Title: Variation of Hodge Structure for Generalized Complex Manifolds

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Abstract: Generalized complex manifolds, like complex manifolds, admit a decomposition of the bundle of di
erential forms. When an analogue of the @ @ lemma holds there is a corresponding Hodge decomposition in twisted cohomology. We look at some aspects of this decomposition, in particular its behavior under deformations of generalized complex structure. We de ne period maps and show a Gri ths transversality result. We use Courant algebroids to develop the notion of a holomorphic family of generalized complex structures and show the period maps for such families are holomorphic.

Variation of Hodge structure for generalized complex manifolds

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The Australian National University
Canberra, Australia

GAP2012 (Geometry And Physics)
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Objective

Generalized complex geometry:

- is a unification of complex and symplectic geometry
- related to physics: string theory backgrounds, supersymmetry, . . .
- is a possible setting for mirror symmetry
- incorporates twisting by gerbes.

Can think of a generalized complex manifold as a part complex part symplectic hybrid.

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Let H be a closed 3-form on M (curvature of a gerbe on M). The ***twisted differential*** $d_H : \Omega^*(M) \to \Omega^*(M)$ is given by

$$d_H\omega = d\omega + H \wedge \omega.$$

The ($\mathbb{Z}_2$-graded) cohomology of d_H is called ***twisted cohomology***, $H^i(M, H)$.

Generalized complex manifolds which satisfy a $\partial\bar{\partial}$-lemma admit a Hodge decomposition

$$H^*(M, H)_{\mathbb{C}} = \bigoplus_{k=-n}^{n} H^k_{\bar{\partial}}(M)$$

in twisted cohomology. Aim is to understand how the Hodge decomposition varies with the generalized complex structure.

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Relation to mirror symmetry

Let M be compact Calabi-Yau n-fold. The Hochschild cohomology and homology are:

$$HH^k(M) = \bigoplus_{q+p=k} H^q(\wedge^p T^{1,0} M)$$

$$HH_k(M) = \bigoplus_{q-p=k} H^q(\wedge^{p,0} T^* M).$$

$HH^*(M)$ is a graded ring and $HH_*(M)$ is a graded module over $HH^*(M)$.

The holomorphic volume form Ω induces

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Relation to mirror symmetry 2

The B-model correlation functions are encoded by the module structure

$$HH^j(M) \otimes HH_k(M) \to HH_{j+k}(M).$$

The special case

$$H^1(T^{1,0}M) \otimes H^q(\wedge^{p,0}T^*M) \to H^{q+1}(\wedge^{p-1,0}T^*M)$$

corresponds to variation of Hodge structure.

Kapustin+Li: generalized Calabi-Yau manifolds give rise to topological field theories generalizing A and B models.
$HH_*(M) \Longrightarrow$ twisted cohomology.
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Generalized tangent bundle

The ***generalized tangent bundle*** is $E = TM \oplus T^*M$

Natural pairing $\langle\ ,\ \rangle : E \otimes E \to \mathbb{R}$

$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2}(\eta(X) + \xi(Y)).$$

Dorfman bracket

$$[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X \eta - i_Y d\xi.$$

Courant bracket is the skew-symmetrization.

If H is a closed 3-form, we have a twisted Dorfman (and Courant) bracket

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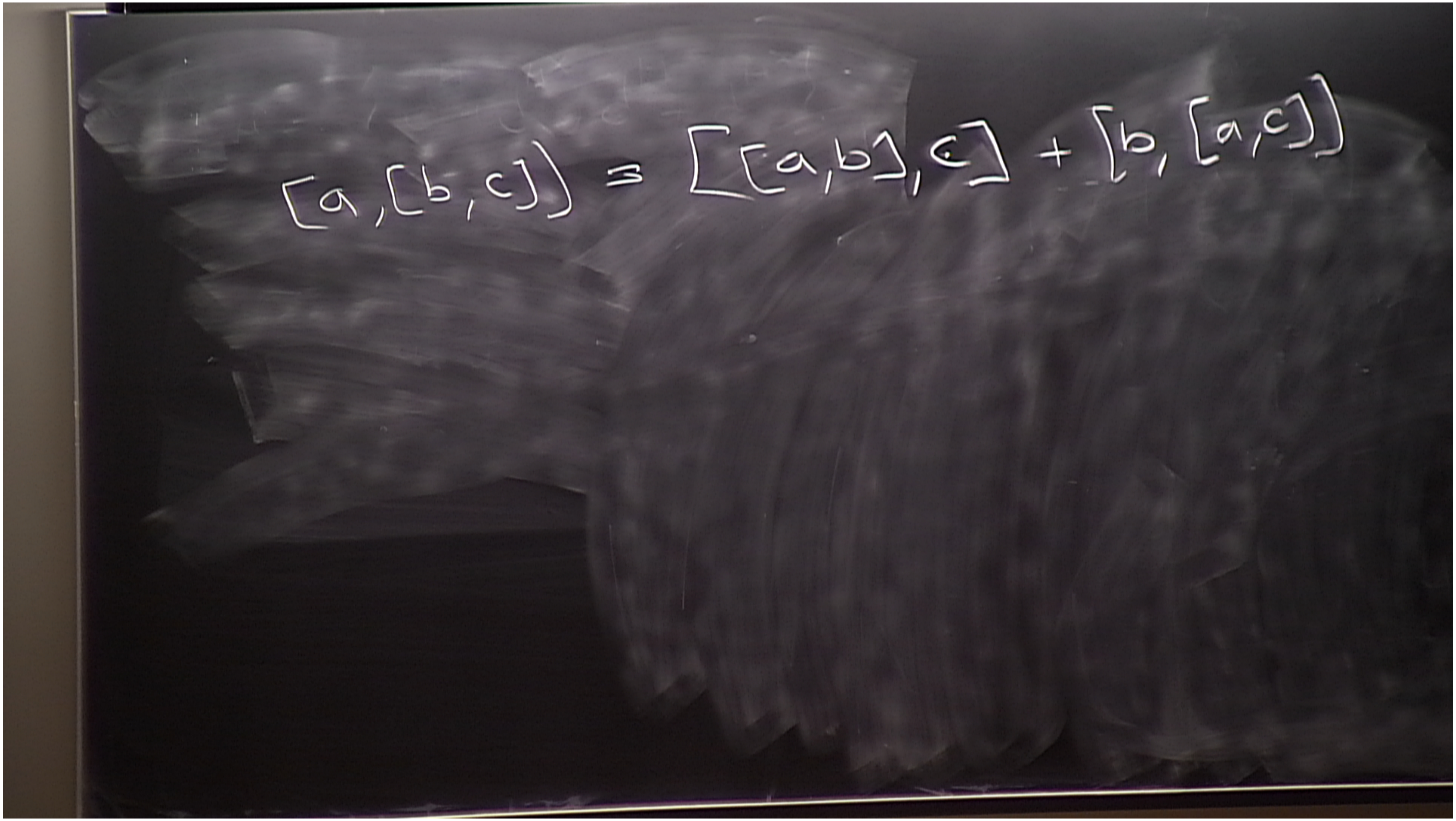
[a,[b,c]] = [[a,b],c] + [b,[a,c]]

Generalized complex structures

A ***generalized almost complex structure*** is $J : E \to E$ such that

- $J^2 = -1$
- $\langle Ja, Jb \rangle = \langle a, b \rangle$.

Then $E_{\mathbb{C}} = E \otimes \mathbb{C} = L \oplus \overline{L}$, where L is the i-eigenspace of J. Note for later: using $\langle \, , \, \rangle$ we have $\overline{L} \simeq L^*$.

We say that J is H***-integrable*** and say J is a ***generalized complex structure*** if L is closed under the (H-twisted) Dorfman (or Courant) bracket.

Either bracket restricted to L makes L a Lie algebroid. The Lie algebroid cohomology $H^*(L)$ will be important later.

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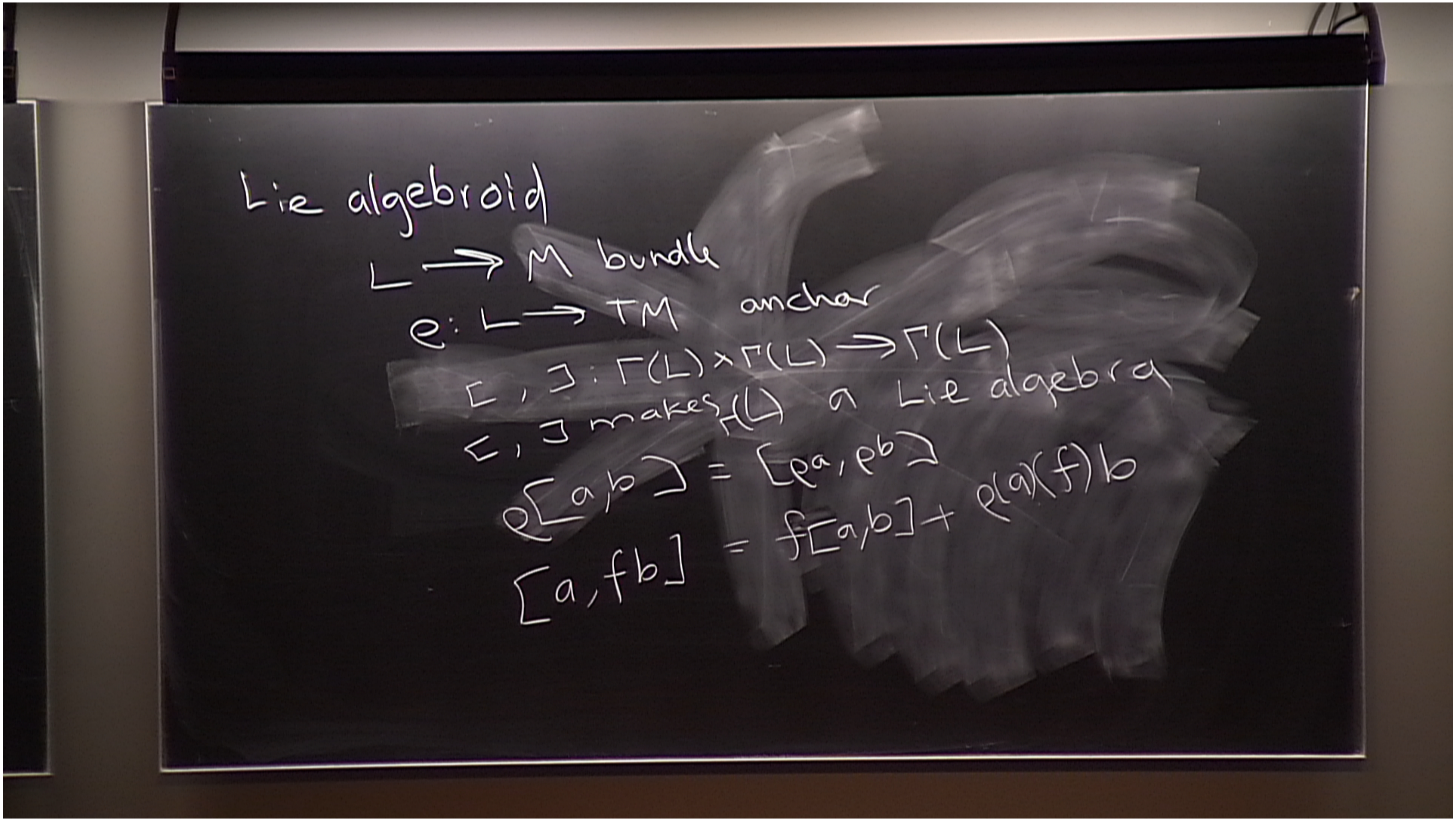
Lie algebroid
L → M bundle
ρ: L → TM anchor
[,]: Γ(L) × Γ(L) → Γ(L)
[,] makes Γ(L) a Lie algebra
ρ[a,b] = [ρa, ρb]
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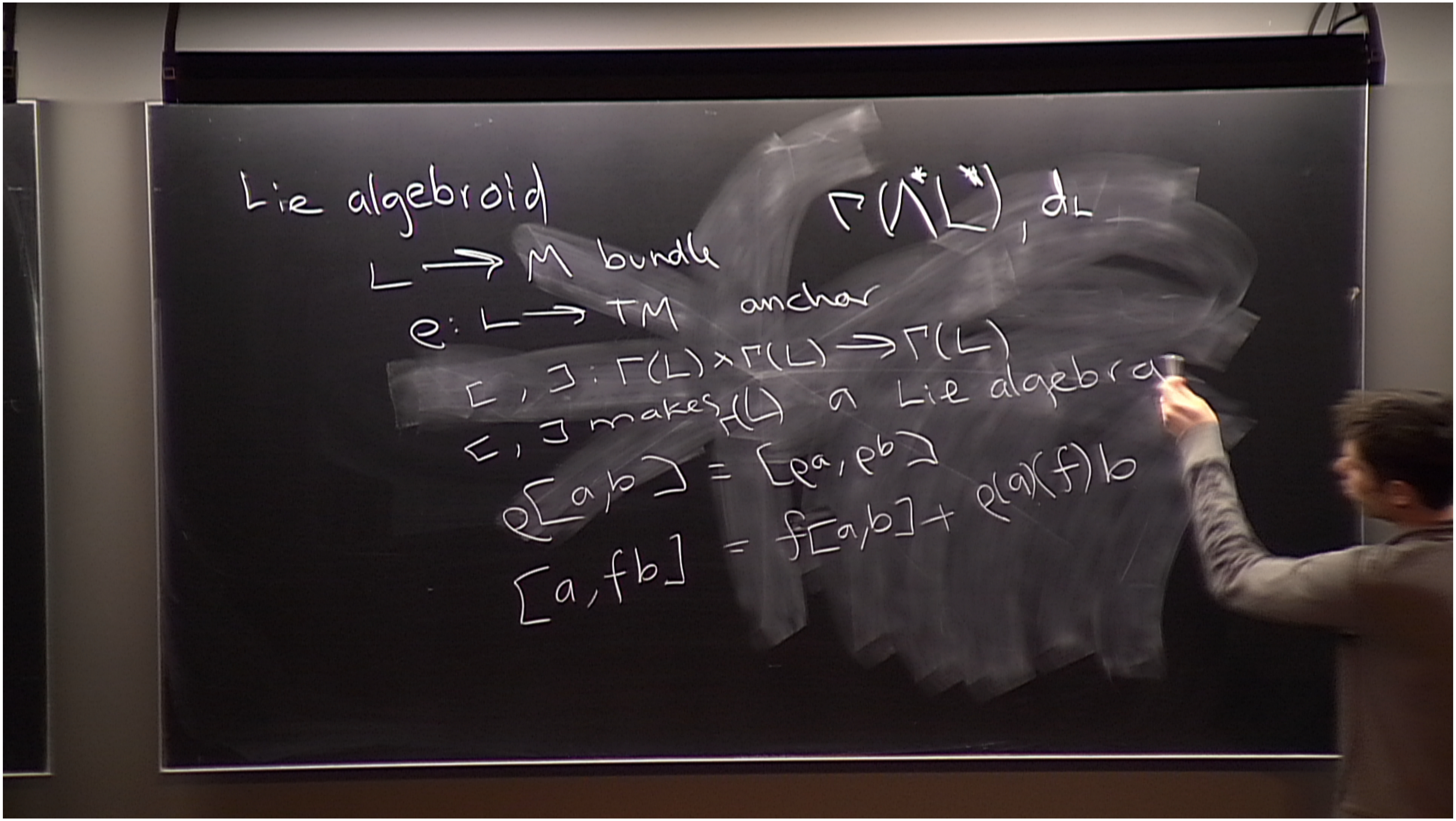

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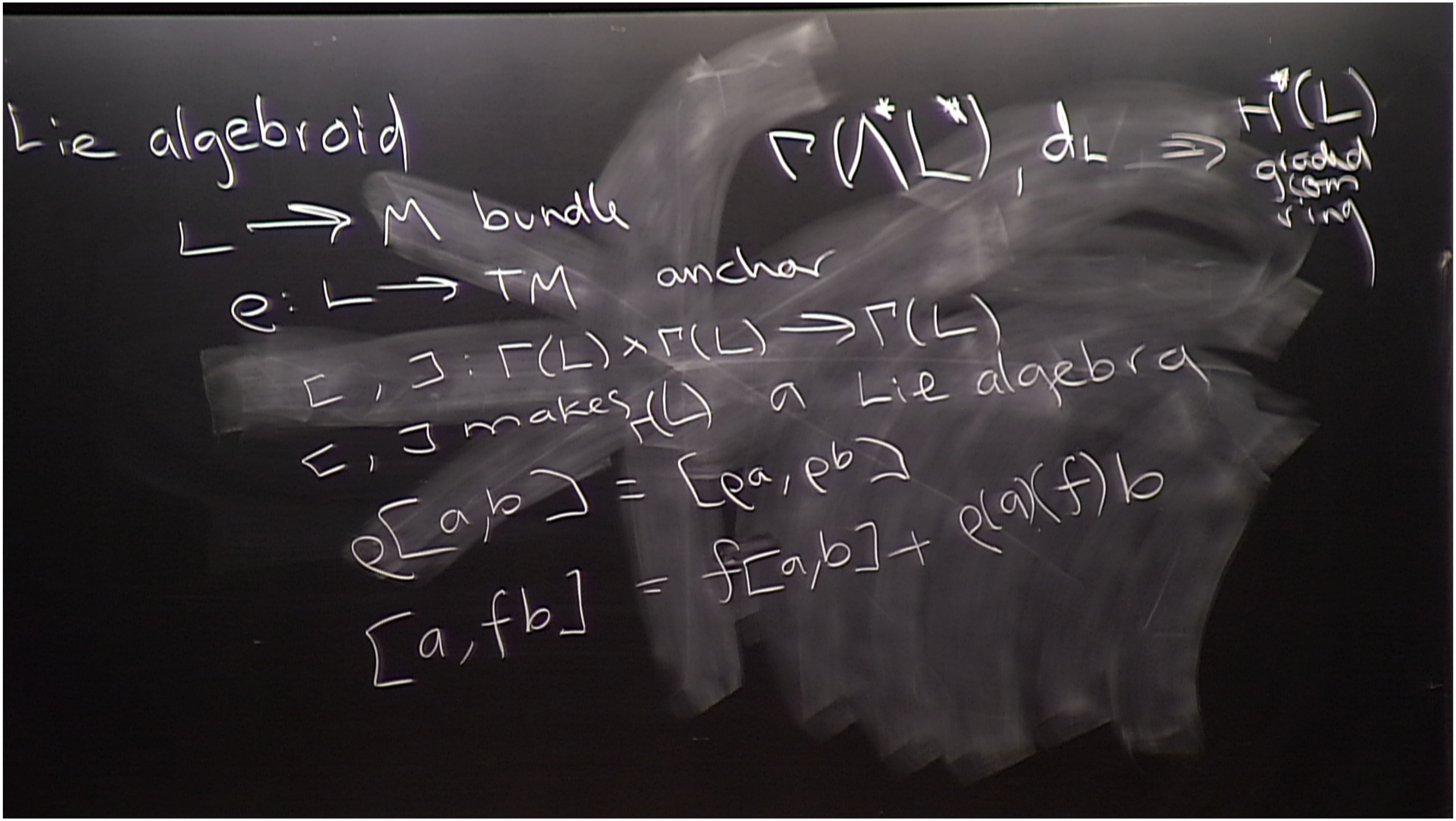
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Differential forms and twisted cohomology

$S = \wedge^* T^* M$ is a Clifford module for $E = TM \oplus T^* M$:

$$(X + \xi) \bullet \omega = i_X \omega + \xi \wedge \omega.$$

$S = S^0 \oplus S^1$, where

$$\begin{aligned} S^0 &= \wedge^{ev} T^* M \\ S^1 &= \wedge^{odd} T^* M. \end{aligned}$$

Recall the twisted differential $d_H : \Gamma(S^i) \to \Gamma(S^{i+1})$:

$$d_H \omega = d\omega + H \wedge \omega.$$

Since H is closed, $d_H^2 = 0$. The $\mathbb{Z}_2$-graded cohomology of d_H is called ***twisted cohomology***, $H^i(M, H)$.

Differential forms and twisted cohomology 2

The twisted differential d_H enters into generalized geometry because the Dorfman bracket is a ***derived bracket***:

$$[a, b] \bullet \omega = [[d_H, a\bullet], b\bullet]\, \omega$$

brackets on the RHS are graded commutators. In other words

$$[a, b] \bullet \omega = d_H(a \bullet b \bullet \omega) - a \bullet d_H(b \bullet \omega) + b \bullet d_H(a \bullet \omega) - b \bullet a \bullet d_H \omega$$

for all $a, b \in \Gamma(E)$, $\omega \in \Omega^*(M)$.

Pure spinors

$\rho \in \Gamma(S_{\mathbb{C}})$ is a ***pure spinor*** if ρ is non-vanishing and the annihilator

$$Ann(\rho) = \{a \in E_{\mathbb{C}} \,|\, a \bullet \rho = 0\}$$

is maximal isotropic.

If $\mathcal{J}$ is a generalized complex structure, the i-eigenspace L is maximal isotropic. There exists a line bundle $K_{\mathcal{J}} \subset S$ whose non-vanishing sections are precisely the pure spinors for L. $K_{\mathcal{J}}$ is the ***canonical bundle*** of $\mathcal{J}$.

Pure spinors have definite parity: $K_{\mathcal{J}} \subset S^{\tau}$ for some $\tau \in \mathbb{Z}_2$.

Get an isomorphism

$$S = \wedge^* L^* \otimes K_{\mathcal{J}}$$

by letting $L^* \simeq \bar{L}$ act on $K_{\mathcal{J}}$ by the Clifford action.

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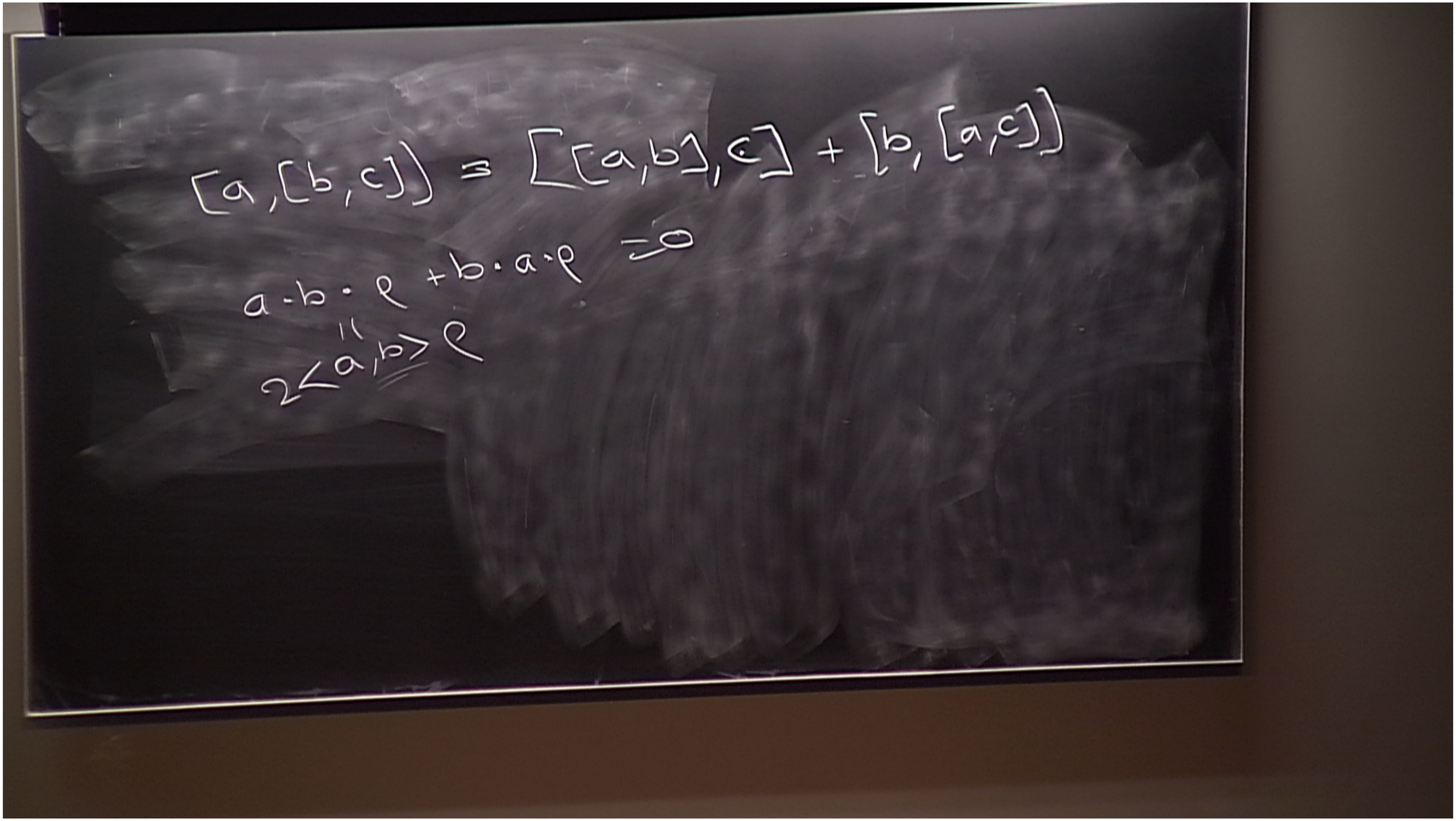
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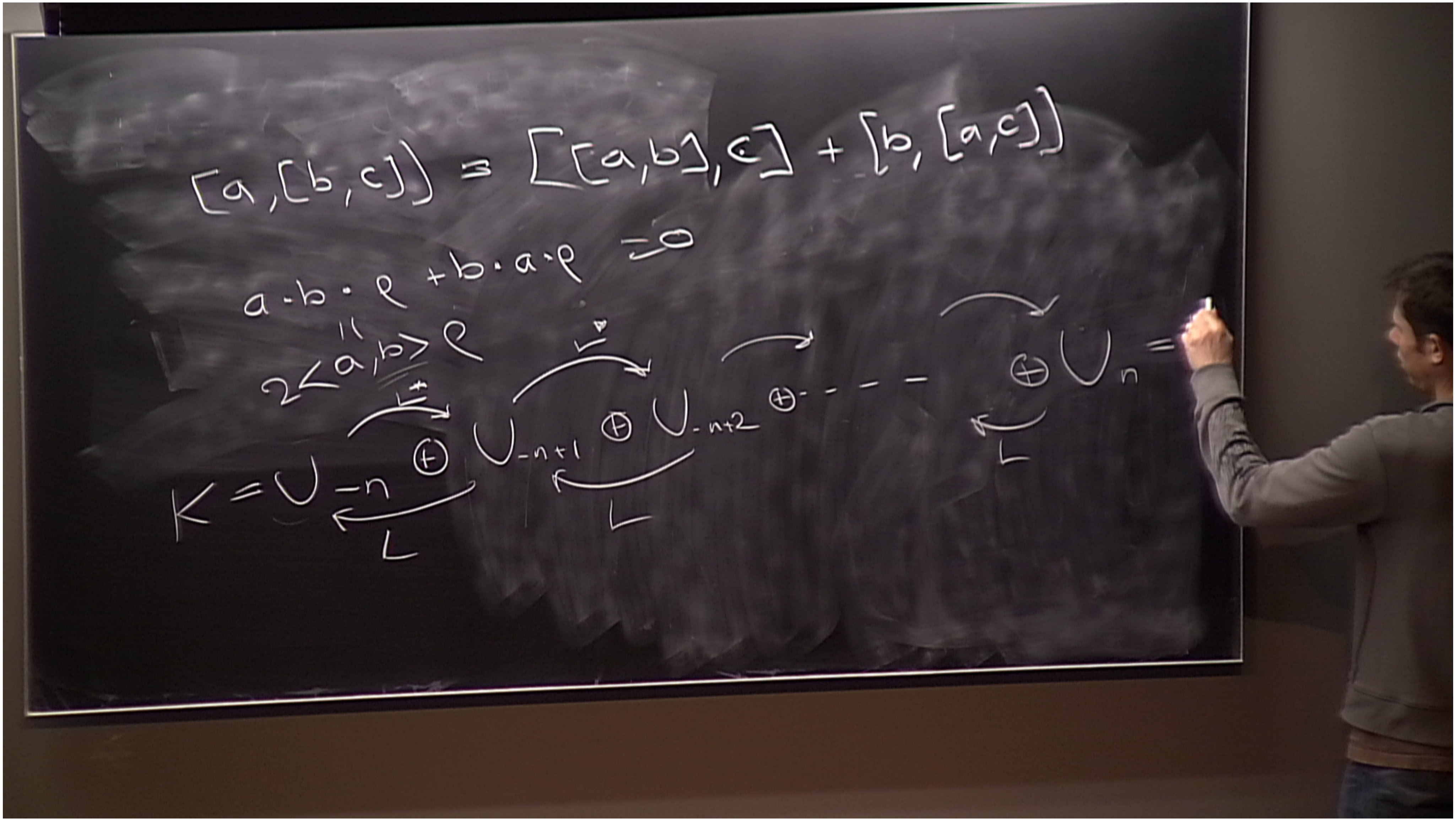
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K = V-n ⊕ V-n+1 ⊕ V-n+2 ⊕ - - - - ⊕ Vn =
L
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Decomposition of forms

Let $\dim(M) = 2n$.

$$S = U_{-n} \oplus U_{-n+1} \oplus \cdots \oplus U_{n-1} \oplus U_n$$

where

$$U_i = (\wedge^{i+n} L^*) \bullet K_J.$$

Let $\partial : \Gamma(U_i) \to \Gamma(U_{i-1})$ be the degree -1 part of d_H and $\overline{\partial} : \Gamma(U_i) \to \Gamma(U_{i+1})$ the degree $+1$ part.

Theorem

J is H-integrable if and only if

$$d_H = \partial + \overline{\partial}.$$

If J is integrable then $\partial^2 = \overline{\partial}^2 = \partial\overline{\partial} + \overline{\partial}\partial = 0$.

Hodge to de Rham spectral sequence

Assume J integrable. Then we have $\overline{\partial}$-cohomology groups

$$H^k_{\overline{\partial}}(M) = \frac{\mathrm{Ker}(\overline{\partial} : \Gamma(U_k) \to \Gamma(U_{k+1}))}{\mathrm{Im}(\overline{\partial} : \Gamma(U_{k-1}) \to \Gamma(U_k))}.$$

The decomposition $d_H = \partial + \overline{\partial}$ together with grading $\{S_i\}$ gives rise to a Hodge to de Rham spectral sequence (E^k_r, d_r) converging to $H^*(M, H)_{\mathbb{C}}$ and such that

$$E^k_1 = H^k_{\overline{\partial}}(M).$$

Let $F^p H(M)$ denote the associated filtration on $H^{T+n-p}(M, H)_{\mathbb{C}}$. In our notation $F^p H(M)$ are the classes represented in degrees $p, p-2, p-4, \ldots$.

$$F^{-n} H(M) \subseteq F^{-n+2} H(M) \subseteq \cdots \subseteq F^n H(M) = H^T(M, H)_{\mathbb{C}}$$

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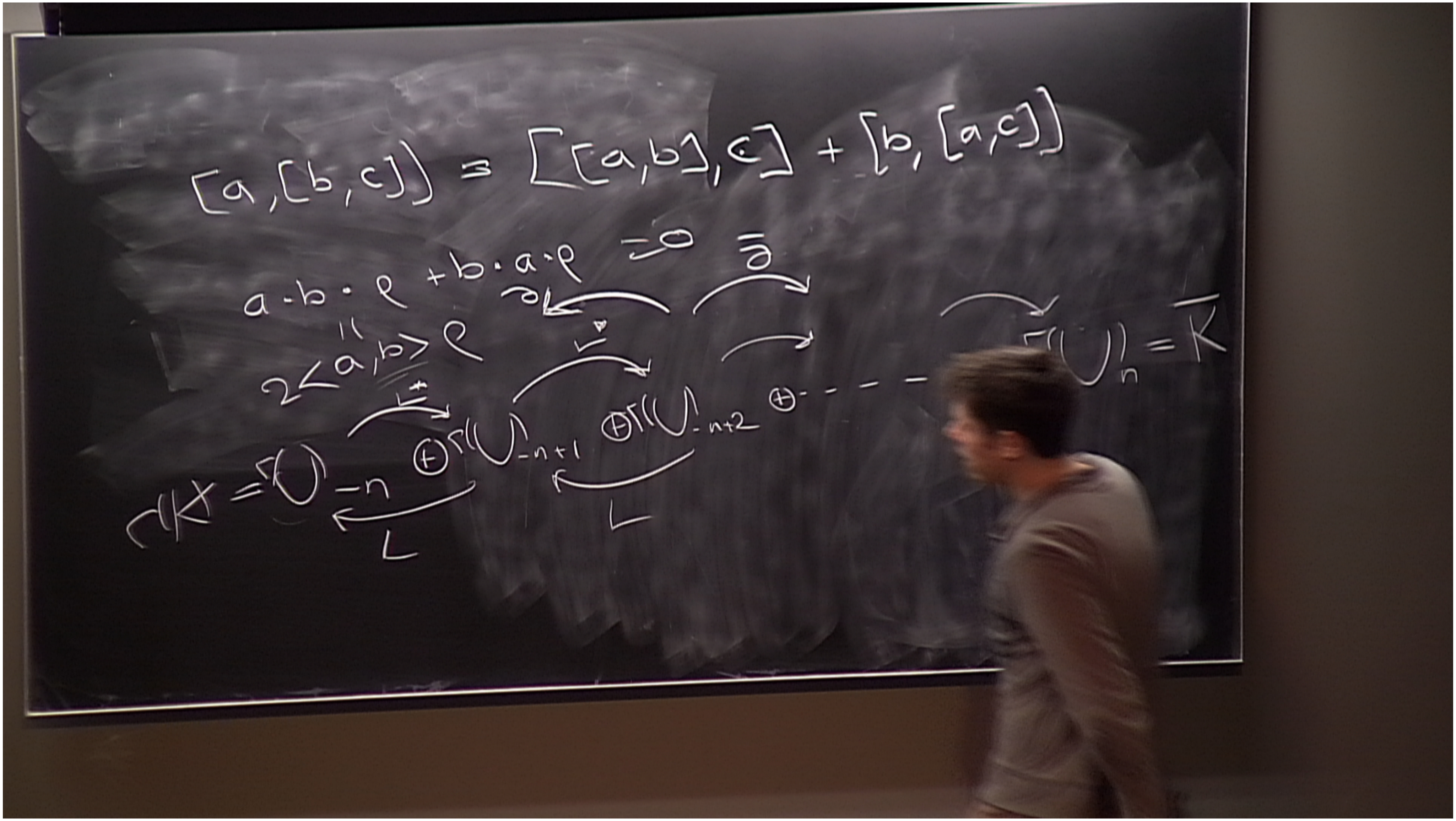
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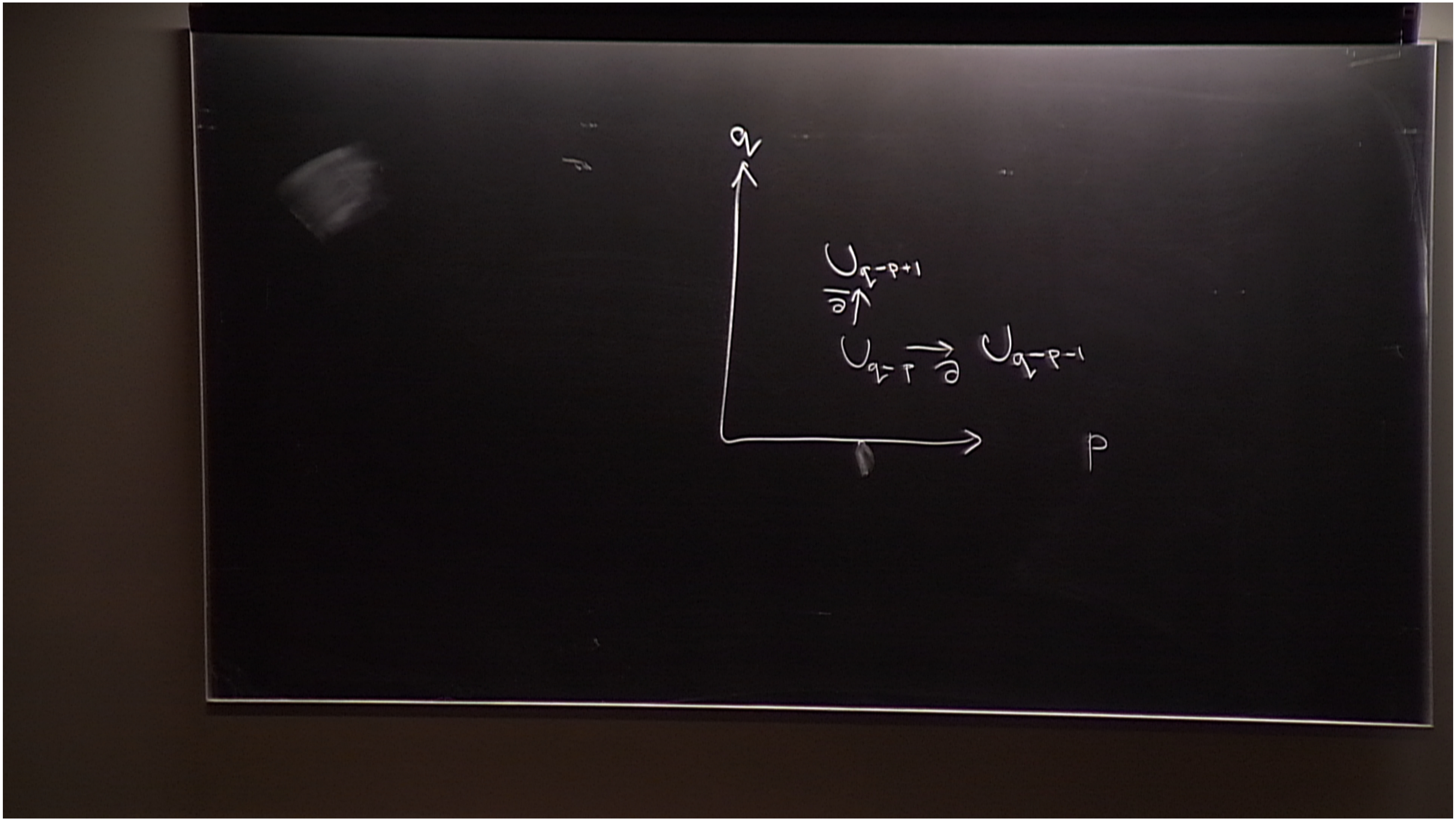
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$$F^{-n+1}H(M) \subseteq F^{-n+3}H(M) \subseteq \cdots \subseteq F^{n-1}H(M) = H^{\tau+1}(M, H)_{\mathbb{C}}.$$

Hodge structure

We say that there is a Hodge structure on twisted cohomology if:

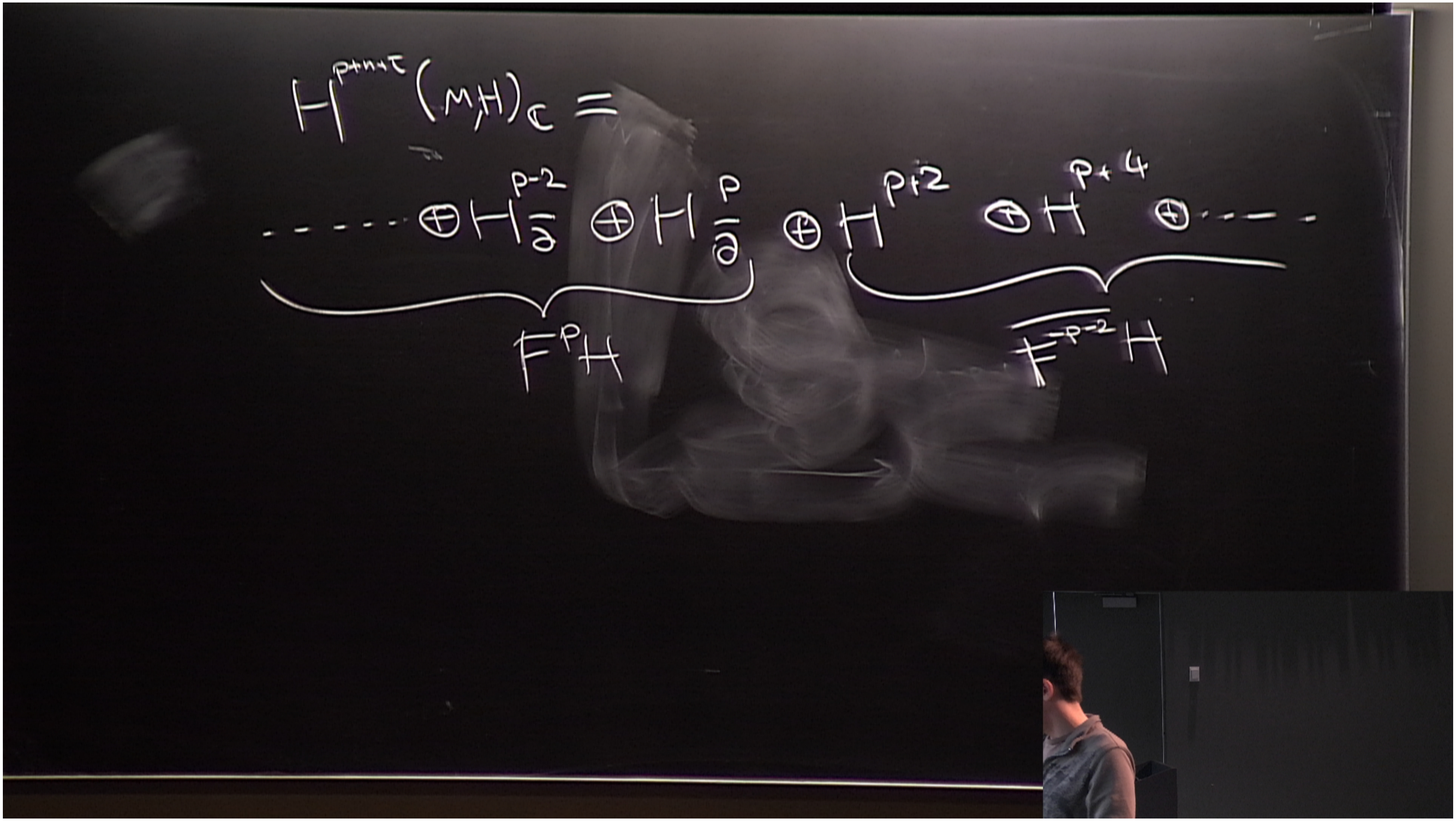
- Hodge to de Rham degenerates at E_1,
- $F^p H(M) \oplus \overline{F^{-p-2} H(M)} \to H^k(M, H)_{\mathbb{C}}$ is an isomorphism.

So there is a decomposition

$$H^{k+n+\tau}(M, H)_{\mathbb{C}} = \bigoplus_{j=k\,(\mathrm{mod}2)} H^j_{\overline{\partial}}(M)$$

such that $\overline{H^j_{\overline{\partial}}(M)} = H^{-j}_{\overline{\partial}}(M)$ and

$$F^p H(M) = \bigoplus_{k=p\,(\mathrm{mod}2),\, k \leq p} H^k_{\overline{\partial}}(M).$$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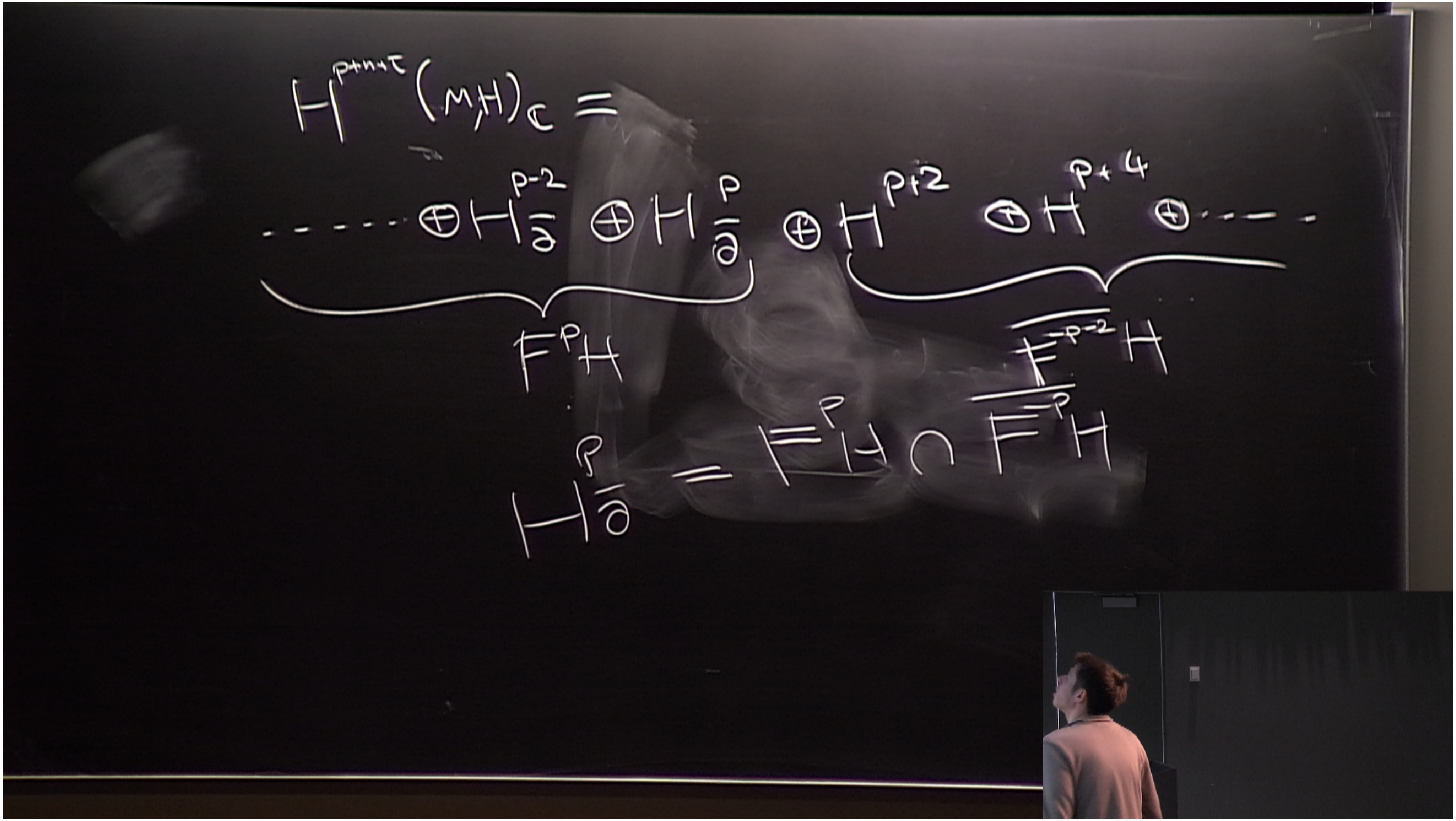
$H^{p+n+r}(M,H)_{\mathbb{C}} =$
$\cdots \oplus H^{p-2}_{\bar\partial} \oplus H^{p}_{\bar\partial} \oplus H^{p+2} \oplus H^{p+4} \oplus \cdots$
$F^{p}H$
$\overline{F^{-p-2}H}$
$H^{p}_{\bar\partial} = F^{p}H \cap \overline{F^{-p}H}$

The $\partial\overline{\partial}$-lemma

Say that (M, J, H) satisfies the $\partial\overline{\partial}$-lemma if:

$$Ker(\partial) \cap Im(\overline{\partial}) = Im(\partial) \cap Ker(\overline{\partial}) = Im(\partial\overline{\partial}).$$

Theorem (Deligne Griffiths Morgan Sullivan)

The $\partial\overline{\partial}$-lemma is equivalent to:

- *Hodge to de Rham degenerates at E_1, and*
- *The induced filtration on twisted cohomology is a Hodge filtration*

i.e. equivalent to a Hodge decomposition in twisted cohomology.

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Families in generalized geometry

How does the Hodge decomposition vary in families?

What do we mean by a family in generalized geometry?

Simplistic version: (M, J_t), where J_t depends on a parameter t.

More geometric version: appeal to Courant algebroids.

Courant algebroids

Definition

A ***Courant algebroid*** on a smooth manifold M consists of

- A vector bundle E,
- A bundle map $\rho : E \to TM$ called the ***anchor***,
- A non-degenerate symmetric bilinear form $\langle\, ,\, \rangle : E \otimes E \to \mathbb{R}$,
- An $\mathbb{R}$-bilinear operation $[\, ,\,] : \Gamma(E) \otimes_{\mathbb{R}} \Gamma(E) \to \Gamma(E)$ on sections of E, the ***Dorfman bracket***,

such that for all $a, b, c \in \Gamma(E)$, $f \in \mathcal{C}^\infty(M)$

CA1 $[a, [b, c]] = [[a, b], c] + [b, [a, c]]$,

CA2 $\rho[a, b] = [\rho(a), \rho(b)]$,

CA3 $[a, fb] = \rho(a)(f)b + f[a, b]$,

CA4 $[a, b] + [b, a] = \rho^* d\langle a, b\rangle$,

CA5 $\rho(a)\langle b, c\rangle = \langle [a, b], c\rangle + \langle a, [b, c]\rangle$

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Exact Courant algebroids

Definition

A Courant algebroid E is ***exact*** if the sequence
$0 \to T^*M \overset{\rho^*}{\to} E \overset{\rho}{\to} TM \to 0$ is exact.

Theorem (Ševera)

Isomorphism classes of exact Courant algebroids on M are in bijection with $H^3(M, \mathbb{R})$. If H is a closed 3-form on M then a representative Courant algebroid for $[H]$ is given by

- *$E = TM \oplus T^*M$ with obvious anchor and symmetric bilinear pairing*
- $[X + \xi, Y + \eta]_H = [X, Y] + \mathcal{L}_X \eta - i_Y d\xi + i_X i_Y H$

Exact Courant algebroids are the ones normally used in generalized geometry.

Localization

Let $\pi : M \to B$ be a fiber bundle and $E \to M$ a family of exact Courant algebroids.

Let $t \in B$ and $M_t = \pi^{-1}(t)$ be the fiber over t. Define an exact Courant algebroid on M_t as follows:

- The underlying bundle is $E_t = E|_{M_t}$.
- Bracket: $[a,b]_t = [\tilde{a},\tilde{b}]|_{M_t}$, where a, b are sections of $E|_{M_t}$ and $\tilde{a}, \tilde{b}$ arbitrary smooth extensions to E.

Claim: this gives an exact Courant algebroid on M_t.

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Smooth family of generalized complex structures

Let $\pi : M \to B$ be a fiber bundle, E a family of exact Courant algebroids over E.

Definition

A ***smooth family of generalized complex structures*** over B is an integrable generalized complex structure J on such a Courant algebroid E.

By restriction J defines a generalized complex structure J_t on each localization E_t. So we really do get a smooth family (M_t, E_t, J_t) of generalized complex structures.

Families by reduction

$\pi : M \to B$ a fiber bundle. Exact Courant algbroids on M give rise to families of exact Courant algebroids as follows:

Let F be an exact Courant algebroid on M. We have

$$A^{\perp} \subset A \subset F$$

where A is the kernel of $F \stackrel{\rho}{\to} TM \stackrel{\pi_*}{\to} \pi^*(TB)$ and $A^{\perp}$ is the annihilator of A.

- $\Gamma(A)$ is a subalgebra of $\Gamma(F)$ and $\Gamma(A^{\perp})$ a two-sided ideal in $\Gamma(A)$.
- The induced bracket on sections of $E = A/A^{\perp}$ makes E into a family of exact Courant algebroids over B.

Holomorphic families

Let F be an exact Courant algebroid on M. Recall we have

$$A^{\perp} \subset A \subset F.$$

Suppose that J is a generalized complex structure on F such that $JA \subseteq A$.

Then $JA^{\perp} \subseteq A^{\perp}$ and J induces a generalized complex structure on $E = A/A^{\perp}$, that is a family of generalized complex structures.

Moreover J induces a complex structure on the bundle $F/A \simeq \pi^*(TB)$. If this coincides with an integrable complex structure I on B then we say that J is a ***holomorphic family*** of generalized complex structures.

Twisted Gauss Manin-connection

Let $\pi : M \to B$ be a fiber bundle and choose $H \in H^3(M, \mathbb{R})$.

Let F be the exact Courant algebroid with Ševera class H and $E = A/A^{\perp}$ the corresponding family of exact Courant algebroids.

The twisted cohomology of the fibers are all isomorphic. In fact the sheaf associated to the presheaf

$$U \mapsto H^*(\pi^{-1}(U), H|_{\pi^{-1}(U)})$$

is a local system with coefficients the twisted cohomology $H^*(M_0, H_0)$ of some fiber M_0.

Alternatively this is a flat vector bundle $(\mathcal{H}^*, \nabla)$. The flat connection ∇ is the ***twisted Gauss-Manin connection***.

Stability of Hodge structures

Let E be a family of exact Courant algebroids associated to $H \in H^3(M, \mathbb{R})$.

Let J be a generalized complex structure on E, i.e. a family of generalized complex structures.

Suppose the $\partial\overline{\partial}$-lemma holds for some fiber $M_0 = \pi^{-1}(0)$. An elliptic semi-continuity argument shows that the $\partial\overline{\partial}$-lemma holds for all fibers sufficiently close to M_0 (assuming the fibers of $\pi : M \to B$ are compact).

So the existence of a Hodge decomposition is stable under all sufficiently small deformations.

Hodge subbundles

Restricting the family if necessary, the $\overline{\partial}$-cohomology groups $H^k_{\overline{\partial}}(M_t)$ define smooth subbundles $H^k_{\overline{\partial}}$ of the bundle $\mathcal{H}^* \otimes \mathbb{C}$ of twisted cohomology groups:

$$\mathcal{H}^*_{\mathbb{C}} = \bigoplus_{k=-n}^{n} H^k_{\overline{\partial}}.$$

Likewise the filtrations $F^p H(M_t)$ define smooth subbundles $F^p\mathcal{H}$ of $\mathcal{H}^{p+n+\tau}_{\mathbb{C}}$.

Recall the filtrations are such that

$$F^p\mathcal{H}/F^{p-2}\mathcal{H} \simeq H^p_{\overline{\partial}}.$$

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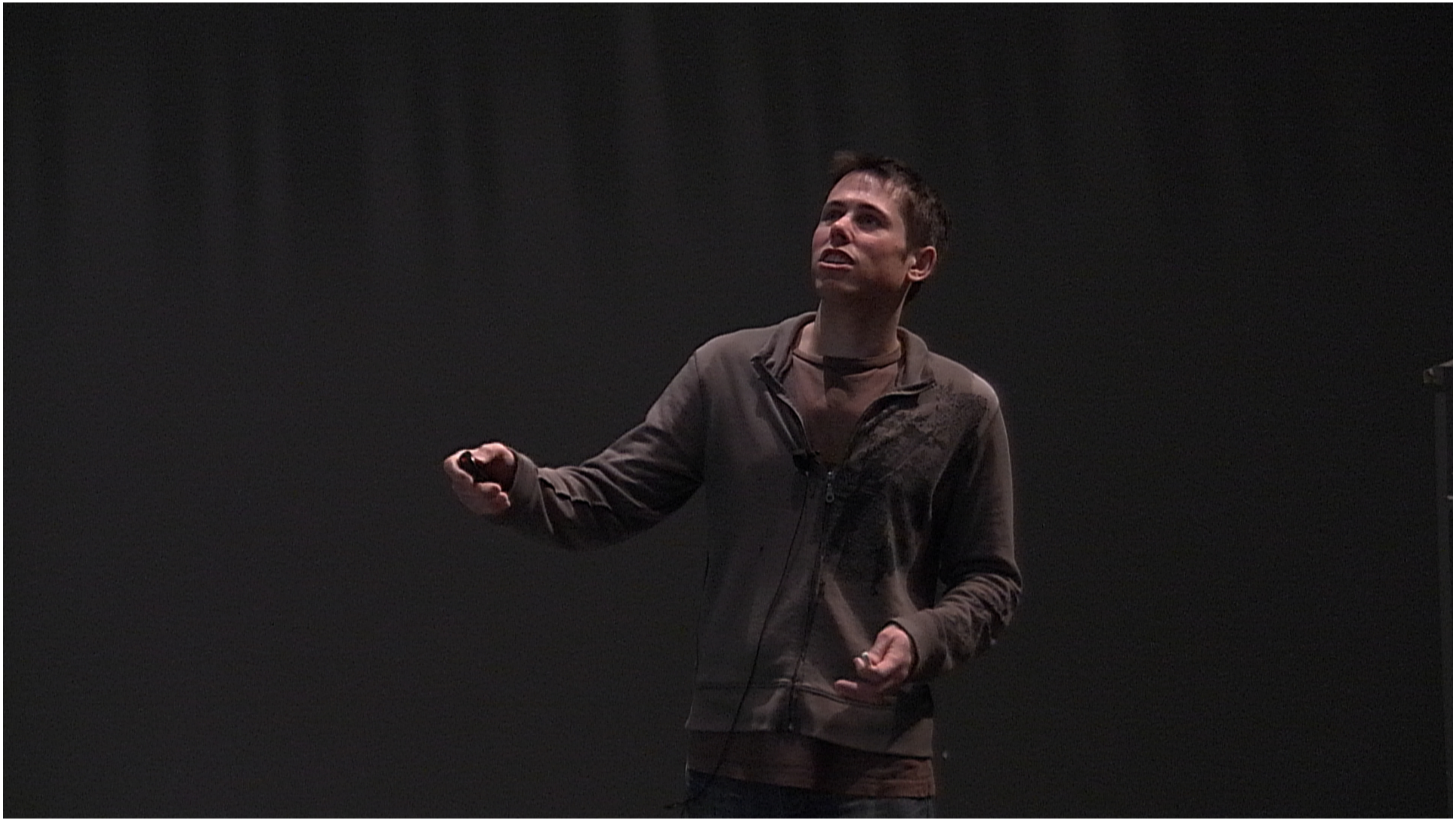


Griffiths transversality

Let X be a vector field on B and s a section of $F^p\mathcal{H}$. Then $\nabla_X s$ is a section of $F^{p+2}\mathcal{H}$. We thus get an induced bundle map

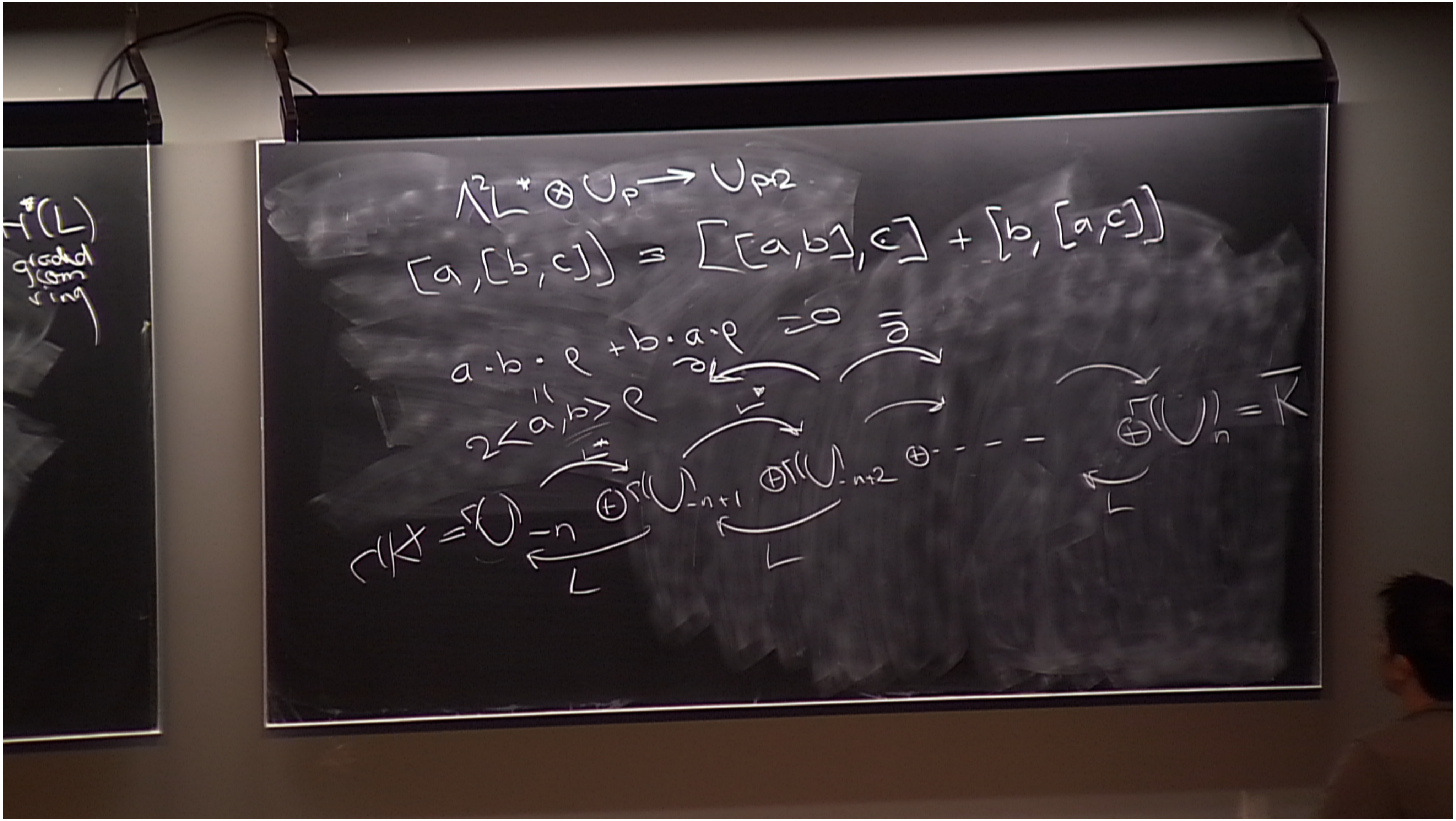
$$t : TB \otimes (F^p\mathcal{H}/F^{p-2}\mathcal{H}) \to (F^{p+2}\mathcal{H}/F^p\mathcal{H}).$$

That is $t_X(s) = \nabla_X(s)(\operatorname{mod} F^p\mathcal{H})$.

For each $b \in B$, let J_b be the induced generalized complex structure on M_b and L_b the i-eigenspace of J_b, which is a Lie algebroid on M_b. There is an element $\kappa_X(b) \in H^2(L_b)$ such that the map

$$t_X(b) : H^p_{\overline{\partial}}(M_b) \to H^{p+2}_{\overline{\partial}}(M_b)$$

is just the cup product (Clifford action) of $\kappa_X(b)$ on twisted cohomology.
$\kappa_X(b)$ is a generalized Kodaira-Spencer class.


Λ²L* ⊗ U_p → U_p+2
[a,[b,c]] = [[a,b],c] + [b,[a,c]]
a·b·ρ + b·a·ρ = 0
2<a,b>ρ
Γ(K) = Γ(U)_-n ⊕ Γ(U)_-n+1 ⊕ Γ(U)_-n+2 ⊕ - - - - ⊕ Γ(U)_n = K̄
graded com ring

Period maps

Assume the base is simply connected and fix a basepoint $0 \in B$. The subbundle $F^p\mathcal{H}$ of the flat bundle $\mathcal{H}_{\mathbb{C}}^{p+n+\tau}$ detemines a period map

$$\mathcal{P}^p : B \to \mathrm{Grass}(d, H^{p+n+\tau}(M_0, H_0)_{\mathbb{C}})$$

into the Grassmannian of $d = \dim(F^p\mathcal{H})$-dimensional subspaces of $H^{p+n+\tau}(M_0, H_0)_{\mathbb{C}}$.

The differential of $\mathcal{P}^p$ is essentially the map

$$t : TB \otimes (F^p\mathcal{H}/F^{p-2}\mathcal{H}) \to (F^{p+2}\mathcal{H}/F^p\mathcal{H}).$$

Period map in holomorphic families

Theorem

If the family of generalized complex structures is holomorphic, then the period maps are holomorphic.

Proof.

The differential of $\mathcal{P}^p$ is given by t, which in turn is given by the Kodaira-Spencer classes $\kappa_X(b)$. One simply shows

$$\kappa_{IX}(b) = i\kappa_X(b)$$

where I is the complex structure on B. □

Generalized Calabi-Yau manifolds

A generalized complex manifold with a globally defined d_H-closed pure spinor ϕ is called a ***generalized Calabi-Yau manifold***.

Theorem (Goto)

If (M, J) is a compact Generalized Calabi-Yau manifold for which the $\partial\bar{\partial}$-lemma holds, then all infinitesimal deformations are unobstructed and there is a smooth (local) moduli space $\mathcal{M}$ whose tangent space at (M, J) is $H^2(L)$.

Period map: if (M, J) is a generalized Calabi-Yau then the pure spinor ϕ is unique up to scale and if M is compact the class $[\phi] \in H^\tau(M, H) \otimes \mathbb{C}$ is non-zero. Therefore we get a period map (at least locally if $\mathcal{M}$ is not simply connected)

$$\mathcal{P} : \mathcal{M} \to \mathbb{P}(H^\tau(M, H)_\mathbb{C}).$$

Note that $\mathcal{P}$ is the period map $\mathcal{P}^{-n}$.

Generalized Calabi-Yau manifolds 2

Theorem

The period map $\mathcal{P} : \mathcal{M} \to \mathbb{P}(H^\tau(M, H)_\mathbb{C})$ is an immersion.

Proof.

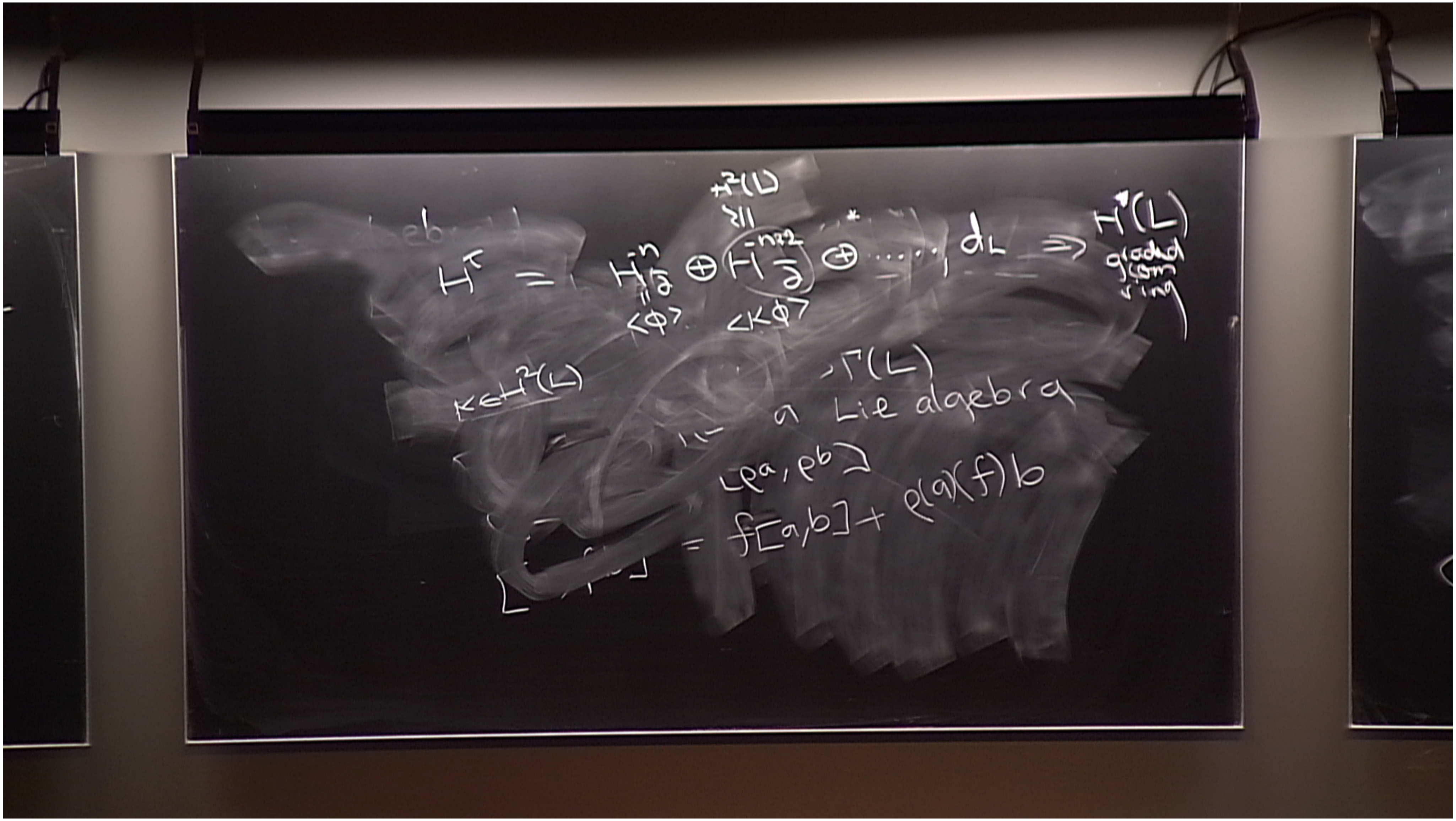
By Griffiths transversality the differential

$$d\mathcal{P}(M, J, \phi) : T_{(M,J,\phi)}\mathcal{M} \to \mathrm{Hom}(\langle[\phi]\rangle, H^\tau(M, H)_\mathbb{C}/\langle[\phi]\rangle)$$

is just the map

$$H^2(L) \otimes \langle[\phi]\rangle \to H_{\overline{\partial}}^{-n+2}(M) \subseteq H^\tau(M, H)_\mathbb{C}/H_{\overline{\partial}}^{-n}(M)$$

given by sending a class $\kappa \in H^2(L)$ to $\kappa \bullet \phi \in H_{\overline{\partial}}^{-n+2}(M)$. This is injective since $H^2(L) \simeq H^{-n+2}(M)_{\overline{\partial}}$ for a Generalized Calabi-Yau. □


a Lie algebra
= f[a,b] + ρ(a)(f)b

Symplectic type

Take $H = 0$. We say that (M, J) is of symplectic type if J has a pure spinor

$$\phi = e^{B+i\omega}$$

where ω is a symplectic form and B is a closed 2-form.

For compact M the $\partial\overline{\partial}$-lemma in this case is equivalent to the strong Lefschetz property.

The Hodge filtration is:

$$F^{2k-n}H(M) = e^{B+i\omega}(H^0(M) \oplus H^2(M) \oplus \cdots \oplus H^{2k}(M))_{\mathbb{C}}$$
$$F^{2k-n+1}H(M) = e^{B+i\omega}(H^1(M) \oplus H^3(M) \oplus \cdots \oplus H^{2k+1}(M))_{\mathbb{C}}$$

so determined completely from the action of $[B + i\omega]$ on $H^*(M)_{\mathbb{C}}$.

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Complex type

Let (M, I) be complex manifold. Then I induces a generalized complex structure which is integrable with respect to the H-twisted Dorfman bracket provided $H = h + \overline{h}$ is of type $(2,1) + (1,2)$.

We have

$$S_k = \bigoplus_{q-p=k} \wedge^{p,q} T^* M.$$

The operator $\overline{\partial}$ is not the usual one but a twisted version

$$\overline{\partial} = \overline{\partial}_0 + \overline{h} \wedge$$

where $\overline{\partial}_0$ is the usual operator.

Complex type 2

There are two filtrations on the cohomology of M:

- The Hodge filtration $F^p H(M)$,
- Filtration by the usual differential form degree W^m

Theorem

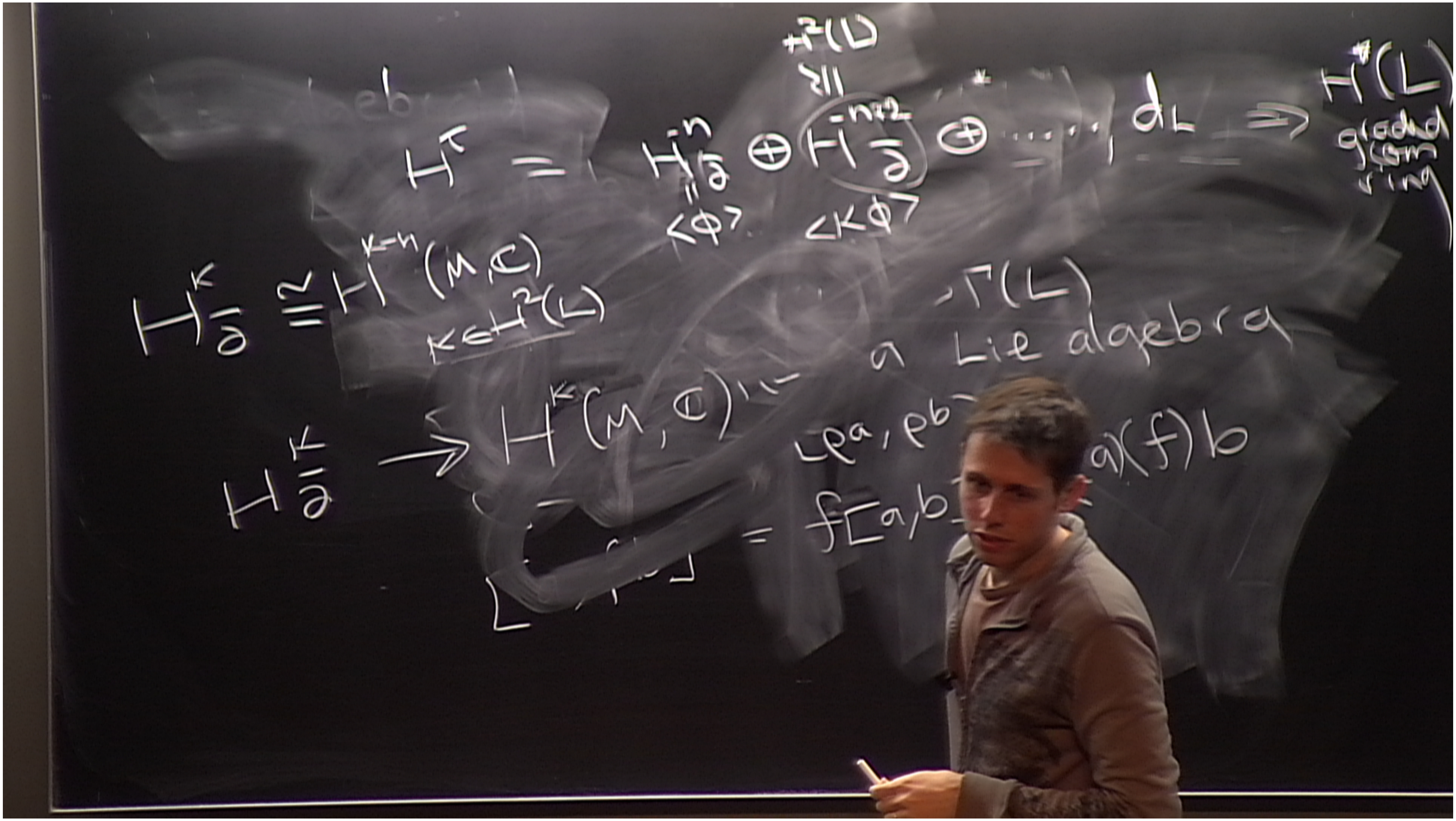
If (M, I, H) satisfies the $\partial\bar{\partial}$-lemma then the filtrations $F^p H(M)$, W^m form a mixed Hodge structure: the $F^p H(M)$ induce on each quotient W^m/W^{m+1} a Hodge structure.

When $H = 0$, this is the obvious mixed Hodge structure on $\bigoplus_{p,q} H^{p,q}(M)$.

What else?

Some things we would like to have (but currently don't):

- Polarizations of the Hodge structures
- Primitive twisted cohomology?
- Generalized complex families with singular fibers?
- Behavior around a punctured disc
- Properties of the monodromy transformations


Γ(L)
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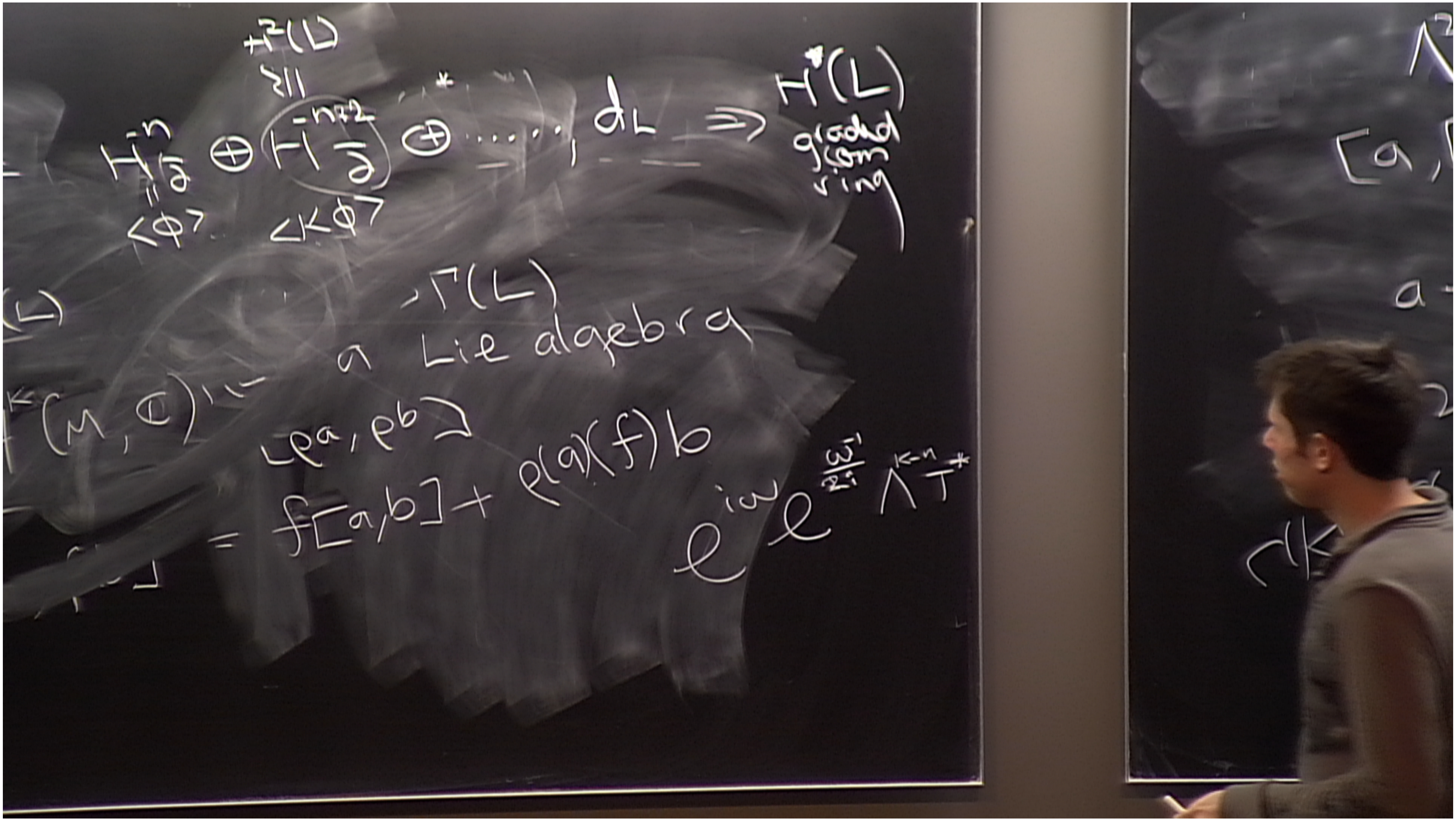
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