Title: A Geometric Framework for Integrable Systems

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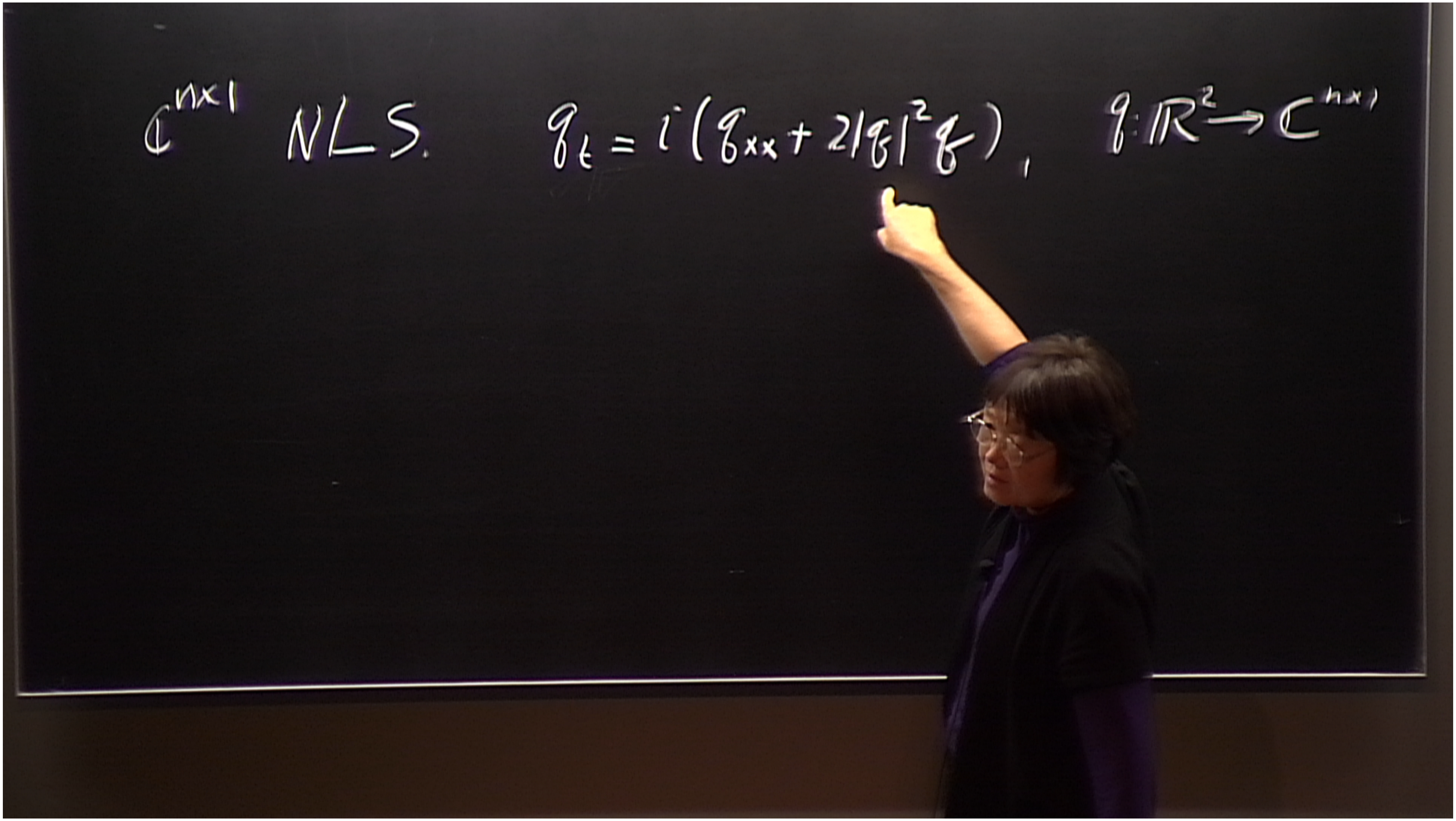
Abstract: I will discuss some joint work with K. Uhlenbeck. There is a general method for constructing soliton hierarchies from a splitting of Lie algebras. We explain how formal scattering and inverse scattering, Hamiltonian structures, commuting conservation laws, B•acklund transformations, tau functions, and Virasoro actions on tau functions can all be constructed in a uni ed way from such splittings.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$q_t = i(q_{xx} + 2|q|^2 q)$,
$q : \mathbb{R}^2 \to \mathbb{C}^{n\times 1}$

A Geometric Framework for Integrable Systems

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Geometry and Physics: GAP 2012

Outline

- Soliton equations in differential geometry
- A general construction of soliton hierarchies from Lie algebra splittings and examples
- Use Lie algebra splittings to derive properties of soliton hierarchies: inverse scattering, Bäcklund transformations, bi-Hamiltonian, tau functions, Virasoro action

(Joint work with Karen Uhlenbeck)

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(Joint work with Karen Uhlenbeck)

(I) Soliton equations in DG

- The Gauss-Codazzi eqs for surfaces in $\mathbb{R}^3$ with $K = -1$, for conformally flat hypersurfaces in $\mathbb{R}^4$, Egroff metrics, isometric immersions of space forms in space forms, ... etc.
- Harmonic maps from $\mathbb{R}^{1,1}$ or $\mathbb{R}^2$ to symmetric spaces.
- Schrödinger map from $\mathbb{R}^1 \times \mathbb{R}^1$ to Hermitian symmetric spaces: $\gamma_t = J_\gamma(\nabla_{\gamma_x}\gamma_x)$. For example, eq for Schrödinger map eq from $\mathbb{R}^1 \times \mathbb{R}^1$ to S^2 or to hyperbolic $\mathbb{H}^2$ is equivalent to the focusing or defocusing NLS $q_t = i(q_{xx} \pm 2|q|^2 q)$ resp.
- YM field on $\mathbb{R}^4$ and $\mathbb{R}^{2,2}$ and monopole equations.
- The generating function of the quantum cohomology of a point is given by the tau function of the KdV that is fixed by the Virasoro action – Witten's conjecture proved by Kontsevich.

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(II) Constructing soliton hierarchies from Lie algebra splittings

Zakharov-Shabat, AKNS, Adler, Adler-van Moerbeke, Gelfand-Dikki, Kuperschmidt-Wilson, Drinfeld-Sokolov, ... developed methods to construct soliton hierarchies from Lie algebras.

Below we give a version given by Terng-Uhlenbeck:

Let L be a formal Lie group with subgroups L_+, L_- such that $L_+ \cap L_- = \{e\}$, and its Lie algebras $\mathcal{L} = \mathcal{L}_+ \oplus \mathcal{L}_-$ as linear subspaces. We call $L_\pm$ a *splitting* of L and $\mathcal{L}_\pm$ a splitting of $\mathcal{L}$.

$\mathcal{J} = \{J_j \mid j \geq 1\} \subset \mathcal{L}_+$ is a *vacuum sequence* if

1. $\mathcal{J}$ is commuting and linearly independent,
2. J_1 generates $\mathcal{J}$ in the enveloping algebra.

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We will construct a soliton hierachy as flows on $C^\infty(\mathbb{R}, \mathcal{M})$ from a splitting $\mathcal{L}_\pm$ and a vacuum sequence $\{J_j \mid j \geq 1\}$, where

$$\mathcal{M} = \{(gJ_1g^{-1})_+ \mid g \in L_-\}.$$

Here for $\xi \in \mathcal{L}$, we write $\xi = \xi_+ + \xi_- \in \mathcal{L}_+ \oplus \mathcal{L}_-$.

Soliton hierarchy

Given a smooth $\xi : \mathbb{R} \to \mathcal{M}$, there is $M : \mathbb{R} \to L_-$ such that $\xi = (MJ_1M^{-1})_+$. This is equivalent to

$$\partial_x - \xi = M(\partial_x - J_1)M^{-1}.$$

The flow on $C^\infty(\mathbb{R}, \mathcal{M})$ defined by J_j is

$$\frac{\partial \xi}{\partial t_j} = [\partial_x - \xi, (MJ_jM^{-1})_+],$$

or equivalently, written as a Lax pair

$$[\partial_x - \xi, \partial_{t_j} - (MJ_jM^{-1})_+] = 0.$$

Theorem. These flows on $C^\infty(\mathbb{R}, \mathcal{M})$ commute.

Soliton hierarchy

Given a smooth $\xi : \mathbb{R} \to \mathcal{M}$, there is $M : \mathbb{R} \to L_{-}$ such that $\xi = (MJ_1M^{-1})_{+}$. This is equivalent to

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The $\mathbb{C}^{n\times 1}$ NLS hierarchy

Splitting:

$$\mathcal{L} = \{A(\lambda) = \sum_{i \le n_0} A_i \lambda^i \text{ for some } n_0 \mid A_i \in u(n+1)\},$$

$$\mathcal{L}_+ = \{A \in \mathcal{L} \mid A(\lambda) = \sum_{j \ge 0} A_j \lambda^j\}, \quad \mathcal{L}_- = \{A \in \mathcal{L} \mid A(\lambda) = \sum_{j<0} A_j \lambda^j\}.$$

Let $a = \mathrm{diag}(i\mathrm{I}_n, -i)$, and $J_j(\lambda) = a\lambda^j$. Then

$$\mathcal{J} = \{J_j \mid j \ge 1\}$$

is a vacuum sequence. The flows in the hierarchy are for maps $u = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}$ with $q \in \mathbb{C}^{n\times 1}$:

$$q_{t_1} = q_x,$$

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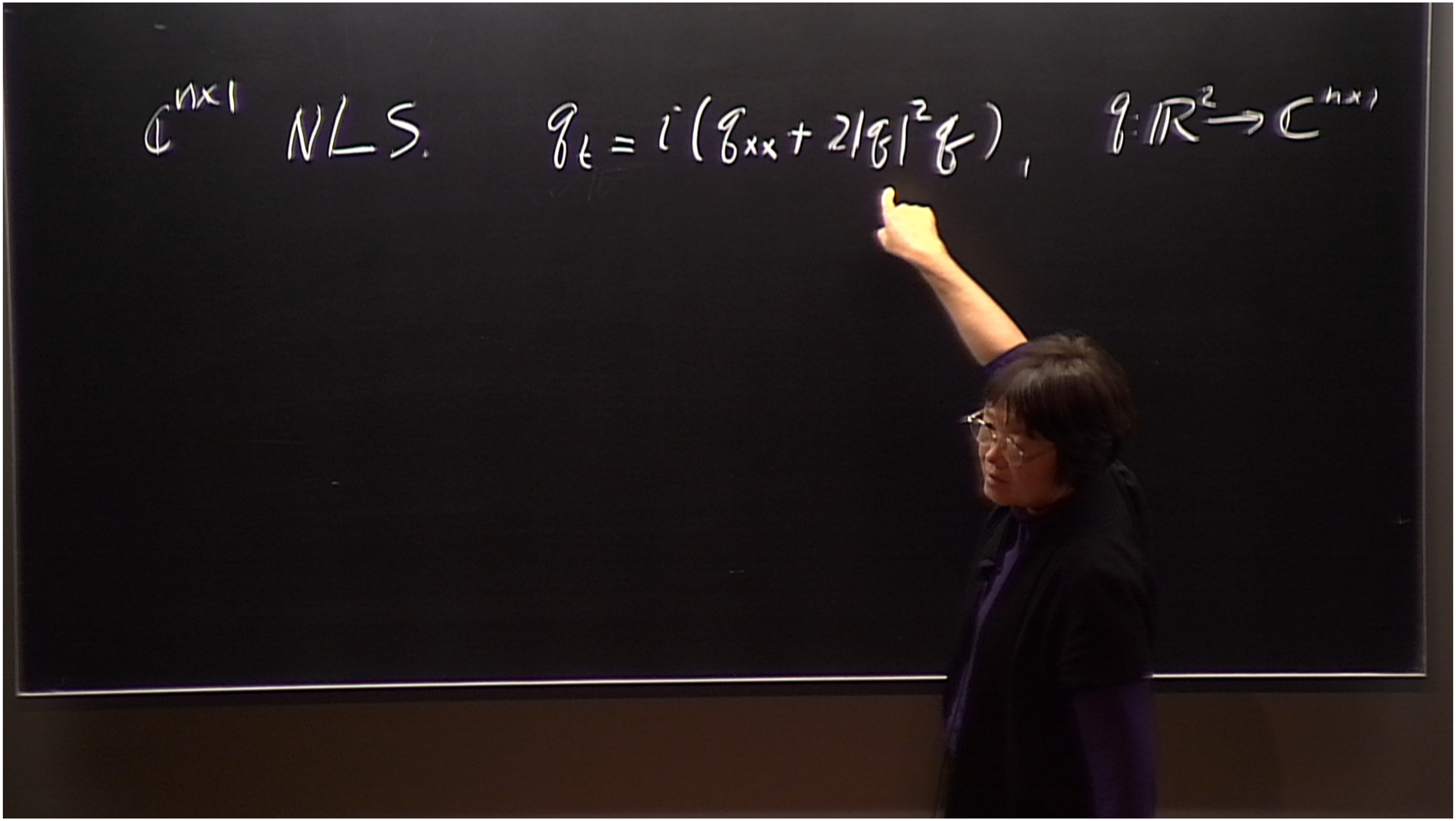
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KdV type hierarchies

Let $\mathcal{N}_-$ and $\mathcal{B}_-$ denote the subalgebras of strictly lower triangular and lower triangular matrices in $sl(n,\mathbb{C})$, and $B: sl(n,\mathbb{C}) \to \mathcal{N}_-$ a linear map such that $\mathrm{Ker}(B) = \mathcal{B}_-$ and

$$B([\xi, B(\eta)] + [B(\xi), \eta]) = B([\xi, \eta])$$

for all $\xi, \eta \in sl(n,\mathbb{C})$.

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$$\mathcal{L}_-^B = \{A = \sum_{i\leq 0} A_i \lambda^i \mid A_0 = B(A_{-1})\}$$

is a subalgebra of $\mathcal{L} = \mathcal{L}(sl(n,\mathbb{C}))$.

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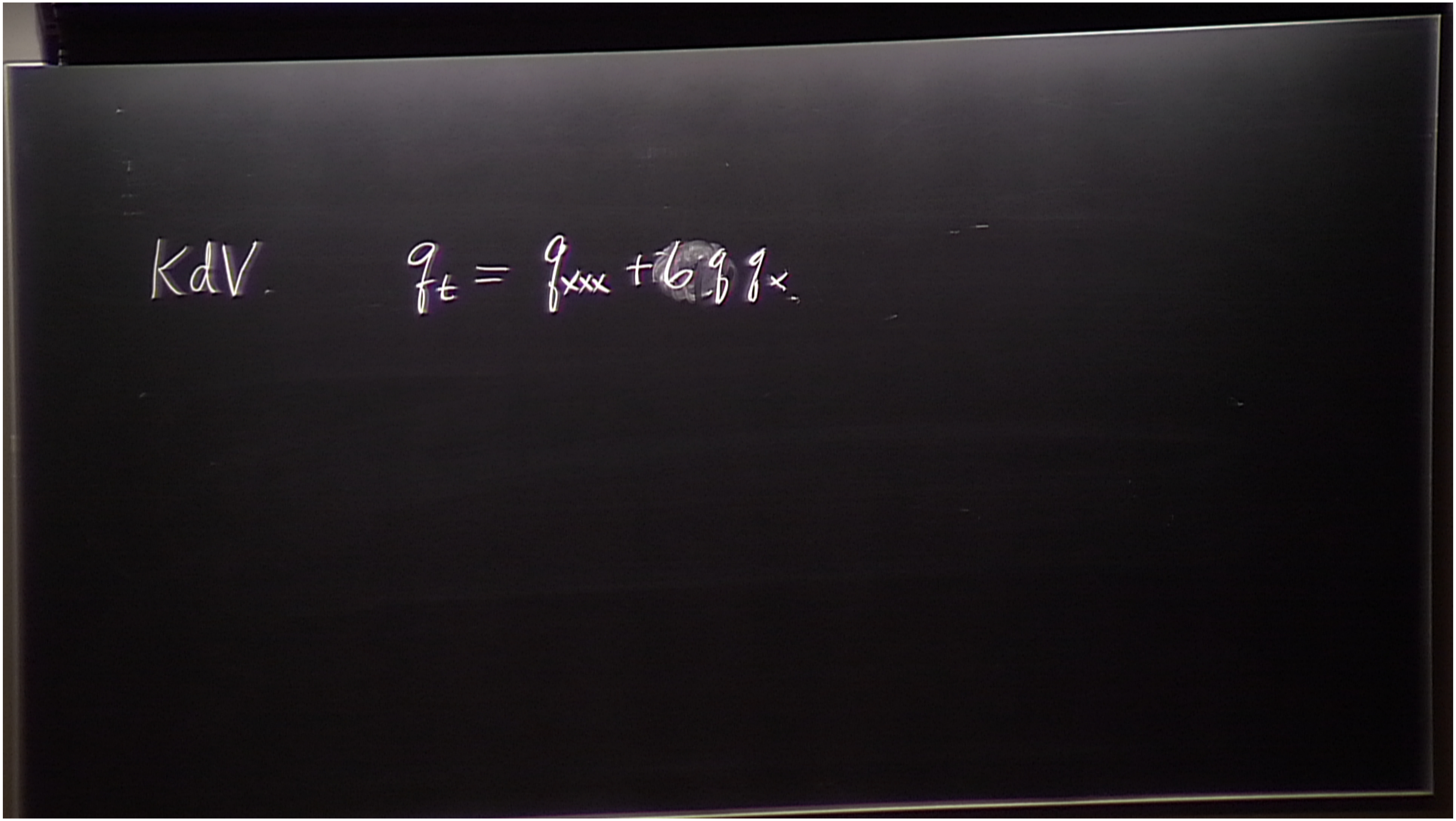
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Set

$$J = e_{n1}\lambda + e_{12} + \cdots + e_{n-1,n}.$$

Then

$$\{J^j \mid j \neq 0 \pmod{n}\}$$

is a vacuum sequence.

We call the hierarchy constructed from these data the $n \times n$ KdV_B hierarchy.

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- When $n = 2$, there are exactly two such B's: $B(e_{12}) = \pm e_{21}$. The flow generated by J^3 is the KdV $q_t = q_{xxx} + 6qq_x$.
- (Di-Terng) When $n = 3$, we found all such B's and there are 13 families. One of these families is:

$$B(e_{12}, e_{23}, e_{13}) = (e_{21}, e_{32}, e_{31}) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ c & 1-c & 0 \end{pmatrix}$$

The flow in this 3×3 KdV$_B$-hierarchy generated by J^2 is

$$\begin{cases} (u_1)_t = \frac{1}{3}(2c-1)(u_1)_{xx} + \frac{2}{3}(u_2)_x, \\ (u_2)_t = \frac{1}{3}(1-2c)(u_2)_{xx} + 6u_1(u_1)_x + \frac{2}{3}(-c^2+c-1)(u_1)_{xxx}. \end{cases}$$

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- When $n = 2$, there are exactly two such B's: $B(e_{12}) = \pm e_{21}$. The flow generated by J^3 is the KdV $q_t = q_{xxx} + 6qq_x$.
- (Di-Terng) When $n = 3$, we found all such B's and there are 13 families. One of these families is:

$$B(e_{12}, e_{23}, e_{13}) = (e_{21}, e_{32}, e_{31}) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ c & 1-c & 0 \end{pmatrix}$$

The flow in this 3×3 KdV$_B$-hierarchy generated by J^2 is

$$\begin{cases} (u_1)_t = \frac{1}{3}(2c-1)(u_1)_{xx} + \frac{2}{3}(u_2)_x, \\ (u_2)_t = \frac{1}{3}(1-2c)(u_2)_{xx} + 6u_1(u_1)_x + \frac{2}{3}(-c^2+c-1)(u_1)_{xxx}. \end{cases}$$

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(III) Derive properties of soliton hierarchy from splittings

Formal inverse scattering

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$$V(x,t_j) f^{-1} = M^{-1}(x,t_j) E(x,t_j)$$

with $E(x,t_j) \in L_+$ and $M(x,t_j) \in L_-$. Then

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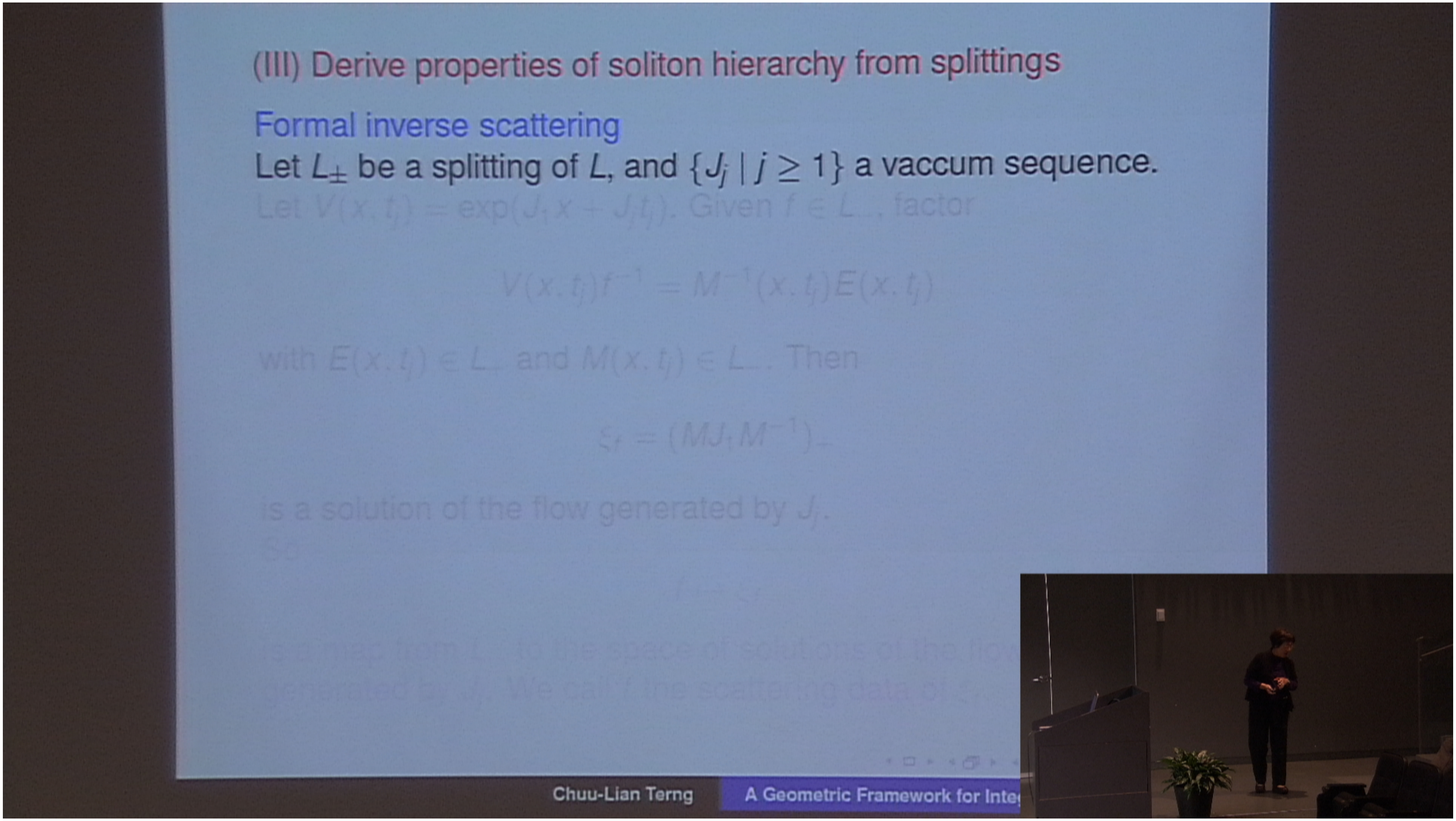
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L_- action on the solution space

The left multiplication of L_- induces an action of L_- on the space of solutions of the flows: For $f, g \in L_-$,

$$g * \xi_f = \xi_{gf}.$$

Bäcklund transformations and pure solitons

If L is a subgroup of the loop group $L(G)$ and g is rational, then this action can be computed using residue calculus. Moreover,

1. the action of a rational loop in L_- with smallest number of poles gives a Bäcklund-Darboux type transformations on the space of solutions,
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Bi-hamiltonian

If $\mathcal{L}$ has a sequence of ad-invariant non-degenerate bilinear form $(\,,\,)_k$, then we can naturally embed $\mathcal{M}$ in $\mathcal{L}_-^*$ via $(\,,\,)_k$ as Poisson submanifolds and the induced Poisson structures are compatible. For example, $(\xi,\eta)_k = \mathrm{res}(\lambda^k \mathrm{tr}(\xi(\lambda)\eta(\lambda))$ is ad-invariant on $\mathcal{L}(sl(n))$.

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Tau functions defined by G. Wilson 1991

Central extension of L

Suppose w is a 2-cocycle on $\mathcal{L}$ such that $w \mid \mathcal{L}_\pm = 0$. Let $\hat{L}$ be the central extension of L given by w. Then $\hat{L}$ is a principal $\mathbb{C}^*$ bundle $\hat{L} \to L$ with $c_1(\hat{L}) = w$.

Since $w \mid L_\pm = 0$, there exists a section S from $L_+ \cup L_-$ to $\hat{L}$.

Let $t = (t_1, \ldots, t_N)$, and $V(t) = \exp(\sum_{j=1}^N t_j J_j)$. Given $f \in L_-$, The tau function is a $\mathbb{C}^*$-valued function defined on an open subset of $t = 0$ in $\mathbb{R}^N$ by

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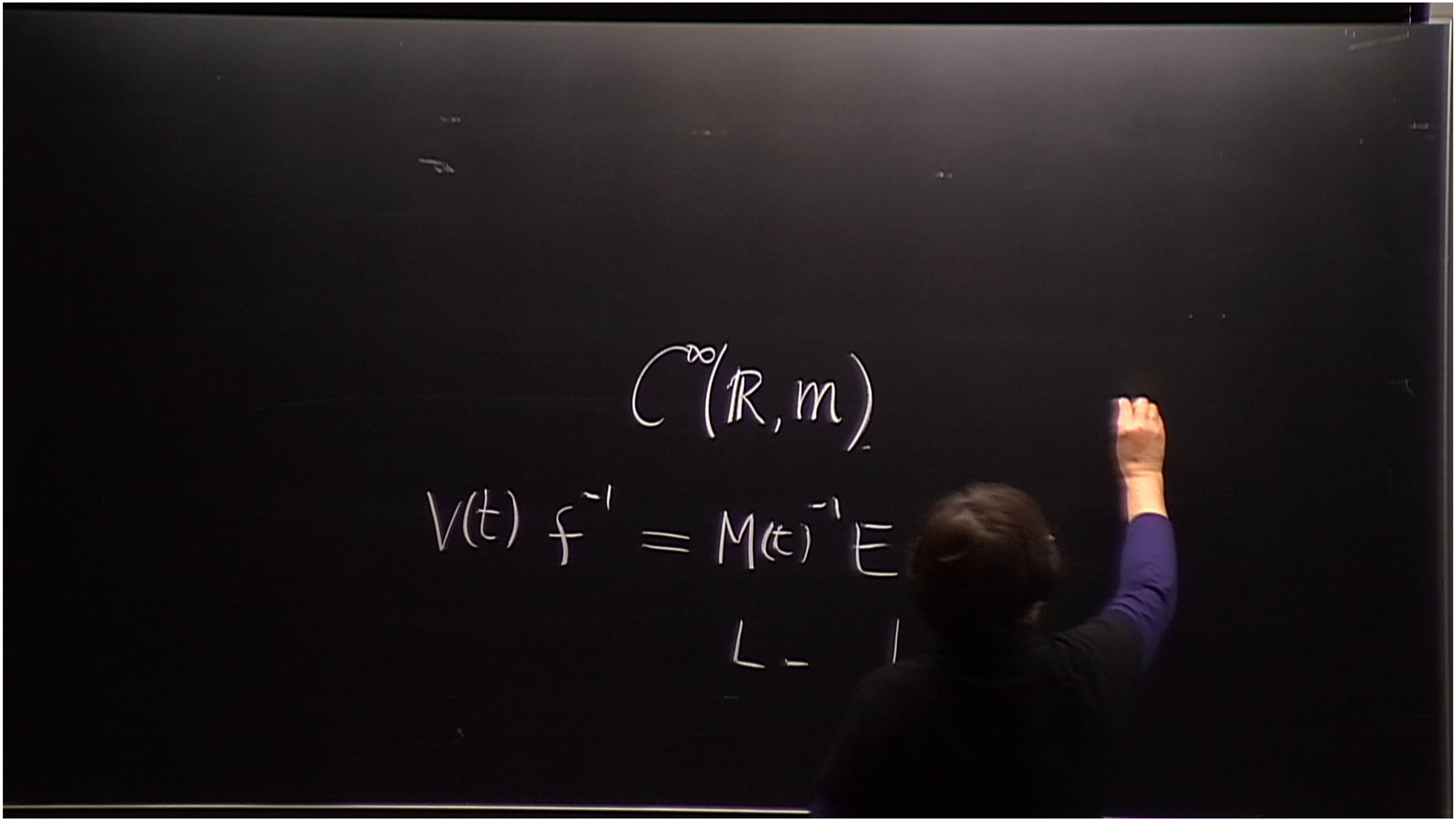
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$f \in L_- \rightsquigarrow \xi_f$ solution of soliton hierarchy

$\tau_f(t)$

$C^\infty(\mathbb{R}, \mathfrak{m})$

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1. For $f \in L_-$, we have
   - $(\ln \tau_f)_{t_j} = \langle M^{-1}M_\lambda, J_j \rangle$,
   - $(\ln \tau_f)_{t_j t_k} = \langle MJ_j M^{-1}, \frac{\partial}{\partial \lambda}((MJ_k M^{-1})_+) \rangle$, where $\langle \xi, \eta \rangle = \mathrm{res}\,(\mathrm{tr}(\xi(\lambda)\eta(\lambda)))$.
2. If $f(s) \in L_-$, then

$$\frac{\partial}{\partial s} \ln \tau_{f(s)} = -\langle M_s M^{-1}, E_\lambda E^{-1} \rangle.$$

Can we recover solution ξ_f from τ_f?

The NLS hierarchy: Let $k = \mathrm{diag}(e^{i\theta}, e^{-i\theta})$ be a constant, and $f \in L_-$. Then $\xi_{kfk^{-1}} = k\xi_f k^{-1}$. In fact, if q is a solution of the NLS $q_t = \frac{i}{2}(q_{xx} + 2|q|^2 q)$, then $e^{2i\theta} q$ is also a solution. But $\tau_{kfk^{-1}} = \tau_f$. So τ_f can only recover the solution ξ_f up to this S^1 symmetry.

KdV hierarchy: $q_f = (\ln \tau_f)_{t_1 t_1}$ is the solution corresponding to $f \in L_-$.

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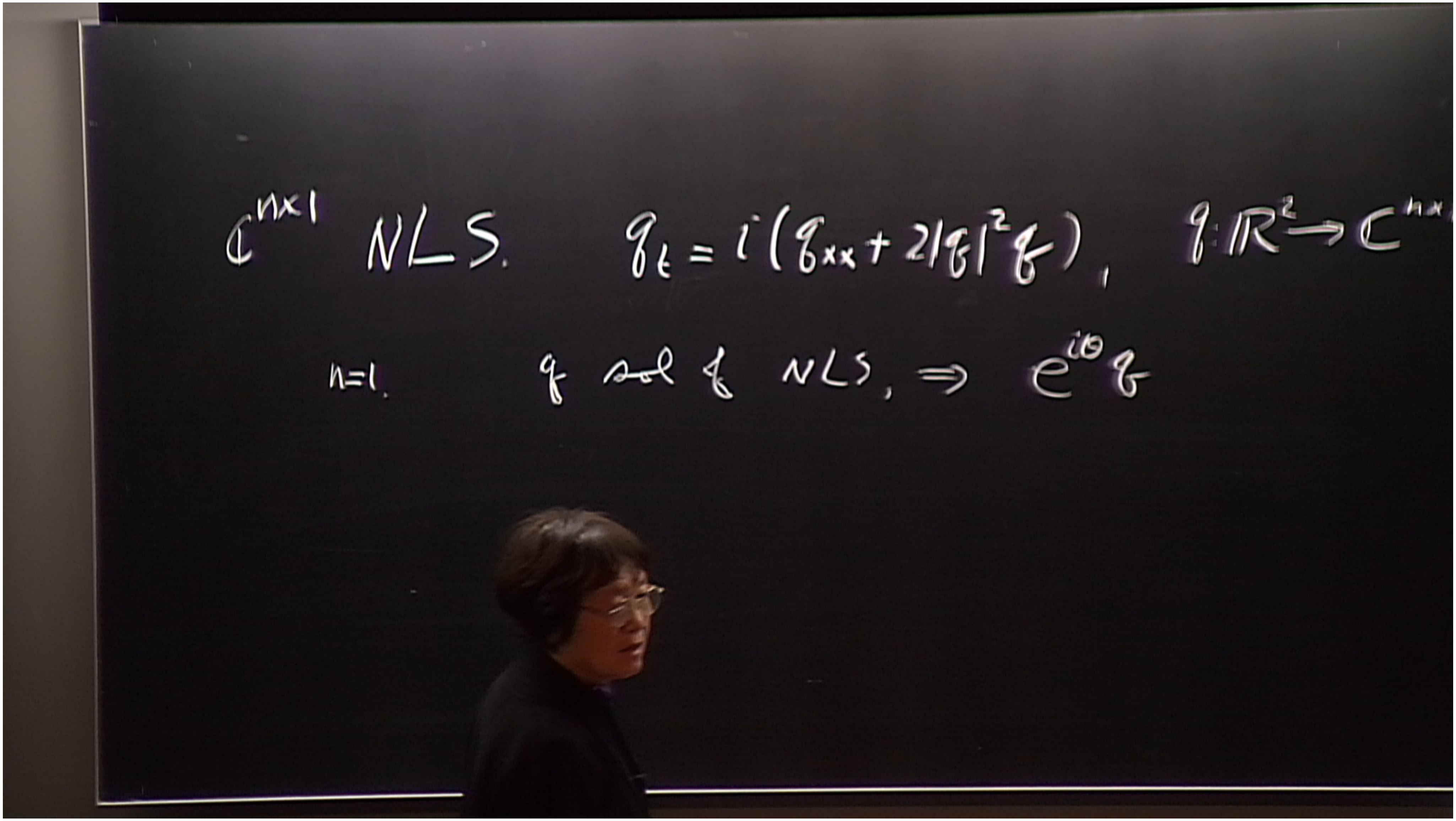
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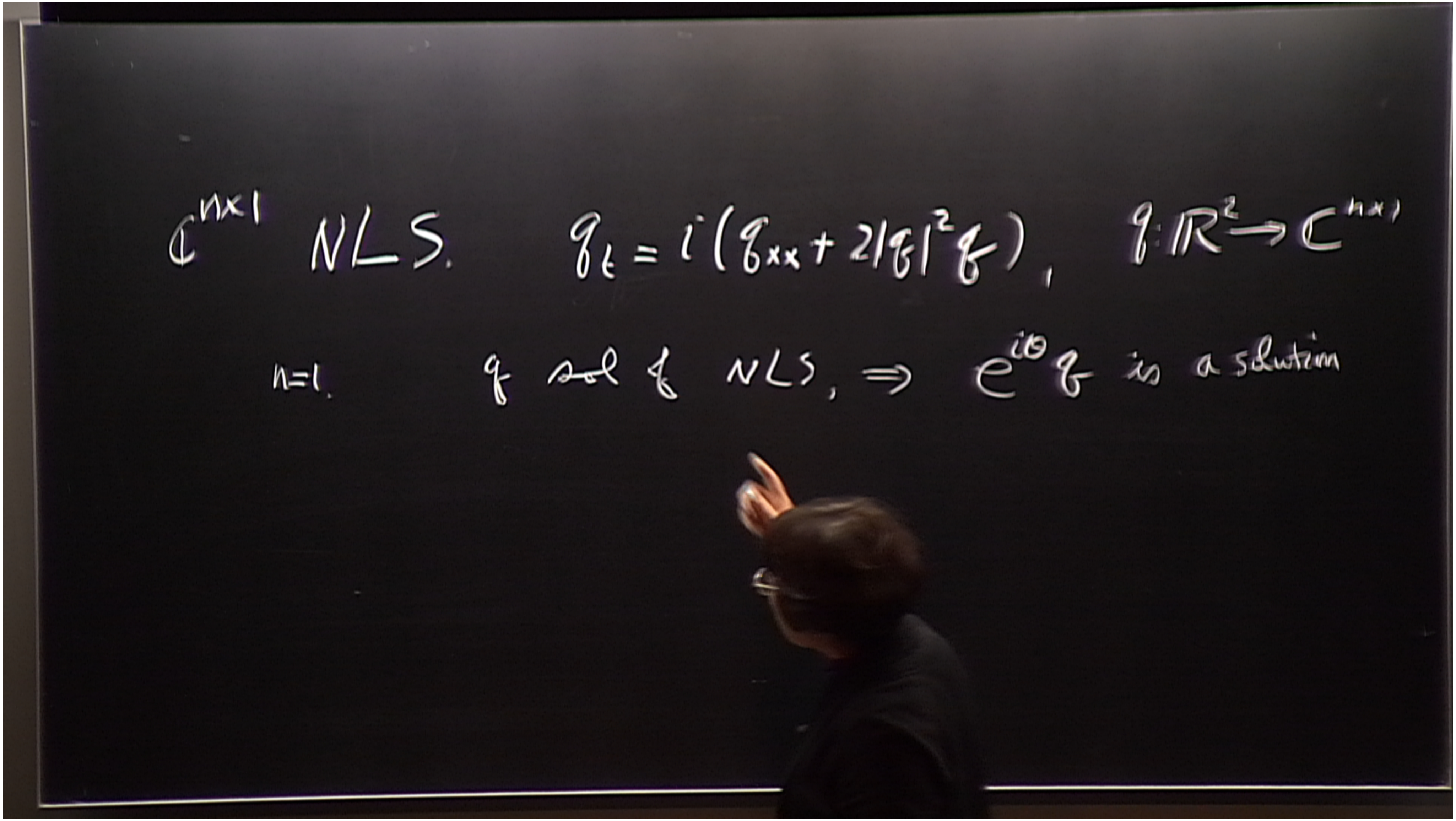
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KdV. $q_t = q_{xxx} + 6qq_x$, $F: C^\infty(\mathbb{R},\mathbb{R}) \to \mathbb{R}$.

$q_t = (\nabla F(q))_x$

$F(q) = \int_{-\infty}^{\infty} \left(\frac{-1}{2} q_x^2 + q^3\right) dx$

$\nabla F(q) = q_{xx} + 3q^2$

$q_t = (q_{xx} + 3q^2)_x$

$q_\xi = (\ln \tau_\xi)_{t_1 t_1}$ sol of KdV.

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Virasoro algebra

Let $D_+(S^1)$ denote the subgroup of diffeomorphisms of S^1 that is the boundary value of a holomorphic map from $|\lambda| < 1$ to $GL(1)$. The Virasoro algebra $\mathcal{V}$ is the Lie subalgebra of the Lie algebra of $D_+(S^1)$ generated by $\{\xi_j \mid j \in \mathbb{Z}\}$, where $\xi_j = \lambda^{j+1}\frac{\partial}{\partial\lambda}$.

Note that

$$[\xi_j, \xi_k] = (k - j)\xi_{k+j}.$$

Let $\mathcal{V}_+$ denote the subalgebra of $\mathcal{V}$ generated by $\{\xi_j \mid j \geq -1\}$.

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Theorem (T-U 2012)
Let $L_\pm$ be a splitting of $L(GL(n))$ with $L_+ = L_+(GL(n))$, and $C : S^1 \to GL(n)$ a group homomorphism. Given $k \in D_+(S^1)$ and $f \in L_-$, define

$$(k \diamond f)(\lambda) = f(k^{-1}(\lambda)) C\left(\frac{k^{-1}(\lambda)}{\lambda}\right),$$

$$(k * f) = g_-, \quad \text{where } k \diamond f = g_+ g_- \in L_+ L_-.$$

Then $\natural$ defines an action of $D_+(S^1)$ on L_-. Moreover, the infinitesimal vector field corresponding to $\xi_j \in \mathcal{V}_+$ is

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We choose homomorphism $C : S^1 \to GL(n)$ so that $\lambda(J_1)_\lambda + [J_1, C'(1)] = J_1$. Then the $\mathcal{V}_+$ action on L_- induces an action on tau functions.

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Some open problems

- Is there a systematic way to decide whether a geometric PDE is an integrable system?
- If a PDE has a Lax pair with a spectral parameter, is there systematic way to find a Lie algebra splitting that gives the PDE? This will give the symmetry group of the PDE.
- Understand the space of solutions of SDYM on $\mathbb{R}^{2,2}$ and their reductions with non-compact real and complex gauge groups.
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