Title: Killing-Yano Symmetry: New Results and Applications

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Abstract: After introducing Killing-Yano tensors and their basic properties, I will concentrate on their applications to black hole physics. Namely, I will focus on two topics: i) symmetries of the Dirac operator in curved background and ii) generalized Killing-Yano tensors in the presence of skew-symmetric torsion and the classification of corresponding Euclidean metrics.


Killing-Yano symmetry: new results and applications
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Symmmetries in GR

- Killing vectors - isometries of spacetime

$$\xi_{(\mu;\nu)} = 0.$$

Noether theorem: conserved quantities

particle: integrals of motion linear in momenta $$C_1 = \xi_\mu u^\mu$$

- Stackel-Killing tensors

M. Walker and R. Penrose, Comm. Math. Phys. 18 , 265 (1970).

$$K_{\mu\nu} = K_{(\mu\nu)}\,, \qquad K_{(\mu\nu;\lambda)} = 0\,.$$

particle: integrals of motion of higher order in momenta $$C_2 = K_{\mu\nu} u^\mu u^\nu$$


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Killing-Yano tensors

for a general differential form

$$\nabla\omega \propto \text{exterior} + \text{divergence} + \text{symmetric parts}$$

Conformal Killing-Yano (CKY) tensor

$$\nabla_X \boldsymbol{k} = \frac{1}{p+1} \boldsymbol{X} \lrcorner \, d\boldsymbol{k} - \frac{1}{D-p+1} \boldsymbol{X}^\flat \wedge \delta \boldsymbol{k} \,.$$

Killing-Yano (KY) tensor: divergence part is missing

closed CKY tensor exterior part is missing

Under Hodge duality divergence part transforms into exterior part and vice versa.

$$*(\text{closed CKY}) \propto \text{KY}.$$

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Principal Killing-Yano (PKY) tensor

= (non-degenerate) closed CKY 2-form

$$\nabla_X \boldsymbol{h} = \boldsymbol{X}^\flat \wedge \boldsymbol{\xi}\,.$$

$$\nabla_X h_{ab} = 2X_{[a}\,\xi_{b]}$$

It follows

$$dh = 0\,,$$

$$\xi_b = \frac{1}{D-1}\nabla_a h^a{}_b$$

Let us postulate the existence of this 2-form and find the consequences (less restrictive than Kahler 2-form)

Canonical metric element

a) Darboux basis:

$$g = \delta_{ab}\omega^{\hat{a}}\omega^{\hat{b}} = \sum_{\mu=1}^{n}(\omega^{\hat{\mu}}\omega^{\hat{\mu}} + \tilde{\omega}^{\hat{\mu}}\tilde{\omega}^{\hat{\mu}}) + \varepsilon\omega^{\hat{0}}\omega^{\hat{0}},$$

$$h = \sum_{\mu=1}^{n} x_\mu \omega^{\hat{\mu}} \wedge \tilde{\omega}^{\hat{\mu}}.$$

$$D = 2n + \varepsilon\,.$$

(Euclidean & PKY is non-degenerate)

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b) Towers of symmetries:

construction based on the following Lemma:

Lemma ([Krtouš *et al.*, 2007b]). *Let $\boldsymbol{k}^{(1)}$ and $\boldsymbol{k}^{(2)}$ be two closed CKY tensors. Then their exterior product $\boldsymbol{k} \equiv \boldsymbol{k}^{(1)} \wedge \boldsymbol{k}^{(2)}$ is also a closed CKY tensor.*

P. Krtouš, DK, D.N. Page, V.P. Frolov, Killing-Yano Tensors, Rank-2 Killing Tensors, and Conserved Quantities in Higher Dimensions, JHEP 0702 (2007) 004.

Towers of hidden symmetries:

closed CKY tensors:

$$h^{(j)} \equiv h^{\wedge j} = \underbrace{h \wedge \ldots \wedge h}_{\text{total of } j \text{ factors}} \,.$$

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Killing-Yano tensors:

$$f^{(j)} \equiv * h^{(j)} .$$

$$\nabla_{(\alpha_1} f_{\alpha_2)\alpha_3 \ldots \alpha_{p+1}} = 0 .$$

Killing tensors:

$$K^{(j)}_{ab} \equiv \frac{1}{(D-2j-1)!(j!)^2} f^{(j)}{}_{ac_1 \ldots c_{D-2j-1}} f^{(j)}{}_{b}{}^{c_1 \ldots c_{D-2j-1}} .$$

$$K^{(j)} = \sum_{\mu=1}^{n} A^{(j)}_{\mu} (\omega^{\hat{\mu}} \omega^{\hat{\mu}} + \tilde{\omega}^{\hat{\mu}} \tilde{\omega}^{\hat{\mu}}) + \varepsilon A^{(j)} \omega^{\hat{0}} \omega^{\hat{0}} .$$

$$\nabla_{(a} K^{(j)}_{bc)} = 0$$

where

$$A^{(j)} = \sum_{\nu_1 < \cdots < \nu_j} x^2_{\nu_1} \ldots x^2_{\nu_j} , \quad A^{(j)}_{\mu} = \sum_{\substack{\nu_1 < \cdots < \nu_j \\ \nu_i \neq \mu}} x^2_{\nu_1} \ldots x^2_{\nu_j} ,$$

Tower of explicit symmetries

Killing co-potentials:

$$\omega^{(j)}_{ab} = \frac{1}{n-2j-1} K^{(j)}_{ac} h^{c}{}_{b}\,, \quad \omega^{(N)} = \frac{\sqrt{-c}}{N!} * b^{(N)}$$

$$\omega^{(j)} = \frac{1}{n-2j-1} \sum_{\mu=1}^{N} x_\mu A^{(j)}_\mu \omega^{\hat\mu} \wedge \tilde\omega^{\hat\mu}$$

Killing vectors:

$$\xi^{(k)} = -\delta\omega^{(k)}$$

$$\xi_{(\mu;\nu)} = 0.$$

(Primary Killing vector $\xi^{(0)} \equiv \xi$)

Kerr-NUT-(A)dS spacetime

$$X_\mu = X_\mu(x_\mu).$$

Einstein space condition:

$$R_{ab} = (-1)^n (D-1) c_n \, g_{ab} \,,$$

implies the specific form of metric functions:

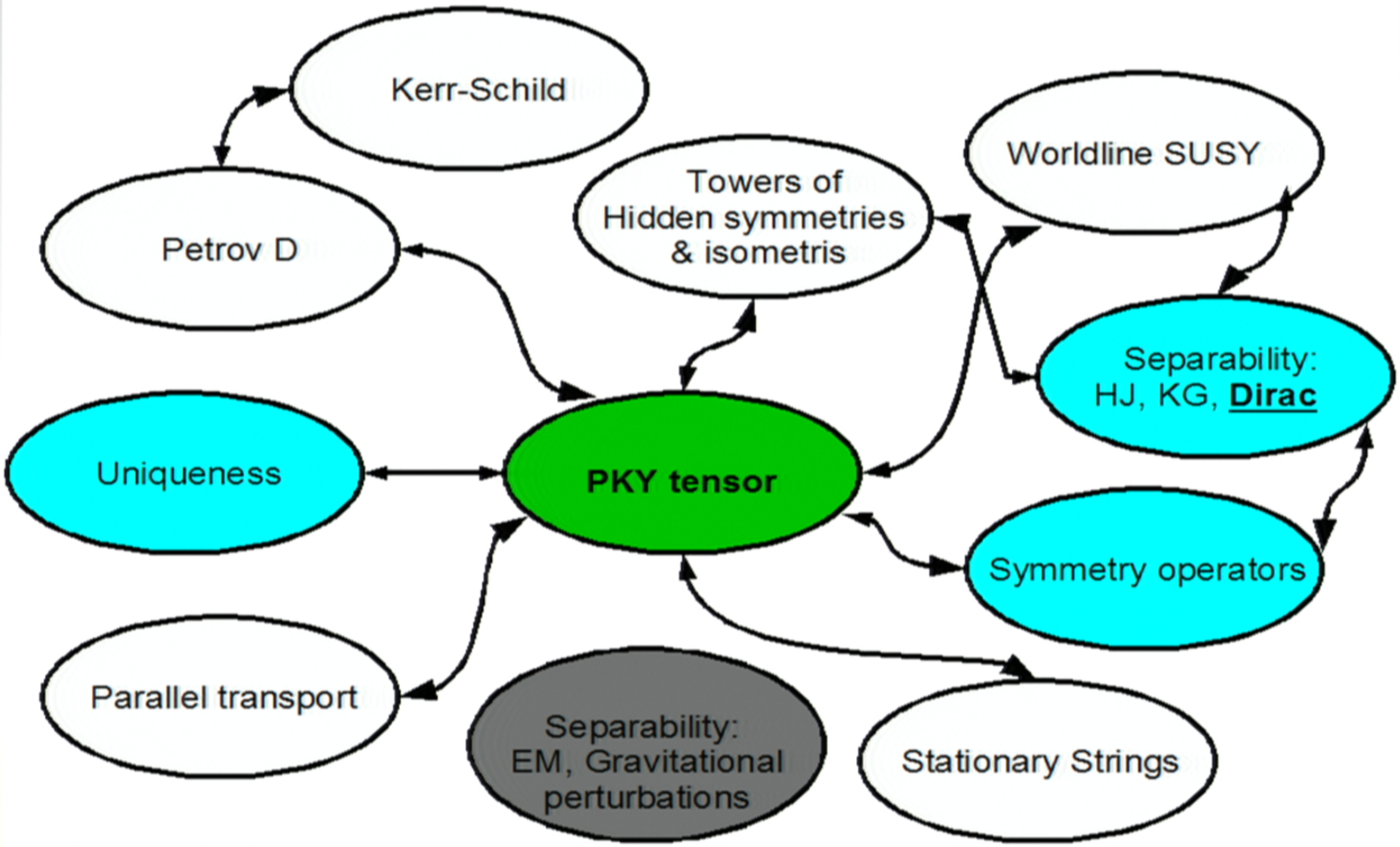
$$X_\mu = \sum_{k=\varepsilon}^{n} c_k x_\mu^{2k} - 2 b_\mu x_\mu^{1-\varepsilon} + \frac{\varepsilon c}{x_\mu^2} \,.$$

W. Chen, H. Lü and C. N. Pope, Class. Quant. Grav. 23 , 5323 (2006).

Constants are related to mass, NUT parameters, rotations, and cosmological constant

- T. Houri, T. Oota, Y. Yasui, Closed conformal Killing-Yano tensor and Kerr-NUT-de Sitter spacetime uniqueness, Phys.Lett.B656 (2007) 214.
- P. Krtouš, V. P. Frolov, DK, Hidden Symmetries of Higher Dimensional Black Holes and Uniqueness of the Kerr-NUT-(A)dS spacetime, Phys. Rev. D 78 (2008) 064022.

Miraculous properties of Kerr-NUT-AdS spacetimes



II) Dirac equation and its symmetry operators

Notations

- orthonormal basis for TM, T*M: $\{X_a\}$ $\{e^a\}$
- contracted wedge product

$$\alpha \underset{0}{\wedge} \beta = \alpha \wedge \beta \,, \qquad \alpha \underset{k}{\wedge} \beta = (X^a \lrcorner\, \alpha) \underset{k-1}{\wedge} (X_a \lrcorner\, \beta)$$

- Levi-Civita connection $\nabla_a \equiv \nabla_{X_a}$ $d = e^a \wedge \nabla_a \,,$ $\delta = -X^a \lrcorner\, \nabla_a$

- Inhomogeneous forms:

$$\alpha = \sum_{p=0}^{n} \alpha_p \,, \quad \alpha_p \in \Lambda^p(M)$$

we define the following 2 operations:

$$\pi\alpha \equiv \sum_{p=0}^{n} p\alpha_p \,, \quad \eta\alpha \equiv \sum_{p=0}^{n} (-1)^p \alpha_p$$

Spinors:

- We identify the elements of the Clifford algebra with differential forms

$$\omega = \frac{1}{p!}\omega_{a_1\ldots a_p} e^{a_1} \wedge \cdots \wedge e^{a_p} \longrightarrow \frac{1}{p!}\omega_{a_1\ldots a_p}\gamma^{[a_1} \ldots \gamma^{a_p]}.$$

- and denote the Clifford multiplication by juxtaposition. For a 1-form α and p-form ω this is defined as

$$\alpha\omega = \alpha \wedge \omega + \alpha \underset{1}{\wedge} \omega\,, \qquad \omega\alpha = (-1)^p \big(\alpha \wedge \omega - \alpha \underset{1}{\wedge} \omega\big)$$

(equivalent to $\gamma^a\gamma^b + \gamma^b\gamma^a = 2\eta^{ab}$)

I. M. Benn and R. W. Tucker, An introduction to spinors and geometry with applications in physics. Adam Hilger, Bristol, 1987.

Dirac equation (with fluxes)

$$\mathcal{D}\psi = 0$$

generalized Dirac operator:

$$\mathcal{D} = e^a \nabla_a + B\,.$$

(B arbitrary inhomogeneous form)

in γ-matrix notations:

$$\mathcal{D} = \gamma^a \nabla_a + \sum_p \frac{1}{p!} B_{a_1 \dots a_p} \gamma^{a_1} \dots \gamma^{a_p}$$

Motivation: This includes the case of a massive Dirac operator, the Dirac operator minimally coupled to a Maxwell field, the Dirac operator in the presence of torsion, as well as more general operators. (In the backgrounds considered for superstring or supergravity theories, the metric is often supplemented by other fields or fluxes which couple to the spinor field and modify the Dirac equation.)

Dirac symmetry operators

- B. Carter and R. G. McLenaghan, Phys. Rev. D19 (1979) 1093{1097.
- R. McLenaghan and P. Spindel, Phys. Rev. D 20 (1979) 409.
- N. Kamran and R. G. McLenaghan, Phys. Rev. D30 (1984) 357.
- I. Benn & P. Charlton, CQG 14 (1997) 1037; I. Benn & J. Kress, CQG 21 (2004) 427.

Separability of the Dirac equation

Not well established! We need a complete set of (linear?) mutually commuting operators.

Proposition. *The most general first-order operator S which commutes with the Dirac operator D, $[D, S] = 0$, splits into the Clifford even and Clifford odd parts*

$$S = S_e + S_o \,, \tag{2.4}$$

where

$$S_e = K_{\omega_o} \equiv X^a \lrcorner\, \omega_o \nabla_a + \frac{\pi - 1}{2\pi} d\omega_o \,, \qquad \text{with } \omega_o \text{ being an odd KY form} \,, \tag{2.5}$$

$$S_o = M_{\omega_e} \equiv e^a \wedge \omega_e \nabla_a - \frac{n - \pi - 1}{2(n - \pi)} \delta\omega_e \,, \qquad \text{with } \omega_e \text{ being an even CCKY form} \,. \tag{2.6}$$

Application 1: Algebra of KY tensors?

Killing tensors

Killing tensors form an algebra with respect to (symmetric) ***Schouten -Nijenhuis brackets***:

$$[K_p, K_q]^{a_1 a_2 \dots a_{p+q-1}}$$

Example:

$$[K_{(i)}, K_{(j)}]^{abc}_{\mathrm{SN}} \equiv K^{e(a}_{(i)} \nabla_e K^{bc)}_{(j)} - K^{e(a}_{(j)} \nabla_e K^{bc)}_{(i)}$$

$$[\partial_{\psi_j}, K_{(i)}]^{ab}_{\mathrm{SN}} \equiv \mathcal{L}_{\partial_{\psi_j}} K^{ab}_{(i)}$$

(Naïve) idea

- Symmetry operators L do not necessary close on themselves under commutation

Their commutator, $[L_1, L_2] = L_1 L_2 - L_2 L_1$

commutes with D: $[[L_1, L_2], D] = 0$ However, in general, it is not

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  of the first order (This led people to study higher-order symmetry operators of the Dirac equation.)

- The requirement $[L_1, L_2] = L_3$ imposes (unfortunately very strong)

  algebraic conditions on ω.

- When these are satisfied we have:

$$[K_\kappa, K_\lambda] = K_{[\kappa,\lambda]_{\mathrm{KY}}}\ ,\ [K_\mu, M_\omega] = M_{[\mu,\omega]_{\mathrm{KY}}}\ ,\ [M_\alpha, M_\beta] = K_{[\alpha,\beta]_{\mathrm{KY}}}$$

  where new tensors are given in terms of Killing-Yano brackets. Some of them related to SN brackets.

- At the moment we do not know any nontrivial examples, but…

Application 2: Complete set of commuting operators

Proposition. *The most general metric admitting the PKY tensor admits the following complete set of commuting operators:*

$$\{D, K_{\xi^{(0)}}, \dots K_{\xi^{(N-1+\varepsilon)}}, M_{h^{(1)}}, \dots M_{h^{(N-1)}}\} \,. \qquad (3.24)$$

Here, D is the Dirac operator, $K_{\xi^{(k)}}$ are operators (2.5) corresponding to Killing forms $\xi^{(k)}$, and $M_{h^{(i)}}$ are operators (2.6) connected with even CCKY forms $h^{(i)}$. In odd dimensions the canonical spacetimes admit another complete set of commuting operators, given by

$$\{D, K_{\xi^{(0)}}, \dots K_{\xi^{(N-1+\varepsilon)}}, K_{f^{(1)}}, \dots K_{f^{(N-1)}}\} \,. \qquad (3.25)$$

Algebraic conditions are satisfied and KY brackets are trivial!

M. Cariglia, P. Krtous, DK, Commuting symmetry operators of the Dirac equation, Killing-Yano and Schouten-Nijenhuis brackets, Phys. Rev. D84 (2011) 024004.

Consequence: Separability of massive Dirac equation

Common eigenfunction

$$K_k \psi = i\, \Psi_k \psi \ ,$$
$$M_j \psi = \mathcal{X}_j \psi \ ,$$

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R-Separability:

- Choice of Gamma matrices

$$\gamma^{\mu} = \iota_{\langle 1 \ldots \mu-1 \rangle} \sigma_{\langle \mu \rangle} \, , \quad \gamma^{\hat{\mu}} = \iota_{\langle 1 \ldots \mu-1 \rangle} \hat{\sigma}_{\langle \mu \rangle} \, ,$$
$$\gamma^{0} = \iota_{\langle 1 \ldots N \rangle} \, ,$$

- Separation ansatz:

$$\psi = R \, \exp\big(i \sum_k \Psi_k \psi_k\big) \bigotimes_{\nu} \chi_{\nu}$$

$\{\chi_\nu\}$ Is an N-tuple of 2D spinors $\chi_\nu = \chi_\nu(x_\nu)$

$$R = \prod_{\substack{\kappa,\lambda \\ \kappa<\lambda}} \left(x_\kappa + \iota_{\langle \kappa\lambda \rangle} x_\lambda\right)^{-\frac{1}{2}}$$

2D spinors satisfy ODEs:

$$\left[\left(\frac{d}{dx_\nu} + \frac{X'_\nu}{4X_\nu} + \frac{\tilde{\Psi}_\nu}{X_\nu}\iota_{\langle\nu\rangle} + \frac{\varepsilon}{2x_\nu}\right)\sigma_{\langle\nu\rangle} - \frac{\left(-\iota_{\langle\nu\rangle}\right)^{N-\nu}}{\sqrt{|X_\nu|}}\left(\varepsilon\frac{i\sqrt{-c}}{2x_\nu^2} + \tilde{\mathcal{X}}_\nu\right)\right]\chi_\nu = 0\,.$$

Dirac separation: T. Oota, Y. Yasui, Phys. Lett. B659, 688-693, 2008.

Motivation

• All "miraculous" properties connected with KY tensors limited to the canonical spacetime (vacuum, type D)

• One would like to extend to wider class of non-vacuum spacetimes, for example to BHs of various supergravities (presence of 3-form, …)

• It is known that some of these solutions possess Killing tensors and allow separability of HJ and KG equations (D.D.K. Chow, 0811.1264) . In fact that is how some of these solutions were "guessed".

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Systematic derivation

By studying symmetry operators of "flux-modified" Dirac operator

$$\mathcal{D} = e^a \nabla_a + B\,. \qquad L = 2\omega^a \nabla_a + \Omega$$

Generalized conformal Killing-Yano system:

$$K_a \omega + \{B, \omega_a\}_\perp = 0\,.$$

(Coupled system of linear first order partial differential equations for homogeneous parts of inhomogeneous form ω. Decouples if B is a combination of a function, 1-form and 3-form.)

KY tensors in the presence of torsion

$$B = iA - \frac{1}{4}T$$

⇒ Generalized conformal Killing-Yano system becomes:

$$\nabla^T_X \boldsymbol{k} - \frac{1}{p+1} \boldsymbol{X} \lrcorner\, \boldsymbol{d}^T \boldsymbol{k} + \frac{1}{D-p+1} \boldsymbol{X}^\flat \wedge \boldsymbol{\delta}^T \boldsymbol{k} = 0\,.$$

- DK, H.K. Kunduri, Y. Yasui, Phys. Lett. B678 (2009) 240.
- Yano & Bochner, Curvature and Betti numbers, 1952.

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PKY with torsion

non-degenerate rank-2 d^T-closed

$$\nabla^T_X h = X^\flat \wedge \xi \qquad \xi = -\frac{1}{D-1}\delta^T h$$

Classification of Euclidean metrics

- Darboux basis

$$g = \sum_{\mu=1}^{n} (e^{\mu} \otimes e^{\mu} + e^{\hat{\mu}} \otimes e^{\hat{\mu}}) + \varepsilon\, e^0 \otimes e^0$$

$$h = \sum_{\mu=1}^{n} x_{\mu}\, e^{\mu} \wedge e^{\hat{\mu}} \,.$$

+ Torsion

- Fix freedom

$$\xi = \sum_{\mu=1}^{n} \sqrt{Q_{\mu}}\, e^{\hat{\mu}} + \varepsilon \sqrt{Q_0}\, e^0$$

- Restrict connection using integrability conditions of PKY equation, Jacobi identity,...obtain admissible form of torsion and DE restricting the form of basis 1-forms.

Punchline: PKY tensor h still generates all Killing tensors as in torsion-less case, but in there are no Killing vectors!

Explicit example of a metric with no Killing vectors
(T. Houri, D. Kubiznak, C. Warnick, Y. Yasui, arXiv:1203.0393)

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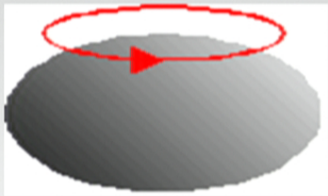
Examples

- **Chong-Cvetic-Lu-Pope black hole**

(Phys. Rev. Lett 95 (2005) 161301)



$$\pounds = *(R+\Lambda) - \frac{1}{2}\boldsymbol{F}\wedge *\boldsymbol{F} + \frac{1}{3\sqrt{3}}\boldsymbol{F}\wedge\boldsymbol{F}\wedge\boldsymbol{A}\,.$$

identify $\boldsymbol{T} = \frac{1}{\sqrt{3}} *\boldsymbol{F}\,.$ $\Rightarrow$ $\boldsymbol{\delta}^T\boldsymbol{T} = 0\,, \quad \boldsymbol{d}^T\boldsymbol{T} = 0\,.$

(Torsion is "harmonic")

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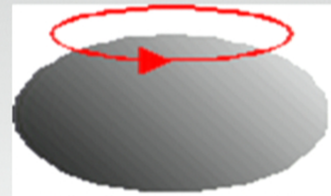

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- **Charged Kerr-NUT-AdS (Kerr-Sen black hole)**

(A. Sen, Phys. Rev. Lett 69 (1992) 1006; Cvetic&Youm; Chow)

$$S = \int e^{\phi}\Big(*\,\mathcal{R} + *d\phi\wedge d\phi - *F\wedge F - \frac{1}{2}*H\wedge H\Big)$$

identify $\boldsymbol{T} = \boldsymbol{H}$

- **Calabi-Yau with torsion metrics**

Summary

1) KY (with torsion) symmetry exists in many important spherical black hole spacetimes, where it guarantees "certain integrability properties" (geodesic, KG, Dirac,...).

2) KY tensors are in 1-1 correspondence with linear symmetry operators of the Dirac operator.

3) Euclidean canonical metric admitting the PKY is known. Is there something more in Lorentzian signature? With higher-order forms?

4) KY symmetry can be extended to the presence of background fluxes. Especially interesting is the **torsion generalization**.

5) Can this be used for constructing new exact solutions?