Title: SKT Geometry

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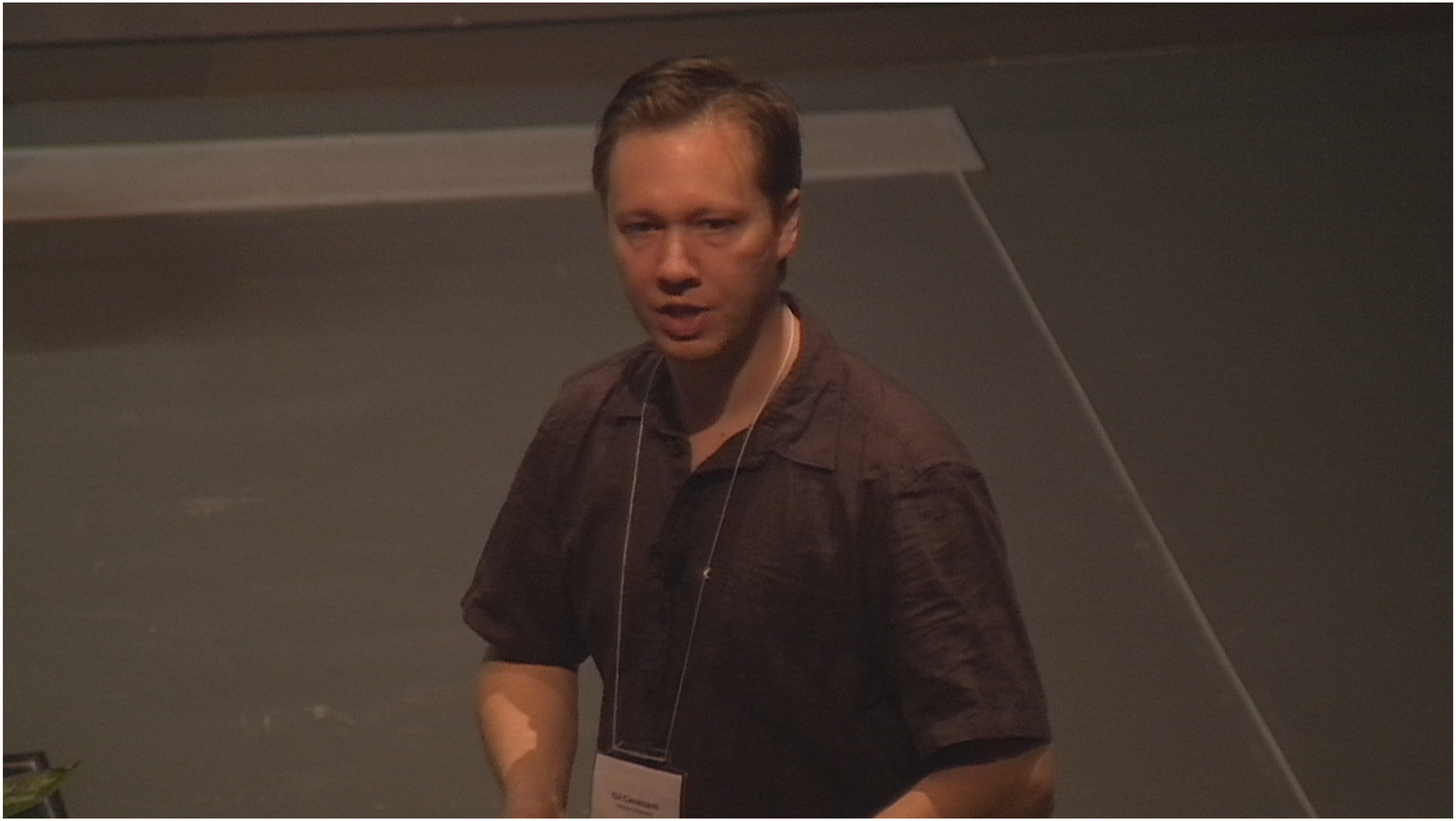
Abstract: In classical terms, an SKT structure is a Hermitian structure for which the Hermitian 2-form is closed with respect to the second order operator ddc. These structures arise naturally in the study of sigma models with (2; 0) or (2; 1)-supersymmetries, much like generalized K•ahler structures arise in the (2; 2)-supersymmetric sigma model. While the introduction of generalized complex geometry has provided the correct framework to study generalized K•ahler structures and great progress has been made in this area in the last few years, SKT structures laid forgotten. We will take a look at what the generalized complex framework can do for SKT structures and in the process dispel some misconceptions that have arisen over the years.



SKT Geometry

arXiv:1203.0493

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GAP 2012
Waterloo

Introduction — SKT strucutures

- Kähler structure with torsion: Hermitian structure (g, I) with a connection ∇ with $\mathrm{Tor}(\nabla) = H \in \Omega^3(M)$ such that

$$\nabla g = \nabla I = 0.$$

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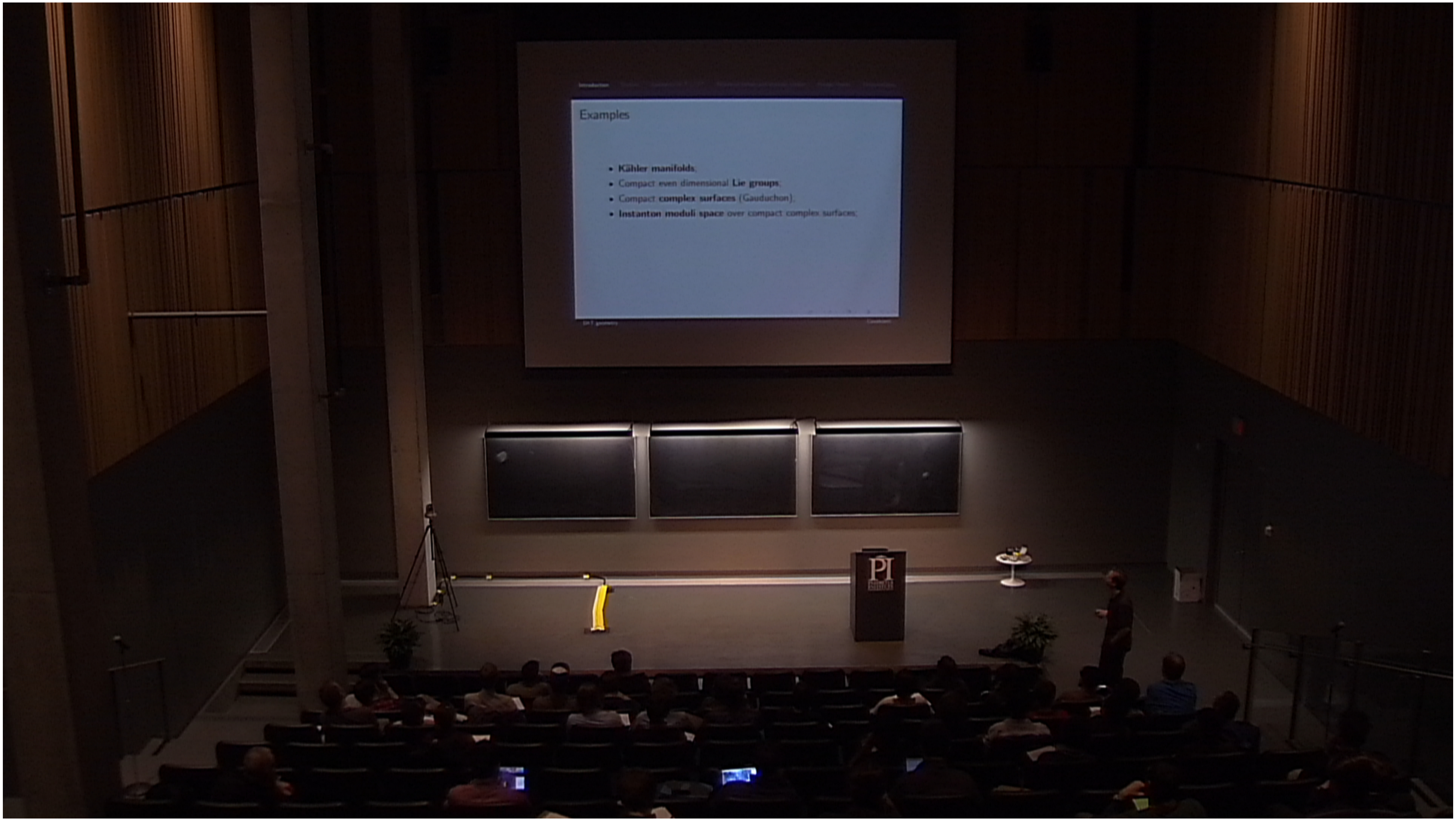
- Kähler structure with strong torsion (SKT): $dH = 0$.
- SKT structure: Hermitian structure (g, I) such that

$$dd^c\omega = 0.$$

- $H = d^c\omega$.



Introduction
Examples
• Kähler manifolds;
• Compact even dimensional Lie groups;
• Compact complex surfaces (Gauduchon);
• Instanton moduli space over compact complex surfaces;

Examples

- **Kähler manifolds**;
- Compact even dimensional **Lie groups**;
- Compact **complex surfaces** (Gauduchon);
- **Instanton moduli space** over compact complex surfaces;
- **Instanton moduli space** over Hermitian manifolds with Gauduchon metrics;

Kähler vs SKT

	Kähler	SKT
Decomposition of cohomology	✓	
Hodge theory	✓	
Frölicher spectral seq. degenerates	✓	
Formality	✓	
Unobstructed deformations	✓	

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Fino, Parton and Salamon. Families of strong KT structures in six dimensions. *Comment. Math. Helv.* 2004. (arXiv:math/0209259)

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Outline of Topics

1. Geometry of $T \oplus T^*$
2. Nijenhuis tensor and intrinsic torsion
3. Hodge theory
4. Deformations

Geometry of $T \oplus T^*$ — Natural pairing

- **Natural pairing**

$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2}(\eta(X) + \xi(Y)).$$

- Action of $T \oplus T^*$ on $\wedge^\bullet T^*$:

$$(X + \xi) \cdot \varphi = \iota_X \varphi + \xi \wedge \varphi.$$

- Extends to an action of $\mathrm{Clif}(T \oplus T^*)$ on $\wedge^\bullet T^*$:

$$v \cdot (v \cdot \varphi) = \langle v, v \rangle \varphi.$$

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- $\wedge^\bullet T^* \rightsquigarrow$ spinors.

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- $\wedge^\bullet T^* \rightsquigarrow$ spinors.
- Spin invariant pairing: $(\cdot, \cdot)_{Ch} : \wedge^\bullet T^* \otimes \wedge^\bullet T^* \longrightarrow \wedge^{top} T^*$.

Geometry of $T \oplus T^*$ — Generalized metric

- A **generalized metric** is an orthogonal, self-adjoint bundle isomorphism:
$$\mathcal{G} : T \oplus T^* \longrightarrow T \oplus T^*;$$
such that
$$\langle \mathcal{G}v, v \rangle > 0.$$
- $\mathcal{G}^{-1} = \mathcal{G}^t = \mathcal{G} \quad \Rightarrow \quad \mathcal{G}^2 = \mathrm{Id}.$

Geometry of $T \oplus T^*$ — Generalized Hodge star

- Generalized metric + orientation $\Rightarrow$ generalized Hodge star $\star$

$$(\varphi, \star\varphi) > 0.$$

- $\star^2 = (-1)^{\frac{m(m-1)}{2}}$.
- SD forms $= -i^{\frac{m(m-1)}{2}}$-eigenspace;
  ASD forms $= i^{\frac{m(m-1)}{2}}$-eigenspace;

Geometry of $T \oplus T^*$ — Gen. almost complex structure

- Generalized almost complex structure:

$$\mathcal{J} : T \oplus T^* \longrightarrow T \oplus T^*; \qquad \mathcal{J}^2 = -\mathrm{Id};$$

  $\mathcal{J}$ is orthogonal.
- $\mathcal{J} \Leftrightarrow L \subset (T \oplus T^*) \otimes \mathbb{C}$, maximal isotropic $L \cap \overline{L} = \{0\}$.
- $\mathcal{J}^t = \mathcal{J}^{-1} = -\mathcal{J} \Rightarrow \mathcal{J} \in \wedge^2 T \oplus T^* = \mathfrak{spin}(T \oplus T^*)$.

Geometry of $T \oplus T^*$ — Gen. almost Hermitian structure

- Generalized almost Hermitian structure: $(\mathcal{G}, \mathcal{J}_1)$

$$\mathcal{G}\mathcal{J}_1 = \mathcal{J}_1\mathcal{G}.$$

- $\mathcal{J}_2 = \mathcal{G}\mathcal{J}_1$ is a gcs and $\mathcal{J}_2\mathcal{J}_1 = \mathcal{J}_1\mathcal{J}_2$.
- $T \oplus T^* = V_+^{1,0} \oplus V_+^{0,1} \oplus V_-^{1,0} \oplus V_-^{0,1}$.

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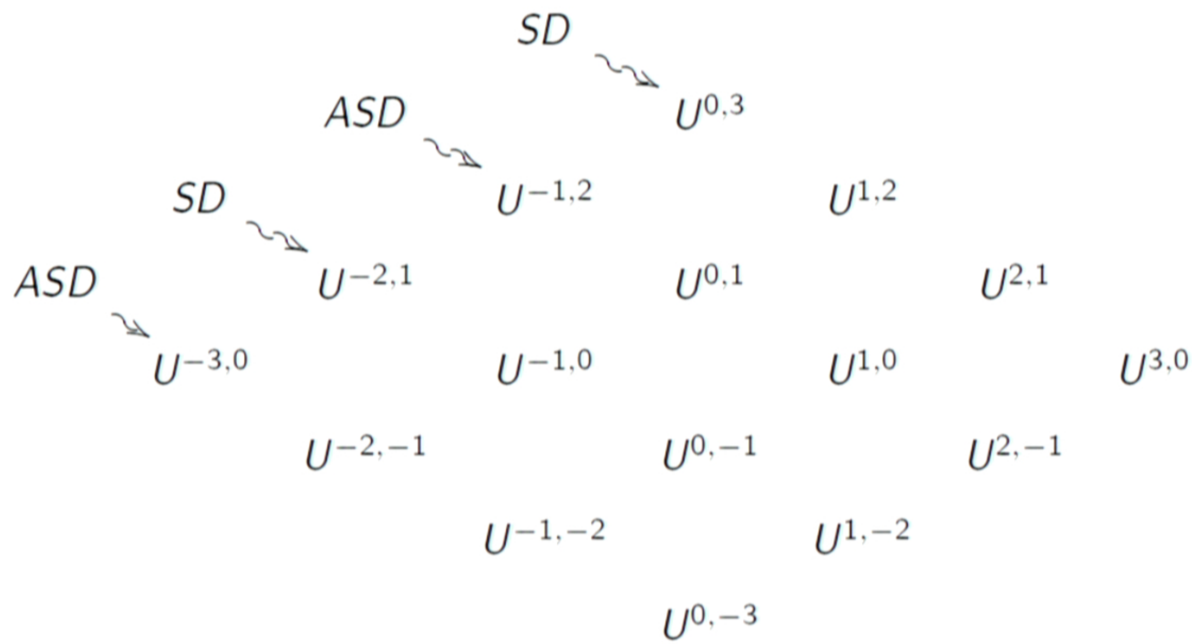
- $T \oplus T^* = V_+^{1,0} \oplus V_+^{0,1} \oplus V_-^{1,0} \oplus V_-^{0,1}$.
- $\wedge^\bullet T^*_{\mathbb{C}} M = \oplus_{p,q} U^p_{\mathcal{J}_1} U^q_{\mathcal{J}_2}$.
- $\star = -e^{\frac{\pi \mathcal{J}_1}{2}} e^{\frac{\pi \mathcal{J}_2}{2}}$

$$\star|_{U^{p,q}} = -i^{p+q}.$$

Geometry of $T \oplus T^*$ — Gen. almost Hermitian structure



Representation of the SD and ASD forms on a 6-dimensional generalized almost Hermitian structure.

Geometry of $T \oplus T^*$ — Courant bracket

- Courant bracket

$$[\![X + \xi, Y + \eta]\!]_H = [X, Y] + \mathcal{L}_X \eta - \iota_Y d\xi - \iota_Y \iota_X H.$$

-

$$[\![v_1, v_2]\!] \cdot \varphi = \{\{v_1, d\}, v_2\} \cdot \varphi.$$

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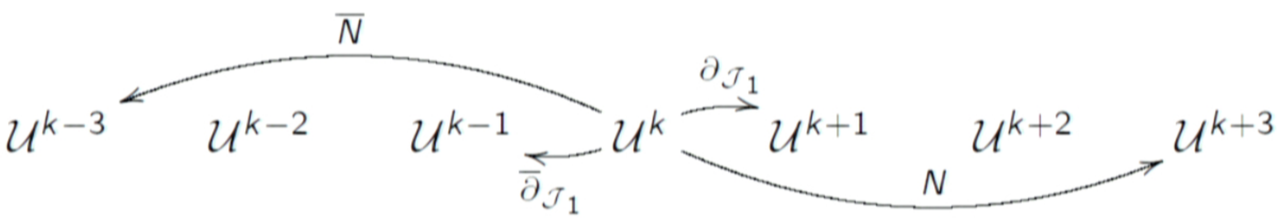
Nijenhuis tensor and intrinsic torsion

- Given a gcs $\mathcal{J}$, define

$$N : \Gamma(\overline{L}) \times \Gamma(\overline{L}) \times \Gamma(\overline{L}) \longrightarrow \Omega^0(M;\mathbb{C})$$

$$N(v_1, v_2, v_3) = \langle [\![v_1, v_2]\!], v_3 \rangle.$$

- $N \in \Gamma(\wedge^3 L)$.
- $\mathcal{J}$ is integrable iff $N \equiv 0$.



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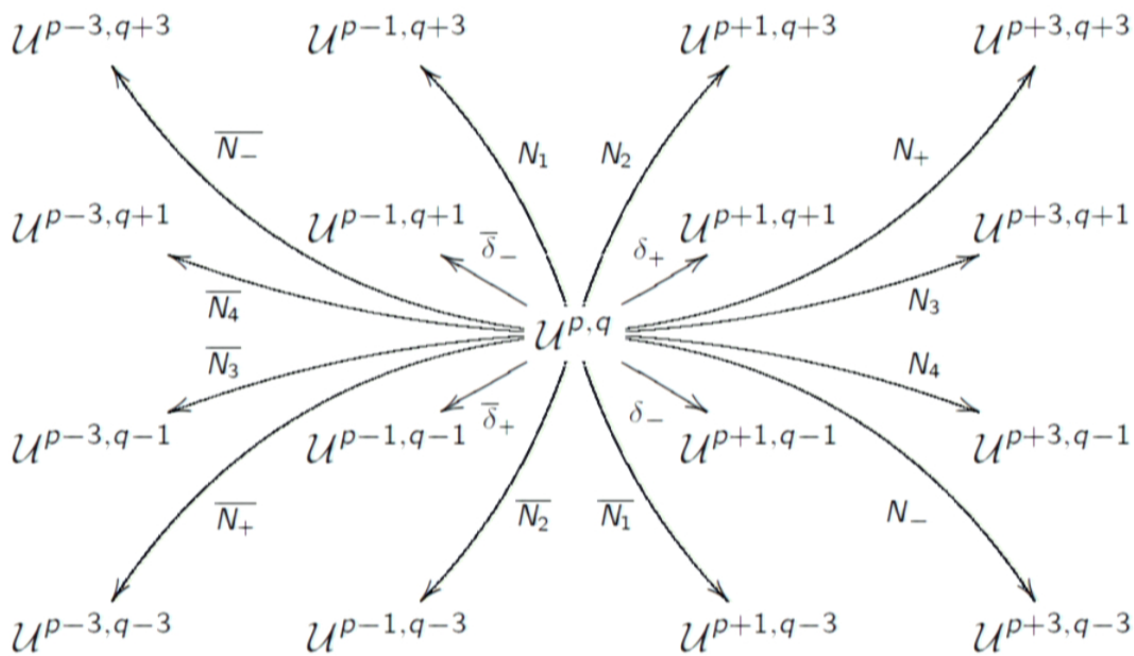
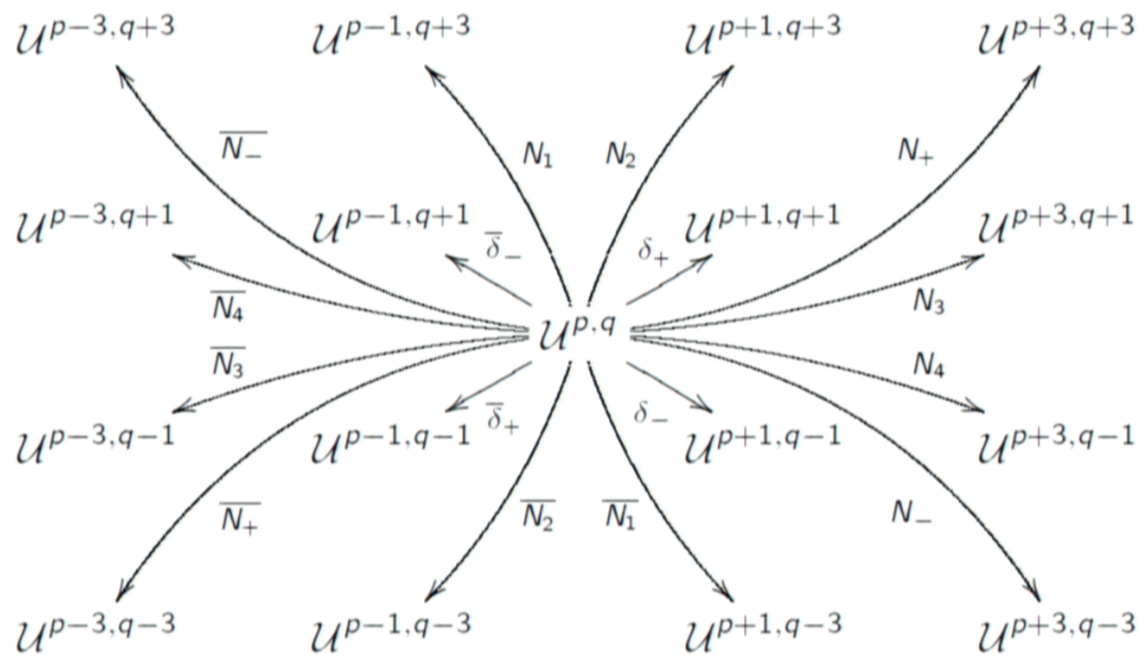


Components of d^H for a generalized almost Hermitian structure.

Nijenhuis tensor and intrinsic torsion



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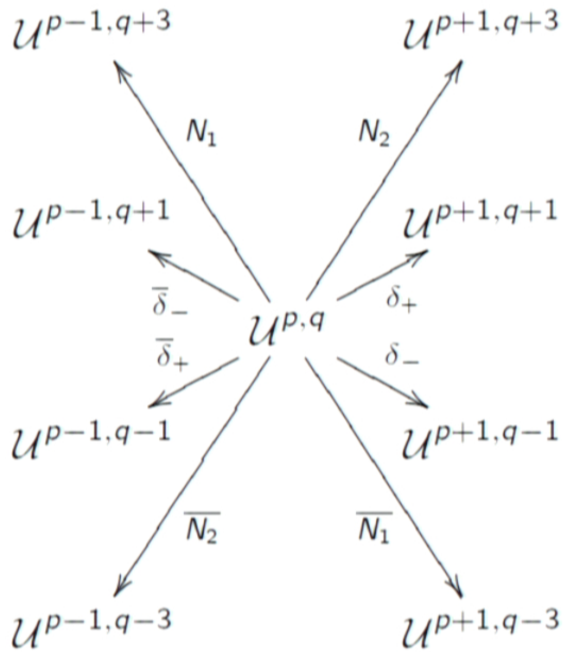

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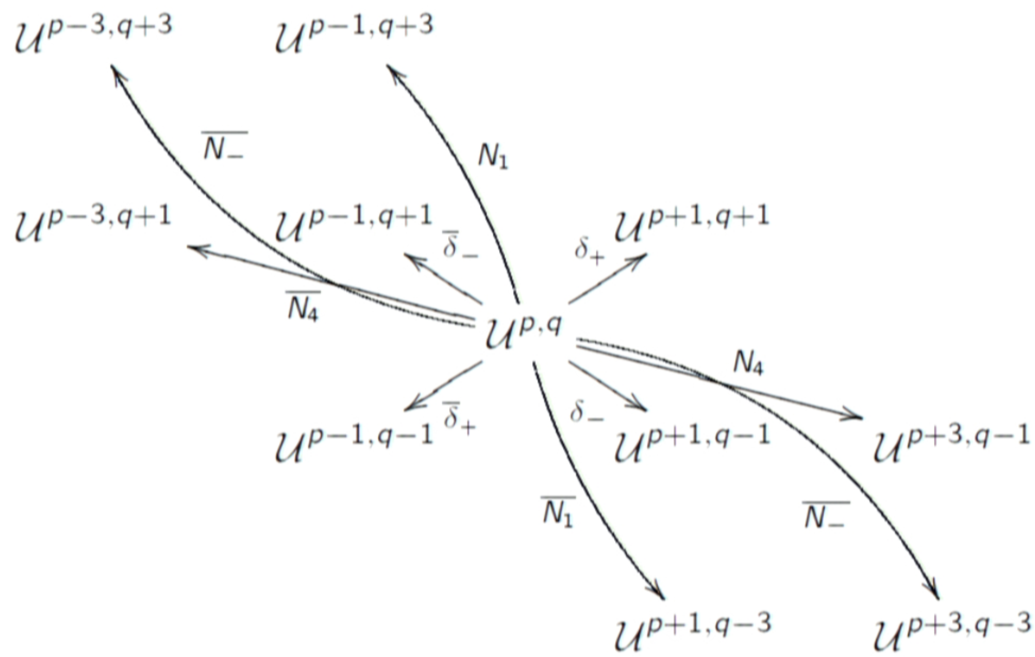
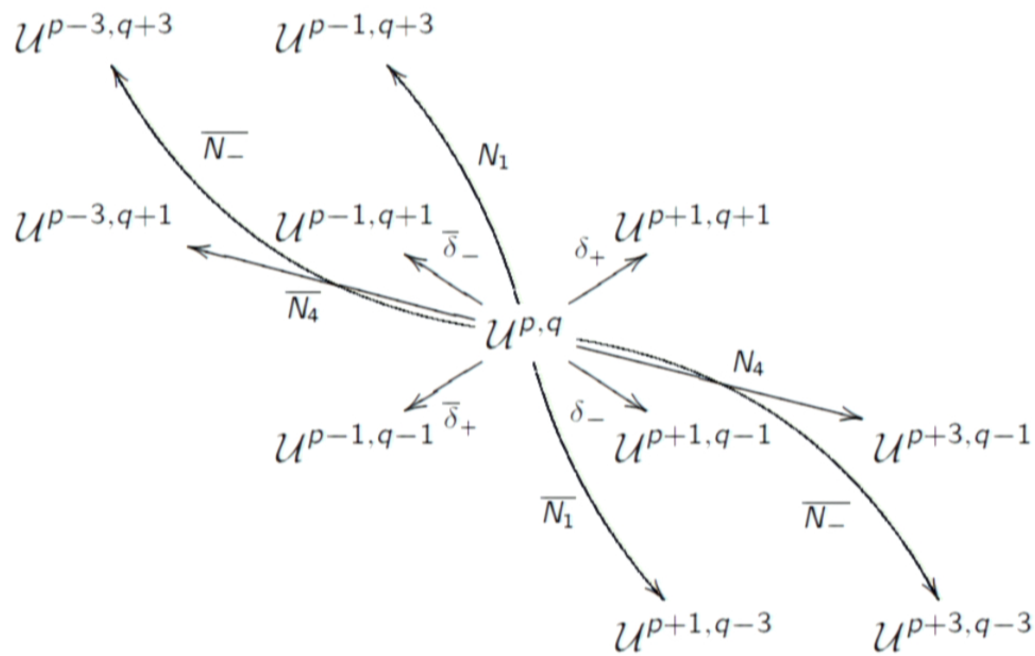


Components of d^H for a (positive) SKT structure.

Nijenhuis tensor and intrinsic torsion



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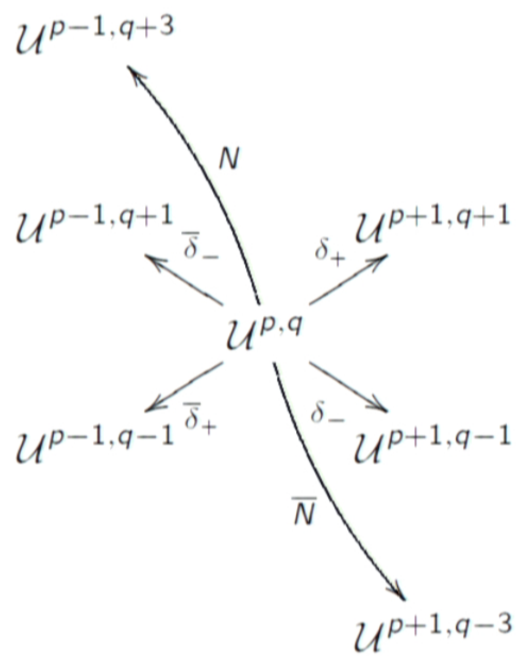
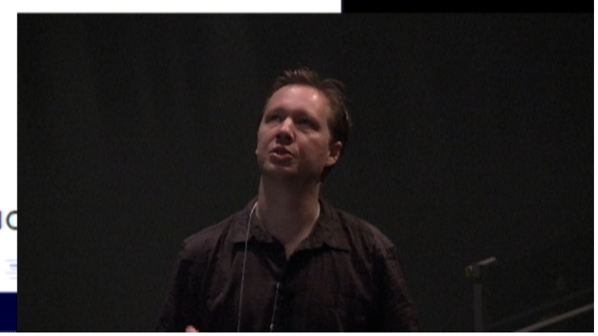


Components of d^H for a gcs extension of an SKT struc



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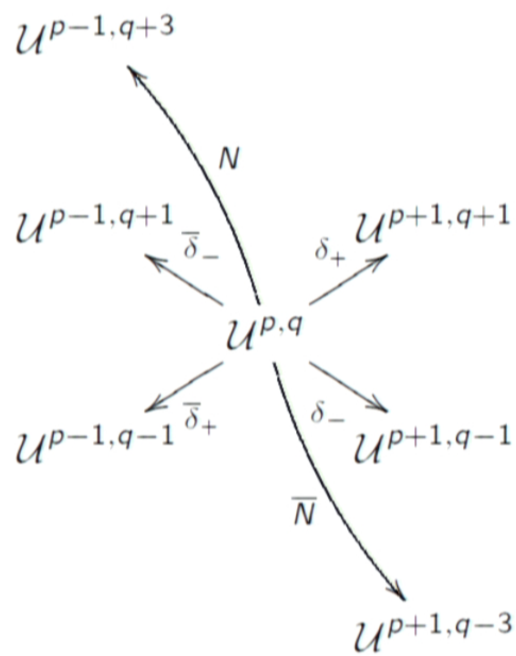

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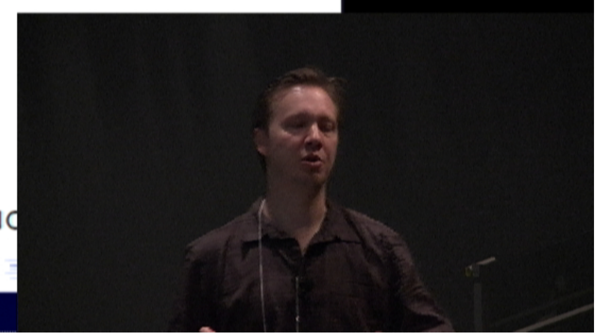

Nijenhuis tensor and intrinsic torsion

Let $W^k = \oplus_{p+q=k}\mathcal{U}^{p,q}$.

Proposition: A generalized almost Hermitian structure is an SKT structure if and only if

$$d^H : \mathcal{W}^k \longrightarrow \mathcal{W}^{k-2} \oplus \mathcal{W}^k \oplus \mathcal{W}^{k+2}.$$

$$\delta_+^N : \mathcal{W}^k \longrightarrow \mathcal{W}^{k+2};$$

$$\overline{\delta_+^N} : \mathcal{W}^k \longrightarrow \mathcal{W}^{k-2};$$

$$\not{\delta}_- : \mathcal{W}^k \longrightarrow \mathcal{W}^k.$$

Nijenhuis tensor and intrinsic torsion

Proposition: A generalized Hermitian structure is an SKT structure if and only if

$$d^H : \mathcal{U}^{p,q} \longrightarrow \mathcal{U}^{p-1,q+3} \oplus \mathcal{U}^{p-1,q+1} \oplus \mathcal{U}^{p-1,q-1} \oplus \mathcal{U}^{p+1,q+1} \oplus \mathcal{U}^{p+1,q-1} \oplus \mathcal{U}^{p+1,q-3}.$$

Hodge theory

In $(M^m, \mathcal{G}, or)$ we define

$$\not{D}_+ = d^H + (-1)^{m+1}(d^H)^*$$

$$\not{D}_- = d^H + (-1)^m(d^H)^*$$

Then:

$$\not{D}_+ : \Omega^\bullet_\pm(M) \longrightarrow \Omega^\bullet_\mp(M)$$

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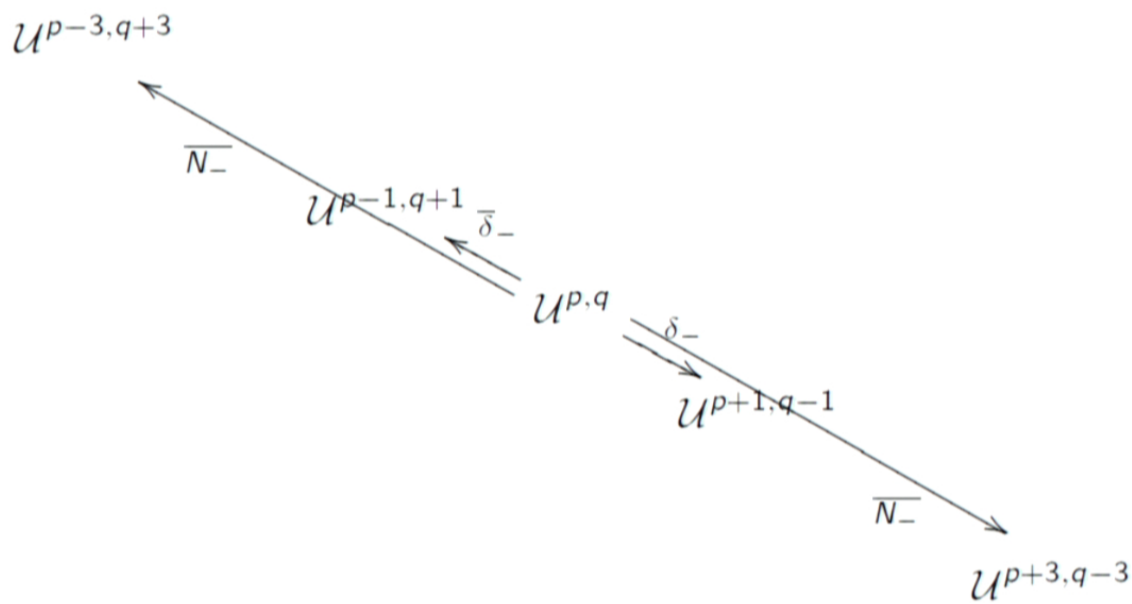
$$(-1)^{m+1}\not{D}_+^2 = (-1)^m \not{D}_-^2 = \frac{1}{4}\triangle_{d^H}.$$

Hodge theory

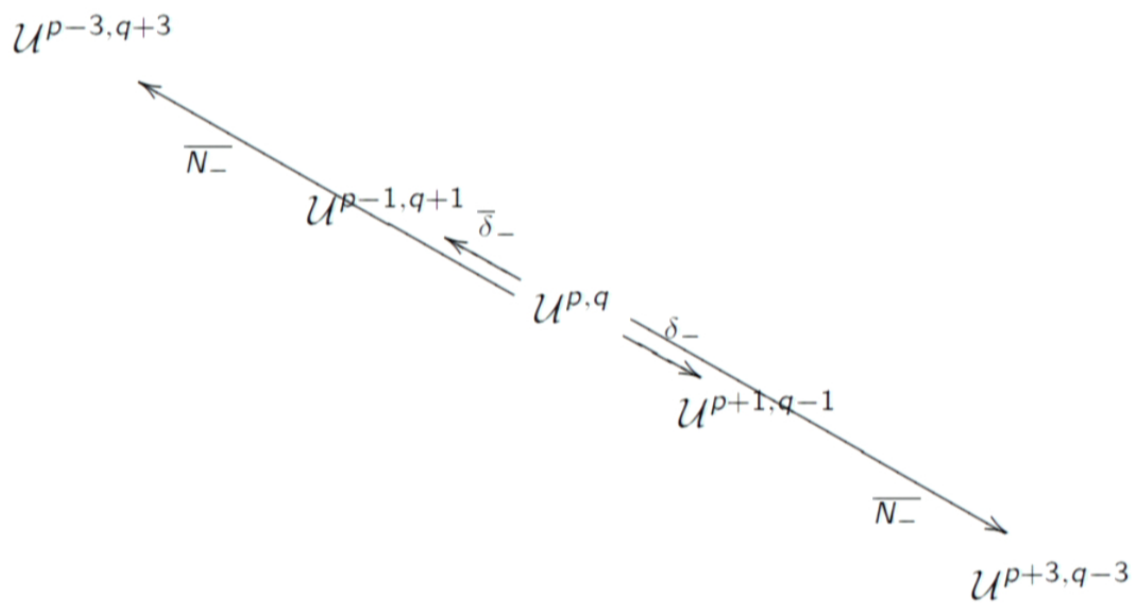
Theorem: *In a compact SKT manifold, the* d^H*-cohomology splits according to the* $\mathcal{W}^k$ *decomposition of forms.*
Remark: *The theorem also holds for parallel (almost) Hermitian structures with closed skew torsion.*

Hodge theory



Components of $\not{D}_-$ for a (positive) SKT structure.

Hodge theory



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Theorem: *In a compact SKT manifold we have*

$$\triangle_{\delta_+^N} = \triangle_{\overline{\delta_+^N}} = \frac{1}{4}\triangle_{d^H}.$$

Proof:
Integration by parts & $\star|_{\mathcal{U}^{p,q}} = -i^{p+q}$ implies that $(\delta_+^N)^* = -\overline{\delta_+^N}$.

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$$\not{D}_+ = \delta_+^N + \overline{\delta_+^N} = \delta_+^N - \delta_+^{N*}$$

$$\frac{1}{4}\triangle_{d^H} = -\not{D}_+^2 = -(\delta_+^N - \delta_+^{N*})^2 = \triangle_{\delta_+^N}.$$

Hodge theory

Theorem: *In a compact SKT manifold* (M, g, I), *the* $\partial + i\overline{\partial}\omega$ *cohomology is isomorphic to the* d^H*-cohomology.*
Proof: There is an isomorphisms of forms

$$\Psi : \Omega^{\bullet}(M; \mathbb{C}) \longrightarrow \Omega^{\bullet}(M; \mathbb{C})$$

such that

$$\Psi(\partial) = \overline{\delta}_+ \qquad \text{and} \qquad \Psi(2i\overline{\partial}\omega) = N$$

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Hodge theory

Corollary: *In a compact SKT manifold (M, g, I), the spectral sequence correspoding to the decomposition*

$$d^H = (\partial + i\overline{\partial}\omega) + (\overline{\partial} - i\partial\omega)$$

degenerates at the second page.

Deformations

- Deformations are given by the action of $\mathrm{SO}(T \oplus T^*)$.
- Small deformations are given by the action of (exponential of) elements in the Lie algebra

$$\Gamma(\mathfrak{spin}(T \oplus T^*)).$$

- It is natural to consider the question of deformations in the context of stability:

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- It is natural to consider the question of deformations in the context of stability:
  Question: *Which deformations of $\mathcal{J}_1$ can be completed with a deformation of $\mathcal{G}$ (or equivalently $\mathcal{J}_2$) so that $(\mathcal{G}, \mathcal{J}_1)$ is a positive SKT structure?*

Deformations

Deformations of $\mathcal{J}_1$ are given by the exponential of sections

$$\alpha \in \Gamma(\wedge^2 \overline{L_{\mathcal{J}_1}})$$

And lead to consideration regarding the behaviour of the operator

$$e^{\alpha} d^H e^{-\alpha} \overset{linearization}{\rightsquigarrow} \{d^H, \alpha\}.$$

with respect to the $\mathcal{U}^{p,q}$ splitting.

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with respect to the $\mathcal{U}^{p,q}$ splitting.Here, the natural differential operators are

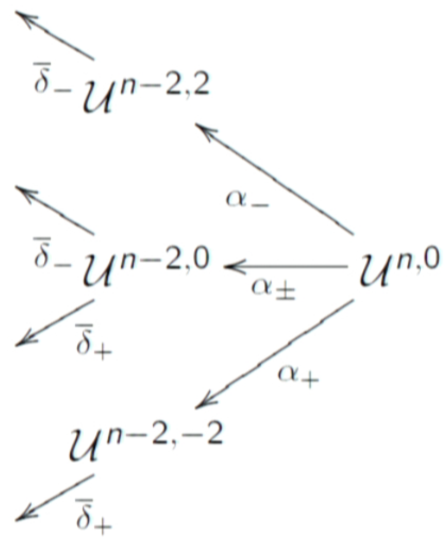
$$\partial_{\pm} = \{\delta_{\pm}, \cdot\};$$

$$\overline{\partial}_{\pm} = \{\overline{\delta}_{\pm}, \cdot\}$$

$$\mathcal{N} = \{N, \cdot\}$$

Deformations

Linear action of α is given by



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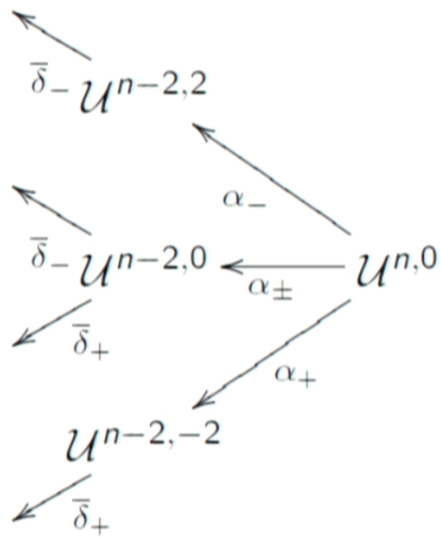
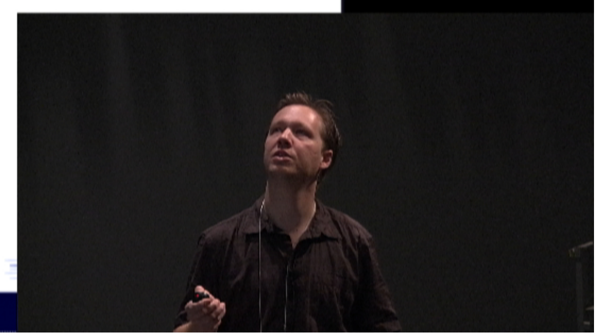
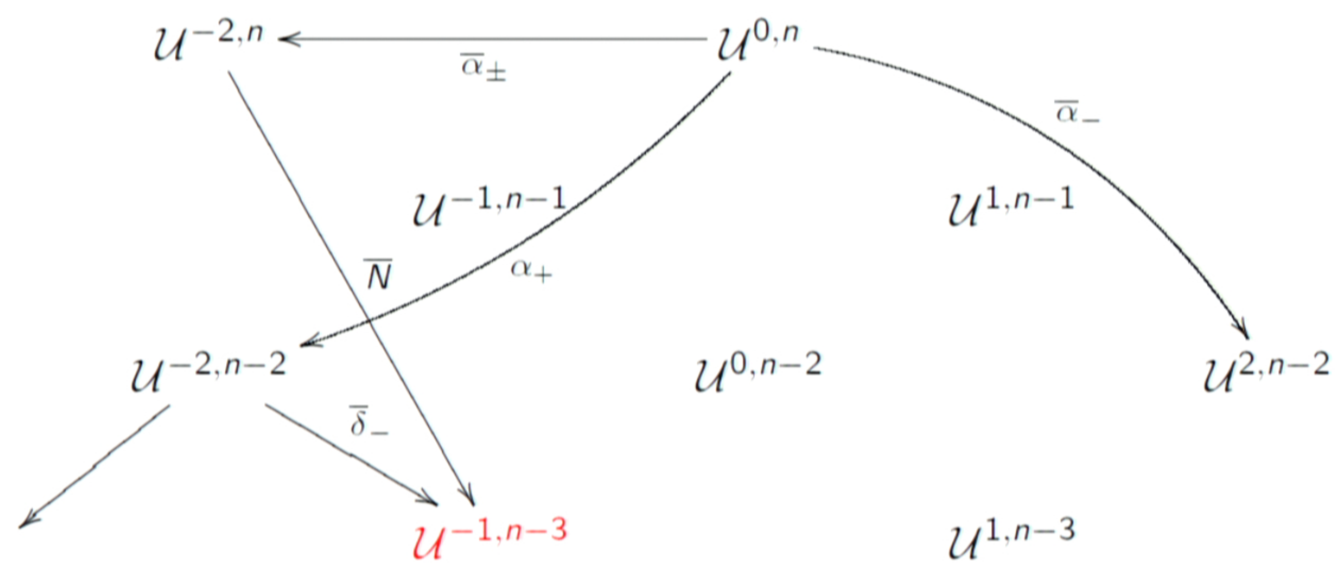




Deformations

For $\mathcal{J}_2$



Deformations

Theorem: *The obstructions to deforming an SKT structure lie in* $H^{2,1}_{\overline{\partial}_+}(M)$.

Theorem: *If* $\alpha = \alpha_- \in \Gamma(\wedge^2 V_-^{0,1})$*, then the deformed structure is still SKT*

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Theorem: *If* $\alpha = \alpha_- \in \Gamma(\wedge^2 V_-^{0,1})$, *then the deformed structure is still SKT*

Corollary: *If* (M, I, ω) *is Kähler, deformations of the symplectic form turn it into an SKT structure.*