)$ dimensional metric,

$$ds^2 = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}(1 + \frac{\lambda}{4}\rho)^2 g_{(0)ij} dx^i dx^j$$

is Einstein with negative cosmological constant, provided $g_{(0)}$ is an Einstein metric in d dimensions,

$$Ric[g_{(0)}] = \lambda(d-1) g_{(0)}, \qquad \lambda = \pm 1, 0$$

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Holographic proof [KS, unpublished (2006)]

- Recall that the obstruction tensor $h_{(d)ij}$ is determined uniquely and locally in terms of $g_{(0)}$. Since the bulk metric does not have a logarithmic term,

$$h_{(d)ij}[g_{(0)}] = 0,$$

for Einstein metrics.

- This proves the claim since $h_{(d)ij}[g_{(0)}] = 0$ are the field equations for the holographic conformal gravity.

➠ Actually $h_{(d)}$ depends only on the conformal structure, so $h_{(d)ij}[g_{(0)}]$ vanishes for conformally Einstein metrics. This is a well known fact in the mathematics literature. In particular, in $d = 4$, $h_{(4)ij}$ is the well-known Bach tensor [Bach (1921)].

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An alternative proof [KS, unpublished (2006)]

I will now present a more direct proof.

- First notice that in both $d = 4$ and $d = 6$ the Lagrangian is at least quadratic in R_{ij},

$$L_4 = R_{ij}R^{ij} - \frac{1}{3}R^2$$

$$L_6 = \frac{1}{2}RR_{ij}R^{ij} - \frac{3}{50}R^3 - R^{ij}R^{kl}R_{ikjl} + \frac{1}{5}R^{ij}D_iD_jR - \frac{1}{2}R^{ij}\Box R_{ij} + \frac{1}{20}R\Box R$$

- So it is manifest that Ricci-flat metrics solve the corresponding field equations.

An alternative proof [KS, unpublished (2006)]

So we now focus on the case $g_{(0)}$ is Einstein with a cosmological constant (of any sign).

- One can show that in both case the Lagrangian can be written in the following way

$$L = E_{ij}K^{ijkl}E_{kl} + L_0$$

  where L_0 is the Einstein-Hilbert Lagrangian (with cosmological constant) and E_{ij} are the corresponding Einstein equations.
- The tensor K^{ijkl} is given by

$$\begin{aligned} d=4 \quad &: \quad K^{ijkl} = d_1 g^{ij} g^{kl} + d_2 g^{ik} g^{jm} \\ d=6 \quad &: \quad K^{ijkl} = c_1 R^{ikjl} + g^{ik} g^{jl}(c_2 R + c_3 \Box + c_4) \\ &\qquad +g^{il} g^{jk}(c_5 R + c_6 \Box + c_7) + c_9 g^{kl} D^i D^j \end{aligned}$$

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Remarks

- It follows that the field equations of holographic conformal gravity are linear in Einstein equations. We thus conclude that Einstein metrics are solution of the holographic conformal gravity.
- Moreover, the on-shell action of the two theories coincide on any Einstein metric.
- One can identify which solutions of conformal gravity are Einstein via an initial value formulation [M. Anderson (2004)] [Maldacena (2012)].
- Our reformulations makes fairly simple to analyze the class of solutions of the holographic conformal gravity that are not Einstein metrics.

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Critical gravity

There has recently been renewed interest in higher derivative gravities,

$$S = \int d^4x\sqrt{g}[R - 2\Lambda + \alpha R^{ij}R_{ij} + \beta R^2]$$

- For general α, β this theory describes the propagation of a massless spin 2 field, a massive spin 2 field and a massive scalar.
- [Lü, Pope (2011)] observed that the spectrum of linearized perturbations around AdS is special when:
- $\alpha = -3\beta$: the massive scalar is absent.
- When in addition, $\beta = -1/(2\Lambda)$ the massive graviton becomes massless. This theory was dubbed "critical gravity".

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Critical gravity as the square of Einstein equations

It is easy to understand these results (and all other special properties of this theory). When $\alpha = -3\beta$,

$$S = \int d^4x \sqrt{g} [R - 2\Lambda - 3\beta(R^{ij}R_{ij} - \frac{1}{3}R^2)]$$

and the higher derivative term is proportional to holographic conformal anomaly. Using our rewriting of the anomaly we obtain

$$S = \int d^4x \left((6\lambda) E_{ij} K^{ijkl} E_{kl} + (1 + \beta\, 2\Lambda) L_0 \right)$$

- In critical gravity, $1 + \beta\, 2\Lambda = 0$, and the theory is the "square" of Einstein field equations.

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Outline

1. The AdS/CFT correspondence
2. Holographic conformal gravity
   - Properties of holographic conformal gravity
   - Critical gravity
3. Adding conformal matter
   - Conformal differential operators
4. Conclusions

Conformal gravity coupled to conformal matter

We would now like to understand how to couple matter to the conformal gravity.

- This can be done simply by considering AdS gravity coupled to matter. Of course, not all bulk matter fields will lead to conformal matter couplings in the boundary.
- Their mass should be such that the boundary field (source of dual operator) has dimension that allows for interaction terms of dimension d.

➢ Example: a scalar field Φ of mass $m^2 = -3$ in AdS_5.

➠ The dual field has dimension $\Delta = 3$ and thus its source $\phi_{(0)}$ has dimension $d - \Delta = 1$.

➠ Terms of the form $\phi_0 \Box \phi_0, \phi_0^4$ have dimension 4.

Holographic construction

The holographic construction is the same as in the case of pure gravity. We focus here in the case the matter is a scalar field.

- We start by finding the general asymptotic solution of gravity coupled to matter. The metric has the same asymptotics as before. The scalar field has the expansion [de Haro, Solodukhin, KS],

$$\Phi(x,\rho) = \rho^{(d-\Delta)/2}(\phi_{(\mathbf{0})}(\mathbf{x}) + \rho\phi_{(2)} + \cdots + \rho^{\Delta-d/2}(\phi_{(2\Delta-d)} + \log\rho\,\psi_{(2\Delta-d)}) + \cdots)$$

- Computing the on-shell action one finds again that contains a logarithmic divergence with coefficients $a_{(d)}$, and $A = \int a_{(d)}$ is conformally invariant.
- Furthermore [de Haro, Solodukhin, KS],

$$h_{(d)ij} \sim \frac{1}{\sqrt{g_{(0)}}}\frac{\delta A}{\delta g^{ij}_{(0)}}, \qquad \psi_{(2\Delta-d)} \sim \frac{1}{\sqrt{g_{(0)}}}\frac{\delta A}{\delta \phi_{(0)}}$$

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Conformal differential operators

- Conformal differential operators are differential operators whose conformal transformations depends on $\sigma(x)$, but not on derivatives of σ.
- An example is the conformal Laplacian we just discussed,

$$P_1 \equiv \Box - \frac{d-2}{4(d-1)} R.$$

It transforms as

$$P_1 \to e^{-(d/2+1)\sigma(x)} P_1 e^{-(d/2-1)\sigma(x)}$$

- An active area of research in mathematics is about the form and properties of such operators [Graham, ...].

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Constructing conformal differential operators

- A scalar conformal operator of weight $(d/2+k)$ transforms as

$$P_k \to e^{-(d/2+k)\sigma(x)} P_k e^{-(d/2-k)\sigma(x)}$$

  This is the conformal differential operator corresponding to the kth power of the Laplacian.
- Up until recently only the $k=1,2$ operators were known explicitly. To explicitly construct the remaining operators one may consider a scalar field of mass $m^2 = -(d/2)^2 + k^2 < 0$ and compute the log divergence of the on-shell action.
- One can use this method to construct non-scalar conformal operators as well. For example, in $d=4$ we get the Maxwell operator from a gauge field in the bulk.

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Conclusions

- We discussed how to obtain conformal gravity coupled to conformal matter using the AdS/CFT correspondence.
  - One starts from AdS gravity coupled to matter fields in $(d+1)$ dimensions and work out the asymptotic solution to the field equations. This amounts to solving algebraic equations.
  - If there is a logarithmic term in the asymptotic solution of a given bulk field then its coefficient is the corresponding field equation of conformal gravity coupled to conformal matter.
  - The logarithmic holographic counterterm is the action for conformal gravity coupled to conformal matter.
- We discuss discussed several special properties of these theories.
  - We derived a relation between holographic conformal gravity and Einstein gravity.
  - We used this to explain the special properties of critical gravity.
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