Title: Causal Sets and Frame-Valued Set Theory

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Abstract: In spacetime physics any set C of eventsâ€”a causal setâ€”is taken to be partially ordered by the relation Â£ of possible causation: for p, q ÃŽ C,   p Â£ q means that q is in pâ€™s future light cone. Fotini Markopoulou has proposed that the causal structure of spacetime itself be represented by â€œsets evolving over Câ€• â€”that is, in essence, by the topos Set ^C of presheaves on C^{op.} In this talk I am going to show how Set ^C may be effectively replaced by a certain model V^(H) of intuitionistic set theoryâ€”frame-valued set theoryâ€”with (I hope) illuminating results. In particular, Markopoulouâ€™s idea of viewing the universe â€œfrom the insideâ€• will amount to placing oneself â€œinsideâ€• V^(H). I will also sketch the role played in this framework by covering schemes and sheaves.

Causal Sets and Frame-Valued Set Theory

John L. Bell

In spacetime physics any set $\mathscr{C}$ of events—a *causal set*—is taken to be partially ordered by the relation $\leq$ of *possible causation*: for $p, q \in \mathscr{C}$, $p \leq q$ means that q is in p's future light cone. Some time ago Fotini Markopoulou proposed that the causal structure of spacetime itself be represented by "sets evolving over $\mathscr{C}$" —that is, in essence, by the topos $\mathscr{S}et^{\mathscr{C}}$ of presheaves on $\mathscr{C}^{op}$. Here I am going to show how the topos-theoretic framework she employed may be effectively replaced by a certain model of intuitionistic set theory, with (I hope) illuminating results. To facilitate the exposition, it will be convenient to *reverse* the causal ordering, that is, $\mathscr{C}$ will be replaced by $\mathscr{C}^{op}$, and the latter written as P—which will, moreover, be required to be no more than a *preordered* set. Specifically, then: P is a set of events preordered by the relation $\leq$, where $q \leq p$ is intended to mean that q is in p's future light cone—that p *could* be the cause of q, or, equally, that q *could* be an effect of p. Notice that, since $\leq$ is reflexive, each event is conventionally regarded as being in its own future light cone. Accordingly for each event p, the set $p\!\downarrow$ = $\{q: q \leq p\}$ may be identified as the *causal future* of p, or the set of *potential effects* of p each. In requiring that $\leq$ be no more

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Accordingly let us fix a preordered set $(P, \leq)$, the *universal causal set*. Markopoulou, in essence, suggests that viewing the universe "from the inside" amounts to placing oneself within the topos of presheaves $\mathcal{Set}^{P^{op}}$.

We shall suppose that we are given a certain relation $\Vdash$ between events p and mathematical sentences σ: we shall think of $p \Vdash \sigma$ as asserting that, given the occurrence of *any event* in the future light cone of p, the sentence σ is *true* or *holds* in the spacetime associated with *P*. It is then clear that the relation $\Vdash$ is *persistent* in the sense that, if $p \Vdash \sigma$ and $q \leq p$, then $q \Vdash \sigma$.

For each sentence σ, the set $[\sigma] = \{p: p \Vdash \sigma\}$ "measures" the degree or extent to which σ holds: the larger $[\sigma]$ is, the "truer" σ is. In particular, when $[\sigma] = P$, σ is "universally" or "absolutely" true, and when $[\sigma] = \varnothing$, σ is "universally" or "absolutely" false. These $[\sigma]$ may accordingly be thought of as "truth values", with *P* corresponding to "absolute truth" and $\varnothing$ to absolute falsity.

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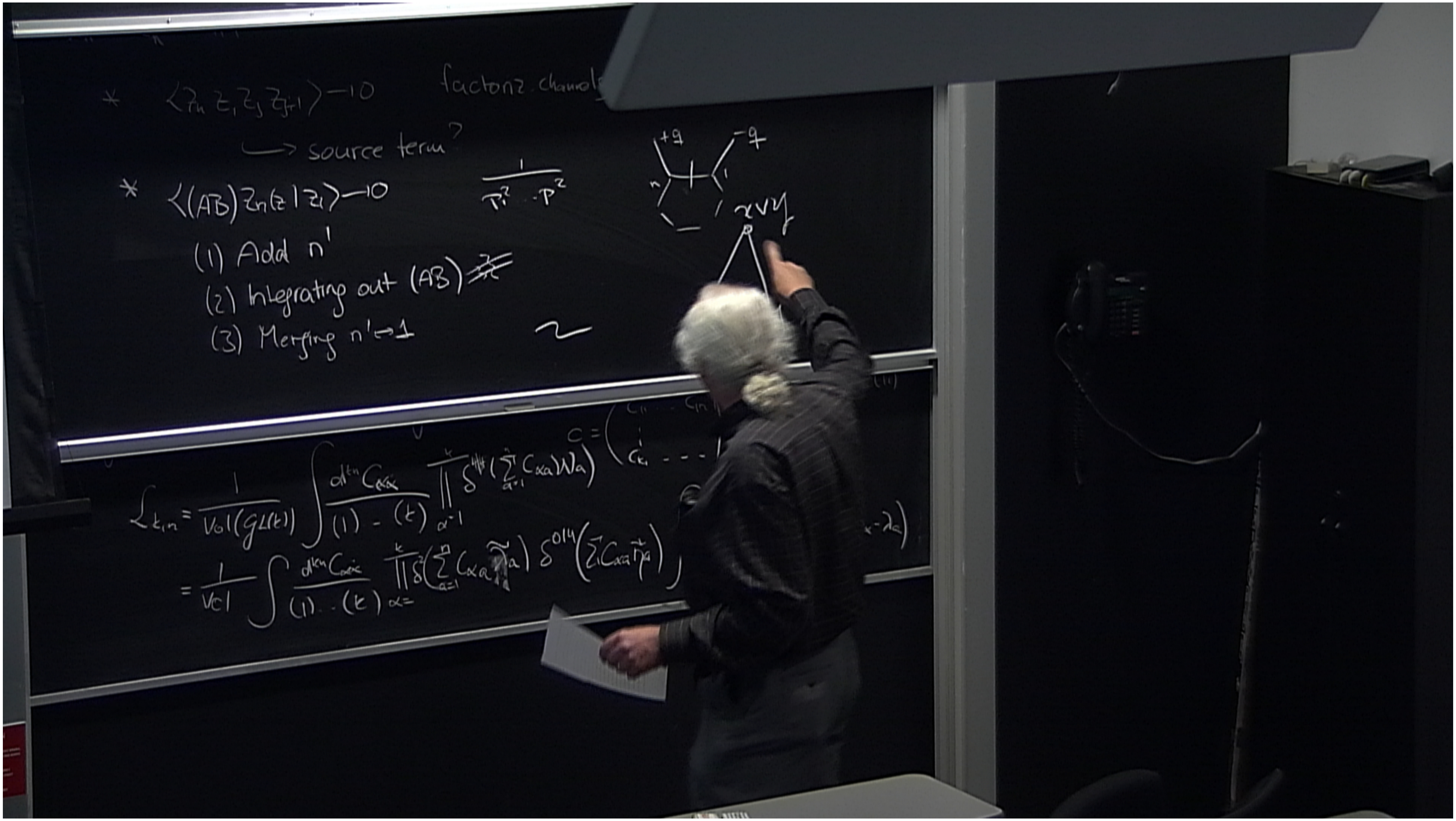
Because of the persistence property, each $[\sigma]$ has the property of being "closed under potential effects", or "causally closed", that is, satisfies $p \in [\sigma]$ and $q \leq p \rightarrow q \in [\sigma]$. A subset of P with this property is called a *sieve*. Sieves serve as generalized "truth values" measuring the degree to which assertions hold. The set $\widehat{P}$ of all sieves, or truth values has a natural logico-algebraic structure —that of a *complete Heyting algebra*, or *frame*. This concept is defined in the following way.

A *lattice* is a partially ordered set L with partial ordering $\leq$ in which each two-element subset $\{x, y\}$ has a supremum or *join*—denoted by $x \vee y$—and an infimum or *meet*—denoted by $x \wedge y$. A lattice L is *complete* if every subset X (including $\varnothing$) has a supremum or *join*—denoted by $\bigvee X$—and an infimum or *meet*—denoted by $\bigwedge X$. Note that $\bigvee \varnothing = 0$, the least or *bottom* element of L, and $\bigwedge \varnothing = 1$, the largest or *top* element of L.

A *Heyting algebra* is a lattice L with top and bottom elements such that, for any elements $x, y \in L$, there is an element—denoted by $x \Rightarrow y$—of L such that, for any $z \in L$,

$$z \leq x \Rightarrow y \text{ iff } z \wedge x \leq y.$$

Thus $x \Rightarrow y$ is the *largest* element z such that $z \wedge x \leq y$. So in particular, if we write $\neg x$ for $x \Rightarrow 0$, then $\neg x$ is the largest element z such that $x \Rightarrow z = 0$; it is called the *pseudocomplement* of x. A *Boolean algebra* is a Heyting algebra



→ source term?
(1) Add n'
(2) Integrating out (AB)
(3) Merging n' ↦ 1

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in which $\neg\neg x = x$ for all x, or equivalently, in which $x \vee \neg x = 1$ for all x.

If we think of the elements of a (complete) Heyting algebra as "truth values", then 0, 1, $\wedge$, $\vee$, $\neg$, $\Rightarrow$, $\bigvee$, $\bigwedge$ represent "true", "false", "and", "or", "not" and "implies", "there exists" and "for all", respectively. The laws satisfied by these operations in a general Heyting algebra correspond to those of *intuitionistic logic*. In Boolean algebras the counterpart of the law of excluded middle also holds.

A basic fact about *complete* Heyting algebras is that the following identity holds in them:

$$(*) \qquad x \wedge \bigvee_{i \in I} y_i = \bigvee_{i \in I} (x \wedge y_i)$$

And conversely, in any complete lattice satisfying (*), defining the operation $\Rightarrow$ by $x \Rightarrow y = \bigvee\{z : z \wedge x \leq y\}$ turns it into a Heyting algebra.

In view of this result a complete Heyting algebra is frequently defined to be a complete lattice satisfying (*). A complete Heyting algebra is briefly called a *frame*.

In the frame $\hat{P}$ $\leq$ is $\subseteq$, joins and meets are just set-theoretic unions and intersections, and the operations $\Rightarrow$ and $\neg$ are given by

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Frames do duty as the "truth-value algebras" of the (current) *language of mathematics*, that is, *set theory*. To be precise, associated with each frame H is a structure $V^{(H)}$—the *universe of H-valued sets*—with the following features.

- Each of the members of $V^{(H)}$—the *H-sets*—is a map from a subset of $V^{(H)}$ to H.
- Corresponding to each sentence σ of the language of set theory (with names for all elements of $V^{(H)}$) is an element $[\![\sigma]\!] = [\![\sigma]\!]^H \in H$ called its *truth value in* $V^{(H)}$. These "truth values" satisfy the following conditions. For $a, b \in V^{(H)}$,

$$[\![b \in a]\!] = \bigvee_{c \in dom(a)} [\![b = c]\!] \wedge a(c) \qquad [\![b = a]\!] = \bigvee_{c \in dom(a) \cup dom(b)} ([\![c \in b]\!] \Leftrightarrow [\![c \in a]\!])$$

$$[\![\sigma \wedge \tau]\!] = [\![\sigma]\!] \wedge [\![\tau]\!], \text{ etc.}$$

$$[\![\exists x \varphi(x)]\!] = \bigvee_{a \in V^{(H)}} [\![\varphi(a)]\!]$$

$$[\![\forall x \varphi(x)]\!] = \bigwedge_{a \in V^{(H)}} [\![\varphi(a)]\!]$$

  A sentence σ is *valid* , or *holds,* in $V^{(H)}$, written $V^{(H)} \models \sigma$, if $[\![\sigma]\!] = 1$, the top element of H.
- The axioms of intuitionistic Zermelo-Fraenkel set theory are valid in $V^{(H)}$. In this sense $V^{(H)}$ is an *H-valued model* of *IZF*. Accordingly the category $\mathcal{S}et^{(H)}$ of sets constructed within $V^{(H)}$ is a topos: in fact $\mathcal{S}et^{(H)}$ can be shown to be equivalent to the topos of canonical sheaves on H.
- There is a canonical embedding $x \mapsto \hat{x}$ of the usual [illegible] H of sets into $V^{(H)}$ satisfying

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- There is a canonical embedding $x \mapsto \hat{x}$ of the usual universe V of sets into $V^{(H)}$ satisfying

$$[\![u \in \hat{x}]\!] = \bigvee_{y \in x} [\![u = \hat{y}]\!] \text{ for } x \in V, u \in V^{(H)}$$

$$x \in y \leftrightarrow V^{(H)} \models \hat{x} \in \hat{y}, \quad x = y \leftrightarrow V^{(H)} \models \hat{x} = \hat{y} \text{ for } x, y \in V$$

$$\varphi(x_1,\ldots,x_n) \leftrightarrow V^{(H)} \models \varphi(\hat{x}_1,\ldots,\hat{x}_n) \text{ for } x_1,\ldots,x_n \in V \text{ and restricted } \varphi$$

(Here a formula φ is *restricted* if all its quantifiers are restricted, i.e. can be put in the form $\forall x \in y$ or $\exists x \in y$.)

We observe that $V^{(2)}$ is essentially just the usual universe of sets.

It follows from the last of these assertions that the canonical representative $\widehat{H}$ of H is a Heyting algebra in $V^{(H)}$. A particularly important H- set is the H-set Φ_H defined by

$$dom(\Phi_H) = \{\hat{a} : a \in H\}, \quad \Phi_H(\hat{a}) = a \text{ for } a \in H .$$

Then $V^{(H)} \models \Phi_H \subseteq \widehat{H}$. Also, for any $a \in H$ we have $[\![\hat{a} \in \Phi_H]\!] = a$, and in particular, for any sentence σ, $[\![\sigma]\!] = [\![\widehat{[\![\sigma]\!]} \in \Phi_H]\!]$. Thus

$$V^{(H)} \models \sigma \leftrightarrow V^{(H)} \models \widehat{[\![\sigma]\!]} \in \Phi_H ;$$

in this sense Φ_H represents the "true" sentences in $V^{(H)}$. Φ_H is called the *canonical truth set* in $V^{(H)}$.

Now let us return to our causal set P. The topos $\mathcal{Set}^{(\widehat{P})}$ of sets in $V^{(\widehat{P})}$ is, as I have observed, equivalent to the topos of canonical sheaves on $\widehat{P}$, which is itself, as is well known, equivalent to the topos $\mathcal{Set}^{P^{op}}$ of presheaves on P. My proposal is then, that we work in $V^{(\hat{P})}$ rather than, as did Markopoulou, within $\mathcal{S}et^{P^{op}}$. Accordingly, describing what the universe looks like "from the inside" will amount to reporting the view from $V^{(\hat{P})}$. The difference in the two descriptions is this. Within $\mathcal{S}et^{P^{op}}$, objects clearly manifest "individual variation". But within $V^{(\hat{P})}$ objects are formally treated as static and their variation is manifested through the fact that they are subject to the laws of intuitionistic logic.

For simplicity let us write H for $\hat{P}$. The "truth value" $[\sigma]$ of a sentence σ in $V^{(H)}$ is a sieve of events in P, and it is natural to think of the events in $[\sigma]$ as those at which σ "holds". So one introduces the *forcing* relation $\Vdash_P$ in $V^{(H)}$ between sentences and elements of P by

$$p \Vdash_P \sigma \leftrightarrow p \in [\![\sigma]\!].$$

This satisfies the standard so-called Kripke rules, viz.,

- $p \Vdash_P \varphi \wedge \psi \leftrightarrow p \Vdash_P \varphi \;\&\; p \Vdash_P \psi$
- $p \Vdash_P \varphi \vee \psi \leftrightarrow p \Vdash_P \varphi$ or $p \Vdash_P \psi$
- $p \Vdash_P \varphi \rightarrow \psi \leftrightarrow \forall q \le p [\, q \Vdash_P \varphi \rightarrow q \Vdash_P \psi]$
- $p \Vdash_P \neg\varphi \leftrightarrow \forall q \le p \; q \nVdash_K \varphi$
- $p \Vdash_P \forall$ [illegible] p [illegible]) for every $a \in V^{(\hat{P})}$
- [illegible] or some $a \in V^{(\hat{P})}$.

[illegible] $V^{(H)}$ by $\mathrm{dom}(K) = \{\hat{p} : p \in P\}$ and [illegible] subset of $\hat{P}$ and for $p \in P$, then, that we work in $V^{(\hat{P})}$ rather than, as did Markopoulou, within $\mathcal{Set}^{P^{op}}$. Accordingly, describing what the universe looks like "from the inside" will amount to reporting the view from $V^{(\hat{P})}$. The difference in the two descriptions is this. Within $\mathcal{Set}^{P^{op}}$, objects clearly manifest "individual variation". But within $V^{(\hat{P})}$ objects are formally treated as static and their variation is manifested through the fact that they are subject to the laws of intuitionistic logic.

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- $p \Vdash_P \forall x \varphi \leftrightarrow p \Vdash_P \varphi(a)$ for every $a \in V^{(\hat{P})}$
- $p \Vdash_P \exists x \varphi \leftrightarrow p \Vdash_P \varphi(a)$ for some $a \in V^{(\hat{P})}$.

Define the set $K \in V^{(H)}$ by $\mathrm{dom}(K) = \{\hat{p} : p \in P\}$ and $K(\hat{p}) = p{\downarrow}$. Then, in $V^{(H)}$, K is a subset of $\hat{P}$ and for $p \in P$, then, that we work in $V^{(\hat{P})}$ rather than, as did Markopoulou, within $\mathcal{Set}^{P^{op}}$. Accordingly, describing what the universe looks like "from the inside" will amount to reporting the view from $V^{(\hat{P})}$. The difference in the two descriptions is this. Within $\mathcal{Set}^{P^{op}}$, objects clearly manifest "individual variation". But within $V^{(\hat{P})}$ objects are formally treated as static and their variation is manifested through the fact that they are subject to the laws of intuitionistic logic.

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$[\![\dot{p} \in K]\!] = p\downarrow$. K is the counterpart in $V^{(\hat{P})}$ of Markopoulou's evolving set *Past*. ($\hat{P}$, incidentally, is the $V^{(H)}$- counterpart of the constant presheaf on P with value P —which Markopoulou calls *World*.) The fact that, for any $p, q \in P$ we have

(*) $$q \Vdash_P \hat{p} \in K \leftrightarrow q \leq p$$

may be construed as asserting that *the events in the causal future of a given event are precisely those forcing (the canonical representative of) that event to be a member of K*. Or, equally, *the events in the causal past of a given event are precisely those forced by that event to be a member of K*. For this reason we shall call K the *causal set in* $V^{(H)}$.

If we identify each $p \in P$ with $p\downarrow \in H$, P may then be regarded as a subset of H so that, in $V^{(H)}$, $\hat{P}$ is a subset of $\hat{H}$. It is not hard to show that $V^{(H)} \models K = \Phi_H \cap \hat{P}$. Moreover, it can be shown that, for any sentence σ, $[\![\sigma]\!] = [\![\exists p \in K. p \leq \widehat{[\![\sigma]\!]}]\!]$, so that, with moderate abuse of notation,

$$V^{(H)} \models [\sigma \leftrightarrow \exists p \in K.\, p \Vdash \sigma].$$

That is, in $V^{(H)}$, *a sentence holds precisely when it is forced to do so at some "causal past stage" in K*. This establishes the centrality of K—and, correspondingly, that of the "evolving" set *Past*— in determining the truth of sentences "from the inside", that is, inside the universe $V^{(H)}$.

Markopoulou also considers the *complement* of *Past*—i.e.,

$[\![\hat{p} \in \neg K]\!] = [\![p \notin K]\!] = \neg p\downarrow = \{q : \forall r \le q . r \not\le p\}$. Markopoulou calls (*mutatis mutandis*) the events in $\neg p\downarrow$ those *beyond p's causal horizon*, in that no observer at p can ever receive "information" from any event in $\neg p\downarrow$. Since clearly we have

$$(\dagger) \qquad q \Vdash_P \hat{p} \in \neg K \;\leftrightarrow\; q \in \neg p\downarrow,$$

it follows that *the events beyond the causal horizon of an event p are precisely those forcing (the canonical representative of) p to be a member of* $\neg K$. In this sense $\neg K$ reflects, or "measures" the causal structure of P.

In this connection it is natural to call $\neg\neg p\downarrow = \{q : \forall r \le q \exists s \le r . s \le p\}$ the *causal horizon* of p: it consists of those events q for which an observer placed at p could, in its future, receive information from any event in the future of an observer placed at q. Since

$$q \Vdash_P \hat{p} \in \neg\neg K \;\leftrightarrow\; q \in \neg\neg p\downarrow,$$

it follows that *the events within the causal horizon of an event are precisely those forcing (the canonical representative of) p to be a member of* $\neg\neg K$.

It is easily shown that $\neg K$ is *empty* (i.e. $V^{(H)} \models \neg K = \varnothing$) if and only if P is *directed downwards*, i.e., for any $p, q \in P$ there is $r \in P$ for which $r \le p$ and $r \le q$. This holds in the case, considered by Markopoulou, of *discrete Newtonian time evolution*—in the present setting, the case in which P is the opposite $\mathbb{N}^{op}$ of the totally ordered set $\mathbb{N}$ of natural numbers.

$[\![\hat{p} \in \neg K]\!] = [\![p \notin K]\!] = \neg p\!\downarrow = \{q : \forall r \leq q . r \nleq p\}$. Markopoulou calls (*mutatis mutandis*) the events in $\neg p\!\downarrow$ those *beyond p's causal horizon*, in that no observer at p can ever receive "information" from any event in $\neg p\!\downarrow$. Since clearly we have

(†) $$q \Vdash_P \hat{p} \in \neg K \;\leftrightarrow\; q \in \neg p\!\downarrow,$$

it follows that *the events beyond the causal horizon of an event p are precisely those forcing (the canonical representative of) p to be a member of* $\neg K$. In this sense $\neg K$ reflects, or "measures" the causal structure of P.

In this connection it is natural to call $\neg\neg p\!\downarrow = \{q : \forall r \leq q \exists s \leq r . s \leq p\}$ the *causal horizon* of p: it consists of those events q for which an observer placed at p could, in its future, receive information from any event in the future of an observer placed at q. Since

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Here the corresponding complete Heyting algebra H is the family of all downward-closed sets of natural numbers. In this case the H-valued set K representing *Past is neither finite nor actually infinite in* $V^{(H)}$.

To see this, observe first that, for any natural number n, we have $[\![\neg(\hat{n} \in \neg K)]\!] = \mathbb{N}$. It follows that $V^{(H)} \models \neg\neg\forall n \in \hat{\mathbb{N}}.\ n \in K$. But, working in $V^{(H)}$, if $\forall n \in.\hat{\mathbb{N}}\ n \in K$, then K is not finite, so if K is finite, then $\neg\forall n \in \hat{\mathbb{N}}.\ n \in K$, and so $\neg\neg\forall n \in \hat{\mathbb{N}}.\ n \in K$ implies the non-finiteness of K.

But, in $V^{(H)}$, K is not actually infinite. For (again working in $V^{(H)}$), if K were actually infinite (i.e., if there existed an injection of $\hat{\mathbb{N}}$ into K), then the statement

$$\forall x \in K\, \exists y \in K.\ x < y$$

would also have to hold in $V^{(H)}$. But calculating that truth value gives:

$$\begin{aligned}
&[\![\forall x \in K \exists y \in K. x < y]\!] \\
&\quad = \bigcap_{m \in \mathbb{N}^{op}} [m \downarrow \Rightarrow \bigcup_{n \in \mathbb{N}^{op}} n \downarrow \cap [\![\hat{m} < \hat{n}]\!]] \\
&\quad = \bigcap_{m} [m \downarrow \Rightarrow \bigcup_{m<n} n \downarrow] \\
&\quad = \bigcap_{m} [m \downarrow \Rightarrow (m+1) \downarrow] \\
&\quad = \bigcap_{m} (m+1) \downarrow = \varnothing
\end{aligned}$$

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So $\forall x \in K\ \exists y \in K.\ x < y$ is false in $V^{(H)}$ and therefore K is not actually infinite. So, in evolving Newtonian spacetime, the set K representing past time is potentially, but not actually infinite: this is, in essence, what Kant asserted of time.

In order to formulate an observable causal *quantum theory* Markopoulou considers the possibility of introducing a *causally evolving algebra of observables*. This amounts to specifying a presheaf $\mathcal{A}$ of C^*-algebras on P, which, in the present framework, corresponds to specifying a set $\mathscr{A}$ in $V^{(H)}$ satisfying

$$V^{(H)} \models \mathscr{A} \text{ is a } C^*\text{-algebra.}$$

The "internal" C^*-algebra $\mathscr{A}$ is then subject to the intuitionistic internal logic of $V^{(H)}$: *any* theorem concerning C^*-algebras—provided only that it be constructively proved—automatically applies to $\mathscr{A}$. Reasoning with $\mathscr{A}$ is more direct and simpler than reasoning with $\mathcal{A}$.

This same procedure of "internalization" can be performed with any causally evolving object: each such object of type $\mathscr{T}$ corresponds to a set S in $V^{(H)}$ satisfying

$$V^{(H)} \models S \text{ is of type } \mathscr{T}.$$

Internalization may also be applied in the case of the presheaves *Antichains* and *Graphs* considered by Markopoulou. Here, for each event p, *Antichains*(p) consists of all sets of causally unrelated events in *Past*(p), while *Graphs*(p) is the set of all graphs supported by elements of *Antichains*(p). In the present framework *Antichains* is represented by the $V^{(H)}$-set $Anti = \{ X \subseteq P: X \text{ is an antichain}\}$ and *Graphs* by the $V^{(H)}$-set $Grph = \{G: \exists X \in A \,.\, G \text{ is a graph supported by } A\}$. Again, both *Anti* and *Grph* can be readily handled using the internal intuitionistic logic of $V^{(H)}$.

Cover schemes or *Grothendieck topologies* may be used to force certain conditions to prevail in the associated models. (This corresponds to the process of *sheafification*.) A *cover scheme* on P is a map $\mathbf{C}$ assigning to each $p \in P$ a family $\mathbf{C}(p)$ of subsets of $p\downarrow = \{q: q \leq p\}$, called ($\mathbf{C}$-)*covers of p*, such that, if $q \leq p$, any cover of p can be sharpened to a cover of q, i.e.,

$$S \in \mathbf{C}(p) \,\&\, q \leq p \rightarrow \exists T \in \mathbf{C}(q)[\forall t \in T \exists s \in S(t \leq s)].$$

A cover S of an event p may be thought of as a "sampling" of the events in p's causal future, a "survey" of p's potenti... effects, in short, a *survey of p*. Using this language the condition immediately above becomes: *for any survey S of a given event p, and any event q which is a potential effect of p, there is a survey of q each event in which is the potential effect of some event in S.*

There are three naturally defined cover schemes on P we shall consider. First, each sieve A in P determines two cover schemes $\mathbf{C}_A$ and $\mathbf{C}^A$ defined by

$$S \in \mathbf{C}_A(p) \leftrightarrow p \in A \cup S \qquad S \in \mathbf{C}^A(p) \leftrightarrow p\downarrow \cap A \subseteq S$$

If $p \in$ A, any part of p's causal future thus counts as a $\mathbf{C}_A$–survey of p, and any part of p's causal future extending the common part of that future with A counts as a $\mathbf{C}^A$–survey of p. Notice that then $\varnothing \in \mathbf{C}_A(p) \leftrightarrow p \in A$ and $\varnothing \in \mathbf{C}^A(p) \leftrightarrow p{\downarrow} \cap A = \varnothing$.

Next, we have the *dense cover scheme* **Den** given by:

$$S \in \mathbf{Den}(p) \leftrightarrow \forall q \leq p \exists s \in S \exists r \leq s(r \leq q):$$

That is, S is a dense survey of p provided that for every potential effect q of p there is an event in S with a potential effect in common with q.

Given a cover scheme $\mathbf{C}$ on P, a sieve I will be said to *encompass* an element $p \in P$ if I includes a $\mathbf{C}$-cover of p. Thus a sieve I encompasses p if it contains all the events in some survey of p. Call I $\mathbf{C}$-*closed* if it contains every element of P that it encompasses, i.e. if

$$\exists S \in \mathbf{C}(p)(S \subseteq I) \rightarrow p \in I .$$

The set $\bar{\mathbf{C}}$ of all $\mathbf{C}$-closed sieves in P, partially ordered by inclusion, can be shown to be a frame—the frame *induced* by $\mathbf{C}$—in which the operations of meet and $\Rightarrow$ coincide with those of $\bar{P}$. Passing from $V^{(P)}$ to $V^{(\mathbf{C})}$ is the process of *sheafification:* essentially, it amounts to replacing the forcing relation $\Vdash_P$ in

$V^{(P)}$ by the new forcing relation $\Vdash_{\mathbf{C}}$ in $V^{(\mathbf{C})}$. For atomic sentences σ these are related by

$$p \Vdash_{\mathbf{C}} \sigma \leftrightarrow \exists S \in \mathbf{C}(p) \forall s \in S.\ s \Vdash_P \sigma\,;$$

i.e., p **C**-*forces the truth of a sentence just the truth of that sentence is P-forced by every event in some* **C**-*survey of* p.

The frame induced by the dense cover scheme **Den** in P turns out to be a complete Boolean algebra B. For the corresponding causal set K_B in $V^{(B)}$ we find that

$$q \Vdash_B \hat{p} \in K_B \leftrightarrow q \in \neg\neg p\!\downarrow$$
$$\leftrightarrow \text{ } q \textit{ is in } p\textit{'s causal horizon.}$$

Comparing this with (*) above, we see that moving to the universe $V^{(B)}$—"Booleanizing" it, so to speak—*amounts to replacing causal futures by causal horizons*. When P is linearly ordered, as for example in the case of Newtonian time, the causal horizon of any event coincides with the whole of P, B is causal future of p. In the associated universe $V^{(\widetilde{C^A})}$ the corresponding causal set K^A satisfies, for every event q

$$q \Vdash_{\widetilde{C^A}} \hat{p} \in K^A.$$

Comparing this with (*), we see that in $V^{(\widetilde{C^A})}$ that every event has been "forced" into p's causal future: in short, that p now marks the "beginning" of the universe as viewed from inside $V^{(\widetilde{C^A})}$.

Similarly, we find that the causal set K_A in the universe $V^{(\widetilde{C_A})}$ has the property

$$q \leq p \rightarrow \forall r [r \Vdash_{\widetilde{C_A}} \hat{q} \in \neg K_A].$$

a comparison with (†) above reveals that, in $V^{(\widetilde{C_A})}$, every event—including p itself—has been placed beyond p's causal horizon. In effect, the event p has been obliterated, effaced from the universe—like the extraordinary scenario in H.G. Wells's *The Man Who Could Work Miracles*, the event p never occurred!

As a final possibility consider the universe $V^{(\tilde{P})}$ associated with the free lower semilattice $\tilde{P}$ generated by P. In this case the elements of $\tilde{P}$ are finite sets of events, preordered by the relation $\sqsubseteq$: for $F, G \in \tilde{P}$, $F \sqsubseteq G$ iff every event in G is in the causal past of an event in F. The empty set of events is the top element of $\tilde{P}$. The causal set causal future of p. In the associated universe $V^{(\overline{C^A})}$ the corresponding causal set K^A satisfies, for every event q

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$$\{q\} \Vdash_{\widetilde{P}} \widehat{\{p\}} \in \widetilde{K} \leftrightarrow q \Vdash_{P} \widehat{p} \in K .$$

In other words, in passing to the new universe the initial event $\widehat{\varnothing}$ and the new light cone overlaps have been "freely adjoined" to the original universe.