Title: Shape Dynamics and General Relativity

Date: May 09, 2012 02:00 PM

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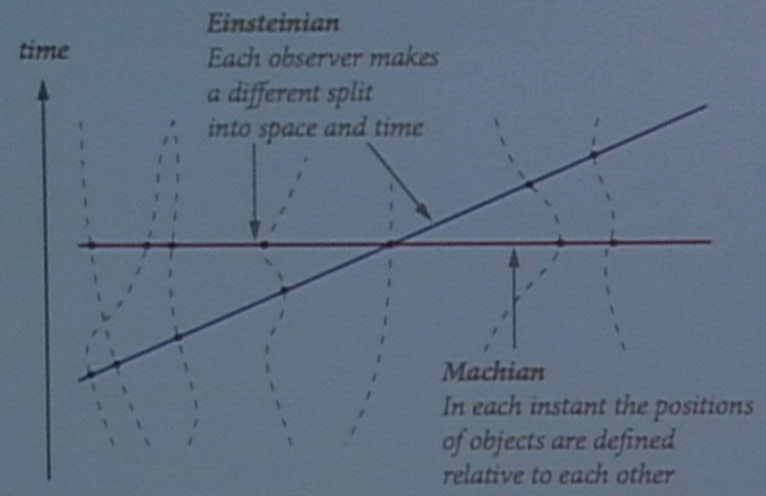
Abstract: Shape Dynamics first arose as a theory of particle interactions formulated without any of Newton's absolute structures. Its fundamental arena is shape space, which is obtained by quotienting Newton's kinematic framework with respect to translations, rotations and dilatations. This leads to a universe defined purely intrinsically in relational terms. It is then postulated that a dynamical history is determined by the specification in shape space of an initial shape and an associated rate of change of shape. There is a very natural way to create a theory that meets such a requirement. It fully implements Mach's principle and shows how time and local inertial frames are determined by the universe as whole. If the same principles are applied to a spatially closed universe in which geometry is dynamical, they lead rather surprisingly to a theory that, modulo some caveats, is dynamically equivalent to general relativity but dual to it in that refoliation invariance is traded for three-dimensional conformal invariance. This shows that there is a hidden three-dimensional conformal symmetry within general relativity. It is in fact what underlies York's crucial method of solution of the initial-value problem in general relativity. It is also remarkable that, as in York's work, shape dynamics inescapably introduces a mathematically distinguished notion of absolute simultaneity, the desirability of which has been found in two currently popular approaches to quantum gravity: causal dynamical triangulations and Horava gravity. I aim to express the key ideas and techniques of shape dynamics as simply as possible.

The Two Notions of Relativity



Systematic Machian relativity leads to a theory of gravity dual - and largely equivalent to - General Relativity

Machian relativity presupposes simultaneity

Preferred terminology: spatial and temporal relationalism (E. Anderson's talk)

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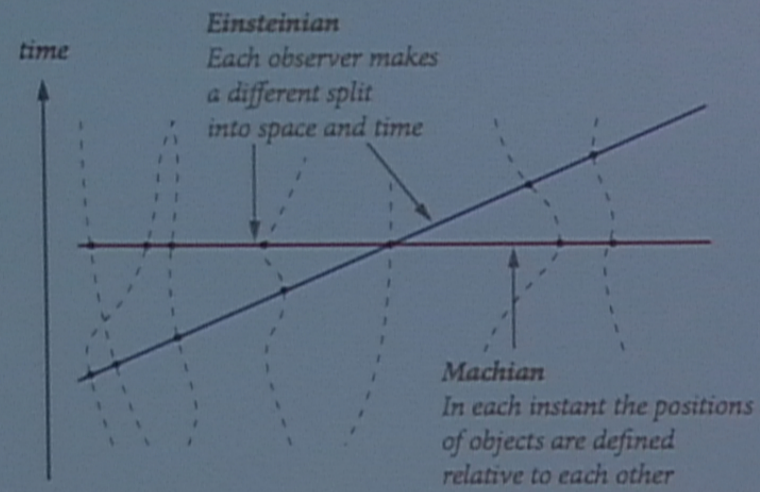

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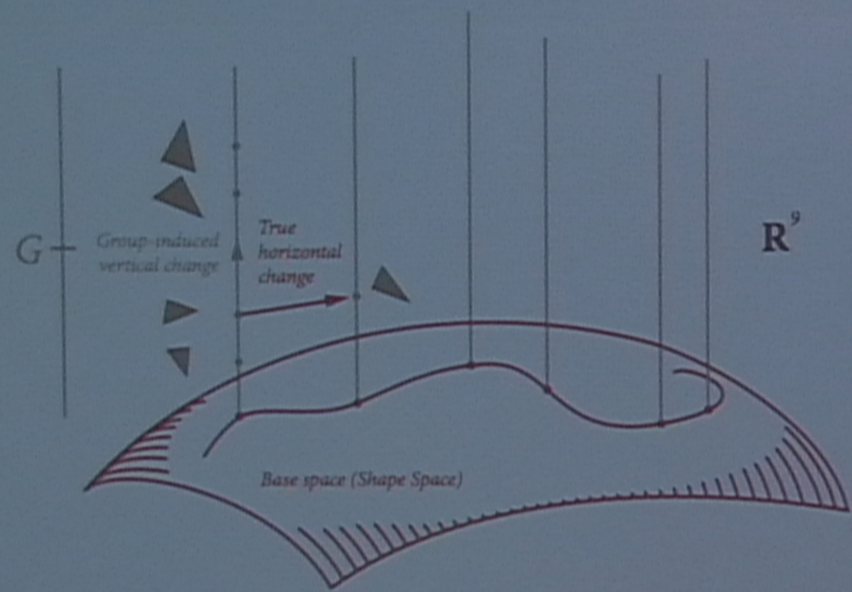
THE 3-PARTICLE CONFIGURATION SPACE AS FIBRE BUNDLE



The *structure group G* generates fictitious vertical change.

Nature generates real horizontal change. Defining it is *the central problem.*

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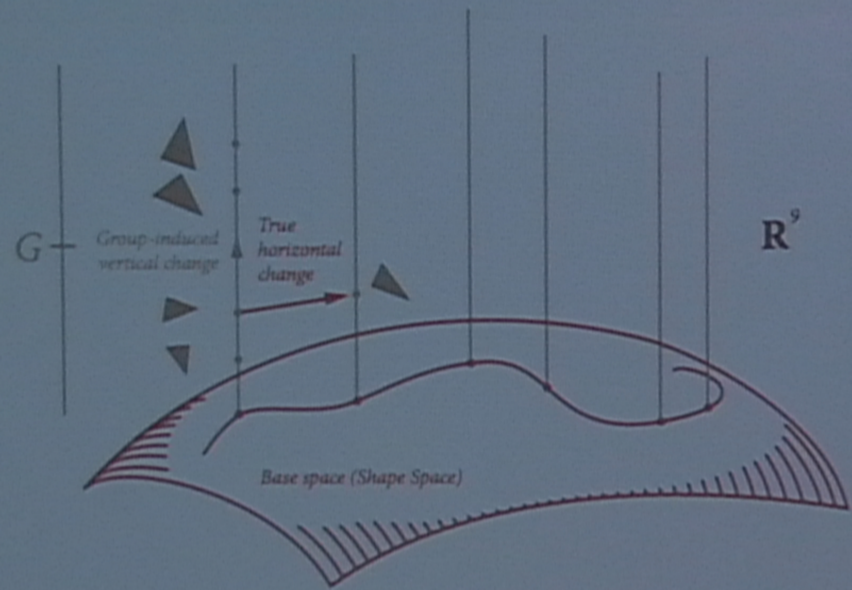

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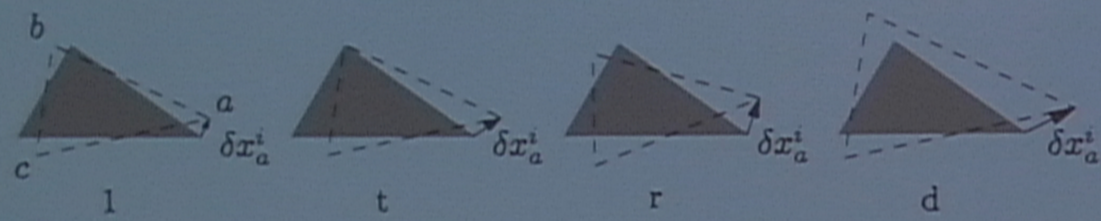
The problems with Newtonian dynamics



Rotations (r) and dilatations (t) relative to the initial placing (1)

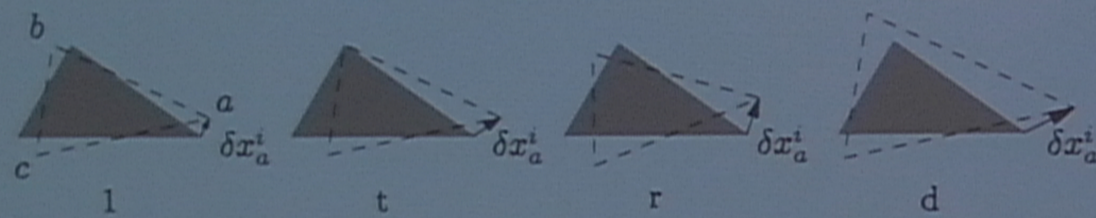
do not change the observable initial data in Shape Space

but do change the subsequent evolution.

By Galilean invariance, translations (t) have no effect.

3

THE PROBLEMS WITH NEWTONIAN DYNAMICS



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The Mach–Poincaré principle

Strong Mach–Poincaré Principle:

A point and a direction in Shape Space determine the evolution uniquely.

Weak Mach–Poincaré Principle:

A point and a tangent vector in Shape Space determine the evolution uniquely.

[Isenberg & Wheeler]

I will call *Shape Dynamics* any theory satisfying either form of that principle.

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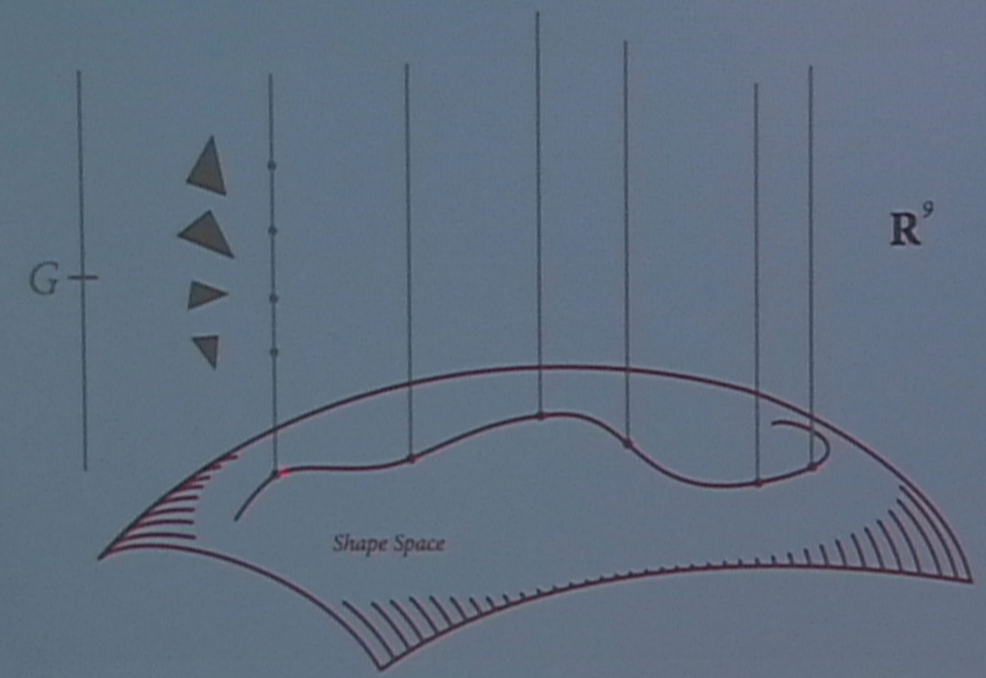
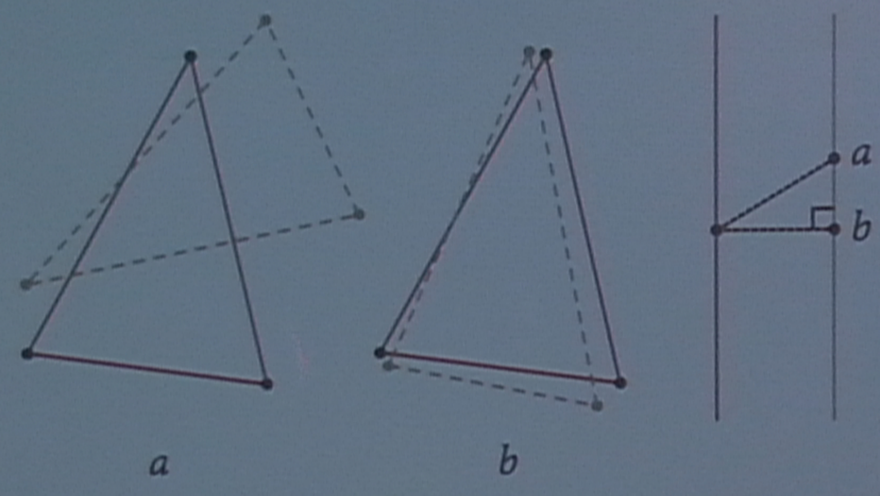
Implementing the Strong Mach–Poincaré Principle



Define a metric and hence geodesic principle on Shape Space using the canonically defined metric on the fibre bundle.

5

Creation of Metric on Shape Space by Best Matching



Arbitrary (a) and best-matched (b) placings of dashed triangle.

$\delta s_{\mathrm{bm}} := \min_{\delta \mathbf{x}_a} \sqrt{W(\mathbf{r}_{ab}) \sum_a \frac{m_a}{2} \delta \mathbf{x}_a \cdot \delta \mathbf{x}_a}$ between orbits. $[W] = l^{-2}$

Background independence: no reliance on position in space or external time

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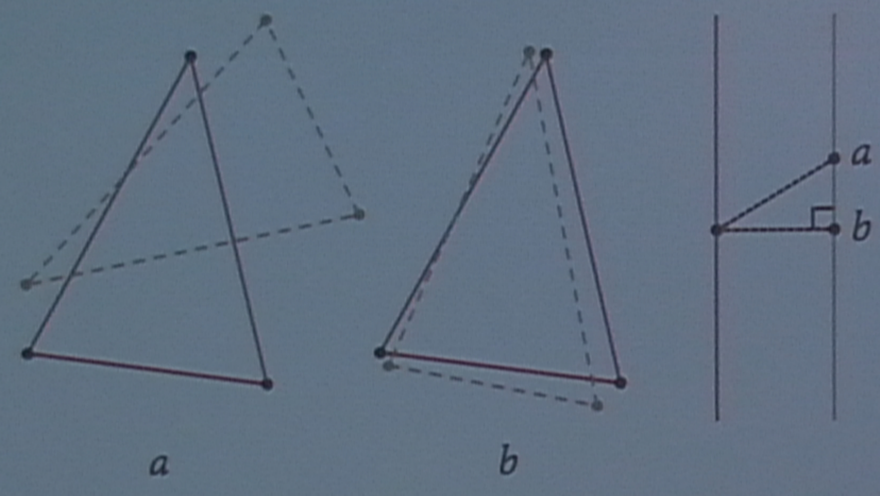

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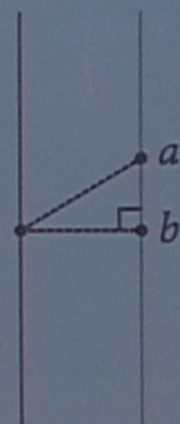
The Shape-Dynamic action

$$A_{\mathrm{SD}} = 2\int d\lambda\, \sqrt{W\,T}$$

$$T = \sum_a \frac{m_a}{2} \left\| \frac{\mathrm{d}\mathbf{x}_a}{\mathrm{d}\lambda} - \mathbf{v} - \boldsymbol{\omega} \times \mathbf{x}_a - \phi\, \mathbf{x}_a \right\|^2$$

$$\mathbf{p}_a = \sqrt{\frac{W}{T}}\, m_a \left(\frac{\mathrm{d}\mathbf{x}_a}{\mathrm{d}\lambda} - \mathbf{v} - \boldsymbol{\omega} \times \mathbf{x}_a - \phi\, \mathbf{x}_a \right)$$



[J. Barbour, CQG **20** (2003), gr-qc/0211021v2]

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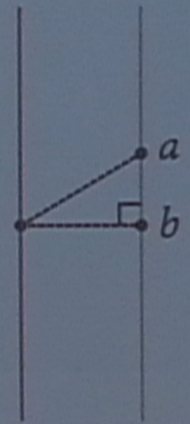

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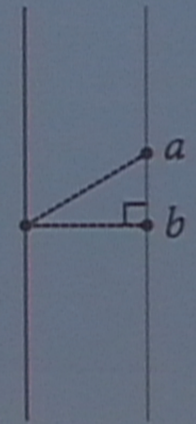

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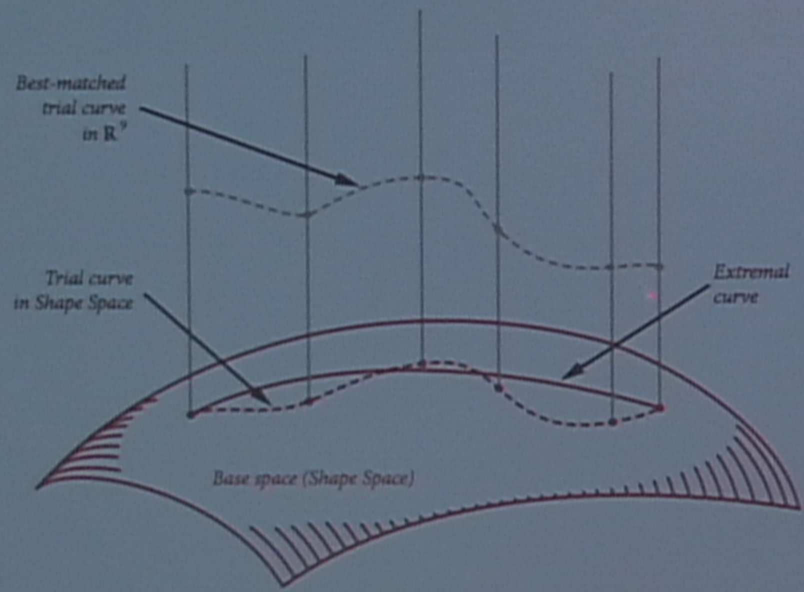
Machian free-end-point variation



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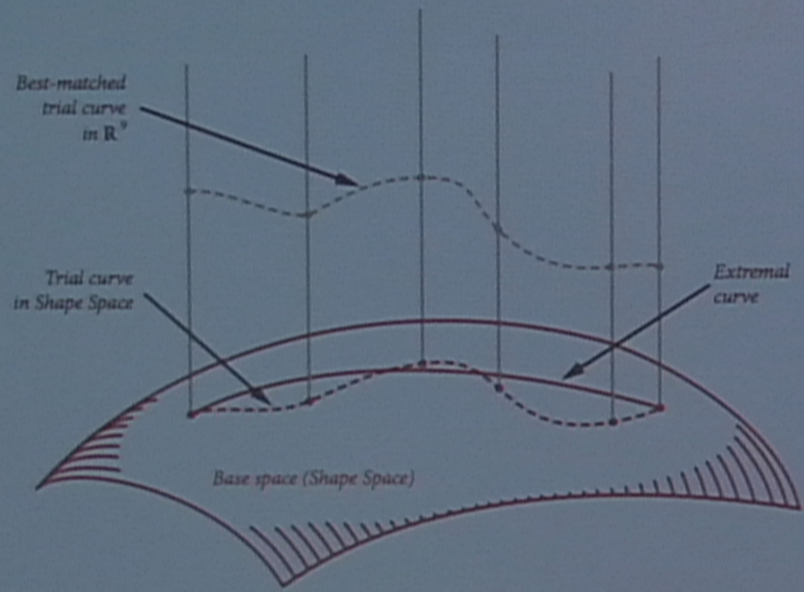

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The Linear and Quadratic Constraints

The best-matched momenta $\mathbf{p}_a^{\mathrm{bm}}$ satisfy

$$\mathbf{P}^{\mathrm{bm}} := \sum_a \mathbf{p}_a^{\mathrm{bm}} = 0 \ \ \text{(translational bm)}$$

$$\mathbf{L}^{\mathrm{bm}} := \sum_a \mathbf{x}_a \times \mathbf{p}_a^{\mathrm{bm}} = 0 \ \ \text{(rotational bm)}$$

$$D^{\mathrm{bm}} := \sum_a \mathbf{x}_a \cdot \mathbf{p}_a^{\mathrm{bm}} = 0 \ \ \text{(dilatational bm)}$$

$$\sum_a \frac{m_a}{2} \mathbf{p}_a^{\mathrm{bm}} \cdot \mathbf{p}_a^{\mathrm{bm}} = W \ \ \text{(quadratic geodesic constraint)}$$

"The Universe is given only once, with its relative motions alone determinable"
(Ernst Mach, 1883).

Enlargement of structure group diminishes true motions.

Choice of structure group converts Mach's intuition into theory.

9

Emergence of Inertial Frames and Newtonian Dynamics

[Barbour & Bertotti, Proc. R. Soc. **A 382** (1982)]

Jacobi-type translationally and rotationally invariant action:

$$A_{\mathrm{J}} = 2\int \mathrm{d}\lambda \sqrt{(E-V)T}, \qquad T = \sum_a \frac{m_a}{2}\frac{\mathrm{d}\mathbf{x}_a^{\mathrm{bm}}}{\mathrm{d}\lambda}\cdot\frac{\mathrm{d}\mathbf{x}_a^{\mathrm{bm}}}{\mathrm{d}\lambda}$$

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Simplify by choosing λ such that always $T = E - V$. Then

$$\frac{\mathrm{d}^2\mathbf{x}_a^{\mathrm{bm}}}{\mathrm{d}t^2} = -\frac{\partial V}{\partial \mathbf{x}_a}$$

Newtonian dynamics emerges from Machian dynamics. For an ‘island’ universe $\mathbf{P} = \mathbf{L} = 0$. This restriction does not apply to subsystems.

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Emergence of Time

Newtonian dynamics emerged through choice of λ such that always $T = E - V$. Then

$$dt := \sqrt{\frac{\sum_a m_a \, d\mathbf{x}_a^{\rm bm} \cdot d\mathbf{x}_a^{\rm bm}}{2(E-V)}}.$$

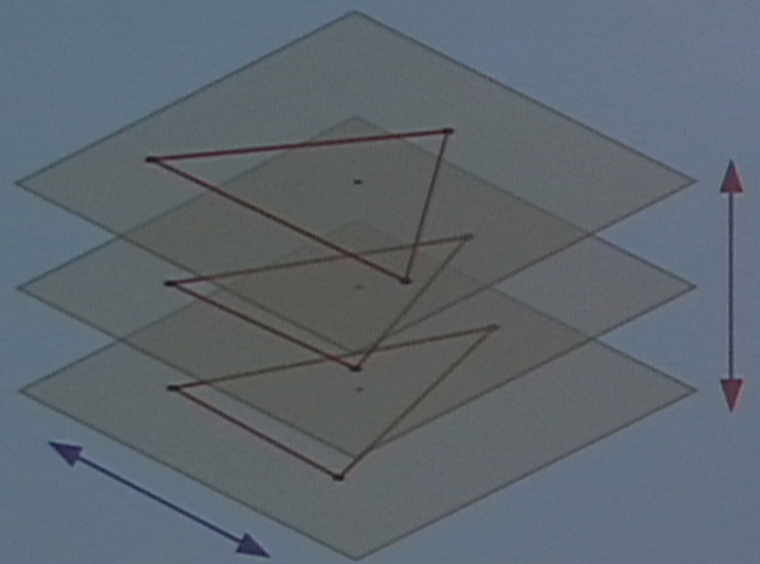
"It is utterly impossible to measure the changes of things by time. Quite the contrary, time is an abstraction at which we arrive from the changes of things." (Ernst Mach, 1883)

Time is the 'distillation' of all the changes in the Universe.

Doubly holistic.

Is E an initial condition or a universal constant?

The Construction of Newtonian Spacetime



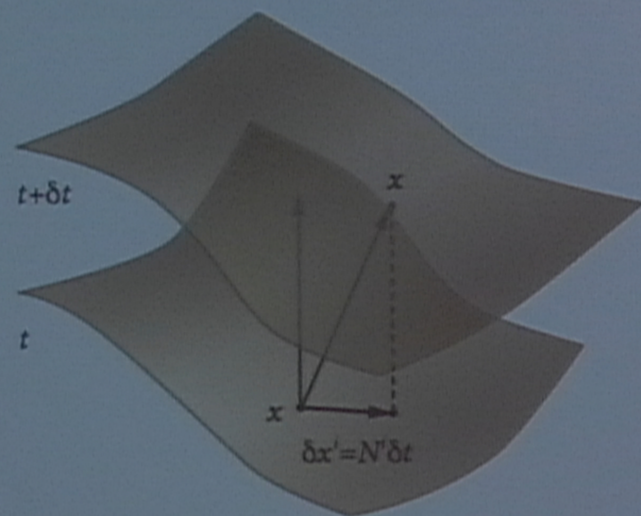
Best matching effects horizontal stacking.

Emergent time fixes vertical separations.

This *distinguished representation* reflects true physics and simplifies the equations.

12

THE ADM 3+1 DECOMPOSITION



$$^{(4)}g_{00} = {}^{(3)}g_{ij}N^iN^j - N^2, \qquad {}^{(4)}g_{0j} = N_k \qquad {}^{(4)}g_{ij} = {}^{(3)}g_{ij}$$

Best-matching wrt 3-diffeos and 3D conformal transformations

13

Two Forms of Scale Invariance

Strong form $\quad \mathbf{P}=\mathbf{L}=0 \;\&\; D=0 \quad \Longrightarrow \quad E=0.$

Point and direction in shape space determine evolution.

Weaker form $\quad \mathbf{P}=\mathbf{L}=0 \;\;\&\;\; E=0$ imposed.

Point and tangent vector in shape space determine evolution.

Evolution curve determined by specification of two points s_1 and s_2 in shape space and dimensionless *ratio* Y_2/Y_1 at them.

$$Y=\frac{D}{\sqrt{I_{\rm cm}}}, \qquad D=\sum_a m_a \mathbf{x}_a\cdot\mathbf{p}_a, \qquad I_{\rm cm}=\frac{1}{M}\sum_{a<b} m_a m_b r_{ab}^2=\sum_a m_a \mathbf{x}_a^{\rm cm}\cdot\mathbf{x}_a^{\rm cm},$$

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THE DYNAMICS OF GEOMETRY

1854: Riemann introduces metric geometry: $g_{ij}(x)$.

1870: Clifford: "I hold that ... variation of the curvature of space is what really happens in that phenomenon which we call the *motion of matter*."

1872 and 1883: Mach insists all motion is relative.

1902: Poincaré formulates aim of a theory of relative motion.

1915: Einstein creates general relativity.

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SUPERSPACE

Riem $(\mathscr{M})$:= Space of all g_{ij} defined on the closed 3-manifold $\mathscr{M}$.

Any two 3-metrics g_{ij} that can be carried into each other by diffeomorphisms represent the same 3-geometry.

$$\text{Superspace} := \frac{\text{Riem}}{\text{3-diffeos}}$$

A 3-vector field $\xi_k(x)$ generates infinitesimal diffeomorphism of g_{ij}:

$$g_{ij}(x) \longrightarrow g_{ij} + \xi_{(i;j)}, \qquad \xi_{(i;j)} = \xi_{i;j} + \xi_{j;i}$$

$$\xi_{(i;j)} = \mathscr{L}_\xi g_{ij} = (K\xi)_{ij}$$

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Conformal Transformations

Full conformal transformations:

$$g_{ij} \to e^{4\phi} g_{ij}, \qquad \phi \in \mathbb{R},$$

Volume-preserving conformal transformations (VPCTs):

$$g_{ij} \to e^{4\hat{\phi}} g_{ij} \qquad \hat{\phi} = \phi + \frac{1}{6} \log \left(\frac{\int \mathrm{d}^3x \sqrt{g}}{\int \mathrm{d}^3x \sqrt{g} \exp 6\phi} \right)$$

VPCTs allow free changes of the local volume element $\sqrt{g(x)}$, $g = \det g_{ij}$ but leave the global volume $V = \int d^3x \sqrt{g}$ unchanged.

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Conformal Superspaces

The Geometrical Group $\mathscr{G}$: The group whose elements consist of combinations of conformal transformations and diffeomorphisms.

Any two 3-metrics g_{ij} that can be carried into each other by elements of $\mathscr{G}$ represent the same conformal 3-geometry.

$$\text{Conformal Superspace (CS)} := \frac{\text{Riem}}{\mathscr{G}}$$

The Restricted Geometrical Group $\mathscr{G}^*$: VPCTs and diffeomorphisms.

$$\text{Conformal Superspace+Volume (CS+V)} := \frac{\text{Riem}}{\mathscr{G}^*}$$

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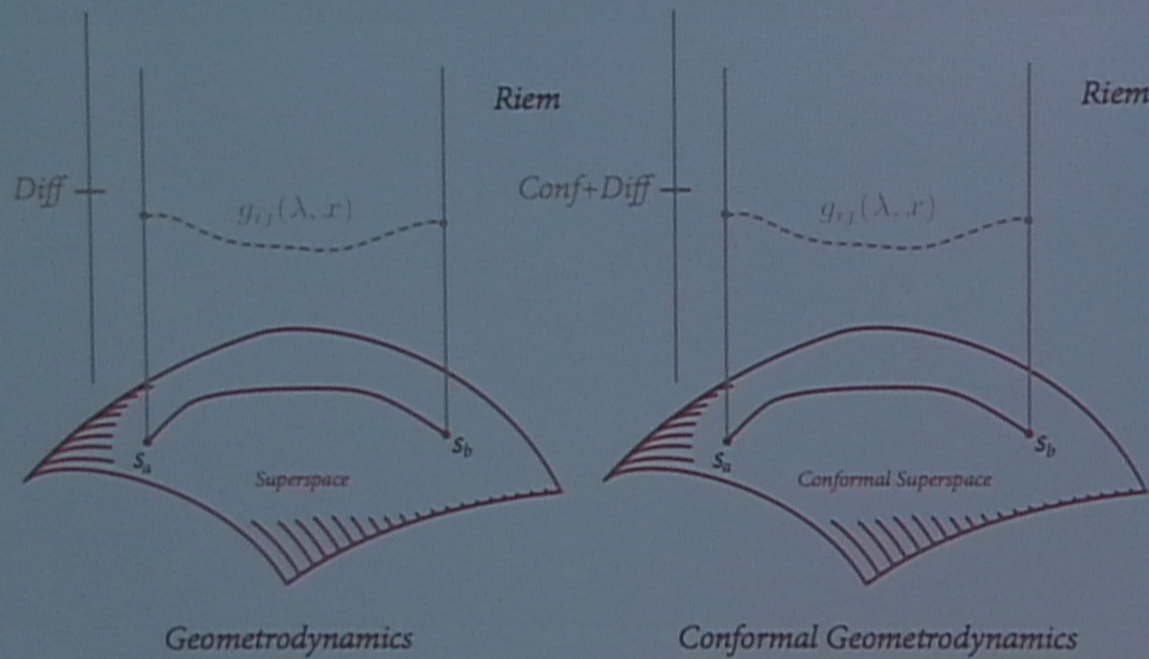
Geometrical Fibre Bundles



Adding matter fields change base space.

Actions for Geometrodynamics

Global square root

$$A_{\mathrm{GSR}} = \int dt \sqrt{\int d^3x \sqrt{g} W \int d^3x \sqrt{g}\, T} \quad \Longrightarrow \quad \text{3 local degrees of freedom}$$

W is a 3-scalar, $\quad g = \det g_{ij}$

$$T = \left(g^{ik} g^{jl} - \lambda\, g^{ij} g^{kl}\right) \frac{\mathrm{d}g_{ij}}{dt} \frac{\mathrm{d}g_{kl}}{dt} \qquad \lambda = 1 \text{ for DeWitt supermetric}$$

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(input from Karel Kuchař and Niall Ó Murchadha)

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$$A_{\mathrm{GSR}} = \int dt \sqrt{\int d^3x \sqrt{g}\, W \int d^3x \sqrt{g}\, T} \quad \Longrightarrow \quad \text{3 local degrees of freedom}$$

W is a 3-scalar, $\quad g = \det g_{ij}$

$$T = \left(g^{ik} g^{jl} - \lambda\, g^{ij} g^{kl}\right) \frac{\mathrm{d}g_{ij}}{dt} \frac{\mathrm{d}g_{kl}}{dt} \qquad \lambda = 1 \text{ for DeWitt supermetric}$$

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(input from Karel Kuchař and Niall Ó Murchadha)

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Constraints for local square roots

Quadratic constraint:

$$\left(g_{ik}g_{jl} - \frac{\lambda}{3\lambda - 1} g_{ij}g_{kl}\right) p^{ij} p^{kl} + W = 0 \quad \text{(identical to GR if} \quad W = 2\Lambda - {}^{3}R \text{ and } \lambda = 1)$$

at each space point.

Linear constraint:

$$p^{ij}{}_{;j} = 0 \qquad \text{(this is the ADM momentum constraint)}$$

from Machian free-end-point variation, equivalent to the introduction of the Lagrange multiplier ξ_i:

$$\frac{\mathrm{d}g_{ij}}{dt} \to \frac{\mathrm{d}g_{ij}}{dt} - \xi_{(i;j)}$$

Local square root as theory selector

Ó Murchadha noted that very few consistent theories with local square root exist. GR, with $W = 2\Lambda - {}^3R$ and $\lambda = 1$ is one of them.

Also possible is:

$$W = 2\Lambda - {}^3R \quad \text{and} \quad \lambda \neq 1$$

at the cost of introducing the further constraints:

$$g_{ij}p^{ij} = \sqrt{g}\,T\,, \quad T = \text{spatial constant}$$

and to perform Dirac's analysis of second-class systems.

Also very interesting is the coupling to matter.

(see F. Mercati's talk in "Conformal Nature of the Universe" workshop)

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Conformally Non-Invariant Actions

$$A_{\mathrm{Conf}} = \int dt \int d^3x \sqrt{g} \sqrt{(2\Lambda - {}^3R)\,T}$$

T is conformally covariant but

$$g_{ij} \to e^{4\phi} g_{ij}, \qquad (R \to \bar{R} = e^{-4\phi} R - 8e^{-5\phi} \nabla^2 e^{\phi}).$$

The conformal non-invariance of R is the basis of York's method for finding solutions to the initial-value problem of general relativity.

Leads to all of the potentially interesting aspects of Conformal Shape Dynamics.

(Input of Niall Ó Murchadha)

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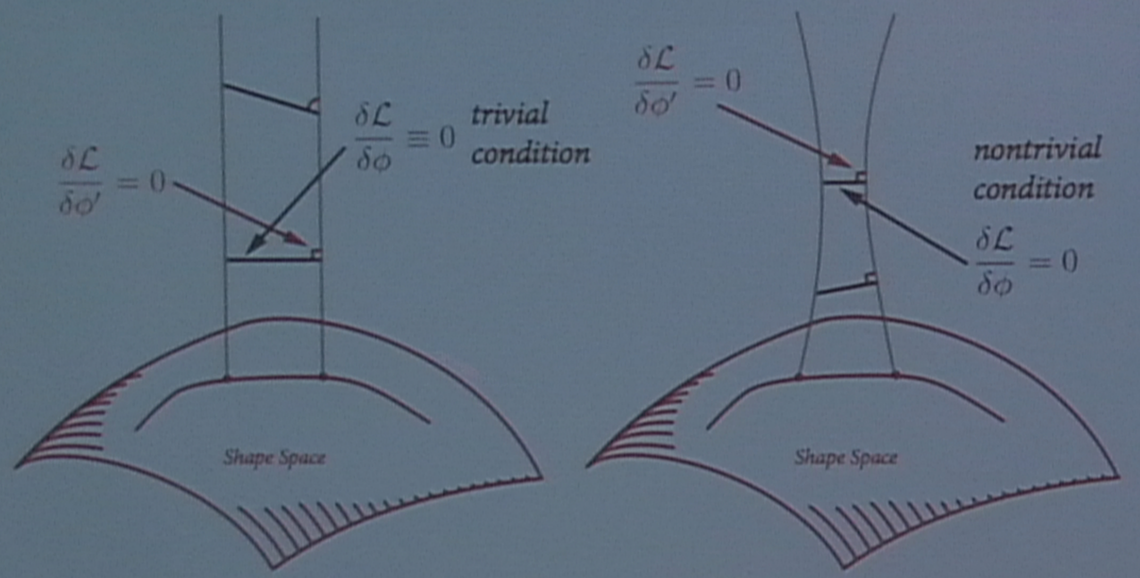
EQUIVARIANT AND NONEQUIVARIANT ACTIONS



Standard Gauge Theory (equivariant) *Generalized Gauge Theory (non equivariant)*

ϕ changes height, ϕ' changes angle

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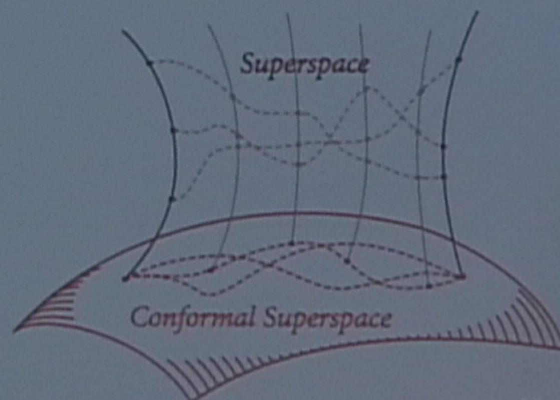
FULLY SCALE-INVARIANT CONFORMAL GEOMETRODYNAMICS

$$A_{\mathrm{Conf}} = \int dt \int d^3x \sqrt{g} \frac{\sqrt{(2\Lambda - {}^3R)T}}{V^{2/3}}$$

$$V = \int d^3x \sqrt{g}, \qquad T = \left(g^{ik}g^{jl} - g^{ij}g^{kl}\right) \frac{dg_{ij}}{dt} \frac{dg_{kl}}{dt},$$

Homogeneous in g. The universe can't expand.

Machian free-end-point variation leads to unique curve in Superspace for fixed end points in Conformal Superspace



Nearly scale-invariant Conformal Geometrodynamics

$$A_{\mathrm{SD}} = \int dt \int d^3x \sqrt{g} \sqrt{(2\Lambda - {}^3R)\,T}\,, \qquad T = \left(g^{ik}g^{jl} - g^{ij}g^{kl}\right) \frac{dg_{ij}}{dt} \frac{dg_{kl}}{dt}\,,$$

Action on Riem identical to lapse-eliminated ADM action but varied more stringently. Everything except conformal geometry and total volume treated as gauge and subject to Machian free-end-point variation

[Anderson, Barbour, Foster, Kelleher, Ó Murchadha, *The physical gravitational degrees of freedom*, CQG **22** (2005)]

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The Constraints of Shape Dynamics

$$p^{ij} := \frac{\delta A_{SD}}{\delta \dot{g}_{ij}} \qquad p = g_{ij} p^{ij}, \qquad \sigma^{ij} := p^{ij} - \frac{1}{3} g^{ij} p$$

Diffeomorphism constraint: $p^{ij}{}_{;j} = 0.$

Conformal constraint: $\dfrac{p}{\sqrt{g}} = C(\lambda)$ (spatial constant).

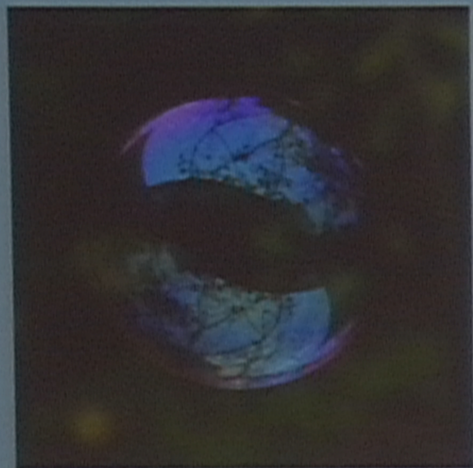
With these two, quadratic constraint leads to modified Lichnerowicz–York eqn:

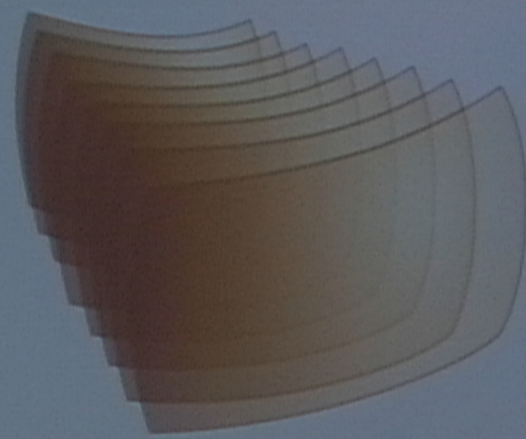
$$\sigma^{ij}\sigma_{ij} - \frac{1}{6} p^2\, \hat{\phi}^{12} - g\,\hat{\phi}^8 \left(R - 8 \frac{\nabla^2 \hat{\phi}}{\hat{\phi}} \right) = 0.$$

Consistency condition $\delta A_{SD}/\delta\phi = 0$ leads to **Lapse-fixing condition:**

$$N\hat{\phi}^{-4}\left(R - 8\frac{\nabla^2\hat{\phi}}{\hat{\phi}} \right) - \hat{\phi}^{-6}\nabla_i\left(\hat{\phi}^2 \nabla^i N\right) + \frac{NC^2}{4} = D \qquad (D \text{ spatial constant})$$

CONSTANT-MEAN-CURVATURE(CMC) FOLIATIONS





Analogous to soap bubbles in three dimensions.

Used by York to solve initial-value problem of GR.

The mean extrinsic curvature C increases monotonically.

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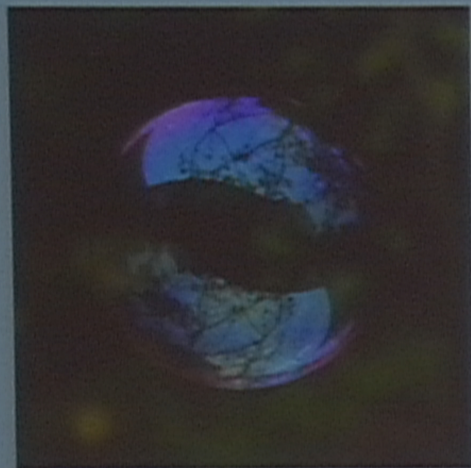

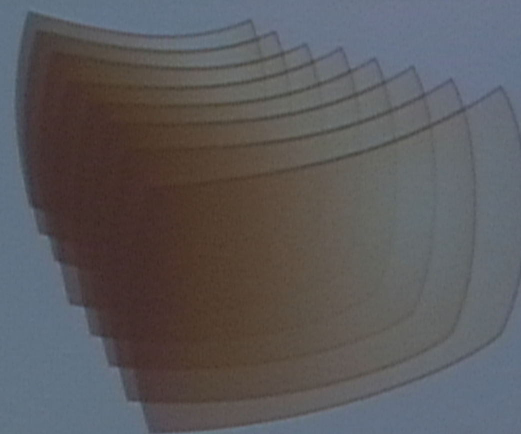

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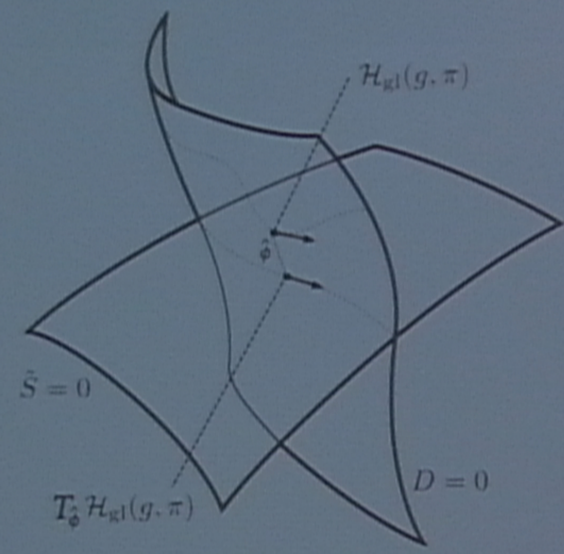
The Two Gauge Theories on One Phase Space



The Theory of Gomes, Gryb and Koslowski.
(see their talks in "Conformal Nature of the Universe" workshop)

The two first-class constraint surfaces gauge-fix each other:
"Doubly General Relativity"

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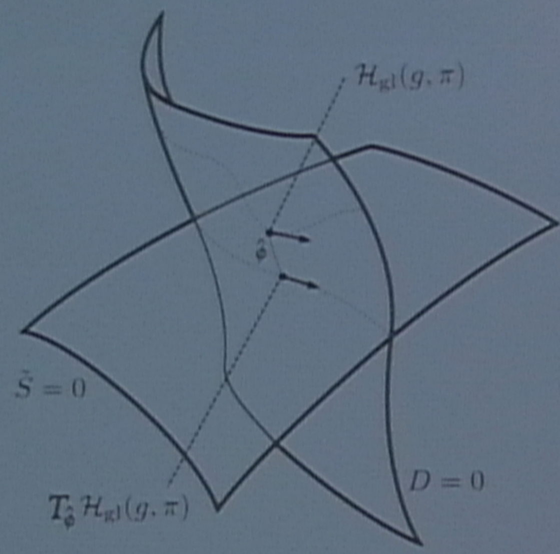

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