Title: An Information-theoretic Approach to Space Dimensionality and Quantum Theory

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Abstract: It is sometimes pointed out as a curiosity that the state space of quantum theory and actual physical space seem related in a surprising way: not only is space three-dimensional and Euclidean, but so is the Bloch ball which describes quantum two-level systems. In the talk, I report on joint work with Lluis Masanes, where we show how this observation can be turned into a mathematical result: suppose that physics takes place in d spatial dimensions, and that some events happen probabilistically (dropping quantum theory and complex amplitudes altogether). Furthermore, suppose there are systems that in some sense behave as â€œbinary units of direction informationâ€•, interacting via some continuous reversible time evolution. We prove that this uniquely determines d=3 and quantum theory, and that it allows observers to infer local spatial geometry from probability measurements.

An information-theoretic approach to space dimensionality and quantum theory

Markus P. Müller
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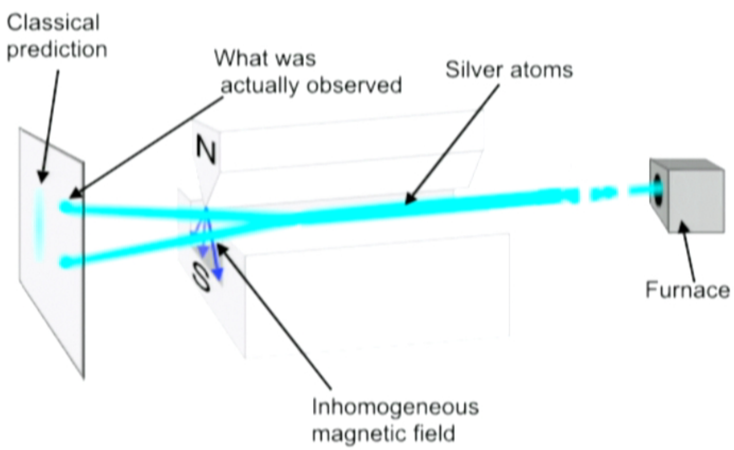

Joint work with Lluís Masanes

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Overview

- The motivation: a curious observation
  - geometry of quantum states vs. physical space; von Weizsäcker's idea
- The framework
  - *d*-dim. physical space; probabilistic events ⟶ convex state spaces
- Three information-theoretic postulates (A,B,C)
  - A+B: *d*-dim. Bloch ball; physical geometry from probability measurements
  - A+B+C: derive that *d*=3, quantum theory, unitary time evolution
  - an impossible generalization
- What does this tell us? Some speculation

1. Motivation: a curious observation

Quantum *n*-level state space: $S_n = \left\{ \rho \in \mathbb{C}^{n\times n}_{s.a.} \;\middle|\; \rho \geq 0,\ \ \mathrm{tr}(\rho) = 1 \right\}.$

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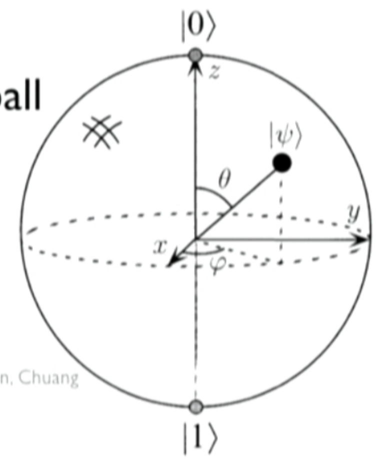
$$S_2 = \left\{ \begin{pmatrix} \frac{1}{2} + r_3 & r_1 - i r_2 \\ r_1 + i r_2 & \frac{1}{2} - r_3 \end{pmatrix} \;\middle|\; |\vec{r}| \leq 1 \right\}$$

Bloch ball



Source: Nielsen, Chuang

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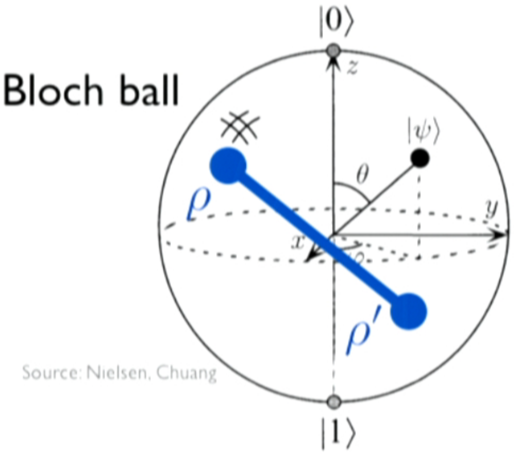

- This is a particularly nice representation:
  $p\rho + (1-p)\rho' \;\;\mapsto\;\; p\vec{r} + (1-p)\vec{r}'$
  statistical mixtures ⟶ convex combinations

- S_2 is Euclidean and 3-dimensional. But so is physical space! Just a coincidence?

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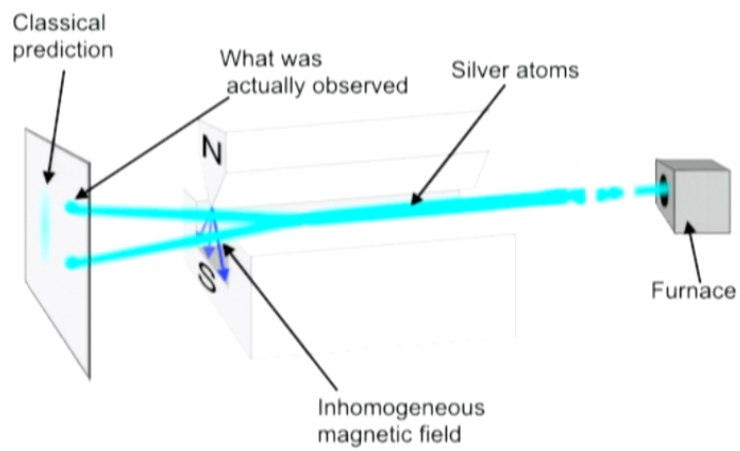
Physical consequence of ballness: 1:1 correspondence between noiseless measurements on 2-level systems and "directions" (of magnetic field)



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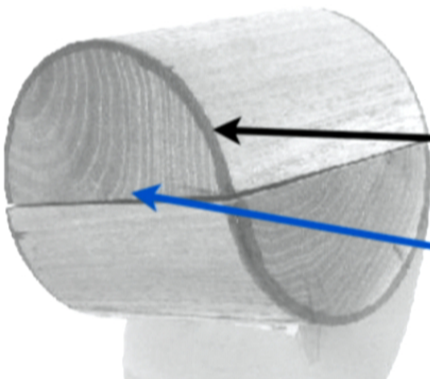
Recall that quantum 3-level systems (and higher) are not balls:



convex set S_n

pure states (extremal points)

mixed states in topological boundary:

$$\rho_{\mathrm{mix}} := \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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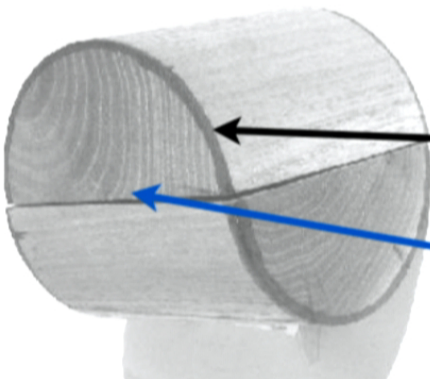

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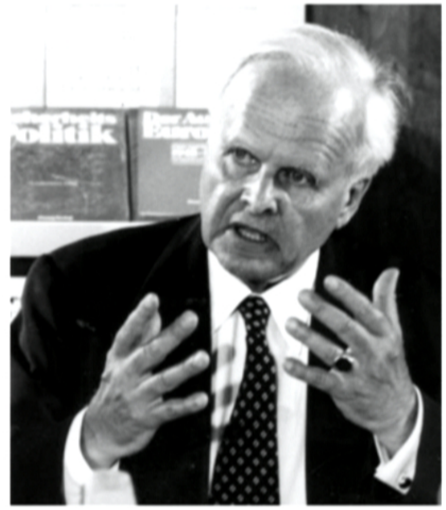

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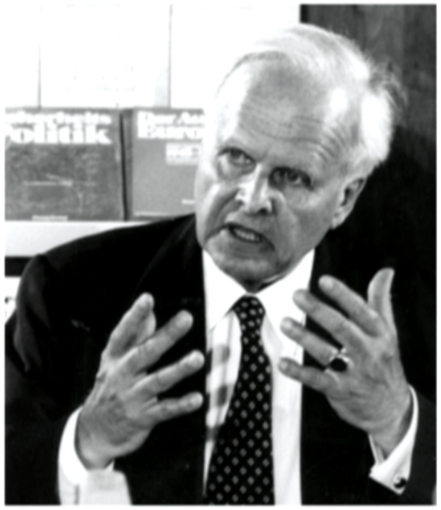
- "ur" = (pure) qubit = quantum 2-level system
- everything is composed of (delocalized) urs
- symmetry group of ur

$$U(2) = SU(2) \otimes U(1) \sim S^3 \times S^1.$$

becomes global symmetry group of universe.



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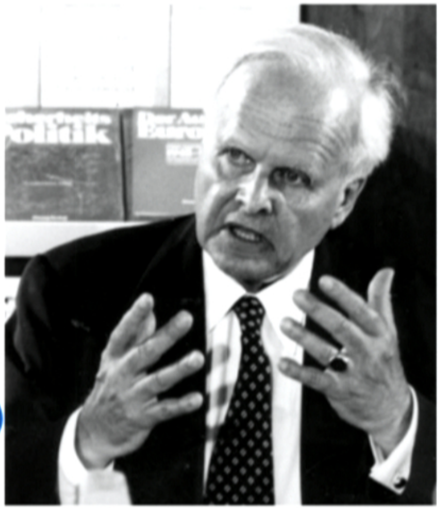

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space (?!)   time (replaced by $\mathbb{R}^1$)

Very vague. What does that mean?

How is decomposition into *delocalized* urs chosen?
Why not ternary ur-alternatives w/ SU(3)?
Why is the result global cosmic space-time?
...

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- Do *not* assume quantum theory; leave spatial dimension d arbitrary.
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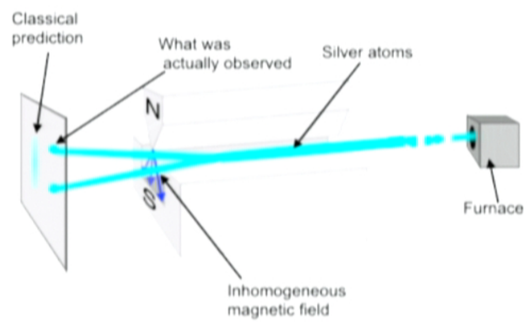

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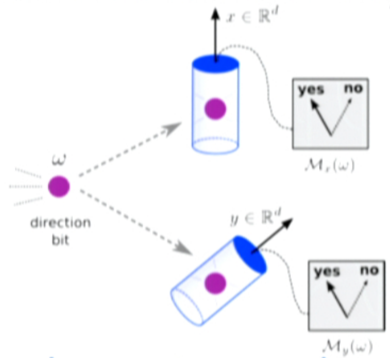
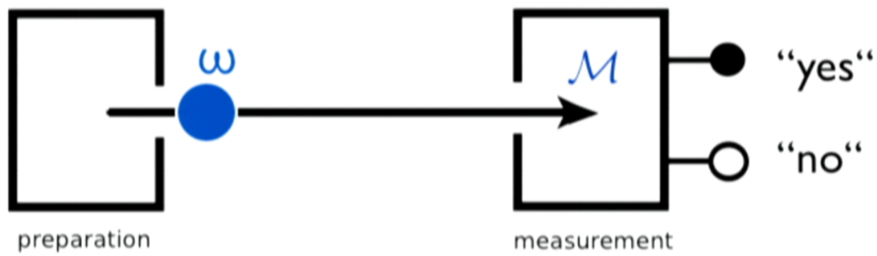


- Give three information-theoretic postulates on how probabilities and rotations are related.
- Prove that we must have *d*=3 and quantum theory necessarily.

M. Müller*, Ll. Masanes

2. The framework

Assumption: there are some events that happen probabilistically.



- Physical systems can be in some state ω. From this, all outcome probabilities of all subsequent events can be computed:

$$\text{Prob(outcome "yes"} \mid \text{meas. } \mathcal{M} \text{ on state } \omega) =: \mathcal{M}(\omega).$$

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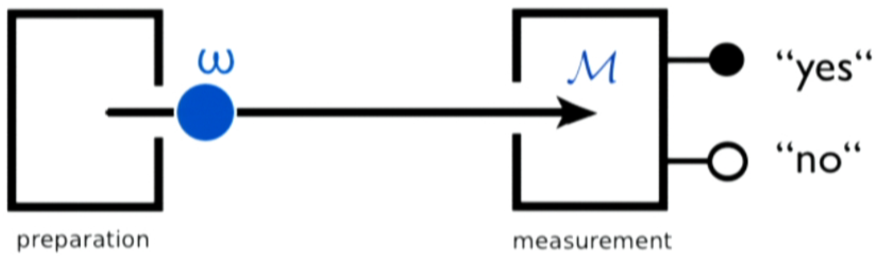

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- Statistical mixtures are described by convex combinations: prepare ω with prob. *p* and state φ with prob. *(1-p)*, result:

$$p\,\omega + (1-p)\varphi$$

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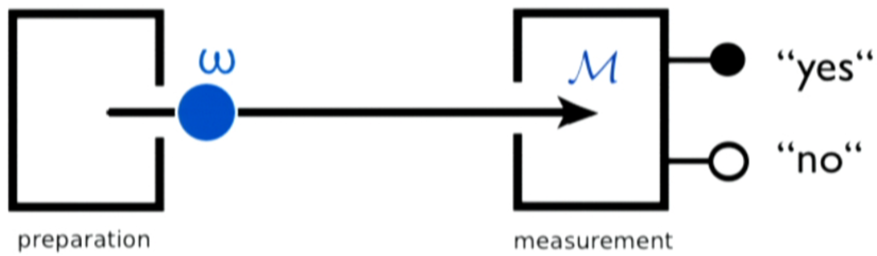
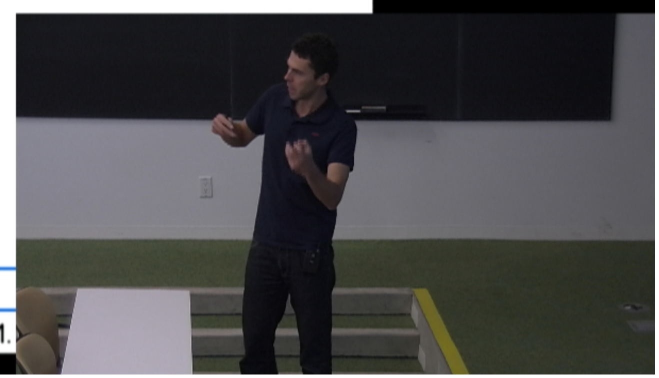


- Consequence: measurements ("effects") $\mathcal{M}$ are affine-linear:

$$\mathcal{M}(p\omega + (1-p)\varphi) = p\mathcal{M}(\omega) + (1-p)\mathcal{M}(\varphi).$$



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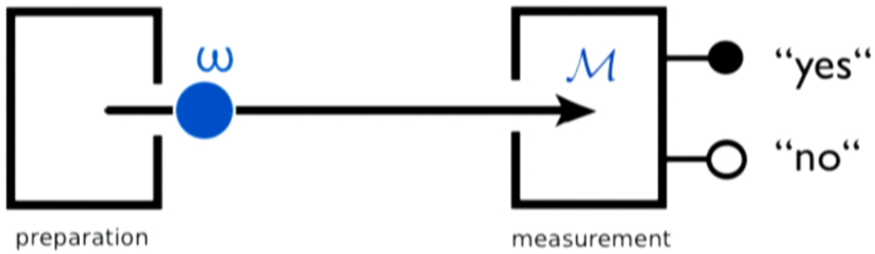

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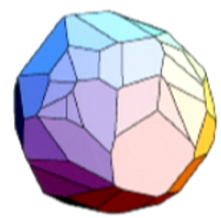
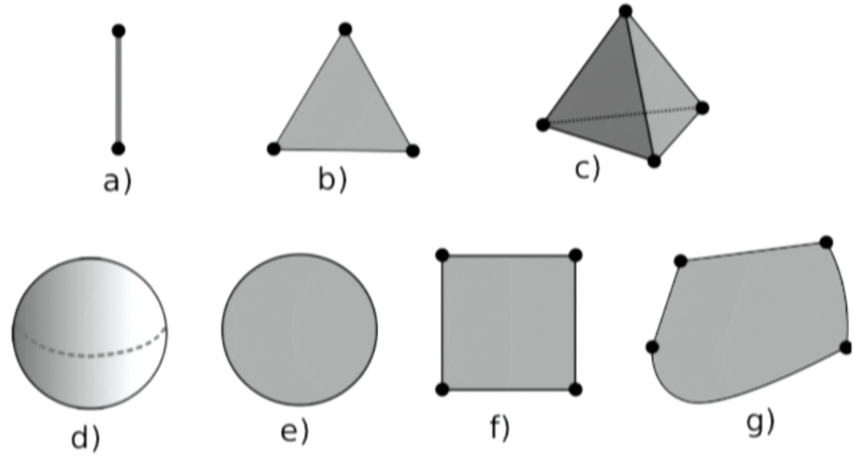
- State space Ω = set of all possible states ω.
  Convex, compact, finite-dimensional.
  Otherwise arbitrary.



Some examples:



- Classical n-level system:

$$\Omega = \{\omega = (p_1, \ldots, p_n) \mid p_i \geq 0, \ \sum_i p_i = 1\}.$$

n pure states: $\omega_1 = (1, 0, \ldots, 0), \ldots, \omega_n = (0, \ldots, 0, 1).$

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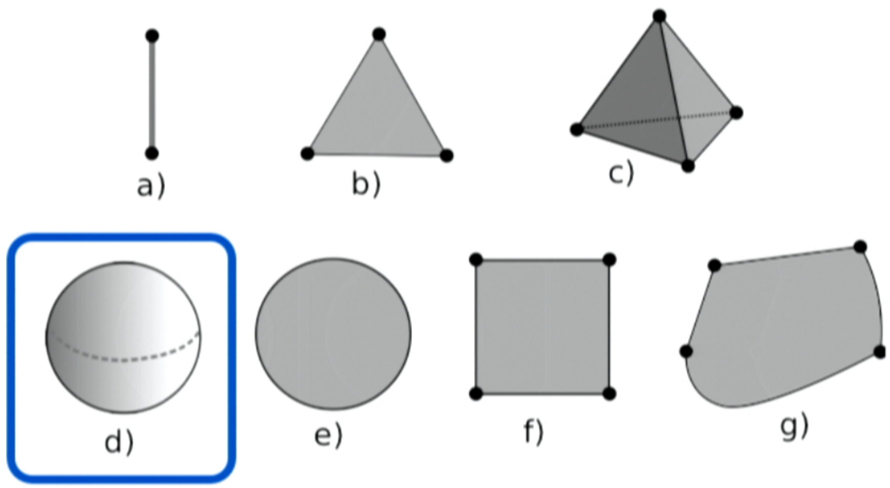

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a), b), c): classical 2-, 3-, 4-level systems.

- d): quantum 2-level system (qubit)

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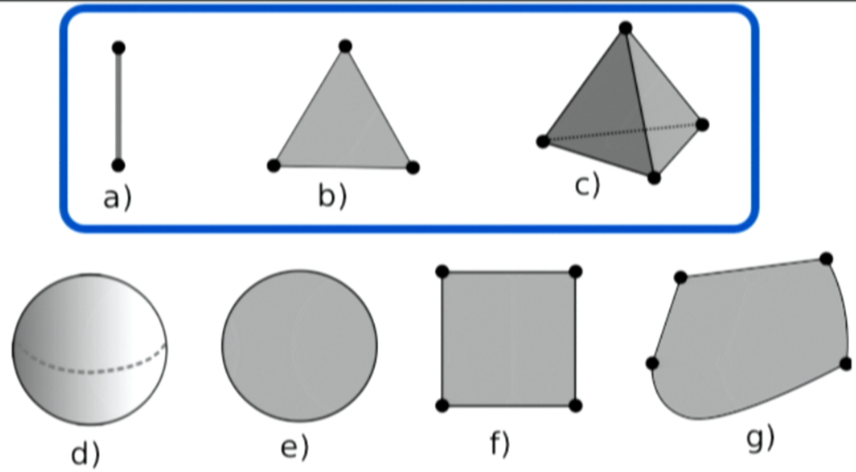

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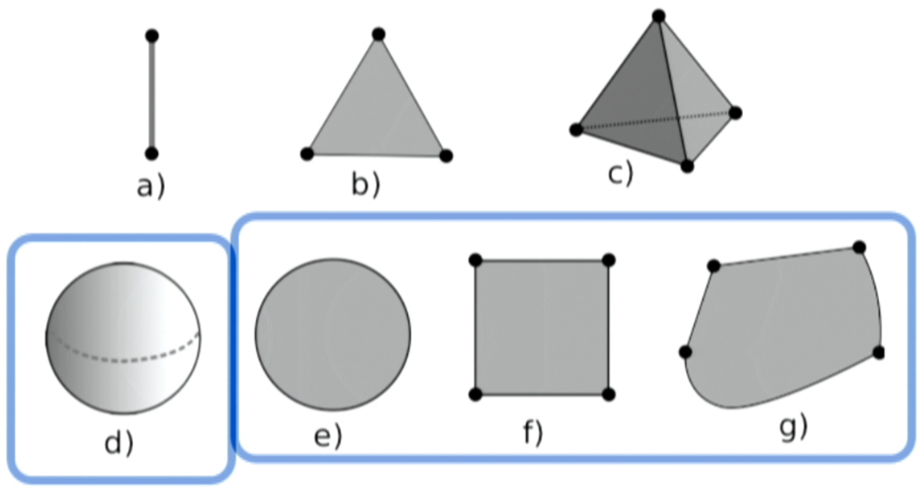

- Classical n-level system:
- e), f), g): neither classical nor quantum.

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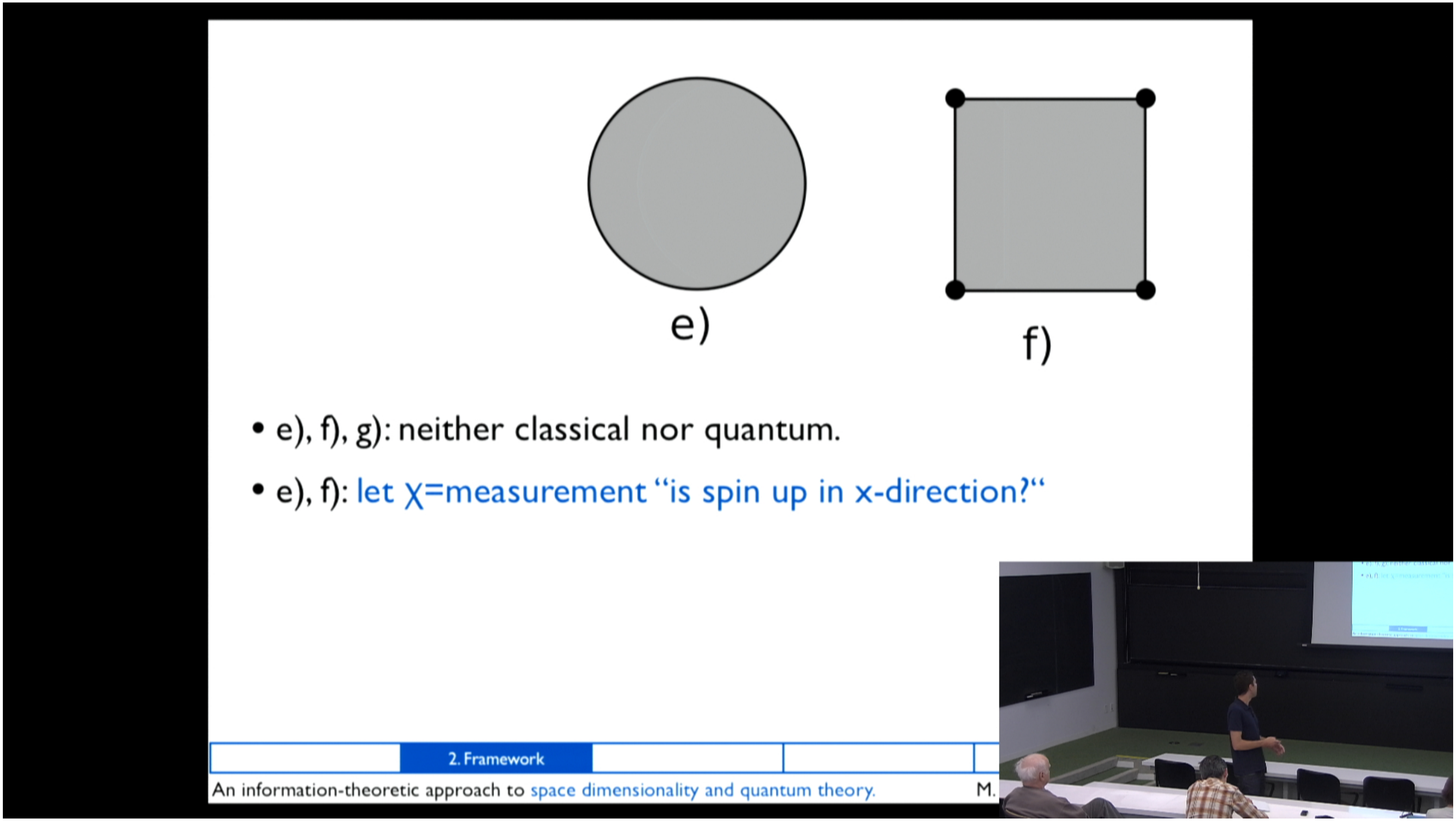
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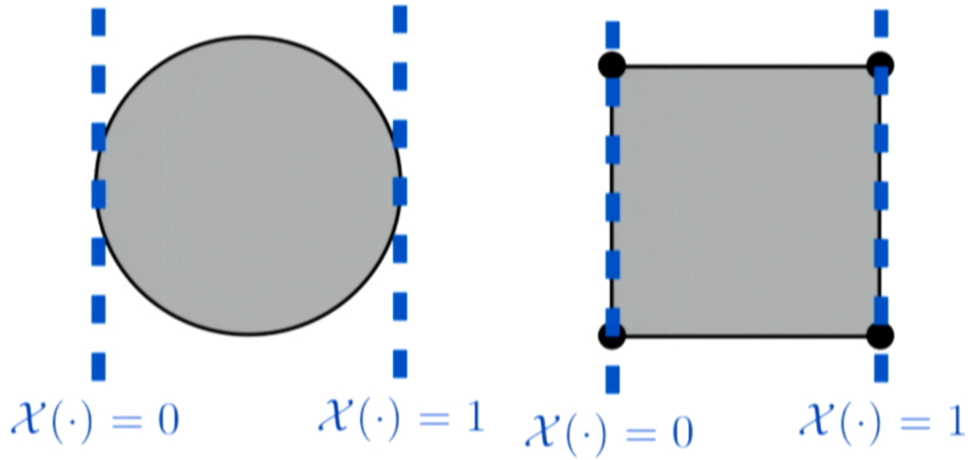

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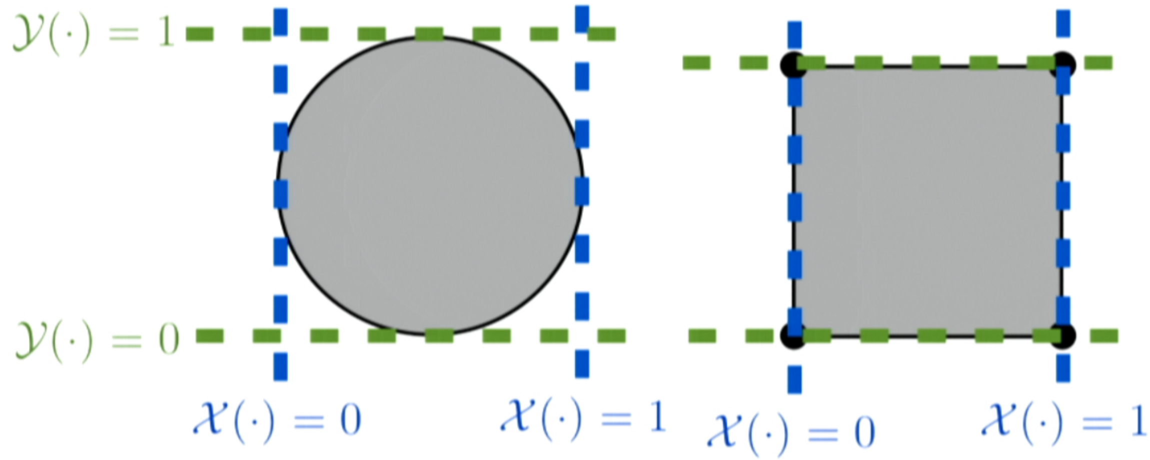

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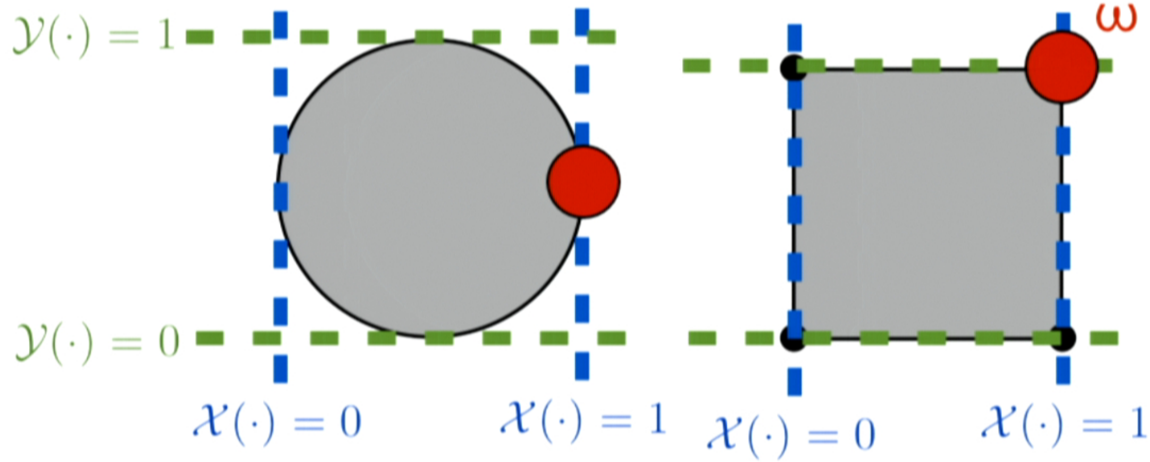
  analogously for Y.



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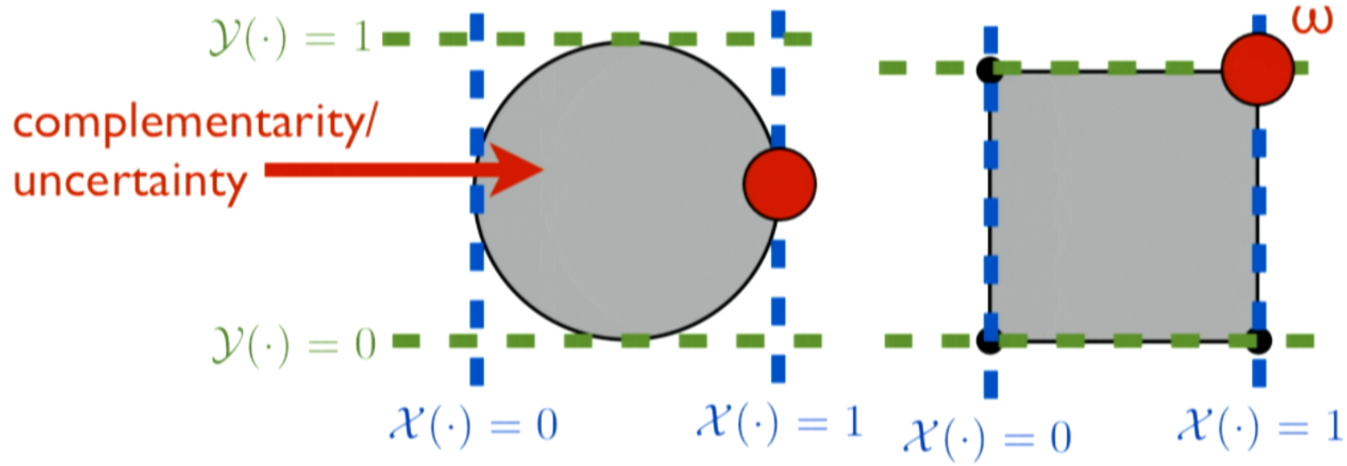
  Square: there is a state ω with $\mathcal{X}(\omega) = \mathcal{Y}(\omega) = 1.$

  Circle: if $\mathcal{X}(\omega) = 1$ then necessarily $\mathcal{Y}(\omega) = 1/2.$



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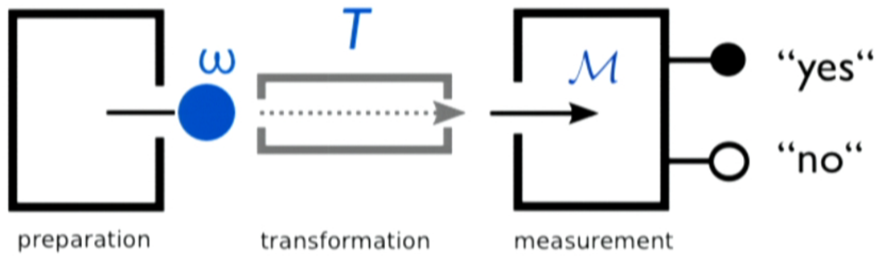

Transformations T map states to states and are linear.

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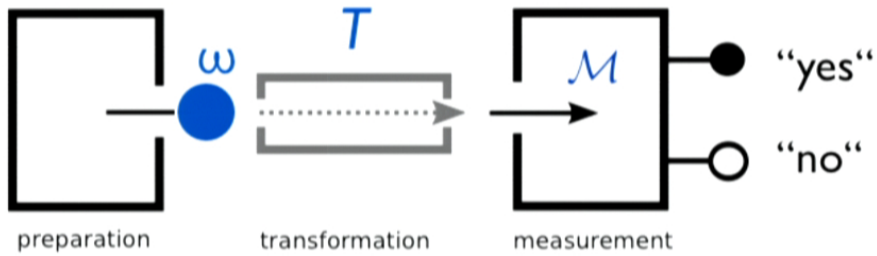

Transformations T map states to states and are linear.

- Here, only interested in reversible transformations T (i.e. invertible).
- They form a compact (maybe finite) group $\mathcal{G}$.
- In quantum theory, these are the unitaries:

$$\rho \mapsto U \rho U^{\dagger}.$$

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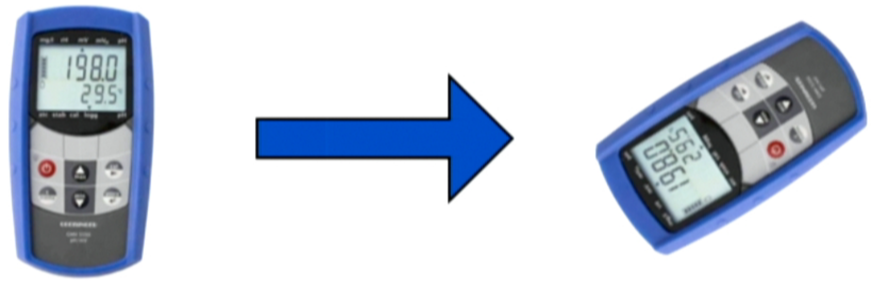
Assumption: physics takes place in d spatial dimensions (+ time).
All we consider happens locally + at rest ⟶ Euclidean space.



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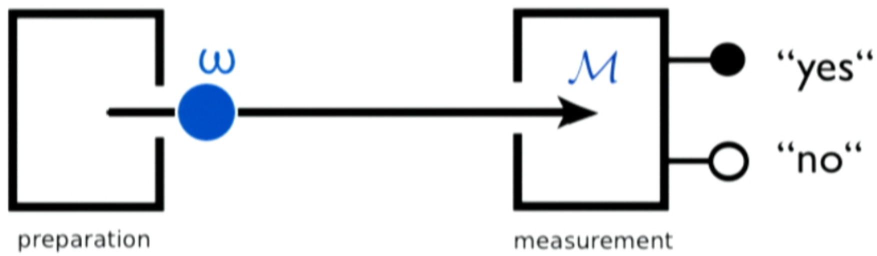

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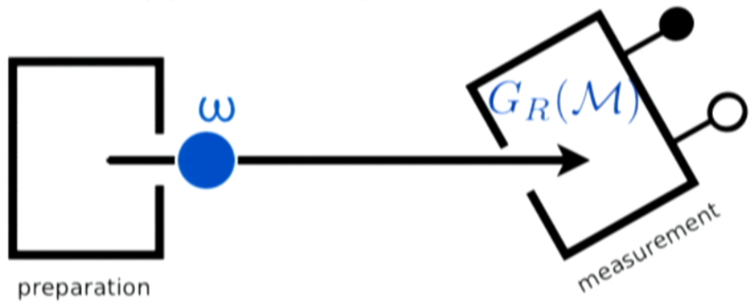

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$$G_R(\mathcal{M})(\omega) = \mathcal{M}(G_R^*(\omega)).$$

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There exist certain systems that behave like "binary units of direction info".

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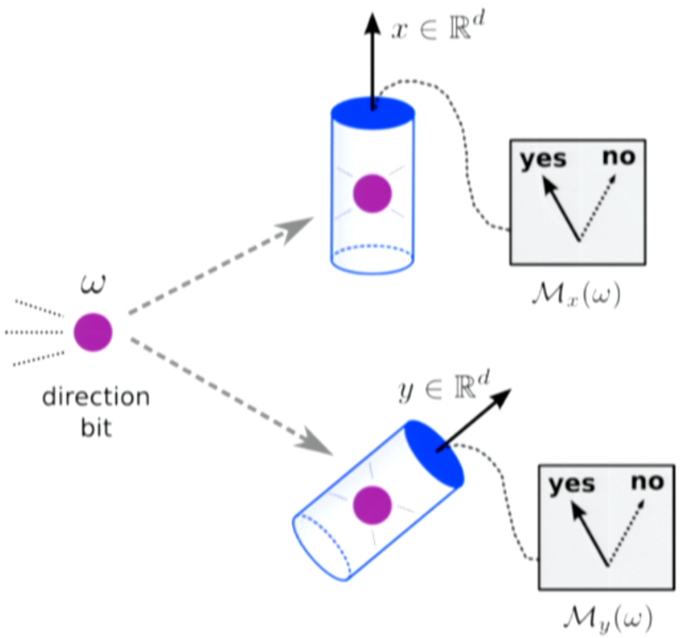

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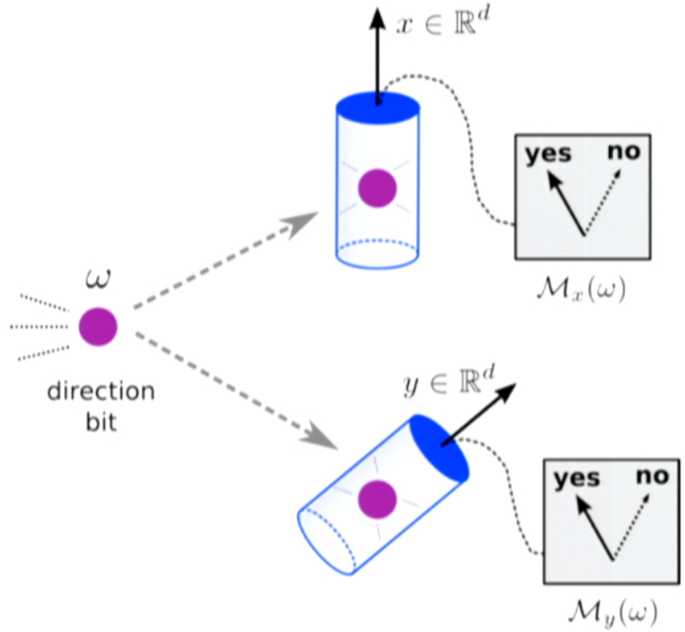

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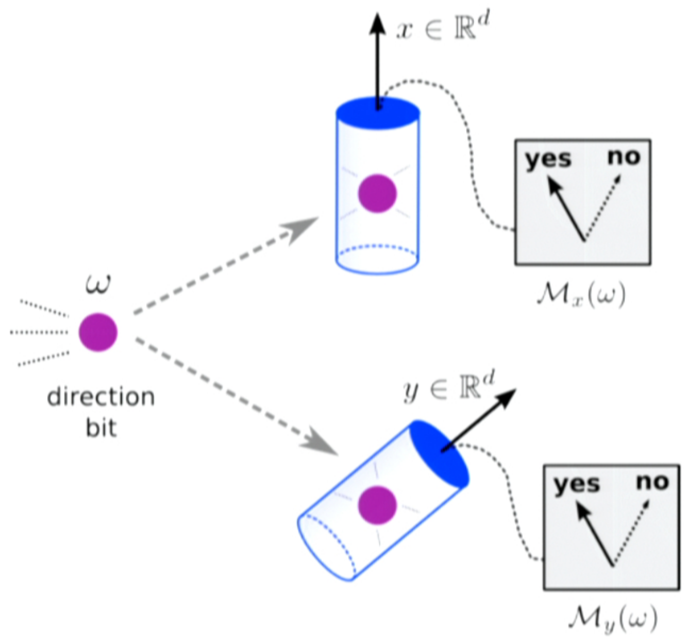

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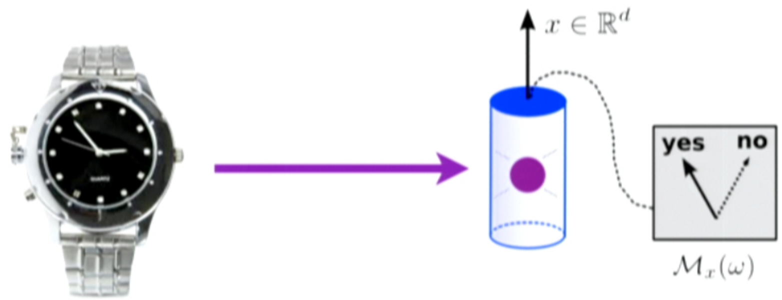
There exists a state ω and a direction $x \in \mathbb{R}^d$ such that

$$\mathcal{M}_x(\omega) = 1$$

but $\mathcal{M}_y(\omega) < 1$ for all $y \neq x$.

A) $\exists\, \omega,\ x \in \mathbb{R}^d$: $\mathcal{M}_x(\omega) = 1$

$\mathcal{M}_y(\omega) < 1 \quad (y \neq x)$



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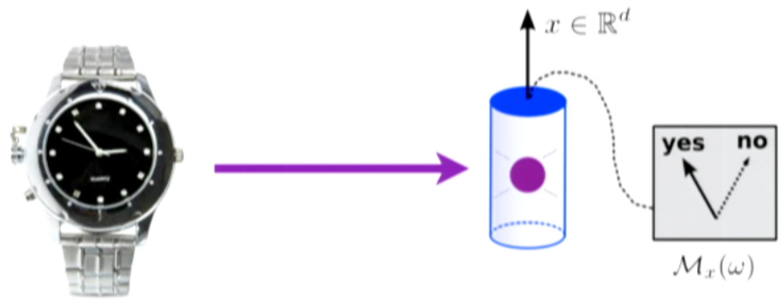

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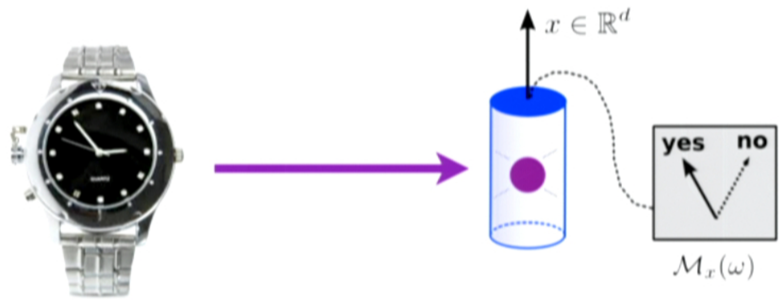
But: this watch carries lots of extra information!



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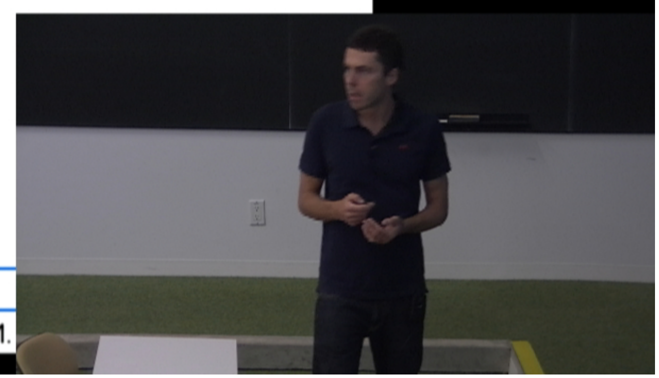

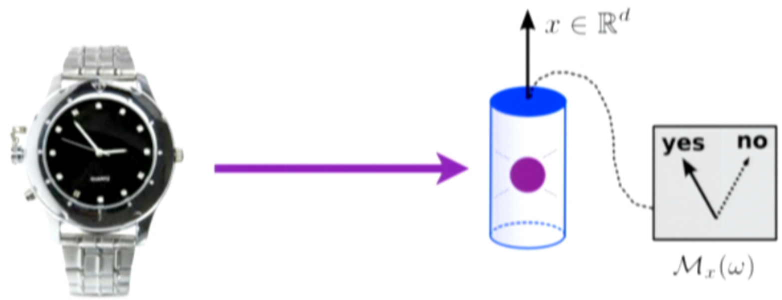

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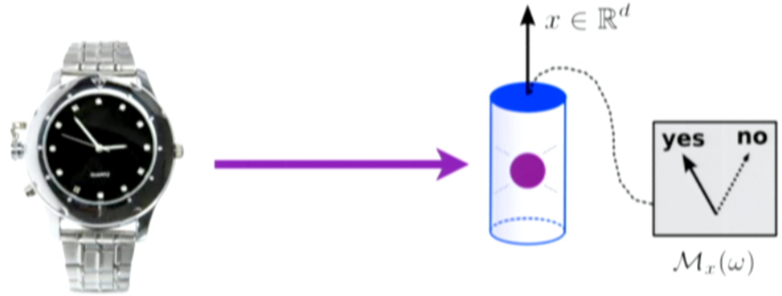

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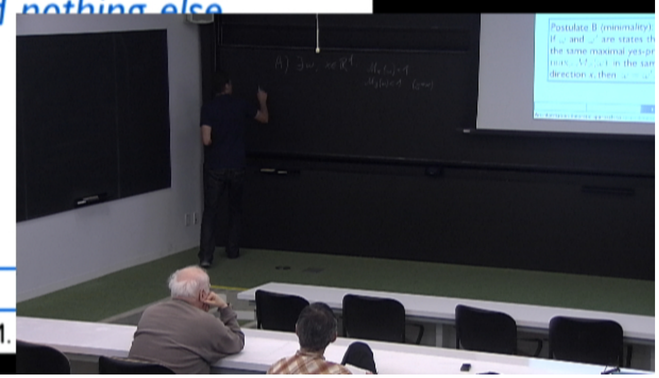

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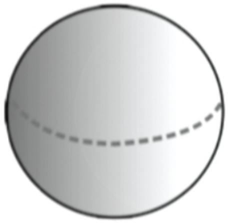

3. Postulates A+B

M. Müller*, Ll. Masanes

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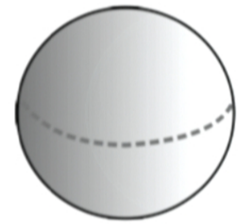

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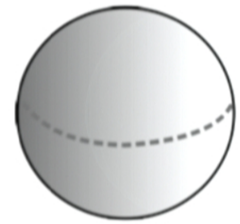

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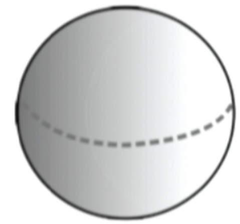

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- Group rep. theory: inner product such that $|\vec{\omega}_y| = 1 \text{ for all } y.$
- Postulate B ⇒ every state can be written $\omega = \lambda \omega_x + (1 - \lambda)\mu.$

⇒ *D*-dim. ball. Dimension counting ⇒ *D*=d.

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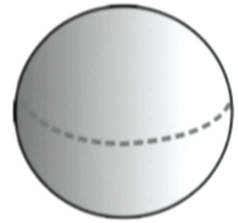

- This is a non-classical state space with *d* independent mutually complementary measurements.
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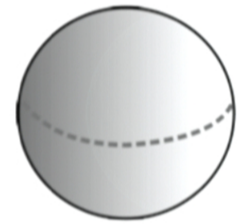

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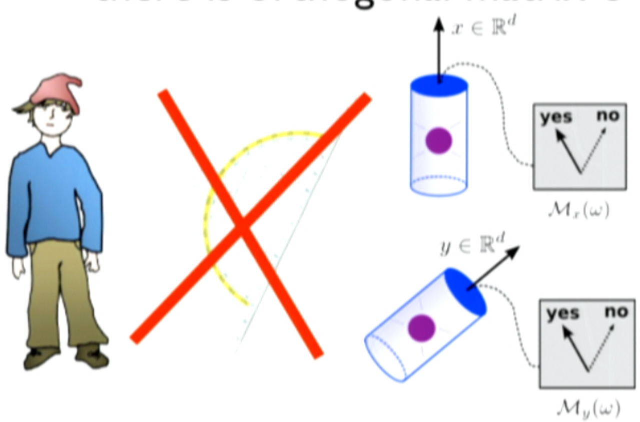

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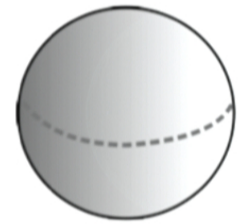

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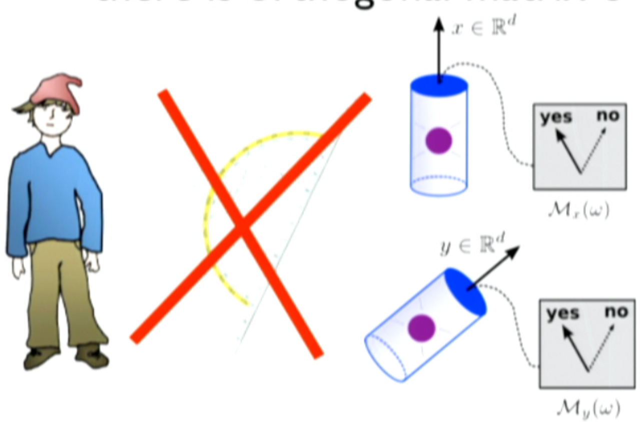

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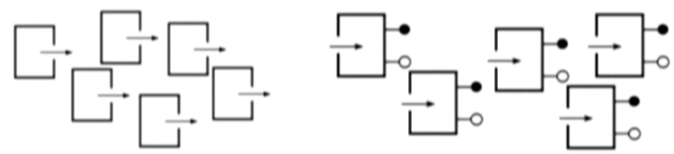
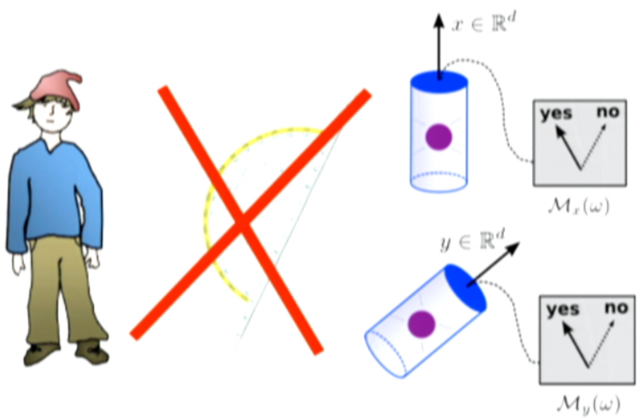


Protocol:

- Select d preparations $\omega_1, \ldots, \omega_d$ with lin. independent Bloch vectors $\vec{\omega}_1, \ldots, \vec{\omega}_d$ (otherwise protocol will fail).
- By trial+error, find $\mathcal{M}_1, \ldots, \mathcal{M}_d$ with $\mathcal{M}_i(\omega_i) \approx 1$.
- Using $\mathcal{M}_i(\omega_j) \approx c + (1-c)\langle\vec{\omega}_i, \vec{\omega}_j\rangle$, determine the matrix $X_{ij} := \langle\vec{\omega}_i, \vec{\omega}_j\rangle$. Compute solution to $S^T S = X$.



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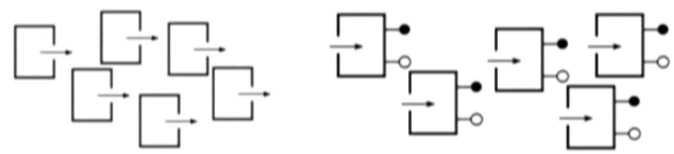

3. Postulates A+B

An information-theoretic approach to space dimensionality and quantum theory. M. Müller*, Ll. Masanes

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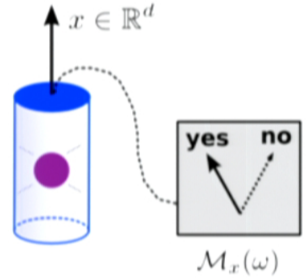

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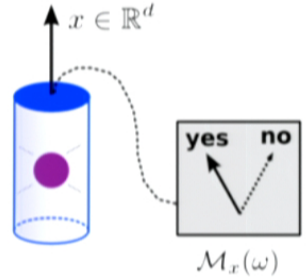

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Orientation characterized by matrix $X \in SO(d)$.

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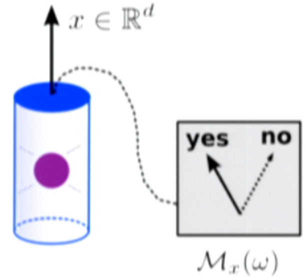

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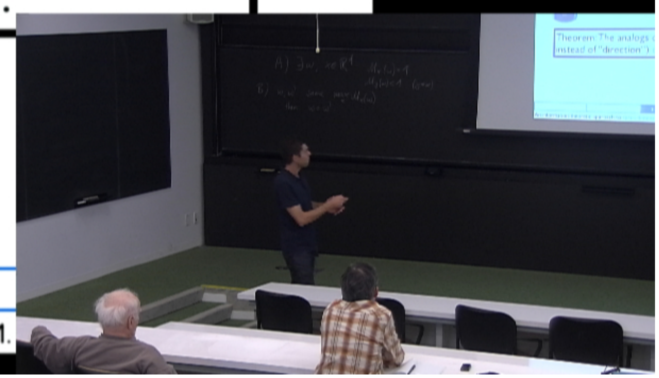

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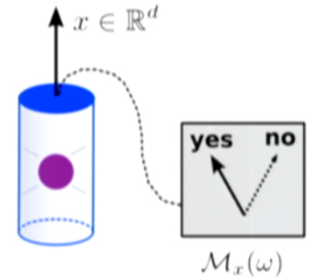

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<u>Proof</u>: State space would again be a unit ball. Pure states: $\{\omega_X\}_{X \in SO(d)}$
But SO(*d*) is not simply connected, and the sphere is.

4. Postulate C

Our final postulate says that two direction bits can interact via some continuous reversible time evolution:

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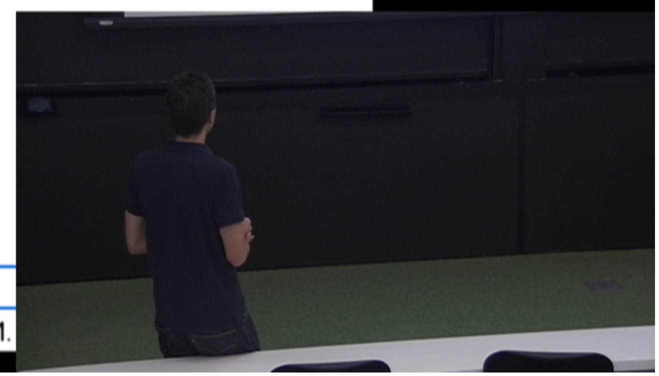

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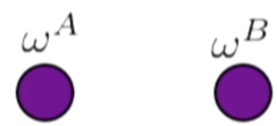

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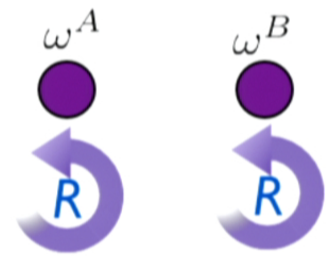
Given $R \in SO(d)$, we want a unique way to specify the global rotation on the composite system.



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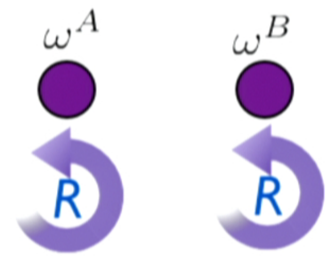

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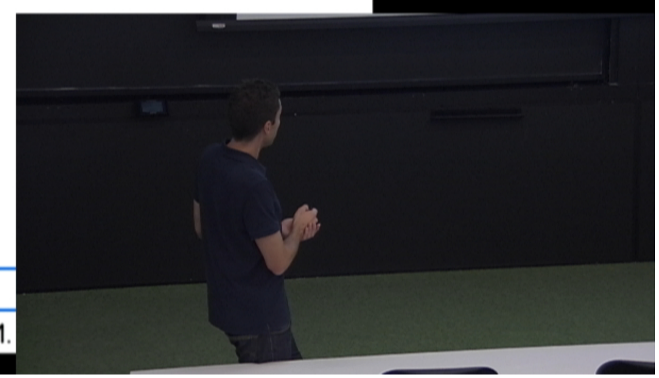

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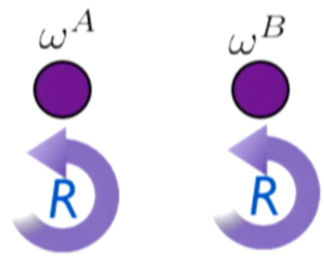

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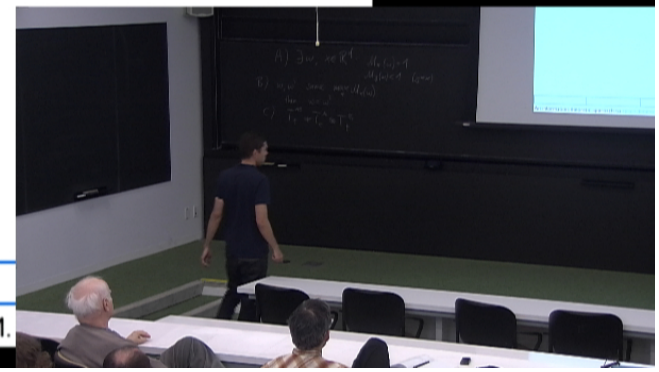

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An information-theoretic approach to space dimensionality and quantum theory. M.

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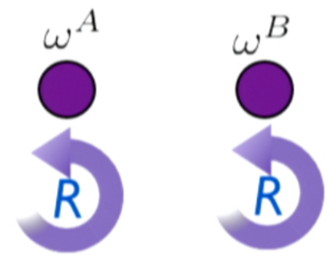

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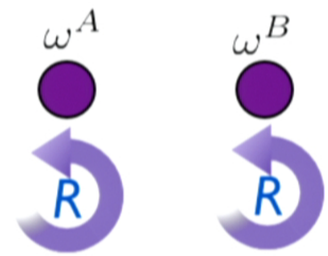

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- True for classical prob. theory, quantum theory, almost all other convex theories studied so far.
- Equivalent to "tomographic locality": global states are uniquely determined by probabilities of local measurements and their correlations.
- Allows to represent product states via tensor product:

$\omega^A \omega^B = \omega^A \otimes \omega^B$. $\qquad \omega^{AB} \mapsto G_R \otimes G_R(\omega^{AB})$.

Theorem: From Postulates A, B and C, it follows that *d*=3.

Proof idea *(Ll. Masanes, MM, D. Pérez-García, R. Augusiak, arXiv:1111.4060)*

- Consider global Lie group $\mathcal{G}^{AB}$ generated by $\{T_t^{AB}\}_{t\in\mathbb{R}}$ and $G^A \otimes G^B$.
- Global Lie algebra element $X \in \mathfrak{g}^{AB}$, then

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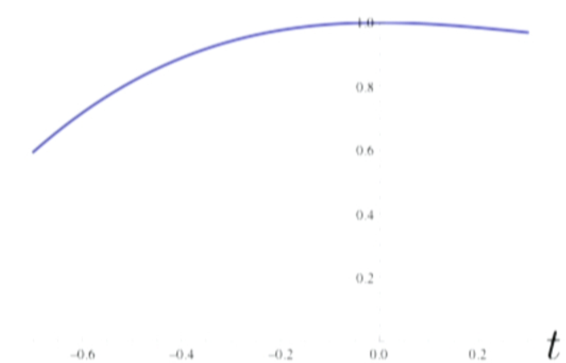

- But this equals 1 for $t=0$, thus

$$\mathcal{M}_x \otimes \mathcal{M}_y \, X \, \omega_x \otimes \omega_y = 0,$$

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Proof idea *(G. de la Torre, Ll. Masanes, A. J. Short, MM, arXiv:1110.5482)*

- We have *d=3*. Embed the *3*-ball in the unit trace matrices of $\mathbb{C}^{2\times 2}_{s.a.}$

$$(r_1, r_2, r_3) \mapsto \begin{pmatrix} \frac{1}{2} + r_3 & r_1 - ir_2 \\ r_1 + ir_2 & \frac{1}{2} - r_3 \end{pmatrix}.$$

- Thus, global states will be unit trace matrices in $\mathbb{C}^{2\times 2}_{s.a.} \otimes \mathbb{C}^{2\times 2}_{s.a.} = \mathbb{C}^{4\times 4}_{s.a.}$

Theorem: From Postulates A, B and C, it follows that the state space of two direction bits is 2-qubit quantum state space (i.e. the set of 4x4 density matrices), and time evolution is given by a one-parameter group of unitaries,

$$\rho \mapsto U(t)\rho U(t)^\dagger.$$

Proof idea *(G. de la Torre, Ll. Masanes, A. J. Short, MM, arXiv:1110.5482)*

- We have *d*=*3*. Embed the *3*-ball in the unit trace matrices of $\mathbb{C}^{2\times 2}_{s.a.}$

$$(r_1, r_2, r_3) \mapsto \begin{pmatrix} \frac{1}{2} + r_3 & r_1 - ir_2 \\ r_1 + ir_2 & \frac{1}{2} - r_3 \end{pmatrix}.$$

- Thus, global states will be unit trace matrices in $\mathbb{C}^{2\times 2}_{s.a.} \otimes \mathbb{C}^{2\times 2}_{s.a.} = \mathbb{C}^{4\times 4}_{s.a.}$
- Now some $X \neq X^A + X^B$ satisfy constraints. But they all generate maps of the form $e^{tX}(\rho) = U\rho U^\dagger$ with $U \in SU(4)$.

Theorem: From Postulates A, B and C, it follows that the state space of two direction bits is 2-qubit quantum state space (i.e. the set of 4x4 density matrices), and time evolution is given by a one-parameter group of unitaries,

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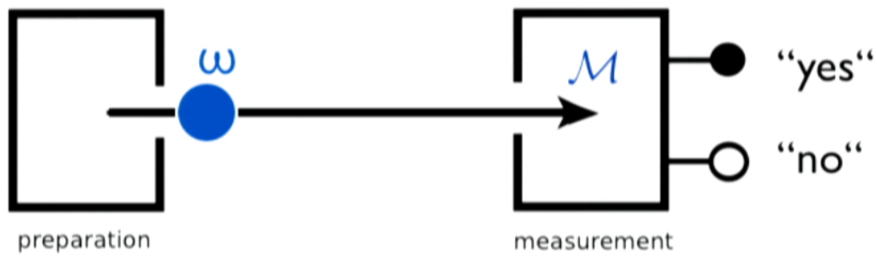
Proof idea *(G. de la Torre, Ll. Masanes, A. J. Short, MM, arXiv:1110.5482)*

- We have at least one entangling unitary (Postulate C) and all local unitaries (rotations). This generates all unitaries!

5. Conclusions

Attempt to clarify the relationship between spatial geometry and the qubit (based on old ideas & new techniques):



- Start with *d* spatial dimensions, not assuming quantum theory.
- Three "information-theoretic" postulates on the relation between spatial geometry (rotations) and probability
- Proof that these determine *d=3* and quantum theory on 2 bits.

5. Conclusions

What does that mean? We don't know...

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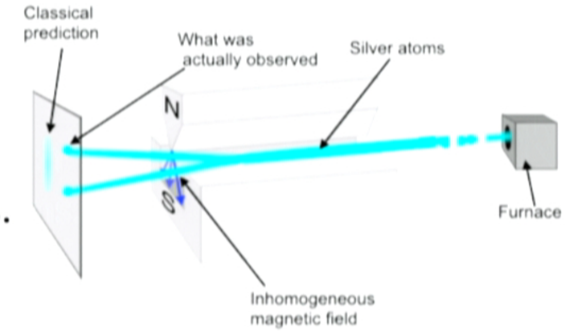
- The “neat“ behaviour of a Stern-Gerlach device is only possible in $d=3$ dimensions.



M. Müller*, Ll. Masanes

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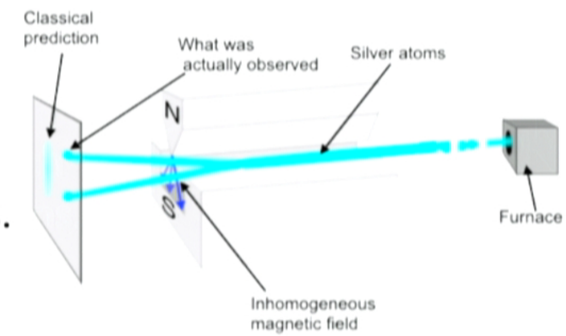
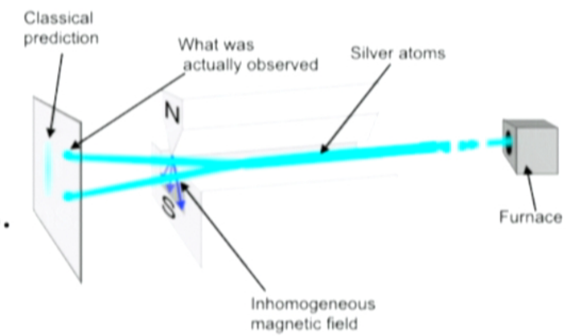


- It is interesting to consider generalizations of quantum theory in the context of fundamental physics.

- Possible (relativistic) generalizations of the result?

5. Conclusions

What does that mean? We don't know...



- The "neat" behaviour of a Stern-Gerlach device is only possible in $d=3$ dimensions.
- It is interesting to consider generalizations of quantum theory in the context of fundamental physics.
- Possible (relativistic) generalizations of the result?
- Speculation: do space(-time) and quantum theory have a *common* information-theoretic origin?

5. Conclusions

Thank you to Lucien Hardy, Lee Smolin; my co-authors; Danny Terno, FJ Schmitt, Hilary Carteret, Mauro d'Ariano, Raymond Lal, Tobias Fritz, ...

- introduction to convex probabilistic theories:
  J. Barrett, arXiv:quant-ph/0508211
- ruling out $d \neq 3$:
  Ll. Masanes, MM, D. Pérez-García, R. Augusiak, arXiv:1111.4060
- $d=3$ implies quantum theory:
  G. de la Torra, Ll. Masanes, A. J. Short, MM, arXiv:1110.5482
- results of this talk:
  MM, Ll. Masanes, arXiv:hopefully.soon

Theorem: From Postulates A, B and C, it follows that *d*=3.

Proof idea (*Ll. Masanes, MM, D. Pérez-García, R. Augusiak, arXiv:1111.4060*)

- We get several constraints on $X \in \mathfrak{g}^{AB}$:

$$\begin{aligned} \mathcal{M}_x \otimes \mathcal{M}_y X \omega_x \otimes \omega_y &= 0, \\ \mathcal{M}_x \otimes \mathcal{M}_y X^2 \omega_x \otimes \omega_y &\leq 0, \qquad \ldots \end{aligned}$$

- If $d \neq 3$, the only *X* satisfying them all are of the form $X = X^A + X^B$ with local rotation generators X^A, X^B.

  These generate non-interacting dynamics.

- For $d \geq 3$, evaluating constraints involves integrals like

$$X \mapsto \int_{SO(d-1)} G^A \otimes \mathbf{1}^B X (G^A)^{-1} \otimes \mathbf{1}^B \, dG^A.$$

This behaves very differently if *SO(d-1)* is Abelian, i.e. iff *d*=3. □