Title: Recent Progress in Compact G2 Manifolds

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Abstract: This talk will give a survey of some recent developments on the construction and classification of compact manifolds with holonomy G2 and their calibrated submanifolds. After reviewing previous work we concentrate on the following three developments: (a) the construction of many new compact G2 manifolds using weak Fano 3-folds; (b) the construction of many compact G2 manifolds containing rigid associative 3-folds; (c) the diffeomorphism classification of (2-connected) G2 manifolds obtained by twisted connect sums. If time permits we will mention new questions
suggested by our results. This work is joint with Alessio Corti, Johannes Nordstrom, and Tommaso Pacini.

Recent progress in G_2 geometry

MARK HASKINS
Imperial College London
with A. Corti, J. Nordström & T. Pacini.

GAP, Waterloo/Perimeter Institute
7 May 2012

Main goal:

• Construct compact G_2 manifolds containing compact rigid associative 3-folds.

Byproducts:

• Construct many new noncompact Calabi-Yau 3-folds.

• Construct many new compact G_2 manifolds.

• Identify the diffeomorphism type of 7-manifold underlying many of our G_2 manifolds; they are the first G_2 manifolds where diffeo type is understood.

• Exhibit different ways to construct G_2 metrics on same underlying smooth 7-manifold; find G_2 metrics with different numbers of (obvious) rigid associative 3-folds.

• Exhibit "geometric transitions" between G_2-metrics on different 7-manifolds.

$SU(3)$ to G_2

$$6 + 1 = 2 * 3 + 1 = 7$$

$$SU(2) \subset SU(3) \subset G_2$$

$$SU(3) + SU(3) + \epsilon = G_2$$

Submanifolds

$$1 + 2 = 3$$

$$\mathbb{S}^1 \times holomorphic = associative$$

Two geometric characterizations of G_2:

(i) *the automorphism group of the octonions* $\mathbb{O}$

(ii) *the stabilizer of a generic* 3*-form in* $\mathbb{R}^7$

Define a vector cross-product on $\mathbb{R}^7 = \mathrm{Im}(\mathbb{O})$

$$u \times v = \mathrm{Im}(uv)$$

where uv denotes octonionic multiplication.

Cross-product has an associated 3-form

$$\phi_0(u, v, w) := \langle u \times v, w \rangle = \langle uv, w \rangle$$

ϕ_0 is a generic 3-form so

$$G_2 = \{A \in \mathrm{GL}(7, \mathbb{R}) | \ A^* \phi_0 = \phi_0\} \subset \mathrm{SO}(7).$$

$6+1 = 2*3+1 = 7$ and $\mathbf{SU}(2) \subset \mathbf{SU}(3) \subset G_2$

- Write $\mathbb{R}^7 = \mathbb{R} \times \mathbb{C}^3$ with $(\mathbb{C}^3, \omega, \Omega)$ the std SU(3) structure then

$$\phi_0 = dt \wedge \omega + \mathrm{Re}\,\Omega$$

Hence stabilizer of $\mathbb{R}$ factor in G_2 is SU(3) $\subset G_2$.
More generally if (X, g) is a Calabi-Yau 3-fold then product metric on $\mathbb{S}^1 \times X$ has holonomy SU(3) $\subset G_2$.

- Write $\mathbb{R}^7 = \mathbb{R}^3 \times \mathbb{C}^2$ with coords (x_1, x_2, x_3) on $\mathbb{R}^3$, with std SU(2) structure $(\mathbb{C}^2, \omega_I, \Omega = \omega_J + i\omega_K)$ then

$$\phi_0 = dx_1 \wedge dx_2 \wedge dx_3 + dx_1 \wedge \omega_I + dx_2 \wedge \omega_J + dx_3 \wedge \omega_K,$$

Hence subgroup of G_2 fixing $\mathbb{R}^3 \subset \mathbb{R}^3 \times \mathbb{C}^2$ is SU(2) $\subset G_2$.

$\exists$ *close relations between G_2 holonomy and Calabi-Yau geometries in 2 and 3 dimensions.*

G_2 structures and G_2 holonomy metrics

Positive 3-forms $\leftrightsquigarrow$ (oriented) G_2-structures

- A 3-form ϕ on an oriented 7-mfd M is *positive* if $\forall\ p \in M\ \exists$ an oriented isomorphism

$$i : T_pM \to \mathbb{R}^7, \text{ such that } i^*\phi_0 = \phi.$$

- Positive 3-forms on $\mathbb{R}^7 \leftrightsquigarrow \mathrm{GL}_+(7,\mathbb{R})/G_2$.
- $\dim(\mathrm{GL}_+(7,\mathbb{R})/G_2) = 35 = \dim \Lambda^3\mathbb{R}^7$.

$\Rightarrow$ Positive 3-forms on M form an *open* subbundle of Λ^3T^*M *i.e.* small perturbations of a G_2 structure are G_2 structures.

Prop: Let (M, ϕ, g) be a G_2 structure on a compact 7-manifold; TFAE

1. $\mathrm{Hol}(g) \subset G_2$ and ϕ is the induced 3-form
2. $\nabla\phi = 0$ where ∇ is Levi-Civita w.r.t g
3. $d\phi = d^*\phi = 0$.

Call such a G_2 structure a *torsion-free* G_2 structure.

NB (3) is nonlinear in ϕ because metric g depends nonlinearly on ϕ.

G_2-structures to G_2-holonomy metrics

Prop:
(a). A compact 7-manifold M admits a G_2-structure iff M is orientable and spinnable.
(b). A compact 7-manifolds M with a torsion-free G_2 structure (ϕ, g) has $\mathrm{Hol}(g) = G_2$ iff $\pi_1 M$ is finite.
(c). A compact 7-manifold (M, g) with $\mathrm{Hol}(g) = G_2$ has nonzero first Pontrjagin class $p_1(M)$.

A strategy to construct G_2-holonomy metrics

1. Find a G_2 structure ϕ with sufficiently small torsion on a 7-manifold with $|\pi_1| < \infty$

2. Perturb to a torsion-free G_2 structure ϕ' close to ϕ.

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Special submanifolds of G_2-manifolds

A *calibrated geometry* is a distinguished class of minimal submanifolds associated with a differential form.

- A *calibrated form* is a closed differential p-form ϕ on a Riemannian manifold (M, g) satisfying $\phi \leq vol_g$.

  $$i.e. \qquad \phi(e_1, \ldots, e_p) \leq 1$$

  for any orthonormal set of p tgt vectors

- For $m \in M$ associate with ϕ the subset $G_m(\phi)$ of oriented p-planes for which equality holds in $(*)$ – the *calibrated* planes.

- A submanifold *calibrated* by ϕ is an oriented p-dim submanifold whose tangent plane at each point m lies in the subset $G_m(\phi)$ of distinguished p-planes.

Lemma: (Harvey–Lawson)
Calibrated submanifolds minimize volume in their homology class.

Associative & coassociative calibrations on G_2-manifolds

3-form ϕ_0 and 4-form $*\phi_0$ on $\mathbb{R}^7$ are G_2-invariant calibrations.

Oriented 3-planes calibrated by ϕ_0 are called *associative* planes.

- $\mathbb{R}^3 \subset \mathbb{R}^3 \times \mathbb{C}^2$ is an associative 3-plane.
- G_2 acts transitively on associative 3-planes.

Oriented 4-planes calibrated by $*\phi_0$ are called *coassociative*.

- 4-plane is coassociative iff its orthogonal complement is associative.

Holonomy/parallel tensor correspondence $\Rightarrow$

on any mfd (M, g) with Hol$(g) \subset G_2$ we have parallel 3 and 4-forms ϕ and $*_g\phi$ modelled on ϕ_0 and $*\phi_0$.

$\Rightarrow$ *associative and coassociative calibrations exist on any mfd with holonomy* G_2.

$1+2=3$ and $\mathbb{S}^1 \times holomorphic = associative$

Recall when we decomposed $\mathbb{R}^7$ as $\mathbb{R} \times \mathbb{C}^3$ we had

$$\phi_0 = dt \wedge \omega + \mathrm{Re}\,\Omega$$

Recall, for X a Calabi-Yau 3-fold $\Rightarrow \mathbb{S}^1 \times X$ has holonomy $\mathrm{SU}(3) \subset G_2$

$\mathbb{S}^1 \times C \subset \mathbb{S}^1 \times X$ is associative iff C is a holomorphic curve in X.
Infinitesimal deformations of $\mathbb{S}^1 \times C$ as an associative 3-fold $\leftrightarrow$ infinitesimal deformations of C as a complex curve in X.

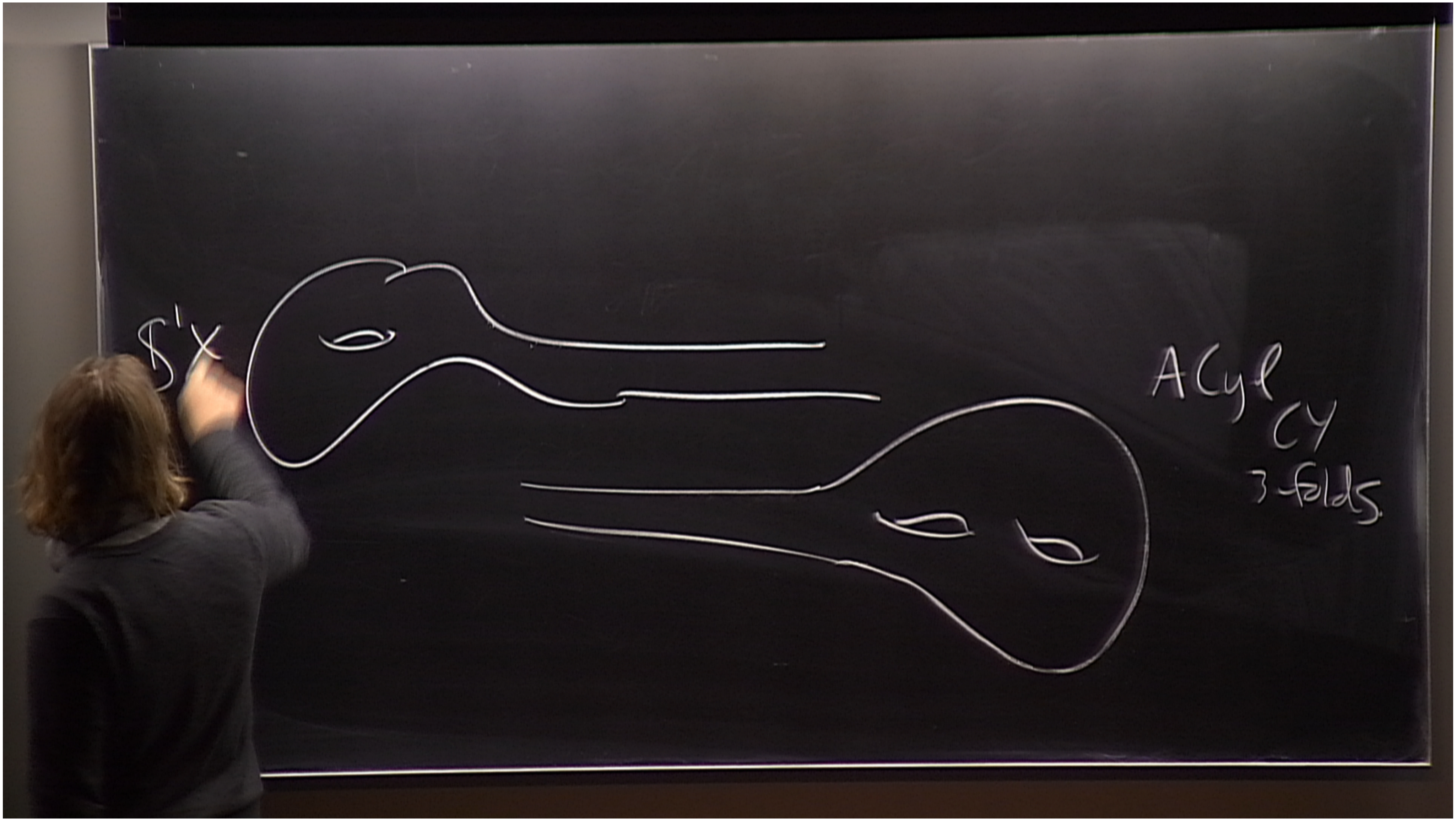
Remark: we also have $\mathbb{S}^1 \times L \subset \mathbb{S}^1 \times X$ is coassociative iff L is a special Lagrangian 3-fold in X.

$\mathbf{SU}(3) + \mathbf{SU}(3) + \epsilon = G_2$

Donaldson suggested constructing compact G_2 manifolds from a pair of asymptotically cylindrical KRF 3-folds via a *neck-stretching* method

- Use noncompact version of Calabi conjecture to construct asymptotically cylindrical Kähler-Ricci-flat (AC KRF) 3-folds X with one end $\sim \mathbb{C}^* \times D$, with D a smooth $K3$
- $M = \mathbb{S}^1 \times X$ is a Riem 7-mfd with $\mathrm{Hol}\, g = \mathrm{SU}(3) \subset G_2$ with end $\sim \mathbb{R}^+ \times T^2 \times K3$.
- Take a *twisted connect sum* of a pair of $M_i = \mathbb{S}^1 \times X_i$
- For $T >> 1$ construct a G_2-structure w/ small torsion (exponentially small in T) and prove it can be corrected to torsion-free.

Kovalev (2003) carried out Donaldson's proposal for AC KRF 3-folds arising from Fano 3-folds.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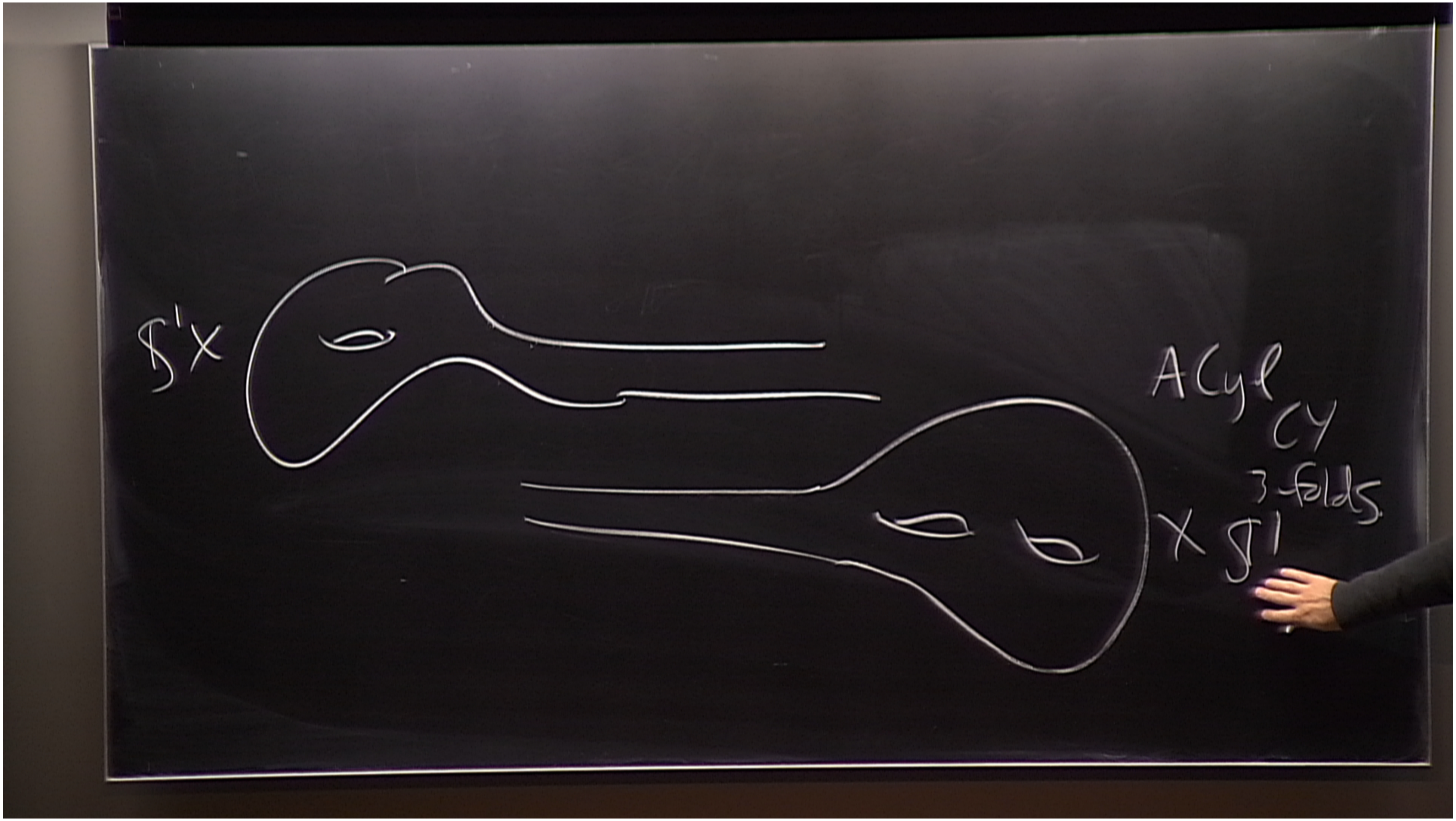
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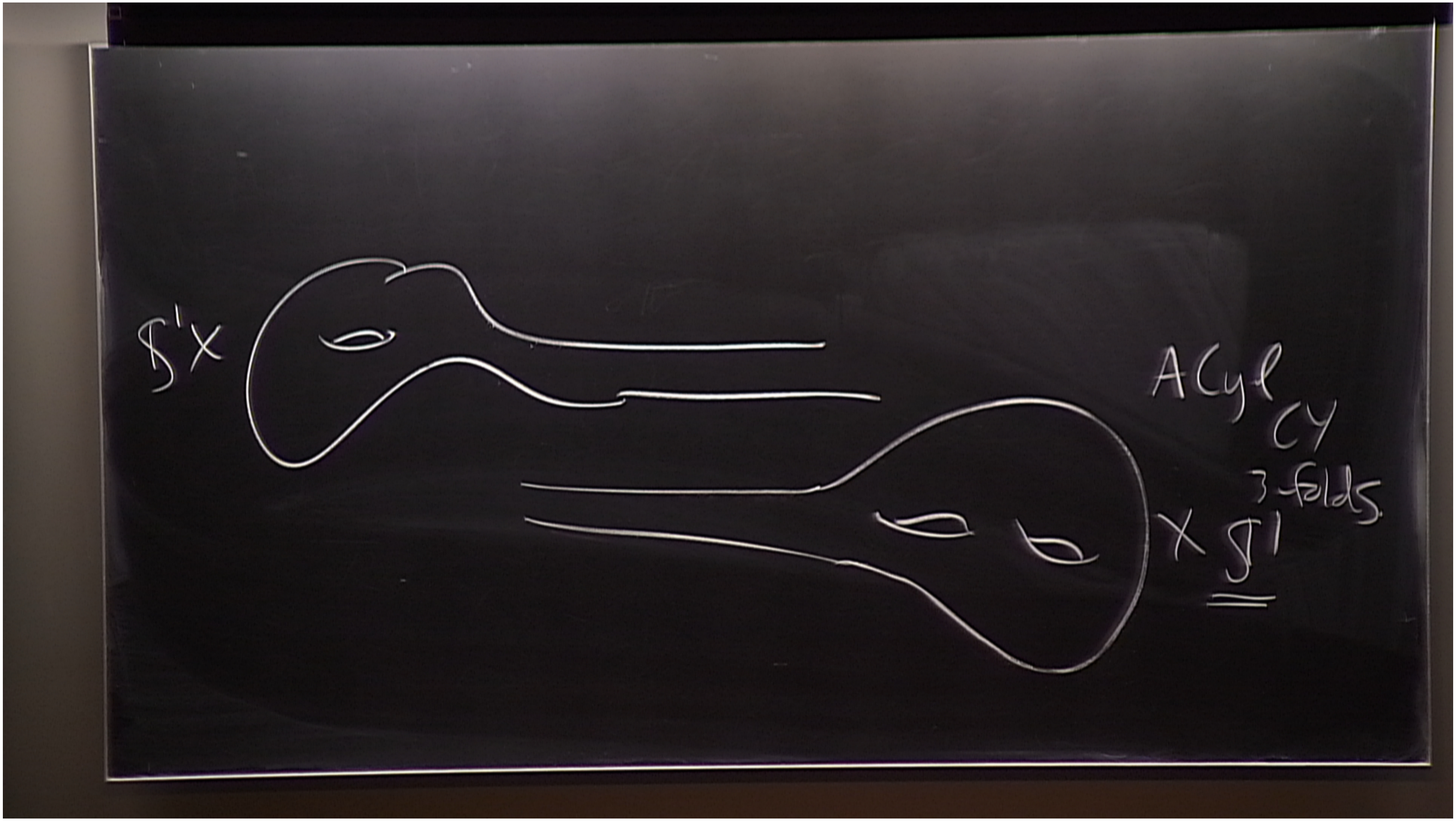
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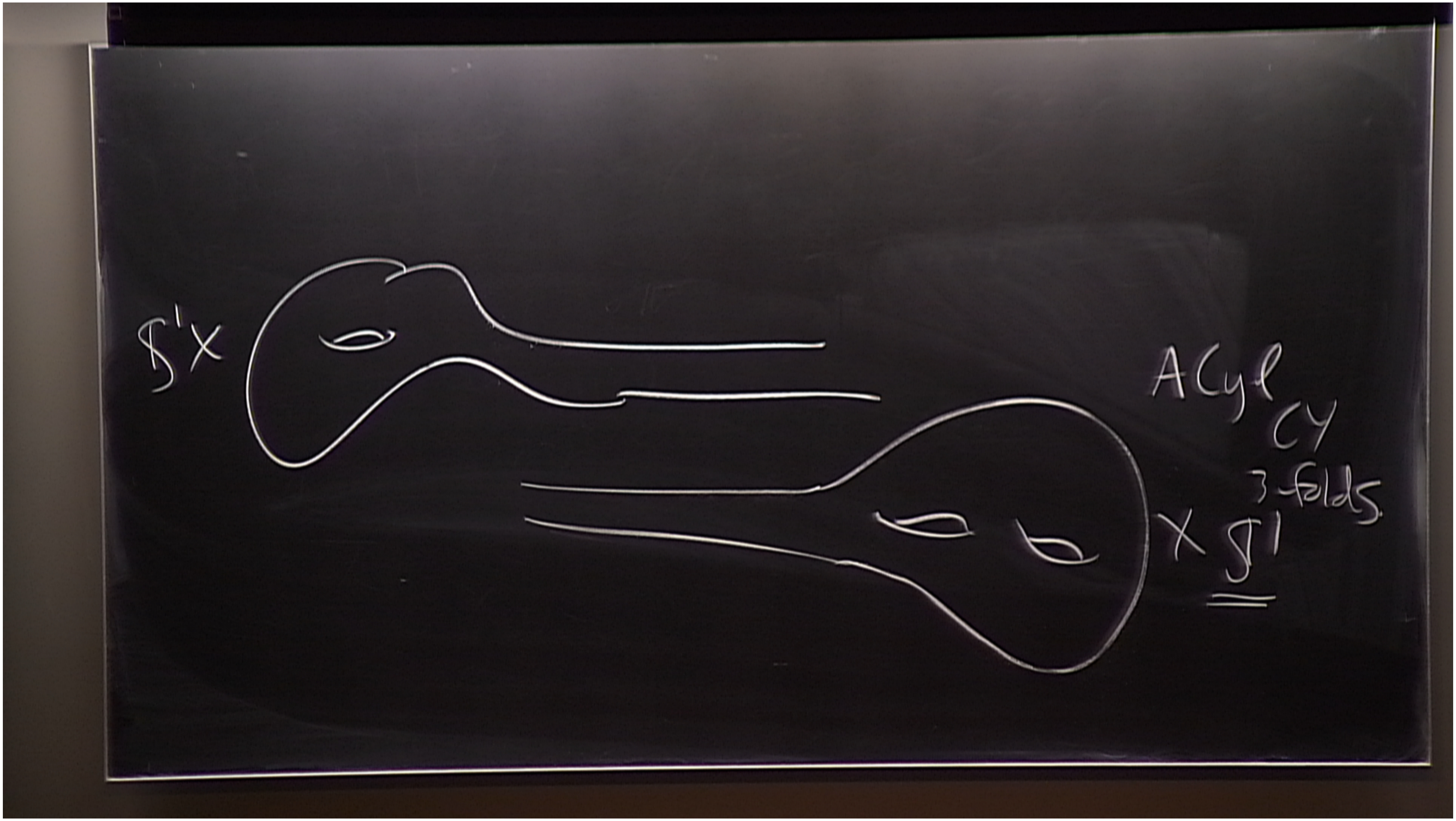

$S^1 \times$
ACyl CY 3-folds
$\times S^1$

Hyperkähler rotation (or matching data)

Product G_2 structure on M_i asymptotic to

$$d\theta_1 \wedge d\theta_2 \wedge dt + d\theta_1 \wedge \omega_I^{(i)} + d\theta_2 \wedge \omega_J^{(i)} + dt \wedge \omega_K^{(i)}$$

- $\omega_I^{(i)}$ denotes Ricci-flat Kähler metric on D_i
- $\omega_J^{(i)} + \sqrt{-1}\,\omega_K^{(i)}$ parallel (2,0)-form on D_i.

To get a well-defined G_2 structure using

$$F : [T-1,T] \times T^2 \times D_1 \to [T-1,T] \times T^2 \times D_2$$

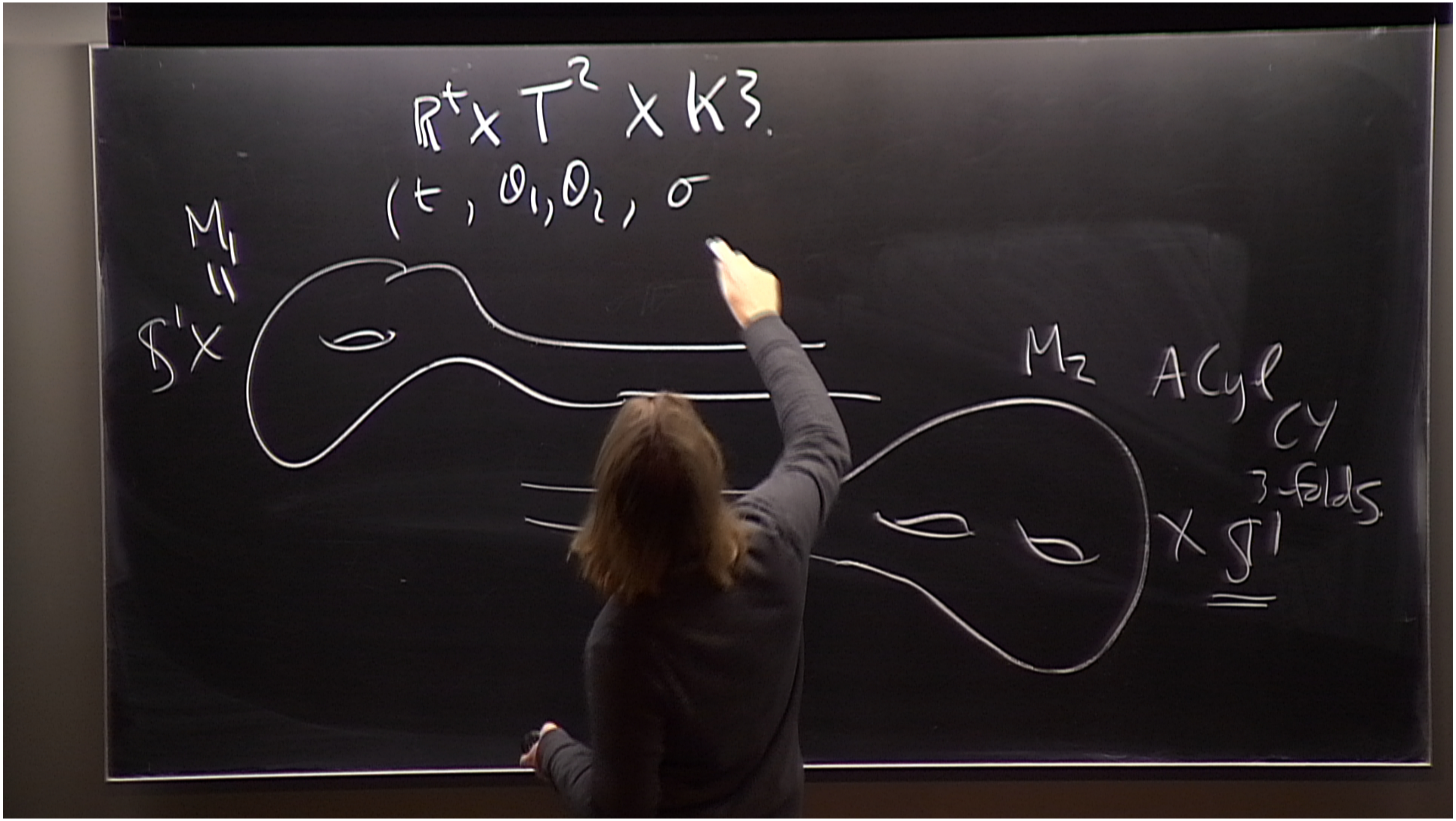
given by

$$(t, \theta_1, \theta_2, y) \mapsto (2T-1-t, \theta_2, \theta_1, f(y))$$

to identify end of M_1 with M_2 we need $f : D_1 \to D_2$ to satisfy

$$f^*\omega_I^{(2)} = \omega_J^{(1)}, \quad f^*\omega_J^{(2)} = \omega_I^{(1)}, \quad f^*\omega_K^{(2)} = -\omega_K^{(1)}$$

Constructing such hyperkähler rotations is nontrivial and a major part of the construction.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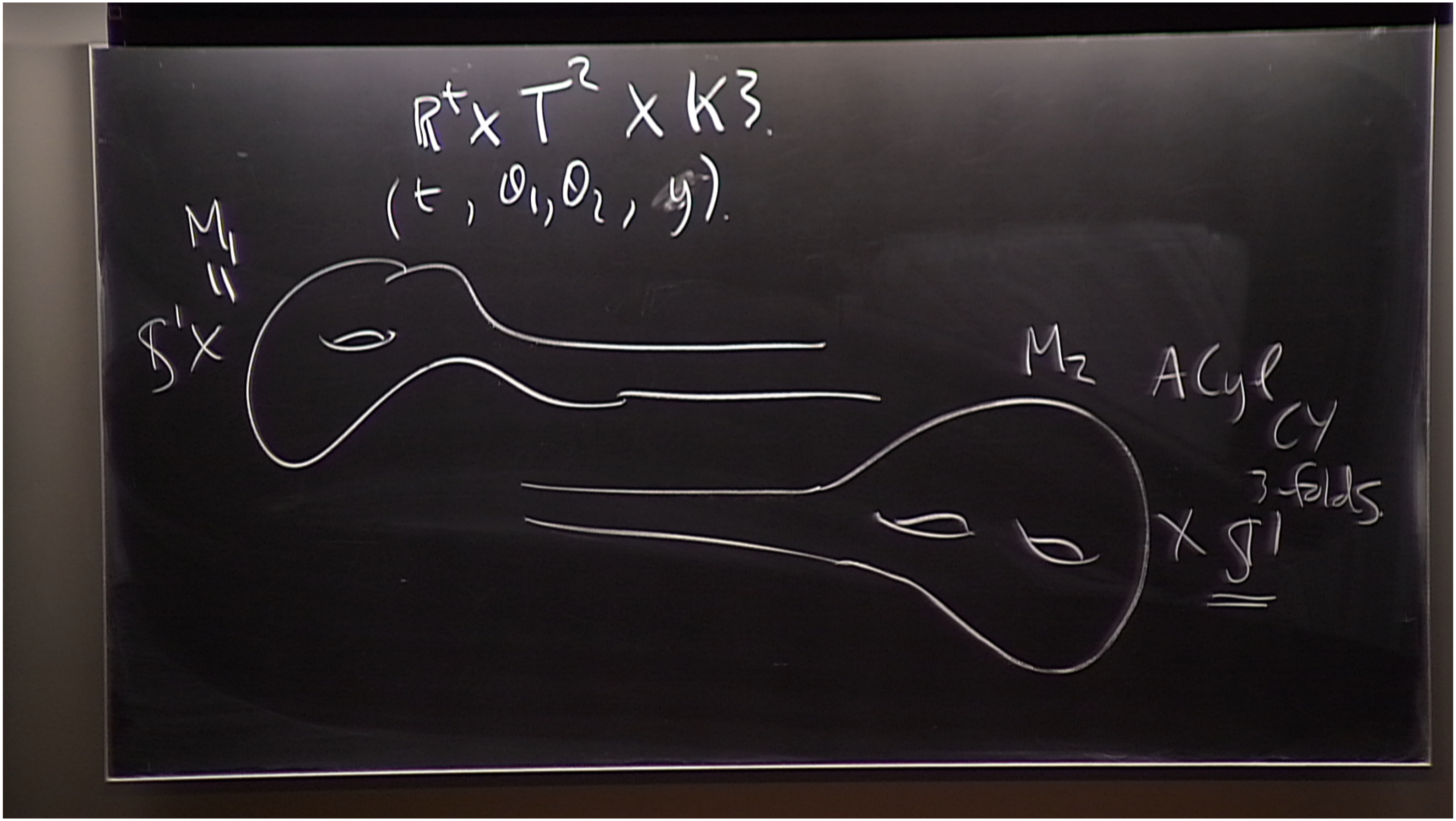

$\mathbb{R}^+ \times T^2 \times K3$.
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M_1
$S^1 \times$
M_2
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Kovalev's twisted connect sum G_2 manifolds

1. Construct suitable asymptotically cylindrical Calabi-Yau 3-folds; Kovalev constructs them from smooth *Fano* 3-*folds*, building on work of Tian-Yau

2. Need to find sufficient conditions for existence of a *hyperkähler rotation* between D_1 and D_2; Use global Torelli theorems and lattice embedding results (Nikulin) to find hyperkähler rotations from suitable initial pairs of Fano 3-folds

3. Given a pair of AC KRF 3-folds X_i and a HK-rotation $f : D_1 \to D_2$ can *always* glue M_1 and M_2 to get a 1-parameter family of cpt manifolds M_T with holonomy G_2.

Given 1 & 2, part 3 always works.

Take-home message: $\Rightarrow$ have reduced solving PDE for G_2-metrics to two problems about complex projective 3-folds.

An asymptotically cylindrical Calabi conjecture

Tian-Yau I (JAMS 1990): The Calabi conjecture on fibred quasiprojective manifolds.

Setup:

- $\overline{X}$ is a projective manifold
- $D \subset \overline{X}$ a divisor
- $\pi : \overline{X} \to \overline{S}$ is a fibre space over a smooth algebraic curve $\overline{S}$ with connected fibres
- $D = \pi^{-1}(D_{\overline{S}})$, $D_{\overline{S}} \subset \overline{S}$ consists of finitely many smooth reduced fibres.

Thm: Let $Z = \overline{X} \setminus D$. Given any (1,1)-form ω representing $c_1(K_{\overline{X}}^{-1} \otimes [D]^{-1})$, there is a complete Kähler metric with ω as its Ricci form and this metric has linear volume growth.

To get Ricci-flat metrics should take $D \in |K_{\overline{X}}^{-1}|$

$\Rightarrow$ *Get complete Calabi-Yau metrics with linear volume growth from anticanonical divisors in fibred quasiprojective varieties.*

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How do we find such $K3$ fibred projective 3-folds?

It suffices to find:

a smooth projective 3*-fold X with $D, D' \in |K_X^{-1}|$ smooth $K3$ surfaces that intersect transversely.*

$D \cap D'$ is a smooth curve C (the base locus of the pencil defined by D and D')

Blowup X along the base locus C to get new projective 3-fold $\overline{X}$

The proper transforms of D and D' are smooth anticanonical divisors on $\overline{X}$ and the pencil they determine gives a morphism $\pi : \overline{X} \to \mathbb{P}^1$ with generic fibre a smooth anticanonical $K3$. Now remove any smooth fibre of π from $\overline{X}$.

Tian-Yau $\Rightarrow \overline{X} \backslash D$ admits a KRF metric with linear volume growth; for gluing purposes Kovalev needed to improve asymptotics of Tian-Yau metric.

Fano and weak Fano 3-folds

Definitions:

1. A smooth Kahler 3-fold X is a *Fano manifold* if K_X^{-1} is ample.

2. A smooth projective 3-fold X is a *weak Fano manifold* if K_X^{-1} is big and nef.

Basic idea: replace condition K_X^{-1} is positive, with K_X^{-1} sufficiently "non-negative".

- A holomorphic line bundle L on X is *nef* if

$$c_1(L).C = \int_C c_1(L) \geq 0$$

for every irreducible (holo) curve $C \subset X$.

- A holomorphic line bundle L on X is *big* if

$$h^0(L^{\otimes m}) \geq Cm^n, \text{ for } m \gg 1, \quad n = \dim_{\mathbb{C}} X.$$

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Fano and weak Fano 3-folds

For *ample* line bundles we have Kodaira vanishing theorem.

For *big and nef* line bundles Kawamata-Viehweg vanishing replaces Kodaira.

Shokurov: on a smooth Fano 3-fold X any sufficiently generic $D \in |K_X^{-1}|$ is a smooth K3 surface.

Reid generalised Shokurov's result to *weak* Fano 3-folds.

Shokurov-Reid + Tian-Yau + Kovalev (improved asymptotics):
$\Rightarrow$ *can construct asymptotically cylindrical KRF metrics from any smooth Fano or weak Fano 3-fold.*

Kovalev used ACyl KRF 3-folds constructed from smooth Fano 3-folds for his twisted connect sum G_2-manifolds; we generalise to (certain classes of) weak Fano 3-folds.

Advantages of weak Fano vs. Fano

1. *Many* more weak Fano than Fano 3-folds
   Fano 3-folds classified: 105 deformation families
   Thousands of weak Fano 3-folds; classification ongoing.

   $\Rightarrow$ get more topological types of ACyl KRF 3-folds and hence (in good cases) compact G_2 mfds

2. In a Fano 3-fold K_X^{-1} is ample:
   $\Rightarrow$ any compact holo curve $C \subset X$ must intersect any anti-canonical divisor

   A *weak Fano* 3-fold can contain holo curves C that do not meet anticanonical divisors.

3. For each smooth rigid $\mathbb{P}^1$ in a weak Fano 3-fold X any G_2 manifold built from X contains a *rigid associative submanifold* w/ topology $S^1 \times S^2$.

Weak-$*$ Fano 3-folds and G_2 manifolds

BUT: need *more* than weak Fano to construct hyperkahler rotation $f : D_1 \to D_2$.

Require a sufficiently good deformation/moduli theory for pair (X, D) where X is a (deformation class of) weak Fano 3-fold and D a smooth anticanonical K3 divisor in X.

Definition: A weak Fano 3-fold is *weak-** if the natural morphism to its anti-canonical model is *small* i.e. contracts only finitely curves to points.

(Also useful to allow larger class of weak Fano 3-folds where AC model is only *semismall*; call these semi-Fano 3-folds).

For weak-* Fano (or semi-Fano) 3-folds can still construct HK rotations. $\Rightarrow$ can use them to construct compact twisted connect sum G_2 manifolds.

Theorem: (Corti-Haskins-Nordström-Pacini) There exist many topological types of compact G_2 manifold which contain *rigid* associative submanifolds diffeomorphic to $S^1 \times S^2$.

Why do we get rigid associatives?

Let C be a cpt holo curve in X not meeting AC divisor D_1

$\rightsquigarrow$ cpt holo curve $C \subset M = \overline{X} \setminus D_1$

$\rightsquigarrow \mathbb{S}^1 \times C$ is cpt associative submfd in $\mathbb{S}^1 \times M$.

• C rigid curve in M iff $\mathbb{S}^1 \times C$ rigid associative 3-fold of $\mathbb{S}^1 \times M$

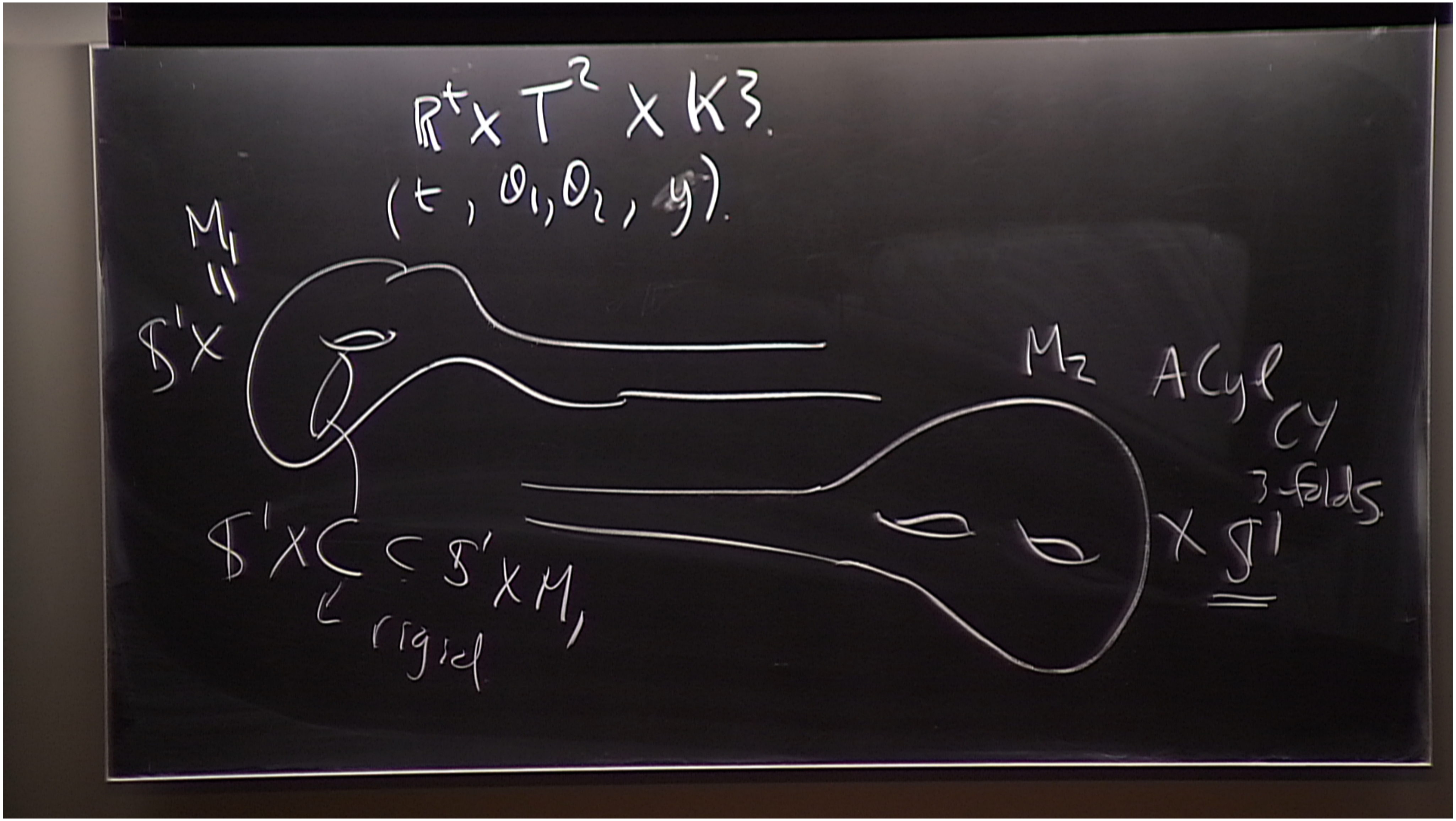
• Since $\mathbb{S}^1 \times C$ is rigid in $\mathbb{S}^1 \times M$, easy to perturb $\mathbb{S}^1 \times C$ to rigid associative 3-fold in glued G_2 structure for $T \gg 1$.

Remarks:

• First examples of *rigid* associative submanifolds in compact G_2 manifolds.

• Infinitesimal deformations of associative submfds $\leftrightsquigarrow$ *twisted harmonic spinors*. Index of twisted Dirac operator is zero since in odd dimension, but hard to control kernel. $\Rightarrow$ deformation theory can be obstructed.

• Can attempt to build invariants of G_2 manifolds by counting associative submfds in a given homology class.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Simple examples of weak-* Fano 3-folds

Example 1: start with a (singular) quartic 3-fold $Y \subset \mathbb{P}^4$ containing a projective plane Π and resolve.

If $\Pi = (x_0 = x_1 = 0)$ then eqn of Y is

$$Y = (x_0 a_3 + x_1 b_3 = 0) \subset \mathbb{P}^4$$

where a_3 and b_3 are homogeneous cubic forms in $(x_0, \ldots, x_4)$. Generically the plane cubics

$$(a_3(0, 0, x_2, x_3, x_4) = 0) \subset \Pi,$$

$$(b_3(0, 0, x_2, x_3, x_4) = 0) \subset \Pi$$

intersect in 9 distinct points, where Y has 9 ordinary double points.

Blowing-up $\Pi \subset Y$ gives a smooth X $f : X \to Y$ is a *projective* small resolution of all 9 nodes of Y.
X is a smooth (projective) weak-* Fano 3-fold; it contains 9 smooth rigid rational curves with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$; X has genus 3 and Picard rank 2.