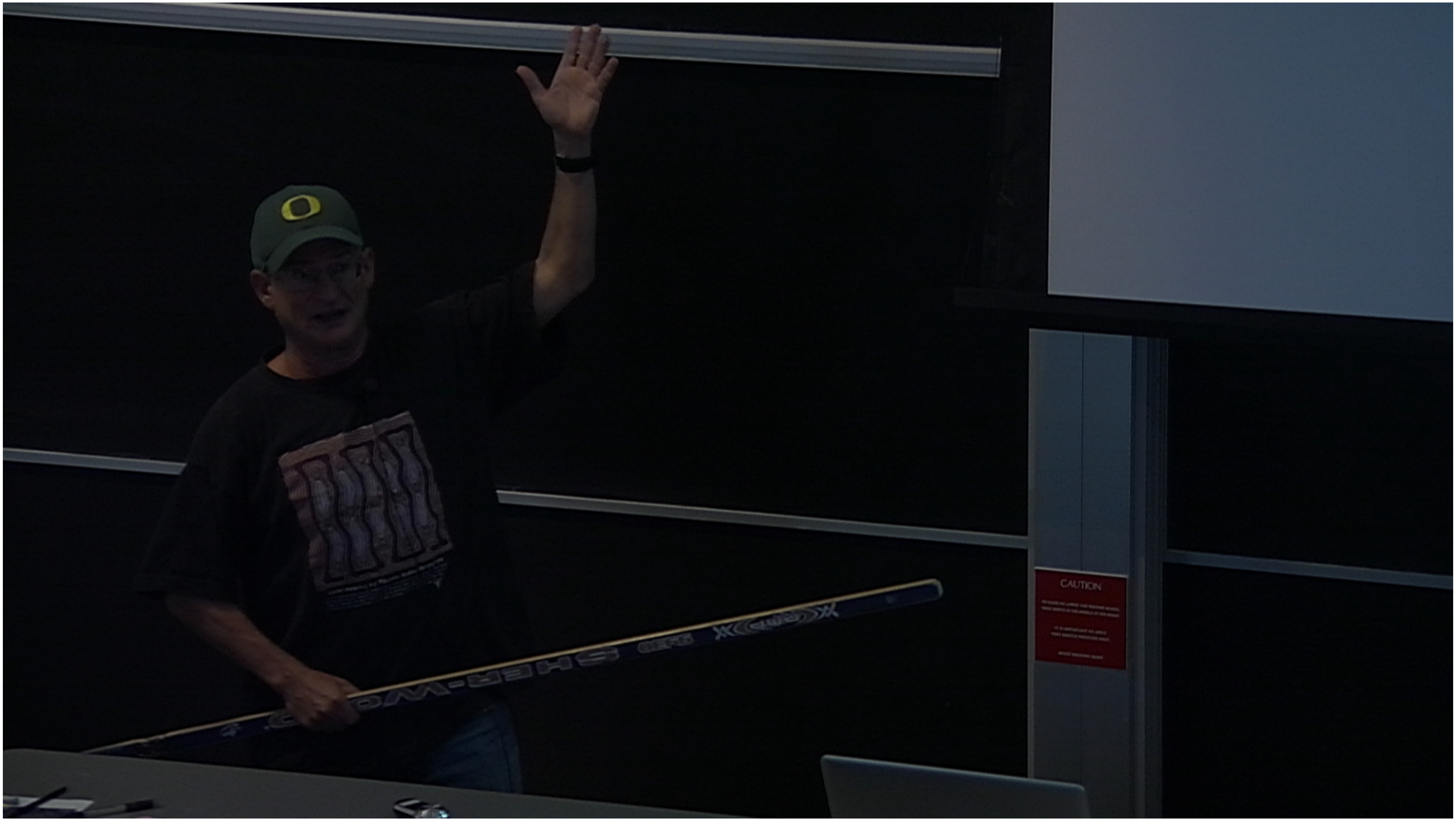
Title: The Conformal Method and Solutions of the Einstein Constraint Equations: A Status Report

Date: May 11, 2012 09:50 AM

URL: http://pirsa.org/12050058

Abstract: The Conformal Method (as well as the closely related Conformal Thin Sandwich Method) has proven to be a very useful procedure both for constructing and for parametrizing solutions of the Einstein initial data constraint equations, for initial data sets with constant mean curvature (CMC). Is this true for non CMC data sets as well? After reviewing the CMC results, we discuss what we know and don't know about non CMC initial data sets and the effectiveness of the Conformal Method in handling them.



The Conformal Method
&
Solutions of the Einstein Constraint Eqn's: A Status Report

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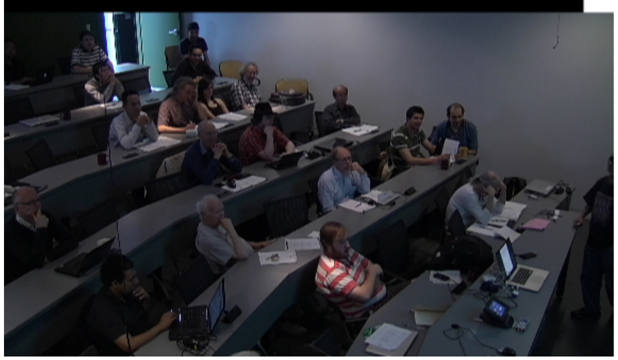
May, 2012



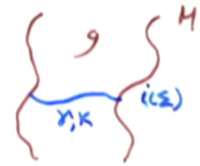
Einstein Constraint Equations

Where They Come From:

If (M,g) is spacetime sol'n

$$G(g) = \kappa T(\psi, g)$$

& $i : \Sigma \to M$ is spacelike



then (Σ, γ, K) satisfy

$$R(\gamma) - K K + (\mathrm{tr} K)^2 = \rho(\psi, \gamma)$$
$$\nabla \cdot K - \nabla(\mathrm{tr} K) = J(\psi, \gamma)$$

← constraints

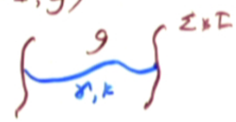
Pf: Gauss-Codazzi

Role in Cauchy Problem

If (Σ, γ, K) satisfies

$$R - KK + (\mathrm{tr} K)^2 = \rho$$
$$\nabla \cdot K - \nabla(\mathrm{tr} K) = J$$

then there exists $(\Sigma \times I, g)$



such that

- g satisfies
  $$G(g) = T(\psi, g)$$
- g induces γ, K

Pf: PDE Theory

Y. Choquet-Bruhat

So:

$\{$Sol'ns of Einstein's Eqns$\}$ ← globally hyperbolic

$\wr\wr$

$\{$Sol'ns of Einstein Constraints$\}$ / Choice of Slicing

¿What Do We know About Sol's $\{\Sigma^3, \gamma, K\}$?

- Parametrizing Them
- Constructing Them
- Local & NonLocal Structures $\{M, A, \ldots\}$

Key Analytical Tools:

- * Conformal Method
  - & Conformal Thin Sandwich
- * Gluing
- * Quasi Spherical

Focus on

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The Conformal Method

for Parametrizing...
Constructing...

Where the Idea Comes From:

Lichnerowicz 1944
Choquet-Bruhat
York

If we write ("decompose")

$$\gamma_{ab} = \phi^4 \lambda_{ab}$$

$$K_{cd} = \phi^{-2}(\sigma_{cd} + LW_{cd}) + \phi^4 \lambda_{cd} \tau$$

for σ_{cd} satisfying $\nabla \cdot \sigma = 0$, $tr\,\sigma = 0$

& $LW_{cd} = \nabla_c W_d + \nabla_d W_c - \frac{2}{3}\lambda_{cd} \nabla \cdot W$

then

$$R - KK + (tr K)^2 = 0$$

$$\nabla \cdot K - \nabla(tr K) = 0$$

becomes

$$\Delta \phi = R\phi - (\sigma + LW)^2 \phi^{-7} + \tau^2 \phi^5$$

$$\nabla^a (LW)_{ab} = \phi^6 \nabla_b \tau$$

Turn this around into

Conformal Procedure for Constructing Sol'ns:

The Conformal Method

for Parametrizing...
Constructing...

Where the Idea Comes From:

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If we write ("decompose")

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Turn this around into

Conformal Procedure for Constructing Sol'ns :

Conformal Method

How to Do It

- Choose Free Data on Σ^3

  λ_{ab} Riem metric

  σ_{cd} Sym 2-tensor

  Trace-free $\lambda^{ab}\sigma_{ab} = 0$

  Div-free $\nabla^a \sigma_{ab} = 0$

  τ function

- Solve on Σ^3

$$\nabla^a (LW)_{ab} = \phi^6 \nabla_b \tau$$

$$\Delta \phi = R\phi - (\sigma_{ab} + LW_{ab})(\sigma^{ab} + LW^{ab})\phi^{-7} + \tau^2 \phi^5$$

  for $\phi > 0$

  W_a

  Determined system

- Reconstruct

$$\gamma_{ab} = \phi^4 \lambda_{ab}$$

$$K_{cd} = \phi^{-2}(\sigma_{cd} + LW_{cd}) + \phi^4 \lambda_{cd} \tau$$

$\Rightarrow$ Soln

¿ Does It Always Work?

No

ex $\Sigma^3 = S^3$

λ = Round

$\sigma = 0$

$\tau = 1$

$\Rightarrow \nabla \cdot LW = 0$

$\Rightarrow LW = 0$

$\Rightarrow \Delta\phi = \phi^5$

No Solution!

Key Question:

¿ For which $(\Sigma^3, \lambda, \sigma, \tau)$ does it work?

Catalog of Many Cases:

Manifold & Asymptotics

- Closed
- Asympt Euclidean
- Asympt hyperbolic
- with Boundary
- Asymp Cylindrical

Regularity

- Analytic
- C^{∞}, C^{k}
- Weak

  $C^{k+\alpha}$, H^{p}_{k}
- Asympt Falloff

  $C^{k+\alpha}_{\delta}$, $H^{p}_{k,\delta}$

Added Fields

- Vacuum
- Einstein Maxwell
- Einstein Fluid
- Einstein Scalar

Conformal Data Classes

- CMC, Near-CMC, Non-CMC
- Yamabe class of λ_{ab}
- $\sigma \equiv 0$ or $\sigma \not\equiv 0$

What's known & Not known:

Roughly

CMC & Near CMC → Known

Non CMC → Unknown,

except for...

Classic Results CMC

(Σ closed, C^∞, CMC, Vacuum)

	$\sigma \equiv 0$, $\tau = 0$	$\sigma \not\equiv 0$, $\tau = 0$	$\sigma \equiv 0$, $\tau \neq 0$	$\sigma \not\equiv 0$, $\tau \neq 0$
Yamabe +	N	Y	N	Y
Yamabe 0	Y	N	N	Y
Yamabe −	N	N	Y	Y

CB
York
O'Murchadha
I

- Tools for Proof

• CMC & Semi-Decoupling

$\nabla \tau = 0$

$\longrightarrow \nabla \cdot L W = 0$ ← or J

linear eqn for W
with no ϕ coupling

$\Rightarrow$ Lichnero Eqn for ϕ

$$\Delta \phi = R\phi - (\sigma + LW)^2 \phi^{-7} + \tau^2 \phi^5$$

• Maximum Principle

$\Delta \phi > 0$ on closed Σ → No Soln

< 0 " " " → No Soln

Classic Results CMC

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	$\sigma \equiv 0$, $\tau = 0$	$\sigma \not\equiv 0$, $\tau = 0$	$\sigma \equiv 0$, $\tau \neq 0$	$\sigma \not\equiv 0$, $\tau \neq 0$
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$\Delta \phi > 0$ on closed Σ $\longrightarrow$ No Soln

< 0 " " " $\longrightarrow$ No Soln

- Yamabe Classification

$\mathcal{Y}^+ \rightarrow \exists\ \psi$ such that
$$R(\psi^4\lambda) > 0$$

$\mathcal{Y}^0 \rightarrow \exists\ \psi$ such that
$$R(\psi^4\lambda) = 0$$

$\mathcal{Y}^- \rightarrow \exists\ \psi$ such that
$$R(\psi^4\lambda) < 0$$

* All metrics in $\mathcal{Y}^+$ <u>or</u> $\mathcal{Y}^0$ <u>or</u> $\mathcal{Y}^-$

* Determine which one by eigvals

Yamabe Invar

- Sub & Super Sol'n Thm

Consider
$$\Delta\phi = F(x,\phi)$$

If $\exists\ \phi_+$ and ϕ_- such that

- $0 < \phi_- \leq \phi_+ < \infty$
- $\Delta\phi_- \geq F(x,\phi_-)$
- $\Delta\phi_+ \leq F(x,\phi_+)$

Then $\exists\ \phi$ such that

- $\phi_- \leq \phi \leq \phi_+$
- $\Delta\phi = F(x,\phi)$

- Yamabe Classification

$\mathcal{Y}^+ \rightarrow \exists\, \psi$ such that $R(\psi^4 \lambda) > 0$

$\mathcal{Y}^0 \rightarrow \exists\, \psi$ such that $R(\psi^4 \lambda) = 0$

$\mathcal{Y}^- \rightarrow \exists\, \psi$ such that $R(\psi^4 \lambda) < 0$

* All metrics in $\mathcal{Y}^+$ or $\mathcal{Y}^0$ or $\mathcal{Y}^-$

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- $\Delta \phi_+ \le F(x, \phi_+)$

Then $\exists\, \phi$ such that

- $\phi_- \le \phi \le \phi_+$
- $\Delta \phi = F(x, \phi)$

- Sketch of Proof

for Σ^3 closed, CMC

- $\nabla \tau = 0$

$\Rightarrow$ Lichnero Only!

$$\Delta\phi = R\phi - (\sigma + LW)^2\phi^{-7} + \tau^2\phi^5$$

- Conformal Covariance for CMC

Soln exists for $(\Sigma^3, \lambda, \sigma, \tau)$

$$\Updownarrow$$

Soln exists for $(\Sigma^3, \theta^4\lambda, \theta^{-2}\sigma, \tau)$

Combine with Yamabe to rewrite Lichero Eq as

$$\Delta\phi = \begin{pmatrix} +1 \\ 0 \\ -1 \end{pmatrix}\phi - (\sigma + LW)^2\phi^{-7} + \tau^2\phi^5$$

- All No Cases

$\Rightarrow \Delta\phi > 0$ or $\Delta\phi < 0 \Rightarrow$ No Solution

ex) $(\Sigma^3, y^+, \sigma \equiv 0, \tau \neq 0)$

$$\Delta\phi = \phi + \tau^2\phi^5$$

$$> 0$$

$\vdots$

- Yes Cases with $\sigma^2 \equiv 0$

$\Rightarrow$ Constant sol'ns

(2✗) $(\Sigma, \gamma^-, \sigma \equiv 0, \tau \neq 0)$

$$\Delta\phi = -\phi + \tau^2\phi^5$$

$\Rightarrow \phi = \sqrt[4]{|\tau|}$ is sol'n

- Yes Cases with $\sigma \not\equiv 0$

$\Rightarrow$ Use Sub & Super Sol'ns

(2✗) $(\Sigma, \gamma^-, \sigma \not\equiv 0, \tau \neq 0)$

$$\Delta\phi = -\phi - (\sigma + LW)^2\phi^{-7} + \tau^2\phi^5$$

ϕ_+ = Big C_+ $\Rightarrow$ Super Soln

ϕ_- = Small C_- $\Rightarrow$ Sub Soln

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Comment on

Parametrization of {Solns} by {Conf Data}

Idea

Would like 1-1 Onto map

Yes

for {CMC Solns on Σ closed}

$$\Updownarrow$$

{Yes Conf Data} / Conf Transfs & Scale Transfs

Conf Transfs

$$\{\lambda, \sigma, \tau\} \Rightarrow \{\theta^{4}\lambda, \theta^{-2}\sigma, \tau\}$$

Scale Transfs

$$\{\lambda, \sigma, \tau\} \Rightarrow \{A^{2}\lambda, A\sigma, \tfrac{1}{A}\tau\}$$

Not known for Non CMC

Comment on

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Not known for Non CMC

Comment on

Other CMC Results

- Asymptotically Euclidean

  known:

  Soln exists for $(\Sigma, \lambda, \sigma, \tau)$ iff $\lambda \in \mathcal{Y}^0$

  metrics conf to $R=0$

- Asympt Hyperbolic

  known

  Sol'n exists

- Einstein-Maxwell $\begin{cases} \text{closed} \\ \text{AE} \\ \text{AH} \end{cases}$

  known

$$\Delta\phi = R\phi - (\sigma + LW)^2 \phi^{-7} - (E^2 + B^2)\phi^{-3} + \tau^2 \phi^5$$

same as Vacuum

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- Einstein-Scalar

  partially known

$$\Delta\phi = (R - |\nabla\psi|^2)\phi - [(\sigma + LW)^2 + \pi^2]\phi^{-7} + (\tau^2 - \mathcal{V}(\psi))\phi^5$$

manageable

tricky if $\tau^2 - \mathcal{V} < 0$

CMC mostly understood

Non CMC

much less

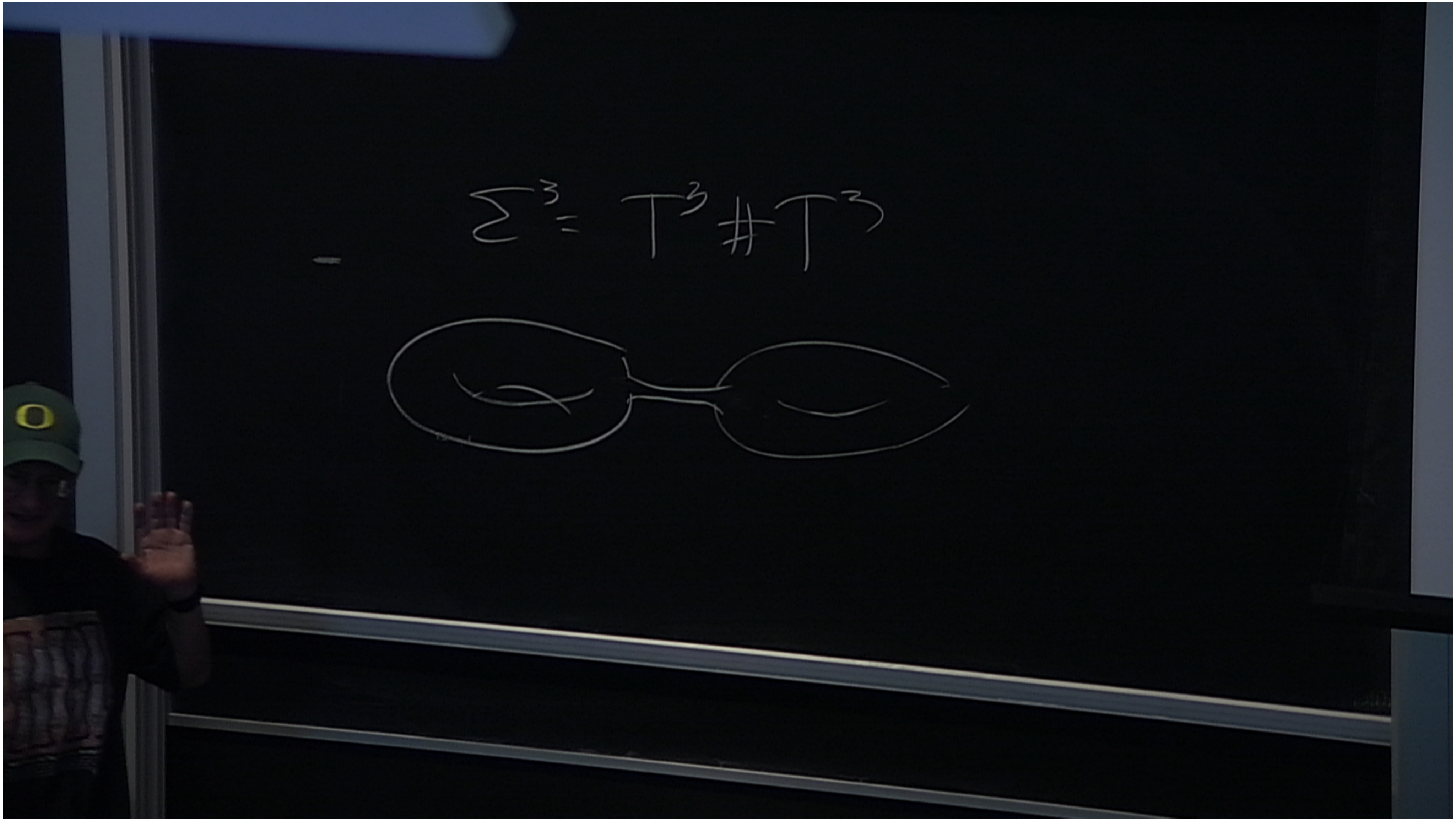
¿Why consider non CMC?

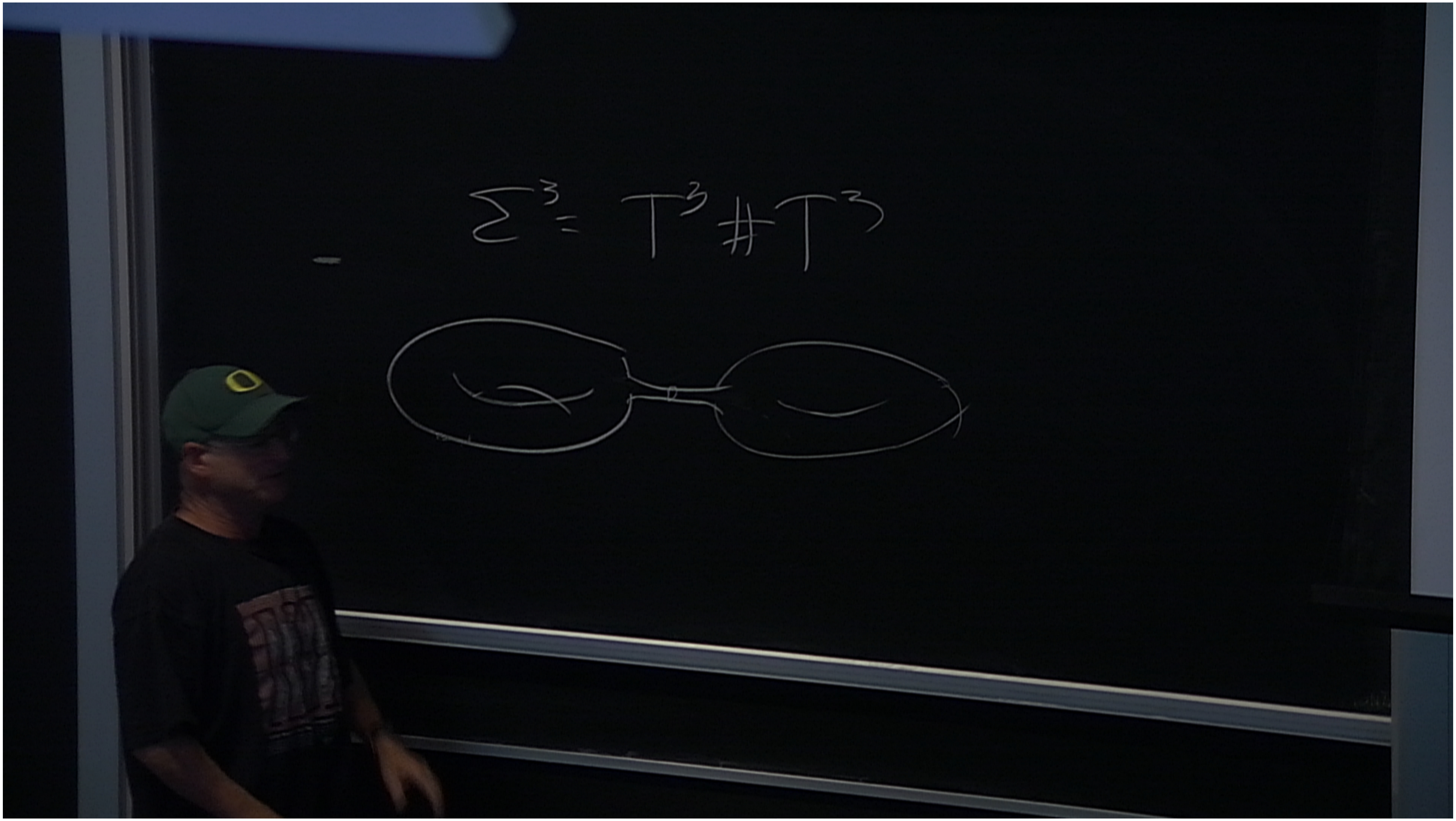
Thm

$\exists$ globally hyperbolic sol'ns with no CMC slices

Start with

Near CMC


$\Sigma^3 = T^3 \# T^3$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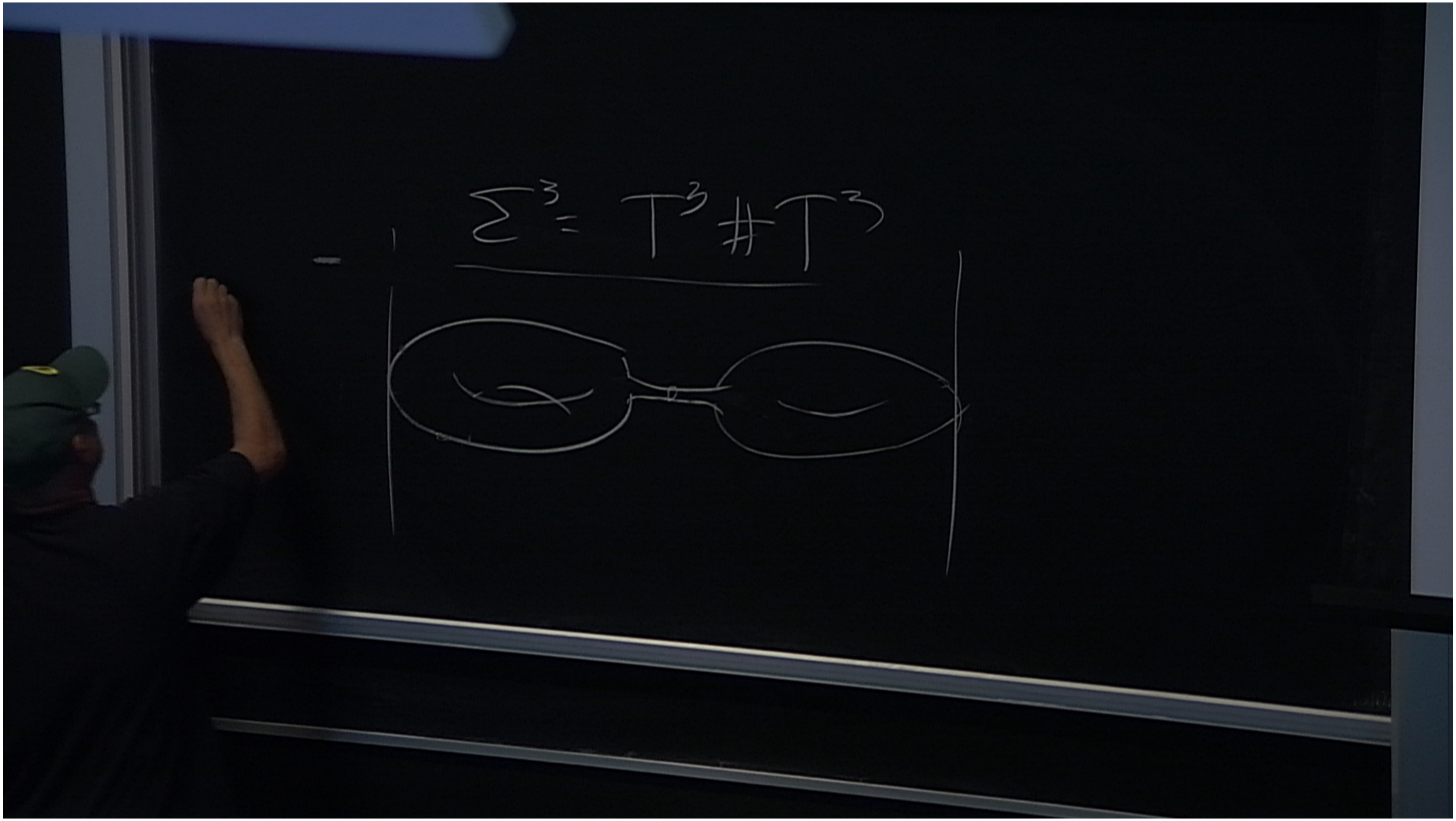
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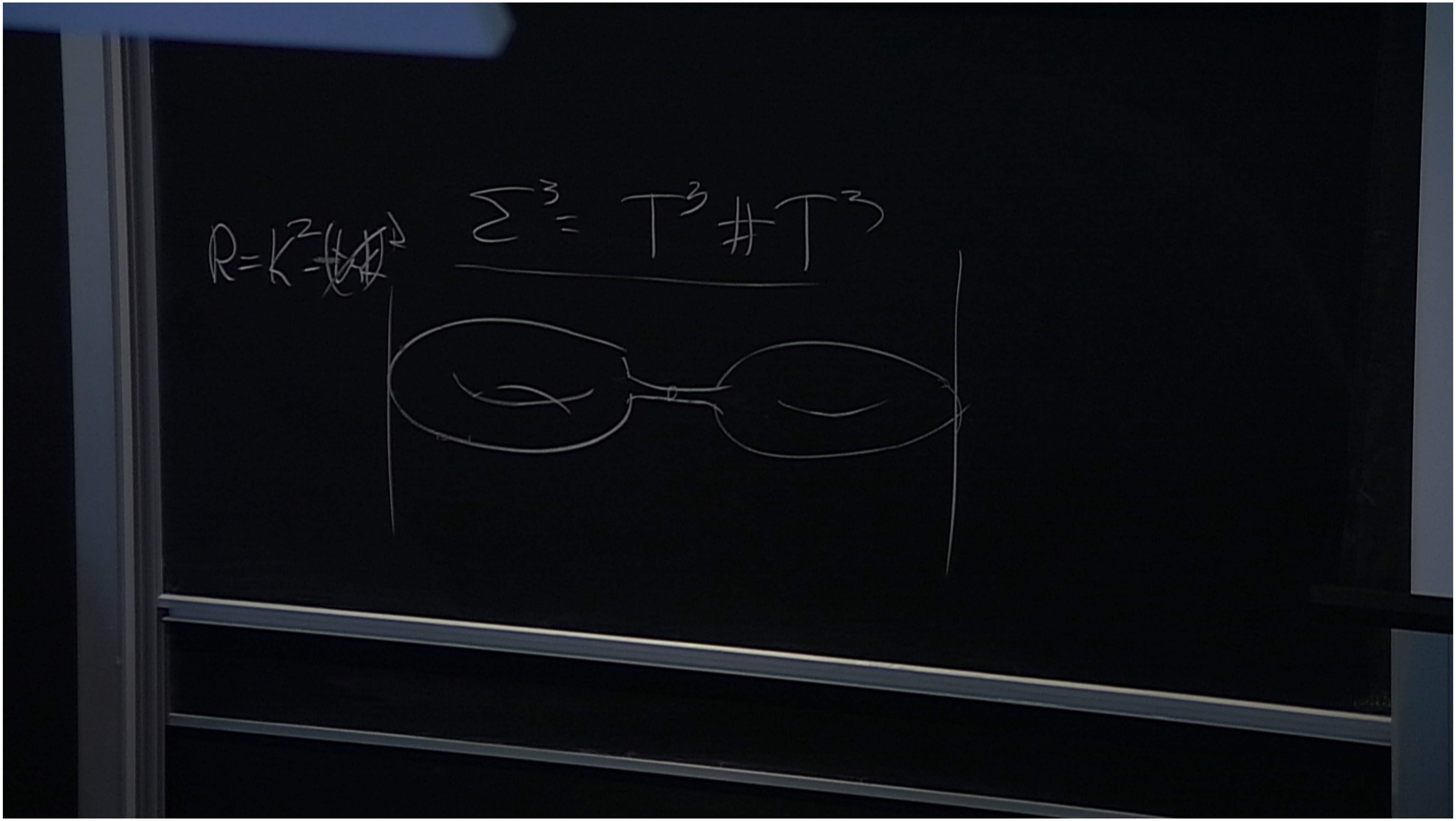
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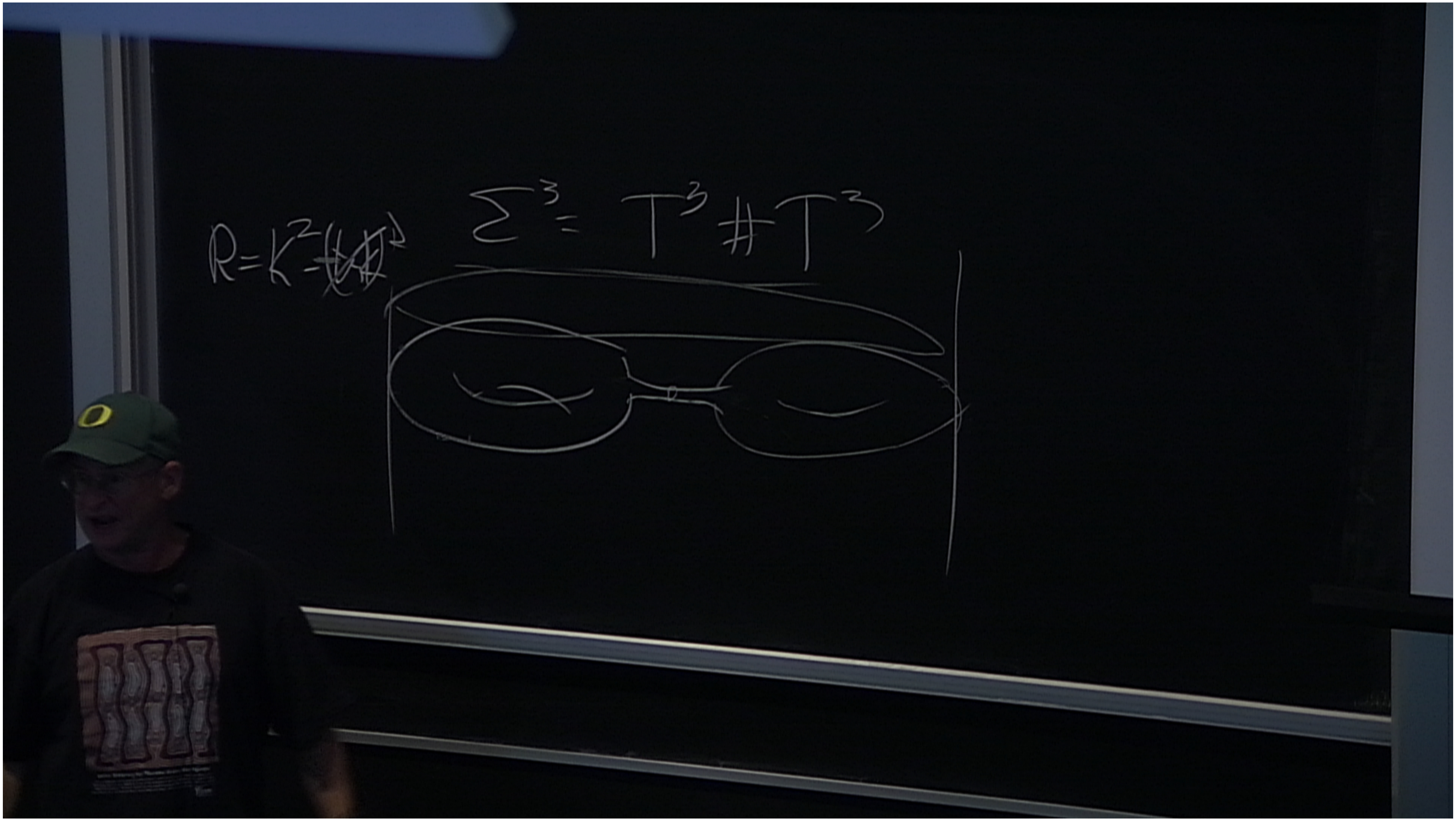
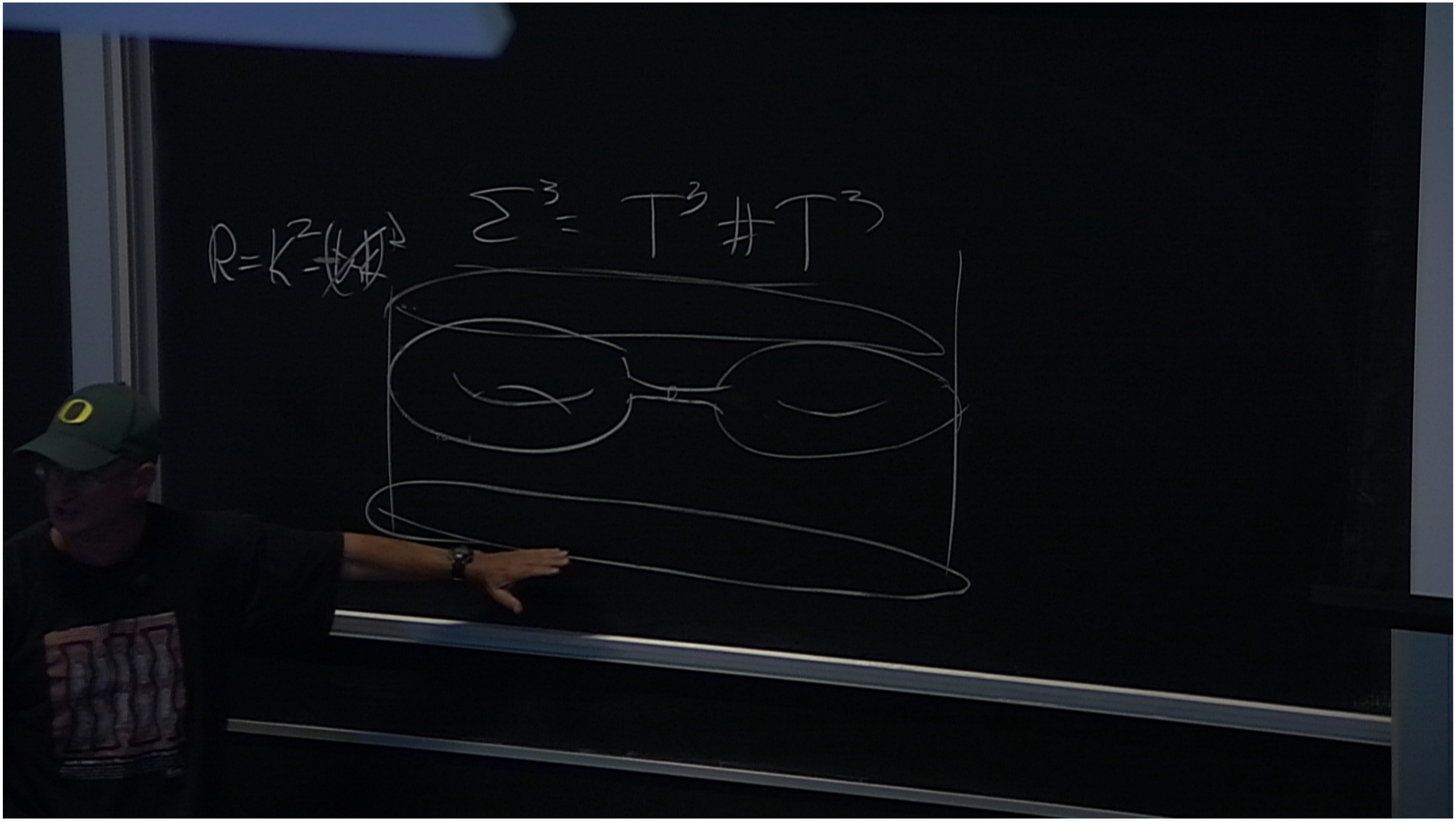


$R = k^2 - \cancel{(k\#)}^2$
$\Sigma^3 = T^3 \# T^3$


$\Sigma^3 = T^3 \# T^3$

- Einstein-Scalar

Partially known

$$\Delta\phi = (R - |\nabla\psi|^2)\phi - \left[(\sigma + LW)^2 + \pi^2\right]\phi^{-7}$$
$$+ (\tau^2 - \mathcal{V}(\psi))\phi^5$$

manageable

tricky if $\tau^2 - \mathcal{V} < 0$

CMC mostly understood

Non CMC
much less

¿Why consider non CMC?

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$\exists$ globally hyperbolic sol'ns with no CMC slices

Start with
Near CMC

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Near-CMC

- Meaning:

  $|\nabla\tau|/\tau$ small

- Why It's Harder

  <u>No</u> deccupling

$$\Delta\phi = +R\phi - (\sigma + LW)^2\phi^{-7} + \tau^2\phi^5$$

$$(\nabla\cdot L)W = \phi^6\,\nabla\tau$$

- Why It's Manageable

$$(\nabla\cdot L)W = \phi^6\,\nabla\tau$$

$$\Rightarrow |LW| < C\phi^6|\nabla\tau|$$

$$\Rightarrow \Delta\phi \doteq R\phi - (\sigma)^2\phi^{-7} + (\underbrace{\tau^2 - C|\nabla\tau|}_{>0})\phi^5$$

- Known & Unknown

  Mostly known <u>if</u> $\sigma^2 \not\equiv 0$

Table: Moncrief - I, O'Murchadha - I, Holst-Nagy-Tsot, Maxwell; Allen-Clausen-I, Dahl-Gicquaud-Humbert

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- Table

(Σ^3 closed, Neaf, CMC, Vac)

	$\sigma\equiv 0$, $\tau^2\not\equiv 0$	$\sigma\equiv 0$, $\tau^2>0$	$\sigma\not\equiv 0$, $\tau^2\not\equiv 0$	$\sigma\not\equiv 0$, $\tau^2>0$
$\mathcal{Y}^+$	?	N	Y	Y
$\mathcal{Y}^0$	?	N	Y	Y
$\mathcal{Y}^-$	Y ?	Y	Y ?	Y

- On the Proof
  - Yes Cases with $\tau^2>0$
    - Use "Constructive Gummel Method"

$$(\nabla\cdot L)W_n = \phi_n^6 \nabla\tau$$

$$\Delta\phi_{n+1} = R\phi_{n+1} - (\sigma + LW_n)^2\phi_{n+1}^{-7} + \tau^2\phi_{n+1}^5$$

&

- Master Sub & Super Solns

$$\Delta\phi_+ \leq R\phi_+ - (\sigma+LW)^2\phi_+^{-7} + \tau^2\phi_+^5$$

for $\underline{\underline{\text{all}}}$ W such that

$$(\nabla\cdot L)W = \phi^6\nabla\tau$$

with $0<\phi\leq\phi_+$

15 $\Rightarrow$ Converge to Soln

- Yes Cases with $\tau^2 \not> 0$

  → Use "Schauder Compactness Procedure"

  for

$$\mathcal{F} : \mathcal{U} \to \mathcal{U}$$

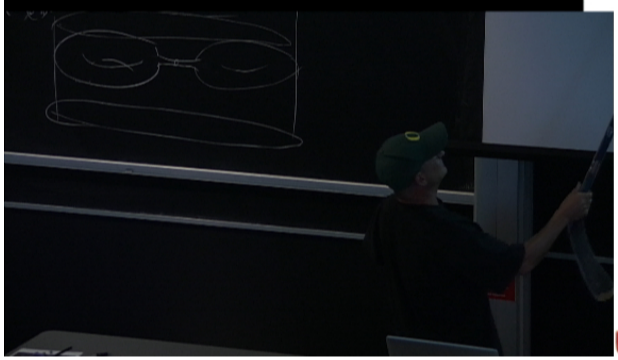
$$\phi \longmapsto \Delta^{-1}\left(R\phi - [\sigma + L(\nabla \cdot L^{-1} \phi^6 \nabla\tau)]^2 \phi^{-7} + \tau^2 \phi^5\right)$$

  Technical Conditions

  $\Rightarrow \exists$ sol'n (not necessarily unique)

- No Cases

  Proof by contradiction . . .



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Comment on

Other Near-CMC

- Asymptotically Euclidean

Known

Sol'n exists for $(\Sigma, \lambda, \sigma, \tau)$ iff $\lambda \in \mathcal{Y}^0$

- Asymptotically Hyperbolic

Known

Sol'n exists

¿ What about

Non CMC

with no restrictions on $\nabla\tau$

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Non CMC

Not much known

But

Some recent progress

- Σ^3 Closed, $\mathcal{Y}^+$, σ^2 small

  swap small $\nabla\tau$

  for small σ^2 (but nonzero)

  Holst
  Nagy
  Tsogtgerel
  Maxwell

  $\Rightarrow$ Sol'n exists

  Proof

  Schauder compactness

- Σ^3 Closed, $\tau^2 > 0$

  $\Rightarrow$ For dense set of λ
  sol'n exists

  Proof

  Dahl
  Gicquaud
  Humbert

  Subcritical Method

  $$\Delta\phi = R\phi - (\sigma + LW)^2\phi^{-7} + \tau^2\phi^5$$

  $$(\nabla\cdot L)W = \phi^{6-\varepsilon}\nabla\tau$$

  $$\varepsilon \to 0$$

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$$(\nabla\cdot L)W = \phi^{6-\varepsilon}\nabla\tau$$

$$\varepsilon \to 0$$

18

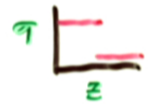
• Maxwell's Horrible Toys

$\Sigma = T^3$

λ = flat

σ is T^2 symmetric

τ is T^2 symmetric step function



$\Longrightarrow$ Cases with

- No Sol'n
- Multiple Sol'n
- 1 Sol'n

¿ Problem for Conformal Method ?

?

14

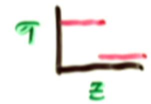
- Maxwell's Horrible Toys

  $\Sigma = T^3$

  λ = flat

  σ is T^2 symmetric

  φ is T^2 symmetric step function



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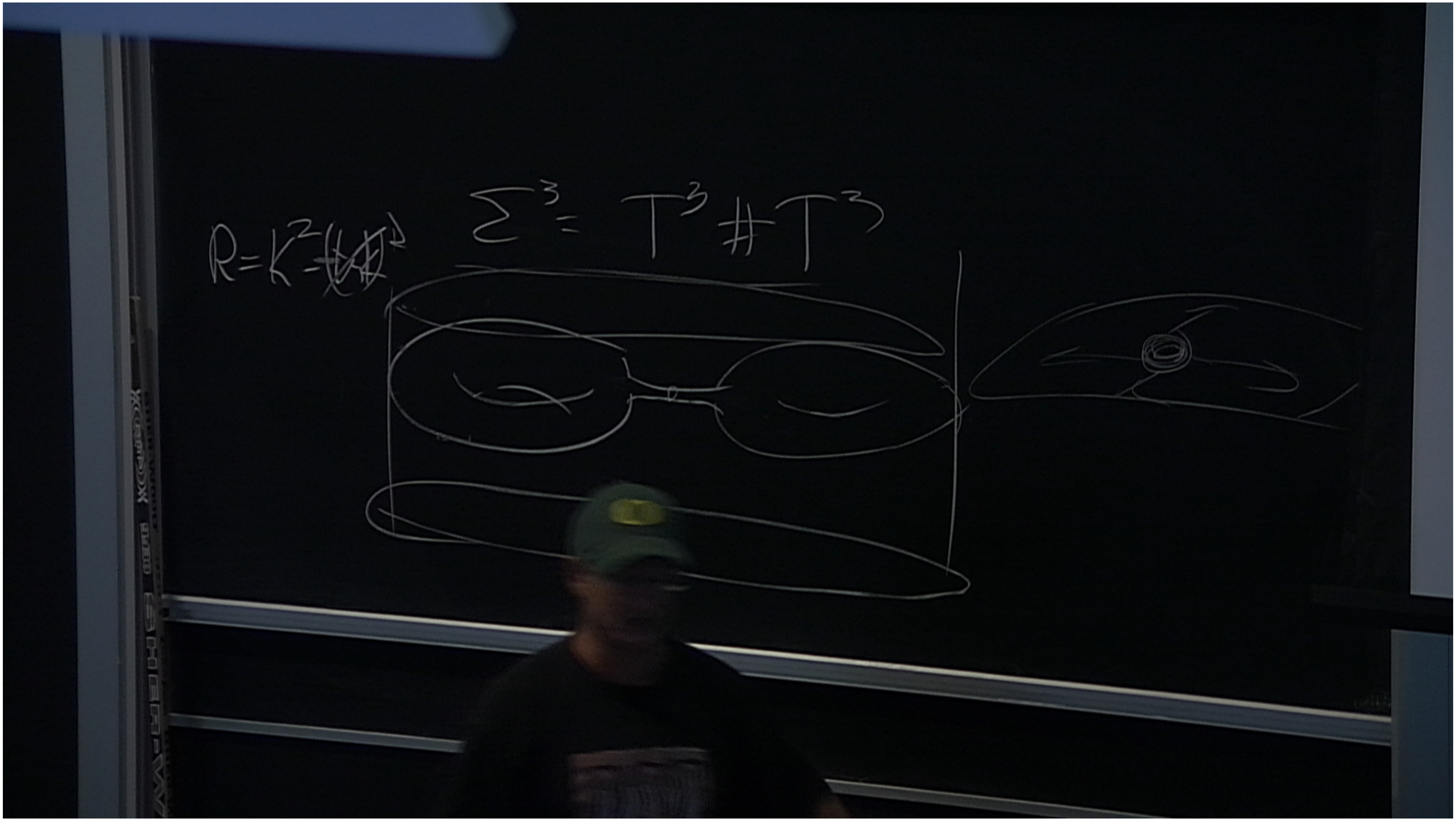
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