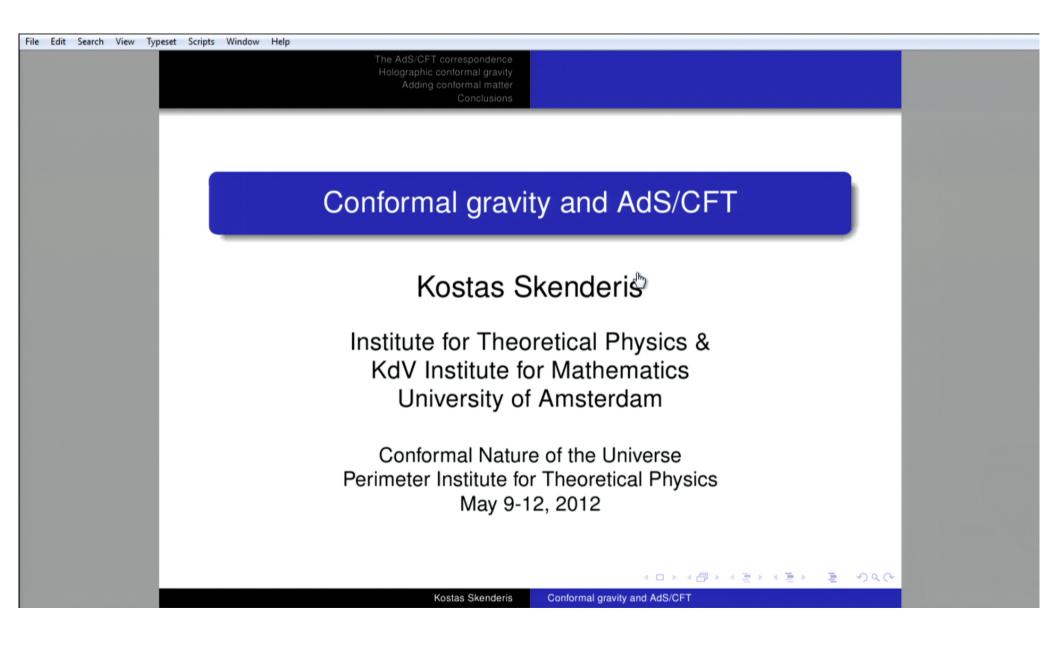
Title: Conformal Gravity and AdS/CFT

Date: May 10, 2012 03:20 PM

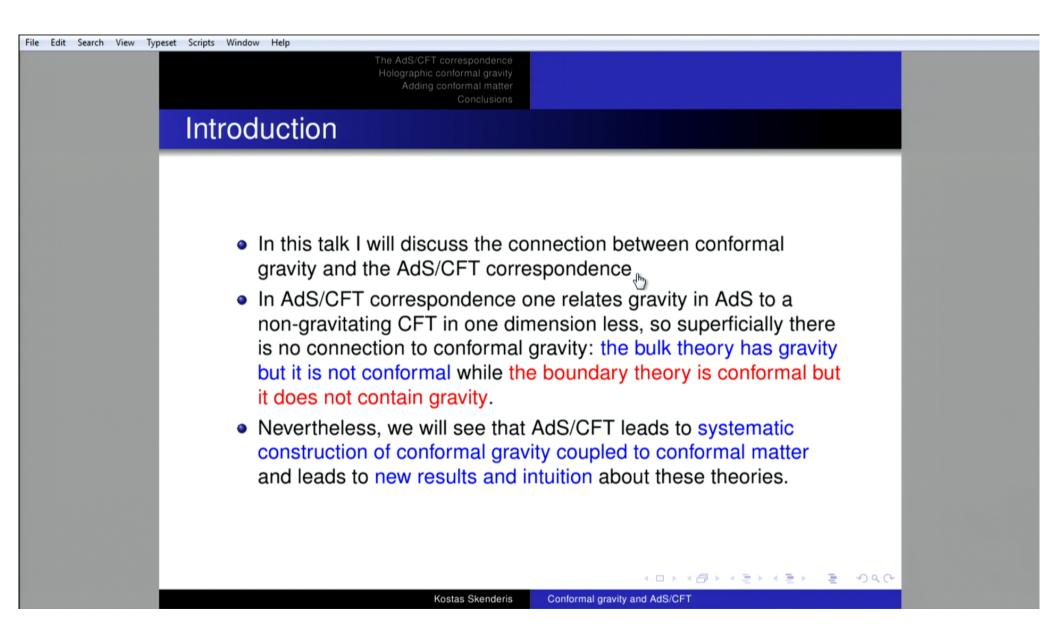
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Abstract: TBA

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#### AdS/CFT: basics

To explain this connection we first need to recall some of the basics of AdS/CFT.

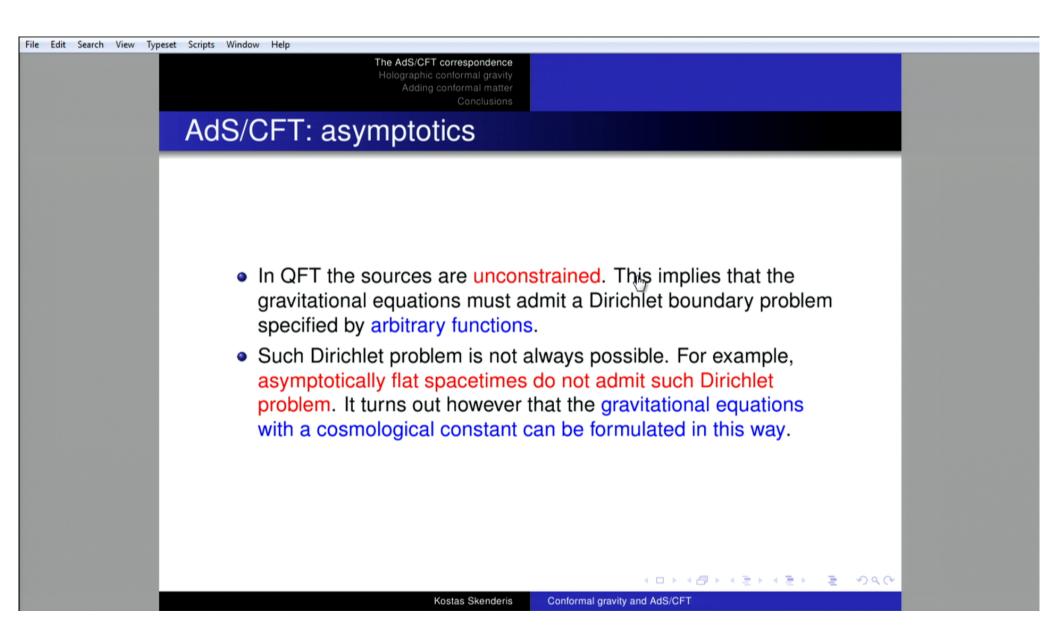
- Bulk fields correspondent to boundary operators. For example,
  - The bulk metric correspond to the boundary stress energy tensor  $T_{ij}$ .
  - A bulk scalar field of mass  $m^2 = \Delta(\Delta d)$  correspond to a boundary scalar operator of dimension  $\Delta$ .
- AdS has a conformal boundary and we need to impose boundary conditions there. The fields parametrizing the boundary conditions are identified with sources for dual operators.
- The on-shell supergravity action, as a function of the fields parametrizing the boundary conditions, is identified with the generating functional of CFT correlators.



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Conformal gravity and AdS/CFT

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#### AdS and its conformal structure

To get some intuition let us first discuss Anti-de Sitter spacetime. The metric in global coordinates is given by

$$ds^{2} = \frac{1}{\cos \theta^{2}} \left( -dt^{2} + d\theta^{2} + \sin^{2} \theta d\Omega_{d-1}^{2} \right)$$

where  $0 \le \theta < \pi/2$ .

- The conformal boundary of  $AdS_{d+1}$  is at  $\theta = \pi/2$ .
- The bulk metric divergences there: there is a second order pole.
   So there is no well-defined boundary metric.
- There is however a well-defined conformal structure, i.e. a metric up to a Weyl transformation.



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### The boundary conformal structure

• To obtain a boundary metric we use a *defining function*, i.e. a function r(x) which is positive in the interior but has a single zero at the boundary. We then define

$$g_{(0)} = \lim_{\theta \to \pi/2} r^2 G$$

This limit exits because the second order pole in G is canceled by the zero of  $r^2$ .

• However, any other  $r'(x) = r(x)e^{\sigma(x)}$  is as good, so what is well-defined here is the conformal class

$$g_{(0)} \sim e^{2\sigma(x)}g_{(0)}$$

• For AdS we may pick  $r = \cos \theta$ , and this leads to the representative of the conformal class

$$ds_0^2 = -dt^2 + d\Omega_{d-1}^2$$

This metric is conformally flat and any other conformally flat metric (for example, the flat metric) is as good.

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Conformal gravity and AdS/CFT

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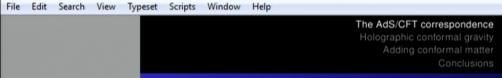
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## The Dirichlet problem for AdS gravity

- Most of the GR literature ([Abbot, Deser],[Henneaux, Teitelboim], [Ashtekar, Magnon] ...) from the 80's discussed Asymptotically AdS spacetimes, i.e. fixed the boundary metric to be  $ds_0^2 = -dt^2 + d\Omega_{d-1}^2$ .
- For AdS/CFT we need to generalize the Dirichlet problem in two ways:
- We would like to keep fixed a a conformal class.
- > We would like this to be a general conformal class.
- We need a general unconstrained  $[g_{(0)}(x)]$  since it will serve as a source for  $T_{ij}$  and QFT sources are always unconstrained (we need to be able to functionally differentiate w.r.t. them).
- Spacetimes of this type were considered in the mathematics literature [Fefferman-Graham (1985)]. We will call them Asymptotically locally AdS.

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Conformal gravity and AdS/CFT

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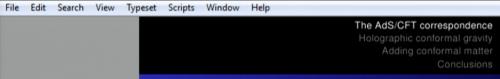


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## Asymptotically locally AdS spacetimes

Let's now consider a representative  $g_{(0)}$  of  $[g_{(0)}]$  and try to obtain a solution of Einstein's equations. It turns out that this can be done in complete generality near the conformal boundary [Fefferman-Graham (1985)]. Here we will focus on odd dimensional spacetimes because these yield the connection with conformal gravity.

 Near conformal infinity one can always choose coordinates such that the metric is

$$ds^2 = \frac{dr^2}{4\rho^2} + \frac{1}{\rho}g_{ij}(x,\rho)dx^idx^j$$

where  $\rho = 0$  is the position of the conformal boundary.

• Einstein equations, solved by expanding in  $\rho$ , become algebraic equations that can be readily solved.



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## Structure of Asymptotically locally AdS spacetimes

The solution is

$$g_{ij}(x,\rho) = \mathbf{g_{(0)}}(\mathbf{x}) + \rho \mathbf{g_{(2)}}_{ij} + \dots + \rho^{d/2} (\mathbf{g_{(d)}}_{ij}) + \log \rho \mathbf{h_{(d)}}_{ij}) + \dots$$

- The blue coefficient  $g_{(2)ij}, \ldots, h_{(\mathbf{d})ij}$  are uniquely and locally determined in terms of  $\mathbf{g}_{(0)}(\mathbf{x})$ .
- h<sub>(d)ij</sub> is called the obstruction tensor in the mathematics literature and it will play an important role in our story.
- Only the trace and divergence of g(d)ij is determined by asymptotics. This coefficient is the most important coefficient for AdS/CFT: it gives the stress energy tensor of the dual theory [de Haro, Solodukhin, KS (2000)] but it will not play an important role today.



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Conformal gravity and AdS/CFT

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Conformal gravity and AdS/CFT

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### Regularizing the on-shell action

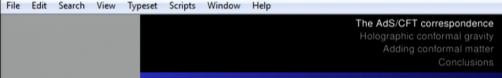
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$$S = \int d^{d+1} \sqrt{G} (R + \Lambda) \sim \int \sqrt{G} 
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• To regulate this infinity we cut-off the spacetime at  $\rho = \epsilon$ , with  $\epsilon$  small. The result is [Henningson, KS (1998)]

$$S_{reg}[g_{(0)}; \epsilon] = \sum_{n=1}^{d/2} \frac{a_n[g_{(0)}]}{\epsilon^n} - \mathbf{a_{(d)}}[\mathbf{g_{(0)}}] \log \epsilon + finite$$

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# Holographic renormalization

 One can now renormalize the on-shell action by adding a set of boundary covariant counterterms [Henningson, KS (1998)] [de Haro, Solodukhin, KS (2000)]

$$S_{ren}[g_{(0)}] = \lim_{\epsilon \to 0} (S_{reg}[g_{(0)}; \epsilon] + S_{ct}[g_{(0)}; \epsilon])$$

The counterterm action is given by

$$S_{ct}[g_{(0)}; \epsilon] = \int_{\rho = \epsilon} d^d x \sqrt{\gamma} \left( 2(1 - d) + \frac{1}{d - 2} R + \dots - \mathbf{a_{(d)}}[\gamma] \log \epsilon \right)$$

where  $\gamma$  is the induced metric at  $\rho = \epsilon$ .



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Conformal gravity and AdS/CFT

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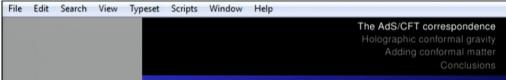
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### The holographic Weyl anomaly [Henningson, KS (1998)]

 The renormalized on-shell action is now finite, but it turns out it depends on the specific representative of the conformal structure we started from

$$S_{ren}[e^{2\sigma(x)}g_{(0)}] = S_{ren}[g_{(0)}] + \mathcal{A}[g_{(0)}, \sigma]$$

• Considering infinitesimal  $\sigma$  one finds

$$\mathcal{A}[g_{(0)}, \sigma] = \int d^d x \sqrt{g_{(0)}} \sigma(x) \mathbf{a_{(d)}}[\mathbf{g_{(0)}}]$$

In even (bulk) dimensions there is no such anomaly.

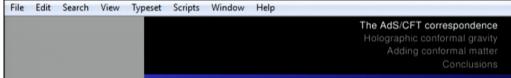


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Conformal gravity and AdS/CFT

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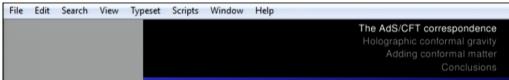
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### Properties of the conformal anomaly

The holographic conformal anomaly is now

$$A[g_{(0)}] = \int d^d x \sqrt{g_{(0)}} \mathbf{a_{(d)}} [\mathbf{g_{(0)}}]$$

It has the following properties:



It is conformally invariant,

$$A[e^{2\sigma(x)}g_{(0)}] = A[g_{(0)}]$$

This means that the fact of whether there is a conformal anomaly or not depends only on the conformal class of  $[g_{(0)}]$ .

The obstruction tensor is given by [de Haro, Solodukhin, KS (2000)]

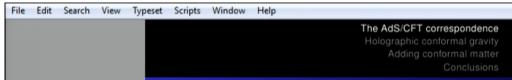
$$h_{(d)ij} \sim \frac{1}{\sqrt{g_{(0)}}} \frac{\delta A}{\delta g_{(0)}^{ij}}$$



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Conformal gravity and AdS/CFT

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Conformal gravity and AdS/CFT

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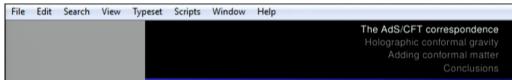
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Conformal gravity and AdS/CFT

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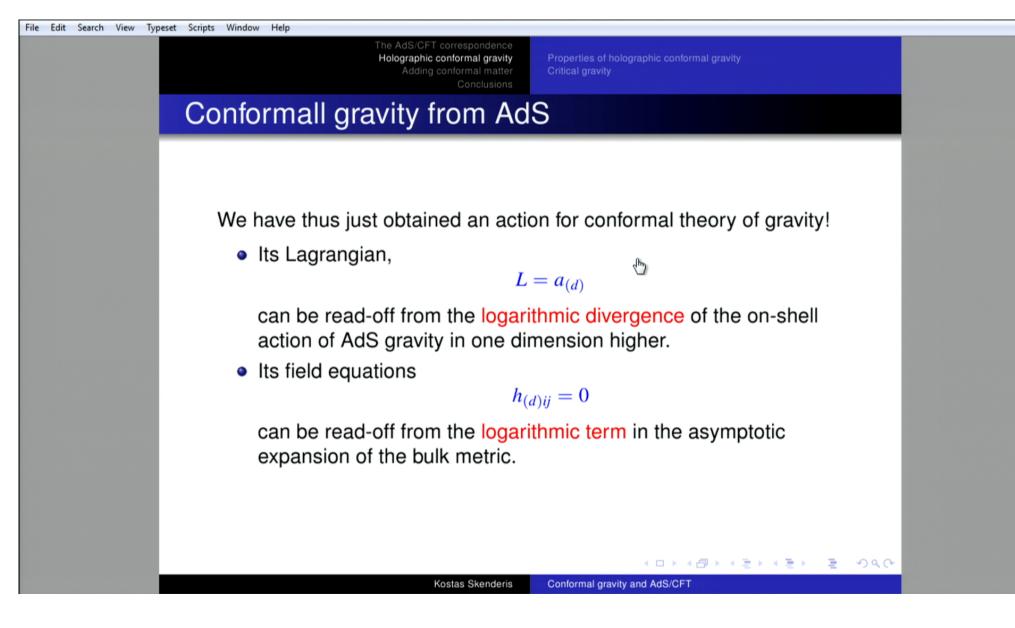
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- ➤ In even dimensions, a well-posed variatior problem where a conformal class is kept fixed, requires additional boundary terms and these are precisely the boundary counterterms.
- In odd dimensions, one must specify a representative  $g_{(0)}$  of the conformal class  $[g_{(0)}]$  for the variational problem to be well-posed. In this case the boundary counterterms ensure that the dependence of the theory on the specific  $g_{(0)}$  is governed only by the anomaly A, which itself depends only on the conformal class  $[g_{(0)}]$ .

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### Holographic conformal gravity in d=4

In this case we get [Henningson, KS (1998)]

$$L=a_{(4)}\sim R_{ij}R^{ij}-\frac{1}{3}R^2$$

This can be expressed also as

$$a_{(4)} \sim E_4 + W_{ijkl} W^{ijkl}$$

where  $E_4$  is the Euler density and  $W_{ijkl}$  is the Weyl tensor.

 This is also exactly equal to the conformal anomaly of N = 4 SYM, providing a highly non-trivial check of the AdS/CFT correspondence.



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Conformal gravity and AdS/CFT

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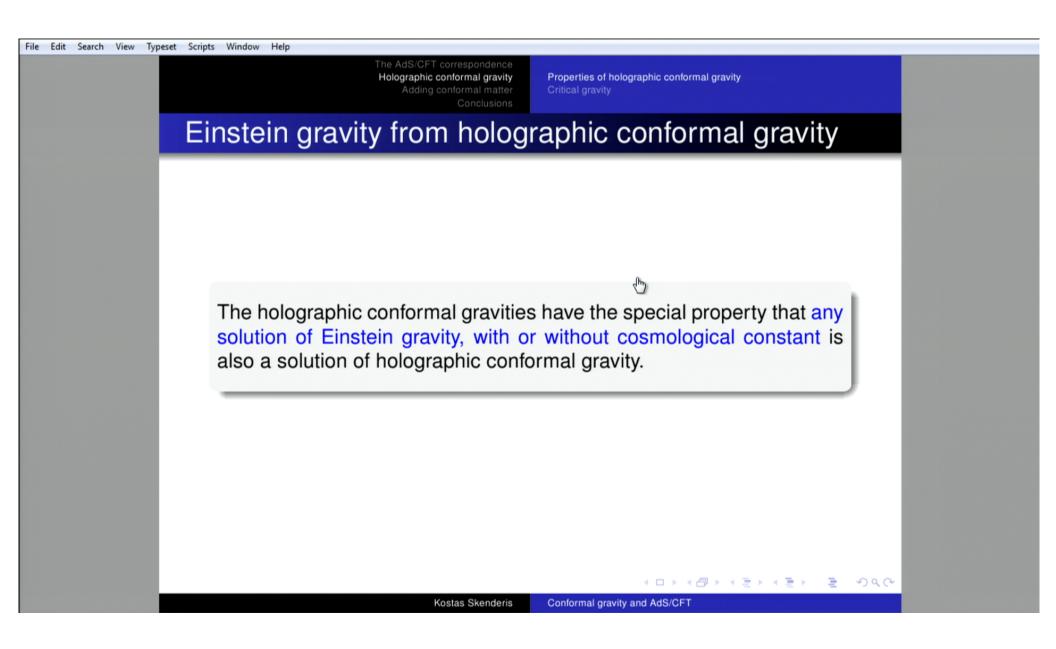
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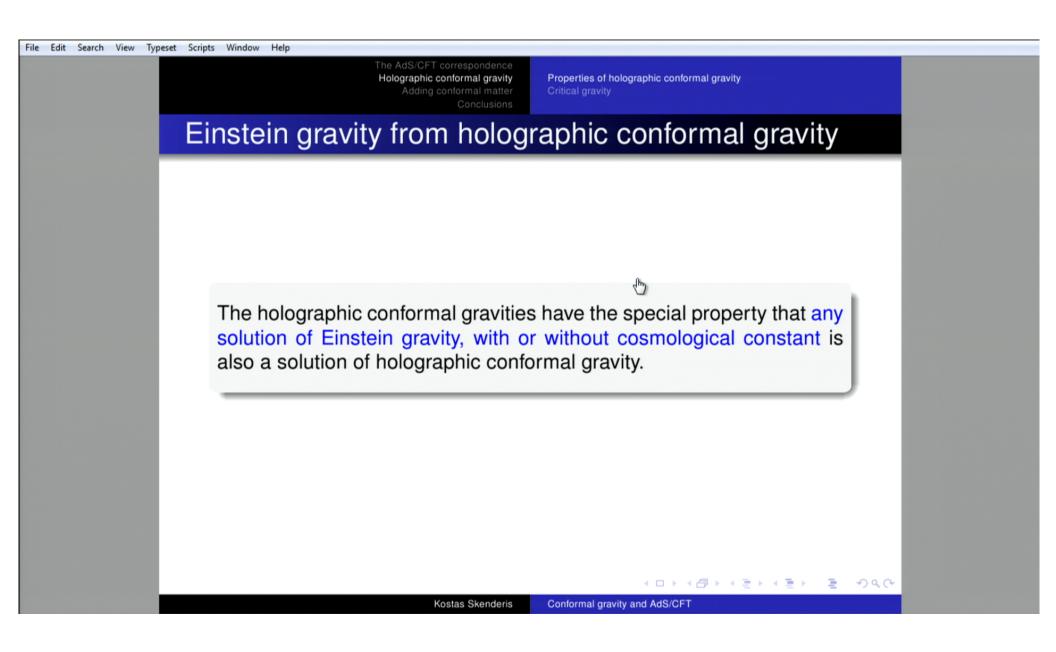
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File Edit Search View Typeset Scripts Window Help The AdS/CFT correspondence Holographic conformal gravity Pheperties of the robbe or malanomaly (2005)] The holographic conformal anomaly is now The counterterms that we just derived requiring finiteness of the on-shell action play another role, perhaps more fundamental. It has the gold was problem where a conformal class is kept fixed, requires additional boundary terms it is conformally invariant, and these are precisely the boundary counterterms. ightharpoonup In odd dimensions, one must specify a representative  $g_{(0)}$  of the conformal class  $[g_{(0)}]$  for the variational problem to be Weils-proseds that its easet to revolve the artheory is tertennis remains and that it threndependemonstrate three cynolouthreasplesision (graphs). governed • The bystraction relief is by the itself [depends only whither (2000)] conformal class  $[g_{(0)}]$ .  $h_{(d)ij} \sim \frac{1}{\sqrt{g_{(0)}}} \frac{\delta A}{\delta g_{(0)}^{ij}}$ 

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Conformal gravity and AdS/CFT

Kostas Skenderis

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## The variational problem [Papadimitriou, KS (2005)]

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#### Holographic proof [KS, unpublished (2006)]

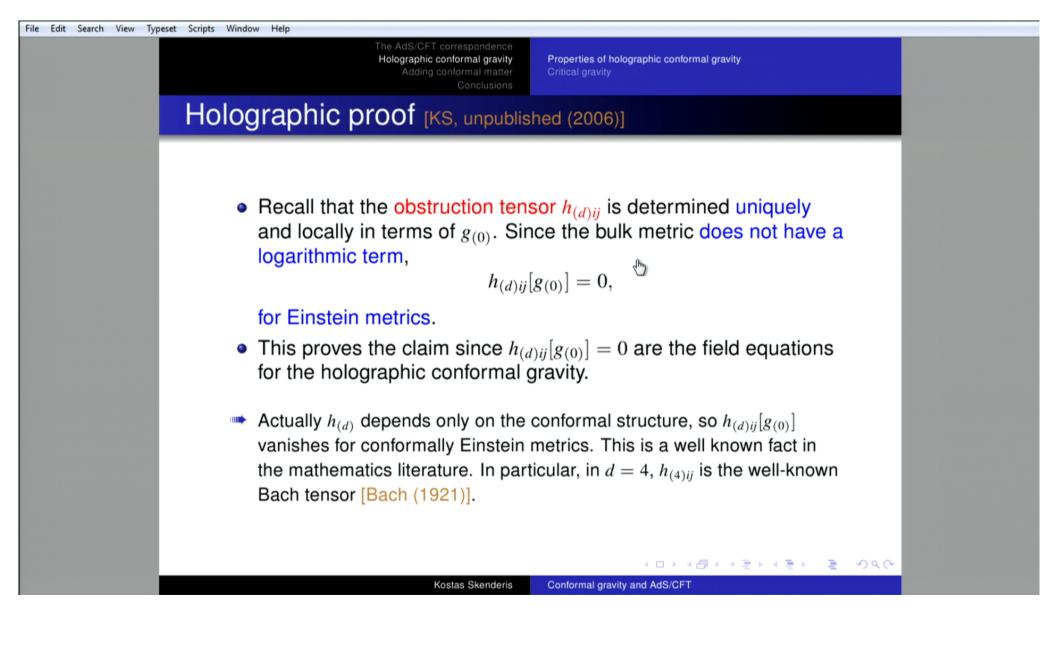
• One can prove by straightforward computation that the (d+1) dimensional metric,

$$ds^{2} = \frac{d\rho^{2}}{4\rho^{2}} + \frac{1}{\rho}(1 + \frac{\lambda}{4}\rho)^{2}g_{(0)ij}dx^{i}dx^{j}$$

is Einstein with negative cosmological constant, provided  $g_{(0)}$  is an Einstein metric in d dimensions,

$$Ric[g_{(0)}] = \lambda(d-1)g_{(0)}, \qquad \lambda = \pm 1, 0$$

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Properties of holographic conformal gravity
Critical gravity

#### An alternative proof [KS, unpublished (2006)]

I will now present a more direct proof.

• First notice that in both d = 4 and d = 6 the Lagrangian is at least quadratic in  $R_{ii}$ ,

$$L_{4} = R_{ij}R^{ij} - \frac{1}{3}R^{2}$$

$$L_{6} = \frac{1}{2}RR_{ij}R^{ij} - \frac{3}{50}R^{3} - R^{ij}R^{kl}R_{ikjl} + \frac{1}{5}R^{ij}D_{i}D_{j}R - \frac{1}{2}R^{ij}\Box R_{ij} + \frac{1}{20}R\Box R$$

 So it is manifest that Ricci-flat metrics solve the corresponding field equations.



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Conformal gravity and AdS/CFT

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Properties of holographic conformal gravity
Critical gravity

#### An alternative proof [KS, unpublished (2006)]

So we now focus on the case  $g_{(0)}$  is Einstein with a cosmological constant (of any sign).

 One can show that in both case the Lagrangian can be written in the following way

 $L = E_{ij}K^{ijkl}E_{kl} + L_0$ 

where  $L_0$  is the Einstein-Hilbert Lagrangian (with cosmological constant) and  $E_{ii}$  are the corresponding Einstein equations.

• The tensor  $K^{ijkl}$  is given by

$$d = 4 : K^{ijkl} = d_1 g^{ij} g^{kl} + d_2 g^{ik} g^{jm}$$

$$d = 6 : K^{ijkl} = c_1 R^{ikjl} + g^{ik} g^{jl} (c_2 R + c_3 \square + c_4)$$

$$+ g^{il} g^{jk} (c_5 R + c_6 \square + c_7) + c_9 g^{kl} D^i D^j$$

where  $d_1, d_2$  and  $c_1, \ldots, c_8$  are specific numerical coefficients.



Kostas Skenderis

Properties of holographic conformal gravity
Critical gravity

#### An alternative proof [KS, unpublished (2006)]

So we now focus on the case  $g_{(0)}$  is Einstein with a cosmological constant (of any sign).

 One can show that in both case the Lagrangian can be written in the following way

 $L = E_{ij}K^{ijkl}E_{kl} + L_0$ 

where  $L_0$  is the Einstein-Hilbert Lagrangian (with cosmological constant) and  $E_{ii}$  are the corresponding Einstein equations.

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Critical gravity

#### An alternative proof [KS, unpublished (2006)]

I will now present a more direct proof.

• First notice that in both d = 4 and d = 6 the Lagrangian is at least quadratic in  $R_{ij}$ ,

$$L_{4} = R_{ij}R^{ij} - \frac{1}{3}R^{2}$$

$$L_{6} = \frac{1}{2}RR_{ij}R^{ij} - \frac{3}{50}R^{3} - R^{ij}R^{kl}R_{ikjl} + \frac{1}{5}R^{ij}D_{i}D_{j}R - \frac{1}{2}R^{ij}\Box R_{ij} + \frac{1}{20}R\Box R$$

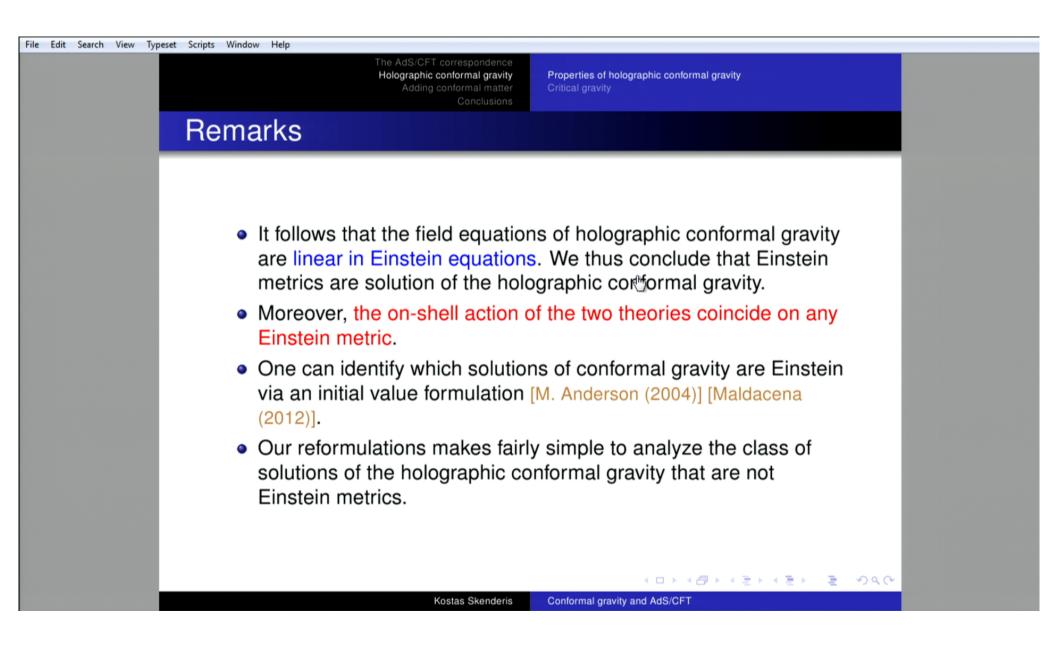
 So it is manifest that Ricci-flat metrics solve the corresponding field equations.



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Conformal gravity and AdS/CFT

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Properties of holographic conformal gravity

Critical gravity

### Critical gravity

There has recently been renewed interest in higher derivative gravities,

$$S = \int d^4x \sqrt{g} [R - 2\Lambda + \alpha R^{ij} R_{ij} + \beta R^2]$$

- For general  $\alpha$ ,  $\beta$  this theory describes the propagation of a massless spin 2 field, a massive spin 2 field and a massive scalar.
- [Lü, Pope (2011)] observed that the spectrum of linearized perturbations around AdS is special when:
- $\alpha = -3\beta$ : the massive scalar is absent.
- When in addition,  $\beta = -1/(2\Lambda)$  the massive graviton becomes massless. This theory was dubbed "critical gravity".



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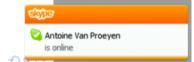


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Conformal gravity and AdS/CFT

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Properties of holographic conformal gravity

Critical gravity

### Critical gravity as the square of Einstein equations

It is easy to understand these results (and all other special properties of this theory). When  $\alpha = -3\beta$ ,

$$S = \int d^4x \sqrt{g} [R - 2\Lambda - 3\beta (R^{ij}R_{ij} - \frac{1}{3}R^2)]$$

and the higher derivative term is proportional to holographic conformal anomaly. Using our rewriting of the anomaly we obtain

$$S = \int d^4x \left( (6\lambda) E_{ij} K^{ijkl} E_{kl} + (1 + \beta 2\Lambda) L_0 \right)$$

In critical gravity,  $1 + \beta 2\Lambda = 0$ , and the theory is the "square" of Einstein field equations.



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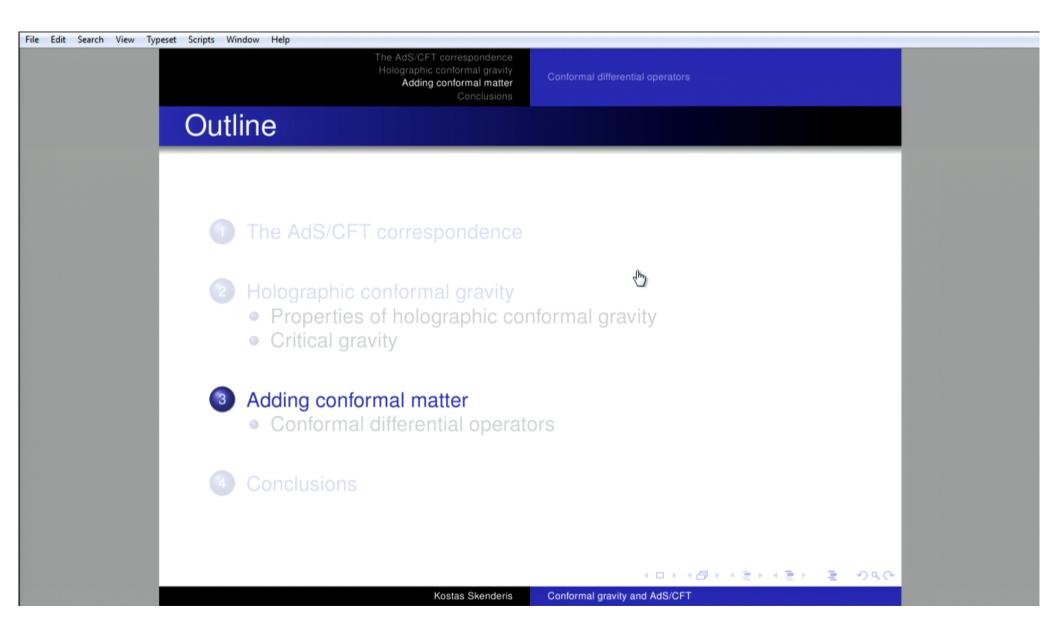
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Conformal differential operators

### Conformal gravity coupled to conformal matter

We would now like to understand how to couple matter to the conformal gravity.

- This can be done simply by considering AdS gravity coupled to matter. Of course, not all bulk matter fields will lead to conformal matter couplings in the boundary.
- Their mass should be such that the boundary field (source of dual operator) has dimension that allows for interaction terms of dimension d.
- ightharpoonup Example: a scalar field  $\Phi$  of mass  $m^2 = -3$  in  $AdS_5$ .
- The dual field has dimension  $\Delta = 3$  and thus its source  $\phi_{(0)}$  has dimension  $d - \Delta = 1$ .
- Terms of the form  $\phi_0 \square \phi_0$ ,  $\phi_0^4$  have dimension 4.

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Conformal differential operators

### Holographic construction

The holographic construction is the same as in the case of pure gravity. We focus here in the case the matter is a scalar field.

 We start by finding the general asymptotic solution of gravity coupled to matter. The metric has the same asymptotics as before. The scalar field has the expansion de Haro, Solodukhin, KS],

$$\Phi(x,\rho) = \rho^{(d-\Delta)/2}(\phi_{(\mathbf{0})}(\mathbf{x}) + \rho\phi_{(\mathbf{2})} + \dots + \rho^{\Delta-d/2}(\phi_{(\mathbf{2}\Delta-d)} + \log \rho\psi_{(\mathbf{2}\Delta-d)}) + \dots)$$

- Computing the on-shell action one finds again that contains a logarithmic divergence with coefficients  $a_{(d)}$ , and  $A = \int a_{(d)}$  is conformally invariant.
- Furthermore [de Haro, Solodukhin, KS],

$$h_{(d)ij} \sim \frac{1}{\sqrt{g_{(0)}}} \frac{\delta A}{\delta g_{(0)}^{ij}}, \qquad \psi_{(2\Delta-d)} \sim \frac{1}{\sqrt{g_{(0)}}} \frac{\delta A}{\delta \phi_{(0)}}$$

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### Conformal differential operators

- Conformal differential operators are differential operators whose conformal transformations depends on  $\sigma(x)$ , but not on derivatives of  $\sigma$ .
- An example is the conformal Laplacian we just discussed,

$$P_1 \equiv \Box - \frac{d-2}{4(d-1)}R.$$

It transforms as

$$P_1 \to e^{-(d/2+1)\sigma(x)} P_1 e^{-(d/2-1)\sigma(x)}$$

• An active area of research in mathematics is about the form and properties of such operators [Graham, ....].



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Conformal gravity and AdS/CFT

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## Constructing conformal differential operators

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• A scalar conformal operator of weight (d/2 + k) transforms as

$$P_k \rightarrow e^{-(d/2+k)\sigma(x)} P_k e^{-(d/2-k)\sigma(x)}$$

This is the conformal differential operator corresponding to the *k*th power of the Laplacian.

- Up until recently only the k=1,2 operators were known explicitly. To explicitly construct the remaining operators one may consider a scalar field of mass  $m^2 = -(d/2)^2 + k^2 < 0$  and compute the log divergence of the on-shell action.
- One can use this method to construct non-scalar conformal operators as well. For example, in d=4 we get the Maxwell operator from a gauge field in the bulk.



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#### Conclusions

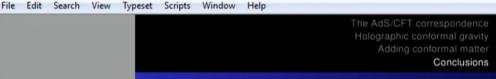
- We discussed how to obtain conformal gravity coupled to conformal matter using the AdS/CFT correspondence.
  - One starts from AdS gravity coupled to matter fields in (d + 1) dimensions and work out the asymptotic solution to the field equations. This amounts to solving algebraic equations.
  - If there is a logarithmic term in the asymptotic solution of a given bulk field then its coefficient is the corresponding field equation of conformal gravity coupled to conformal matter.
  - The logarithmic holographic counterterm is the action for conformal gravity coupled to conformal matter.
- We discuss discussed several special properties of these theories.
  - We derived a relation between holographic conformal gravity and Einstein gravity.
  - We used this to explain the special properties of critical gravity.
  - We showed how to construct conformal differential operators



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