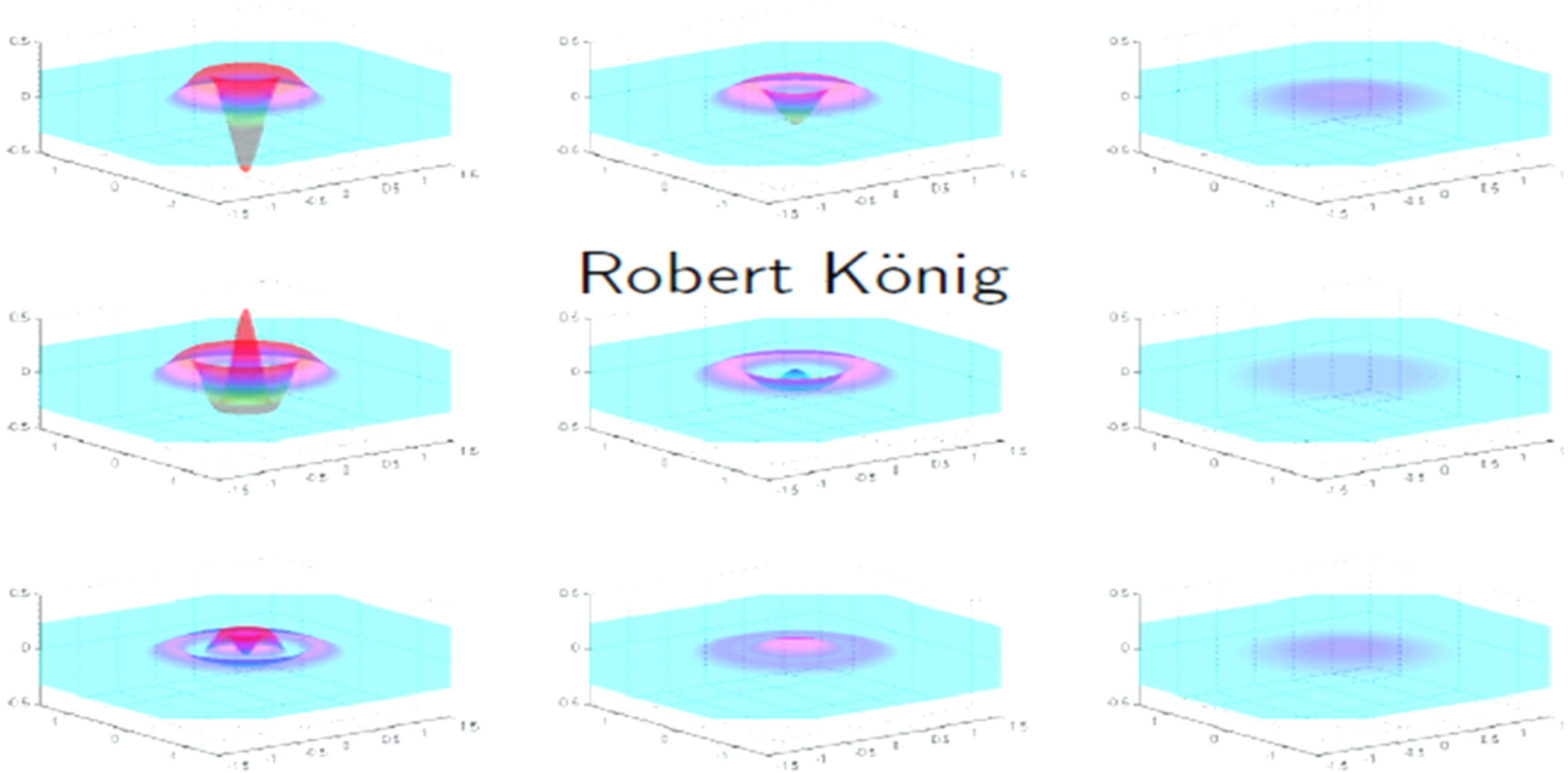
Title: The entropy power inequality for quantum systems

Date: May 16, 2012 04:00 PM

URL: http://pirsa.org/12050018

Abstract: When two independent analog signals, X and Y are added together giving $Z=X+Y$, the entropy of Z, $H(Z)$, is not a simple function of the entropies $H(X)$ and $H(Y)$, but rather depends on the details of X and Y's distributions. Nevertheless, the entropy power inequality (EPI), which states that $e^{2H(Z)} \geq e^{2H(X)} + e^{2H(Y)}$, gives a very tight restriction on the entropy of Z. This inequality has found many applications in information theory and statistics. The quantum analogue of adding two random variables is the combination of two independent bosonic modes at a beam splitter. The purpose of this talk is to give an outline of the proof of two separate generalizations of the entropy power inequality to the quantum regime. These inequalities provide strong new upper bounds for the classical capacity of quantum additive noise channels, including quantum analogues of the additive white Gaussian noise channels.   Our proofs are similar in spirit to standard classical proofs of the EPI, but some new quantities and ideas are needed in the quantum setting. Specifically, we find a new quantum de Bruijin identity relating entropy production under diffusion to a divergence-based quantum Fisher information. Furthermore, this Fisher information exhibits certain convexity properties in the context of beam splitters.   This is joint work with Graeme Smith.

The quantum entropy power inequality

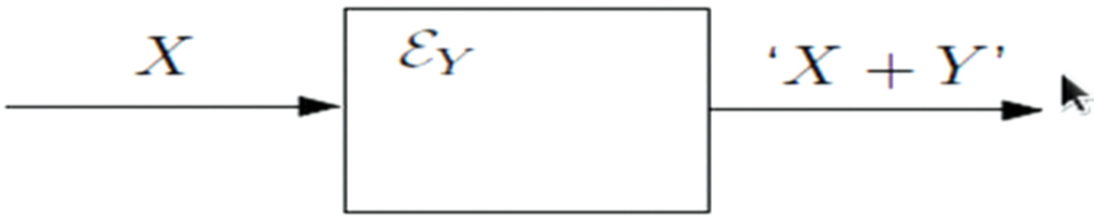


Robert König

joint work with Graeme Smith

Continuous-variable additive noise channels

Goal: understand their capacities



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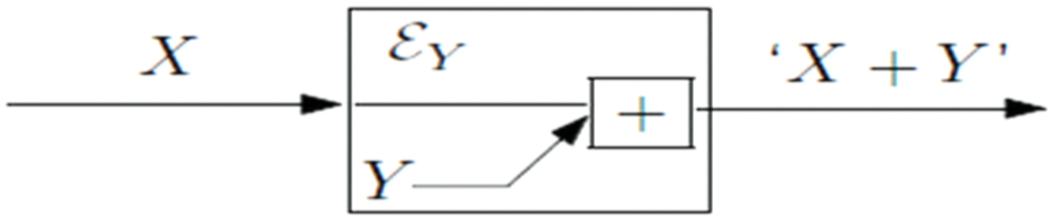

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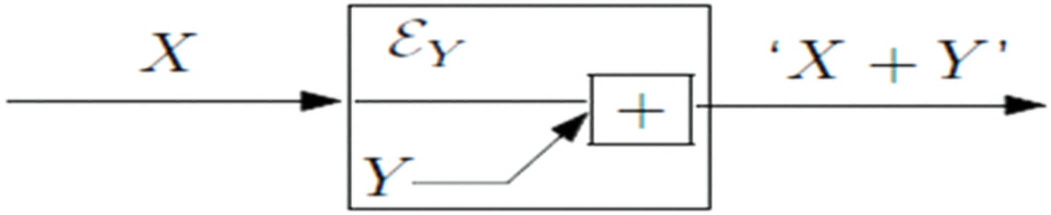


Special examples:

classical

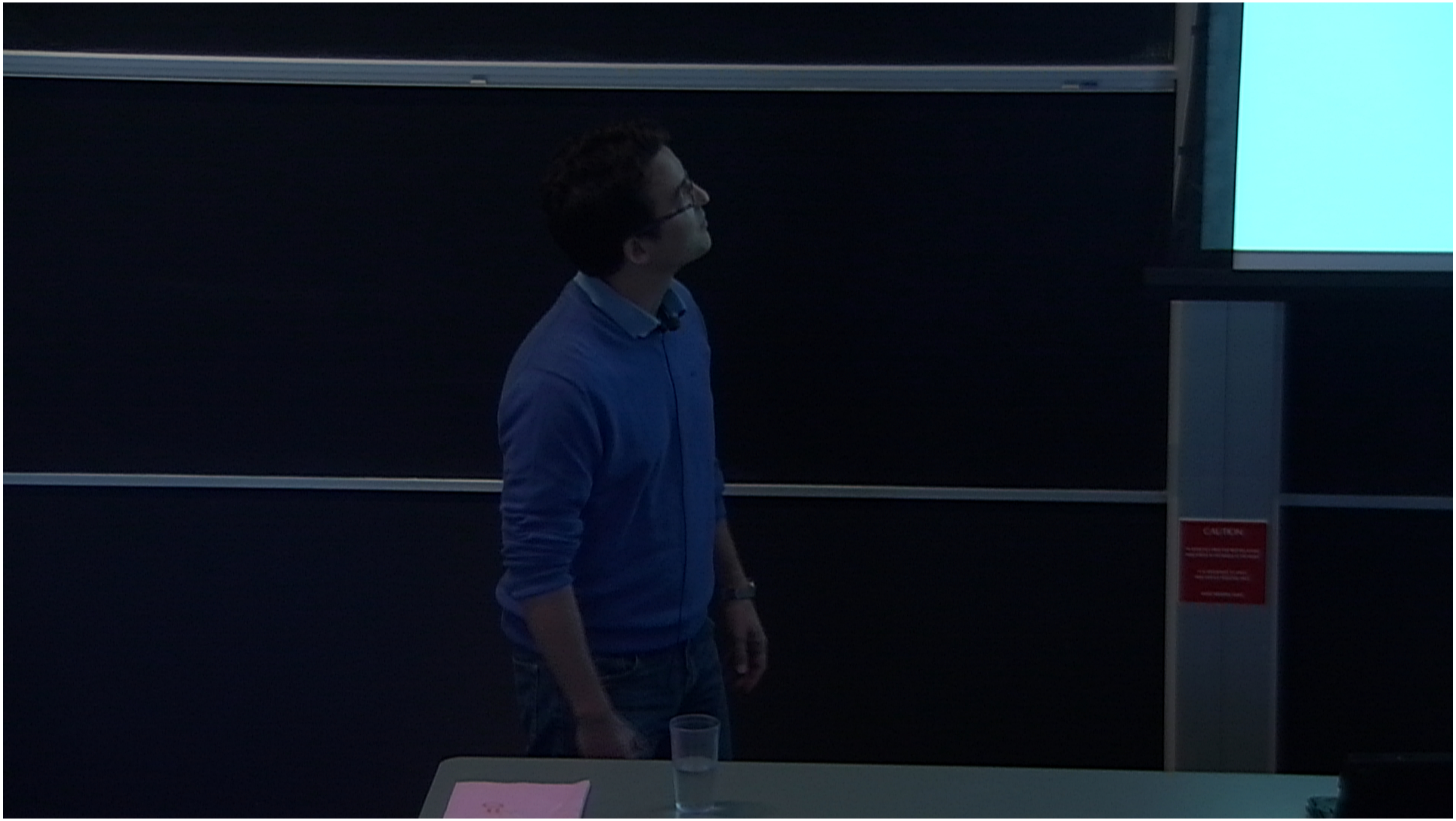
- $f_Y(y) \sim e^{-|y-y_0|^2/2\sigma^2}$
  Gaussian additive noise

quantum

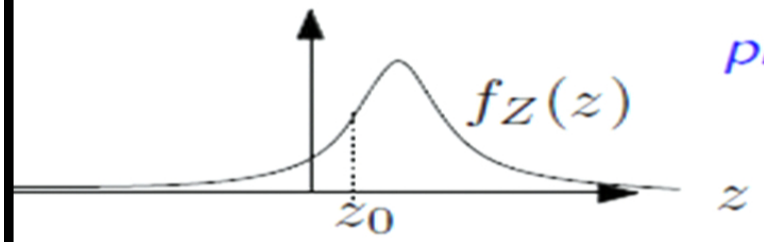
- $\rho_Y \sim e^{-\beta H}$
  thermal noise

Outline

- The classical entropy power inequality
- The quantum entropy power inequality
- application to capacities
- Proof



Continuous variable classical information: convolution



probability density function $f_Z(z)$

$$\Pr[Z \le z_0] = \int_{-\infty}^{z_0} f_Z(z)dz$$

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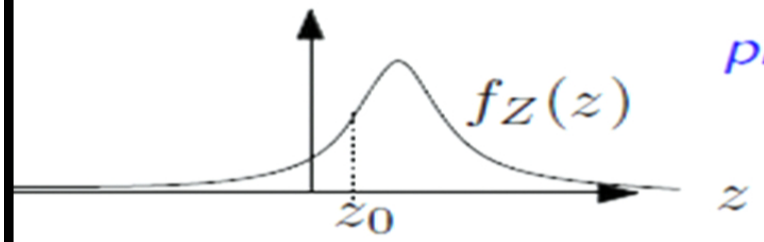

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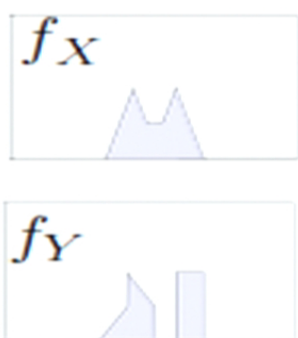

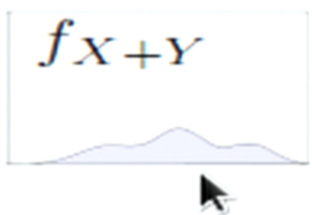

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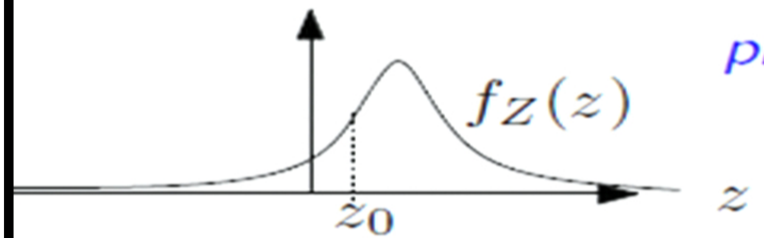

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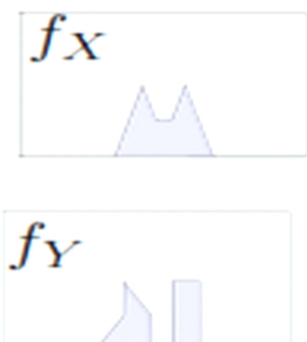
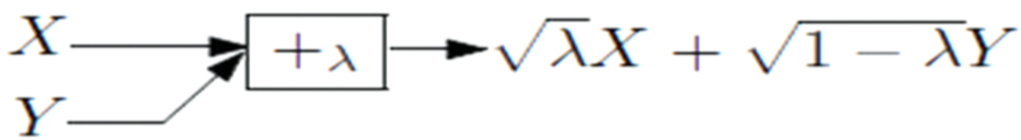
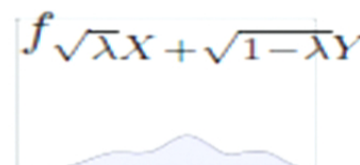
addition=convolution $\circ$ **rescaling:** $\sqrt{\lambda}X + \sqrt{1-\lambda}Y$

$$f_{\sqrt{\lambda}X+\sqrt{1-\lambda}Y}(z) = \int_{-\infty}^{\infty} \frac{f_X(\sqrt{\lambda}(z-s))}{\sqrt{\lambda}} \frac{f_Y(\sqrt{1-\lambda}s)}{\sqrt{1-\lambda}} ds$$

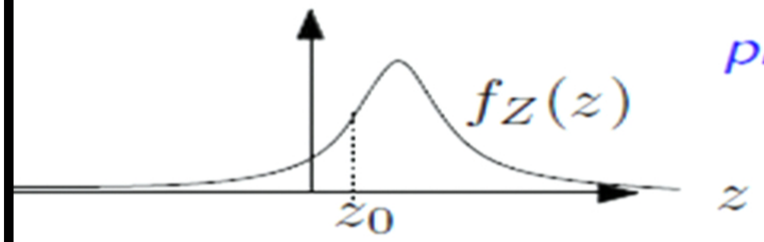






'transmissivity' $0 < \lambda < 1$

Continuous variable classical information: convolution



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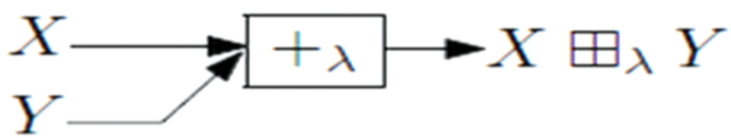
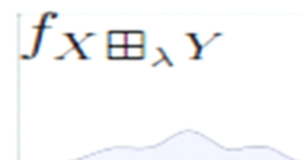
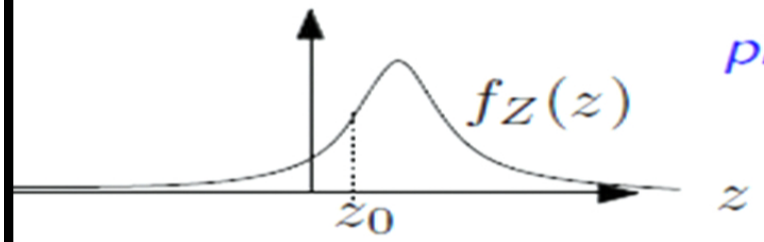




entropy power inequality: relation between the entropies of X, Y *and* $X \boxplus_\lambda Y$

Classical information on $\mathbb{R}^n$: information measures



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$$\Pr[Z \le z_0] = \int_{-\infty}^{z_0} f_Z(z)dz$$

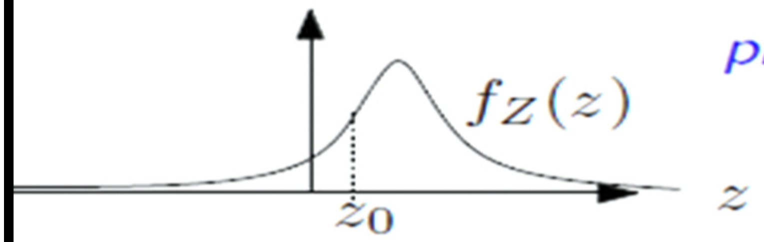
variance ("power"): $\sigma^2(Z) = \mathbb{E}\left[(Z - \mathbb{E}[Z])^2\right]$

(differential) entropy: $H(Z) = -\int f_Z(z) \log f_Z(z) dz$

Example: Gaussian $f_X(x) = \frac{1}{(2\pi\sigma^2)^{n/2}} e^{-\|x\|^2/(2\sigma^2)}$

$$H(X) = \frac{n}{2} \log 2\pi e \sigma^2$$

Classical information on $\mathbb{R}^n$: information measures



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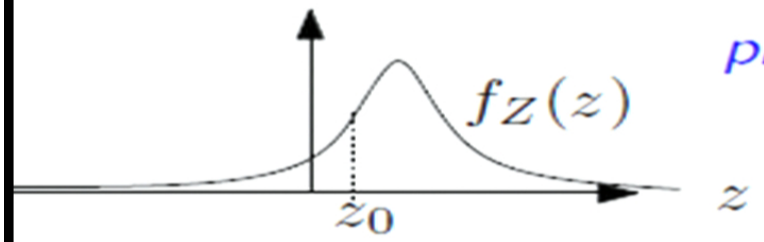

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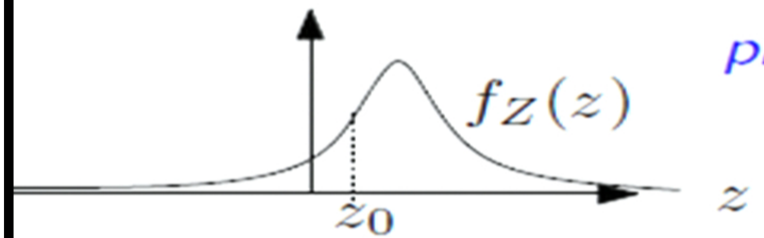

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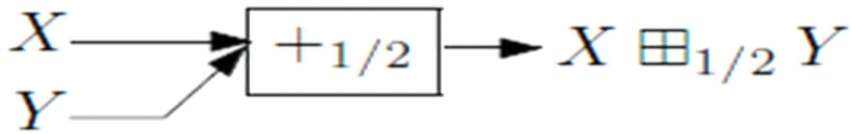
relation to estimation: $\mathbb{E}\left[(Z - \hat{Z})^2\right] \geq \frac{1}{2\pi e} e^{2H(Z)/n}$

The classical entropy power inequality

Shannon 1948

Stam 1959
Blachman 1965
Barron 1986

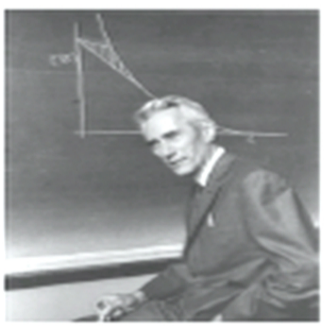


$$\frac{1}{2}\exp(2H(X)/n) + \frac{1}{2}\exp(2H(Y)/n) \leq \exp\left(2H(X \boxplus_{1/2} Y)/n\right)$$

for X, Y independent random variables on $\mathbb{R}^n$

applications:converses for capacity regions of

- Gaussian broadcast channel
- quadratic Gaussian distributed source coding/CEO problem/multiple description coding ...
- partial decode-forward for Gaussian RC
- Gaussian wiretap channel
- ...



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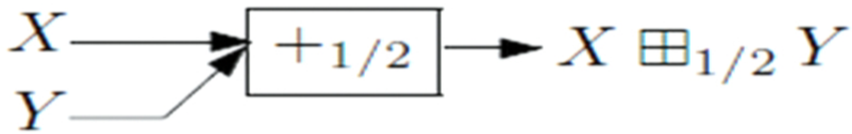

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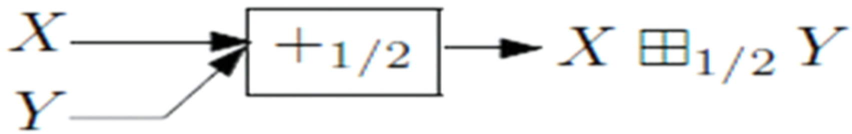

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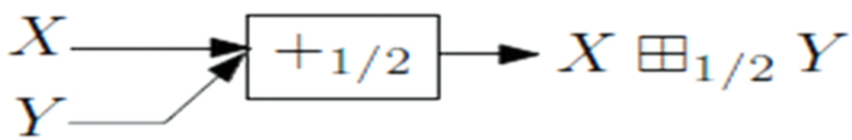
alternative version:



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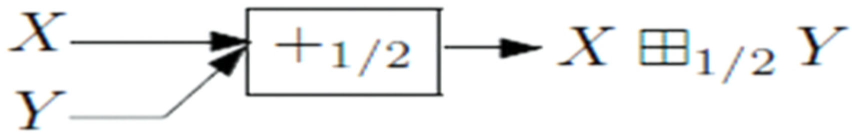

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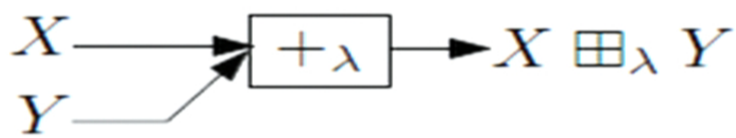

$$\boxed{\lambda H(X) + (1-\lambda)H(Y) \leq H(X \boxplus_\lambda Y)} \qquad 0 < \lambda < 1$$

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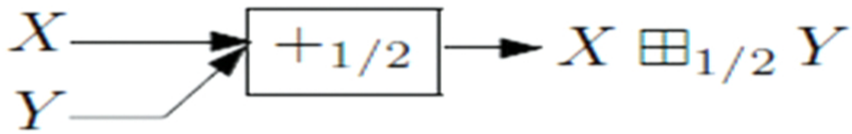

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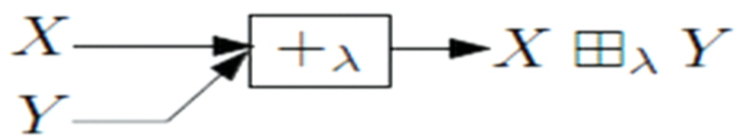

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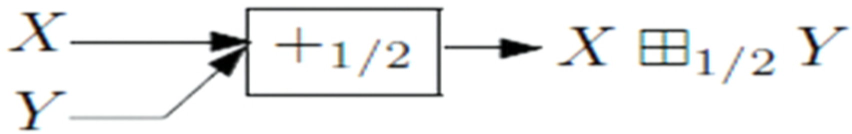

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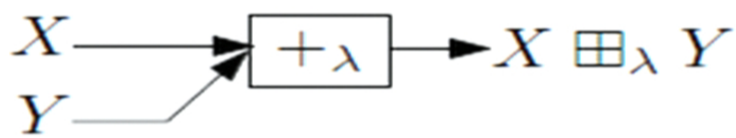

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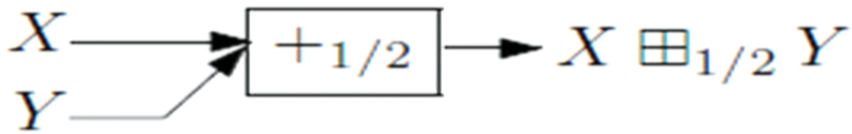

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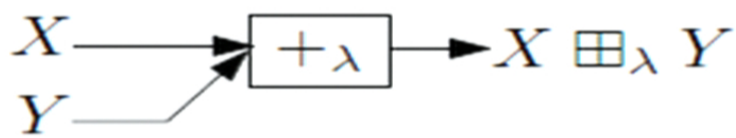


$$\lambda H(X) + (1-\lambda)H(Y) \leq H(X \boxplus_\lambda Y) \qquad 0 < \lambda < 1$$

Continuous variable quantum information: states

single mode: separable Hilbert space $\mathcal{H} = span\{|n\rangle\}_{n\geq 0}$

$|n\rangle$: eigenstate of Hamiltonian $H = Q^2 + P^2$

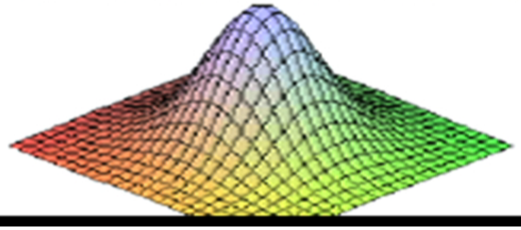
canonical commutation relations (CCR) $[Q,P] = -[P,Q] = iI$
$[Q,Q] = [P,P] = 0$

density operator ρ defined by *characteristic function* $\chi_\rho(\xi)$

$\rho = \frac{1}{2\pi} \int \chi_\rho(\xi) D(-\xi) d^2\xi$ where $D(\xi) = e^{i(\xi_1 P - \xi_2 Q)}$

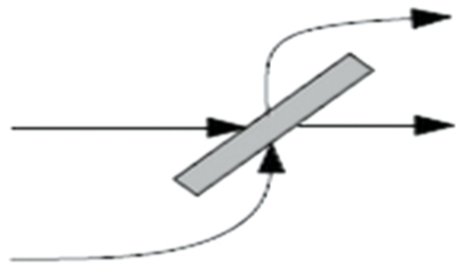
explicit formula: $\chi_\rho(\xi) = \mathrm{tr}\left(D(\xi)\rho\right)$

Special example: $\chi_\rho(\xi) \sim \exp(-(\xi - \xi_0)^T \Gamma(\xi - \xi_0))$
Gaussian state



50:50 beamsplitter: description

$U_{1/2}$: Gaussian with $S = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ I & -I \end{pmatrix}$



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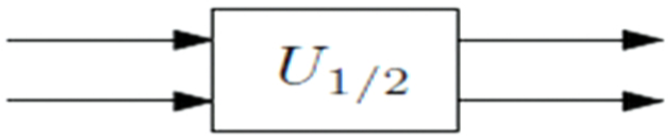

$$(Q_1, P_1) \quad\to\quad \tfrac{1}{\sqrt{2}}(Q_1, P_1) + \tfrac{1}{\sqrt{2}}(Q_2, P_2)$$
$$(Q_2, P_2) \quad\to\quad \tfrac{1}{\sqrt{2}}(Q_1, P_1) - \tfrac{1}{\sqrt{2}}(Q_2, P_2)$$

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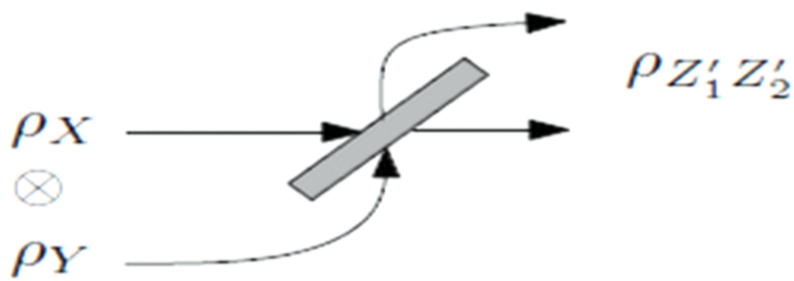
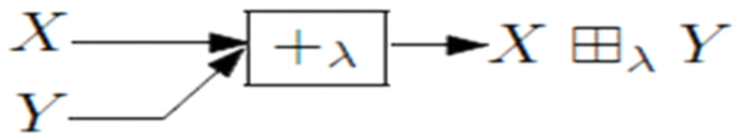
action on product states

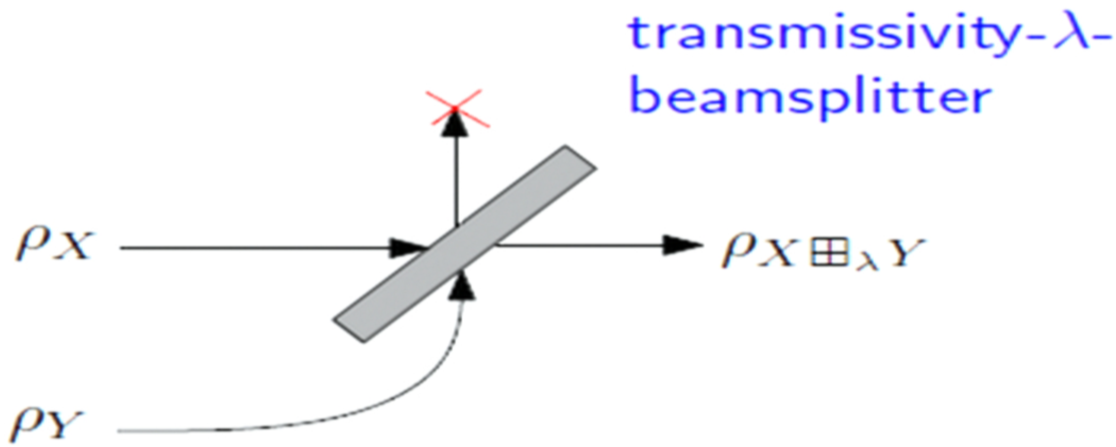


$$\chi_X(\xi_1) \cdot \chi_Y(\xi_2) \quad \mapsto \quad \chi_{Z_1 Z_2}(\xi_1, \xi_2) = \chi_X((\xi_1 + \xi_2)/\sqrt{2}) \cdot \chi_Y((\xi_1 - \xi_2)/\sqrt{2})$$

The quantum entropy power inequality

A quantum addition law





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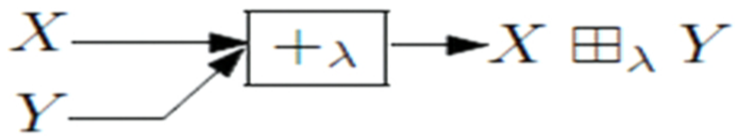

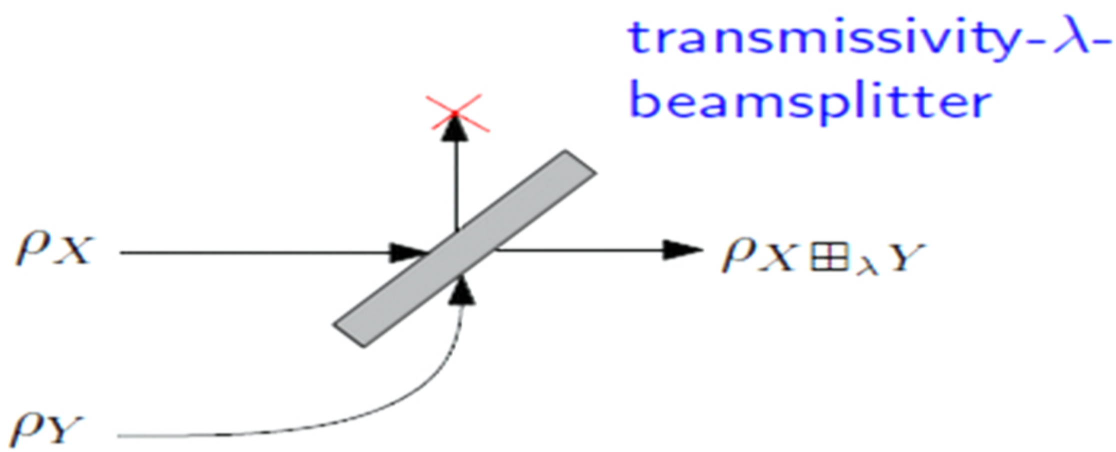

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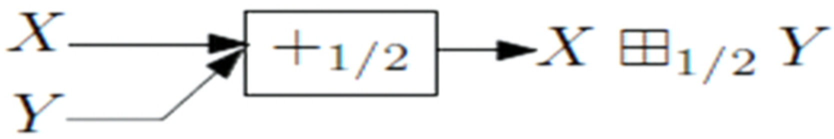


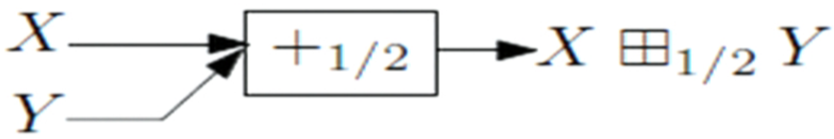
$$\frac{1}{2}\exp(S(X)/n) + \frac{1}{2}\exp(S(Y)/n) \leq \exp\left(S(X \boxplus_{1/2} Y)/n\right)$$

for $\rho_X \otimes \rho_Y$ product of two n-mode states

application(s):

- additive lower bounds on minimum output-entropy of additive continuous-variable channel ($\equiv$ upper bound on capacity)
- ?

The quantum entropy power inequality



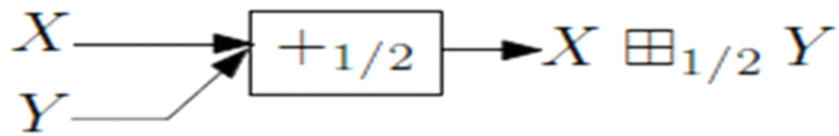
$$\frac{1}{2}\exp(S(X)/n) + \frac{1}{2}\exp(S(Y)/n) \leq \exp\left(S(X \boxplus_{1/2} Y)/n\right)$$

for $\rho_X \otimes \rho_Y$ product of two n-mode states

application(s):

- additive lower bounds on minimum output-entropy of additive continuous-variable channel ($\equiv$ upper bound on capacity)
- ?

The quantum entropy power inequality



$$\tfrac{1}{2}\exp(S(X)/n) + \tfrac{1}{2}\exp(S(Y)/n) \leq \exp\left(S(X \boxplus_{1/2} Y)/n\right)$$

for $\rho_X \otimes \rho_Y$ product of two n-mode states

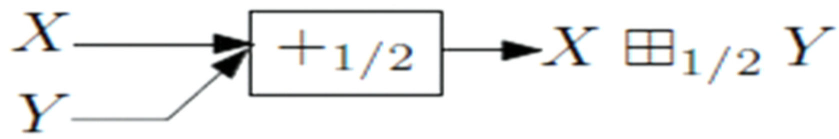
compare to classical inequality

$$\tfrac{1}{2}\exp(2H(X)/n) + \tfrac{1}{2}\exp(2H(Y)/n) \leq \exp\left(2H(X \boxplus_{1/2} Y)/n\right)$$

for X, Y independent random variables on $\mathbb{R}^n$

"explanation": 1 quantum mode $\equiv$ 2 canonical degrees of freedom

The quantum entropy power inequality



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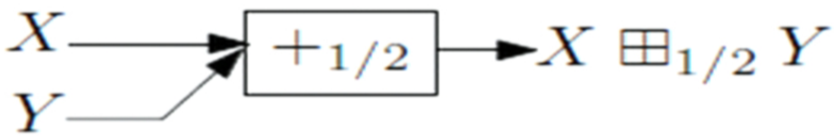
The quantum entropy power inequality

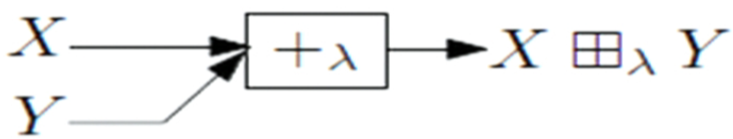
version a:



$$\frac{1}{2}\exp(S(X)/n) + \frac{1}{2}\exp(S(Y)/n) \leq \exp\left(S(X \boxplus_{1/2} Y)/n\right)$$

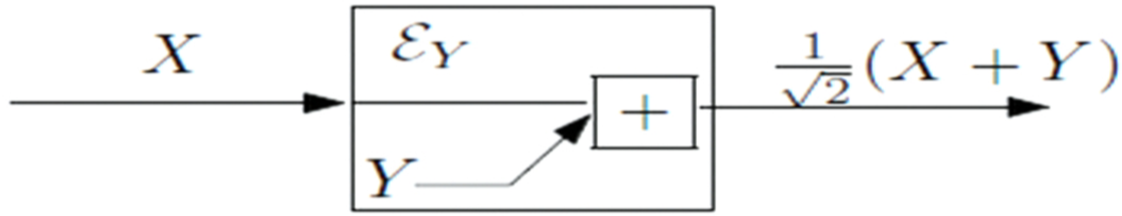
for $\rho_X \otimes \rho_Y$ product of two n-mode states

version b:



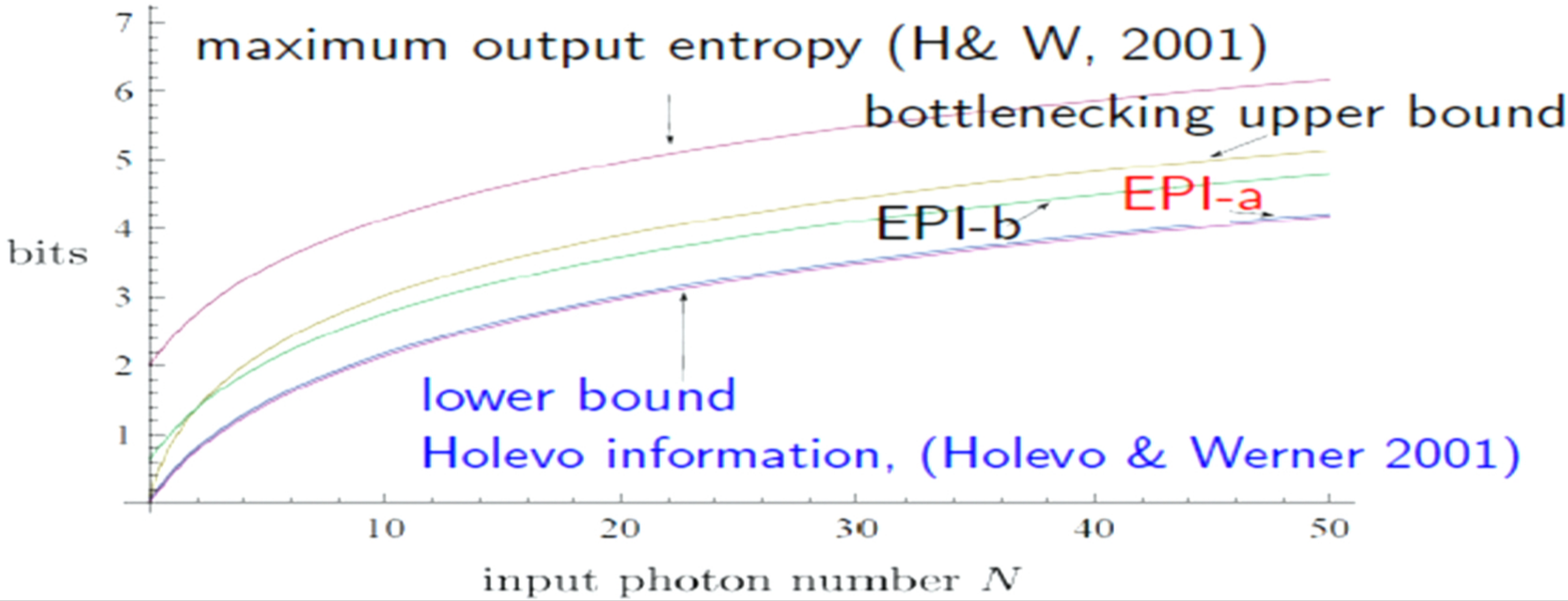
$$\lambda S(X) + (1-\lambda) S(Y) \leq S(X \boxplus_\lambda Y) \qquad 0 < \lambda < 1$$

Thermal noise channel: example application

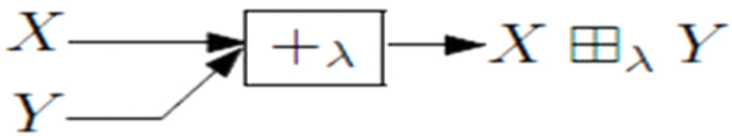


$\lambda = \frac{1}{2}$ and $N_E = 2$

see arXiv:1205.3407



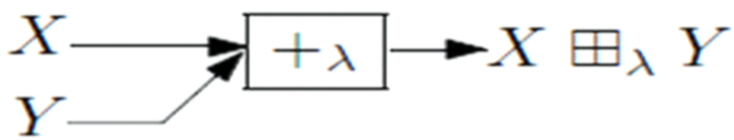
Proof of the classical entropy power inequality

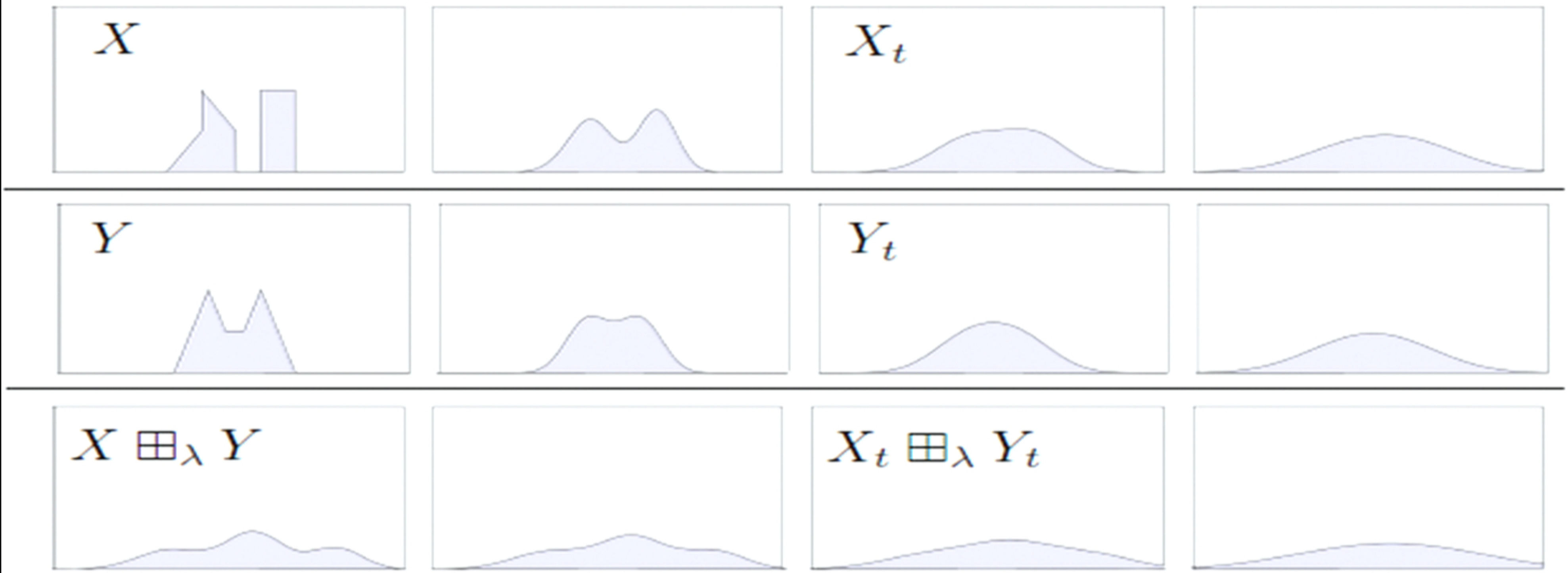


$$\boxed{\lambda H(X) + (1-\lambda) H(Y) \leq H(X \boxplus_\lambda Y)}$$

proof adapted (for easy quantum generalization)
from Dembo, Thomas 1991
& Blachman 1965

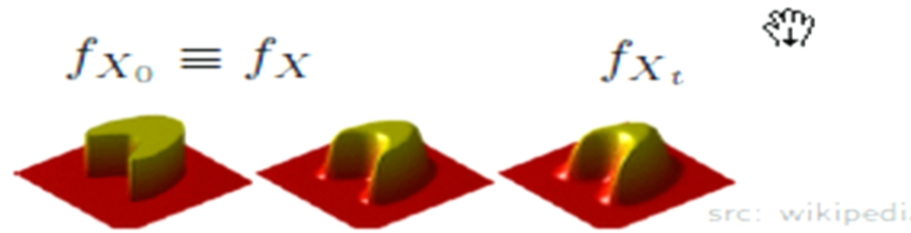
Illustration of proof



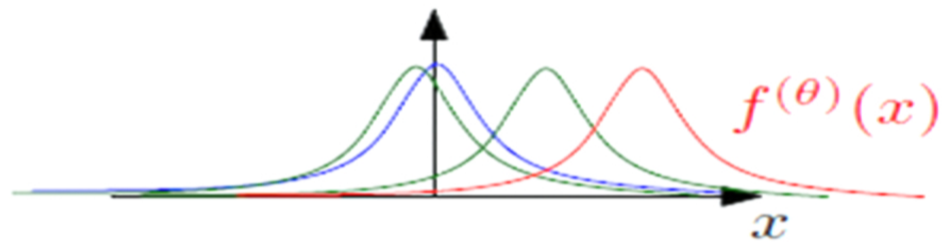


Proof of the classical entropy power inequality

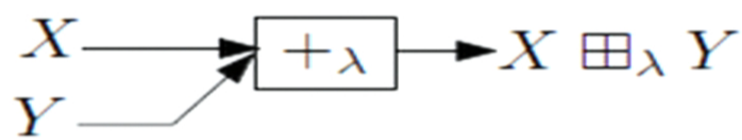
- Gaussian perturbations



- de Bruijin's identity: connects entropies to Fisher information



- convexity inequality for Fisher information

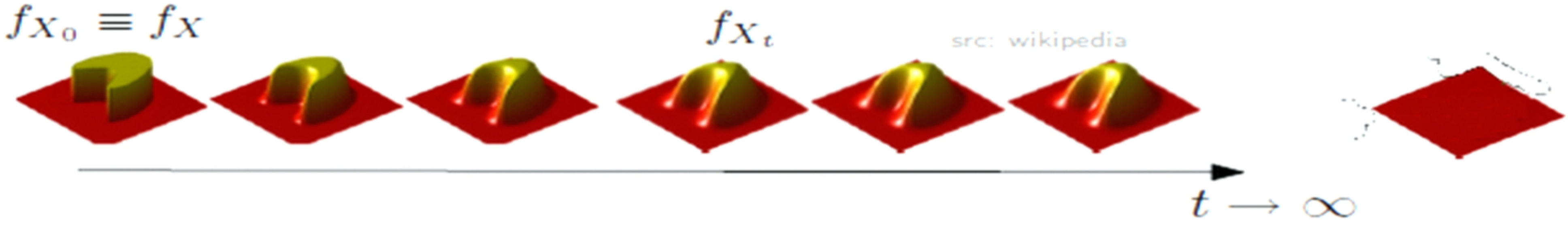


$$\lambda J(X) + (1-\lambda) J(Y) \geq J(X \boxplus_\lambda Y)$$

Gaussian perturbations/heat equation

Consider a *Gaussian perturbation* $Z \sim \mathcal{N}(0, I)$

$$X_t = X + \sqrt{t}Z \qquad \text{where } t \geq 0$$



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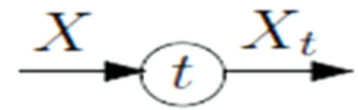
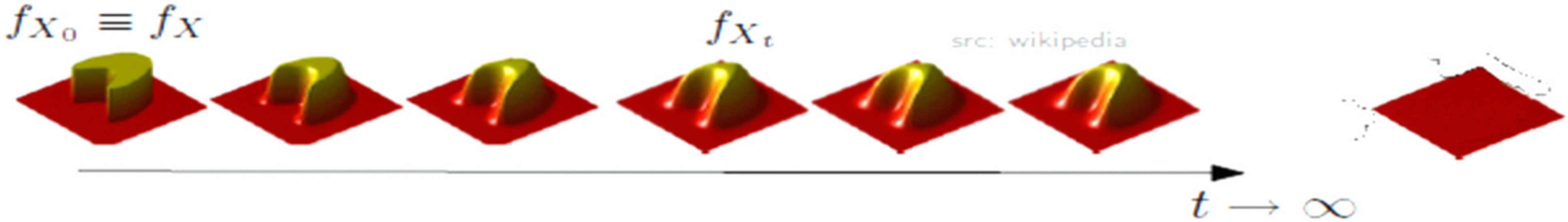


$$X_t = X + \sqrt{t}Z \qquad \text{where } t \geq 0$$



heat equation

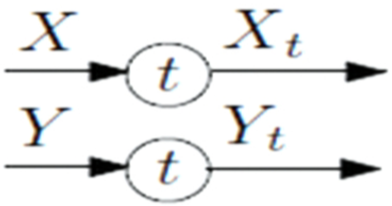
$$\frac{\partial}{\partial t} f_{X_t}(x) = -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} f_{X_t}(x)$$

initial condition $f_{X_0} \equiv f_X$

Application: EPI true in the infinite-diffusion limit $t \longrightarrow \infty$

Consider i.i.d. *Gaussian perturbation* $Z_1, Z_2 \sim \mathcal{N}(0, I)$

$$\begin{aligned} X_t &= X + \sqrt{t} Z_1 \\ Y_t &= Y + \sqrt{t} Z_2 \end{aligned} \qquad \text{where } t \geq 0$$



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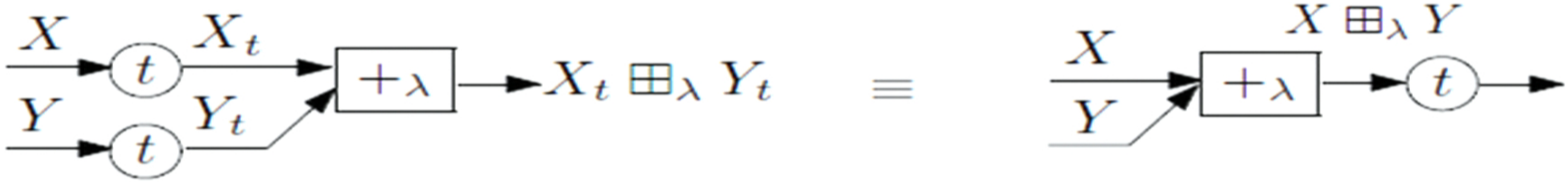
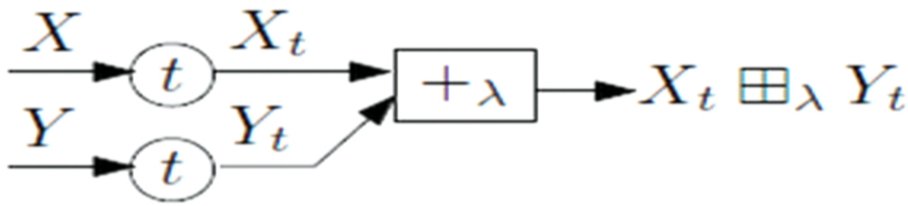


diffusing inputs.... *is equivalent to diffusing output*

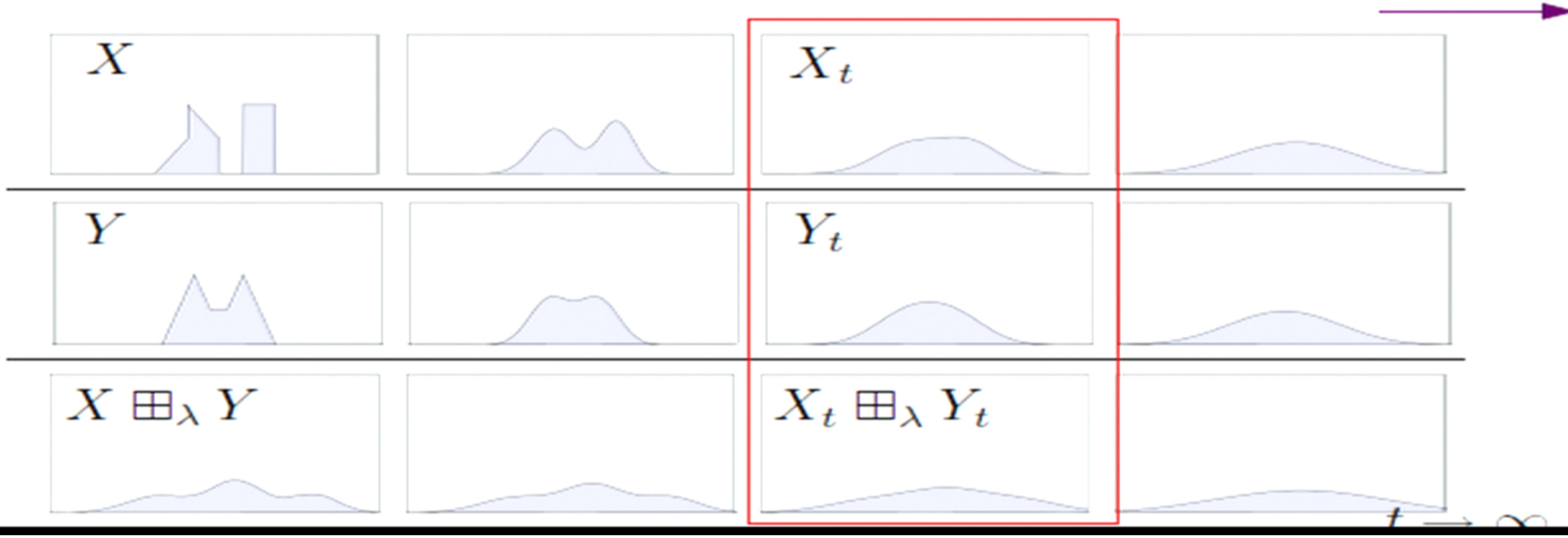
Consequence: $H(X_t) \sim H(Y_t) \sim H(X_t \boxplus_\lambda Y_t) \sim g(t)$ as $t \to \infty$

Illustration of proof

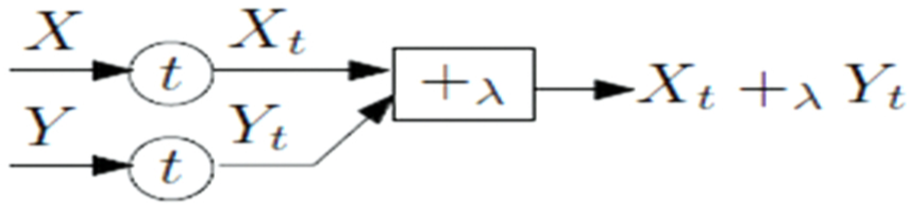


entropy power inequality holds here

?



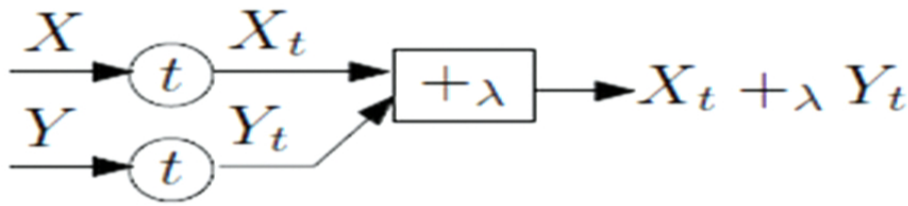
Entropy power inequality: proof by studying diffusion



Set $$\boxed{\delta(t) := H(X_t \boxplus_\lambda Y_t) - \lambda H(X_t) - (1-\lambda) H(Y_t)}$$

- $\delta(\infty) = 0$
- $\delta(0) \geq 0$ iff EPI holds for (X, Y)

Entropy power inequality: proof by studying diffusion

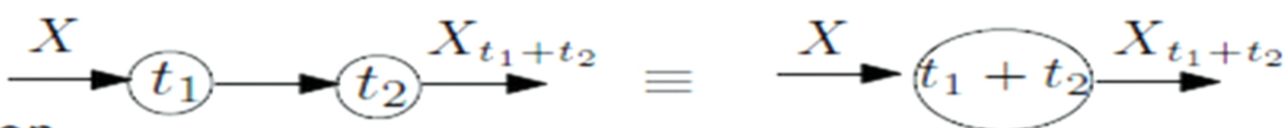


Set $$\delta(t) := H(X_t \boxplus_\lambda Y_t) - \lambda H(X_t) - (1-\lambda) H(Y_t)$$

Claim: to show EPI, it suffices to show $\delta'(t) \leq 0 \quad \forall t \geq 0$

Simplification:
(infinitesimal diffusion) showing $\delta'(0) \leq 0$ suffices

proof: use semi-group property of diffusion



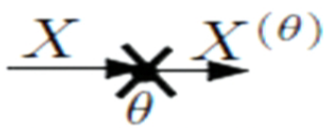
Proof tool: Fisher information

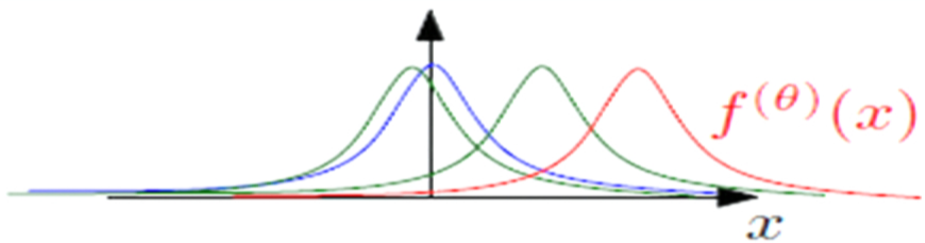
family $\{f^{(\theta)}\}_\theta$ of probability d.f., *unknown parameter* θ

Problem: quantify information about θ, given observations:

example: unknown translation

$$f^{(\theta)}(x) = f(x - \theta)$$





Proof tool: Fisher information

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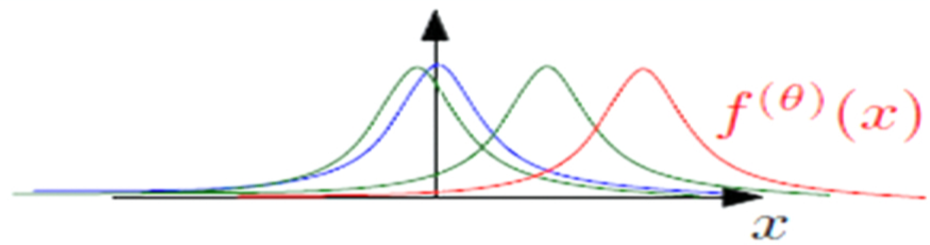
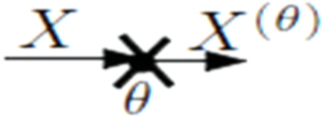
Fisher information: $J(\{f^{(\theta)}\}_\theta)|_{\theta=\theta_0} = \int (\partial_\theta \log f^{(\theta)}(x))^2 dx|_{\theta=\theta_0}$

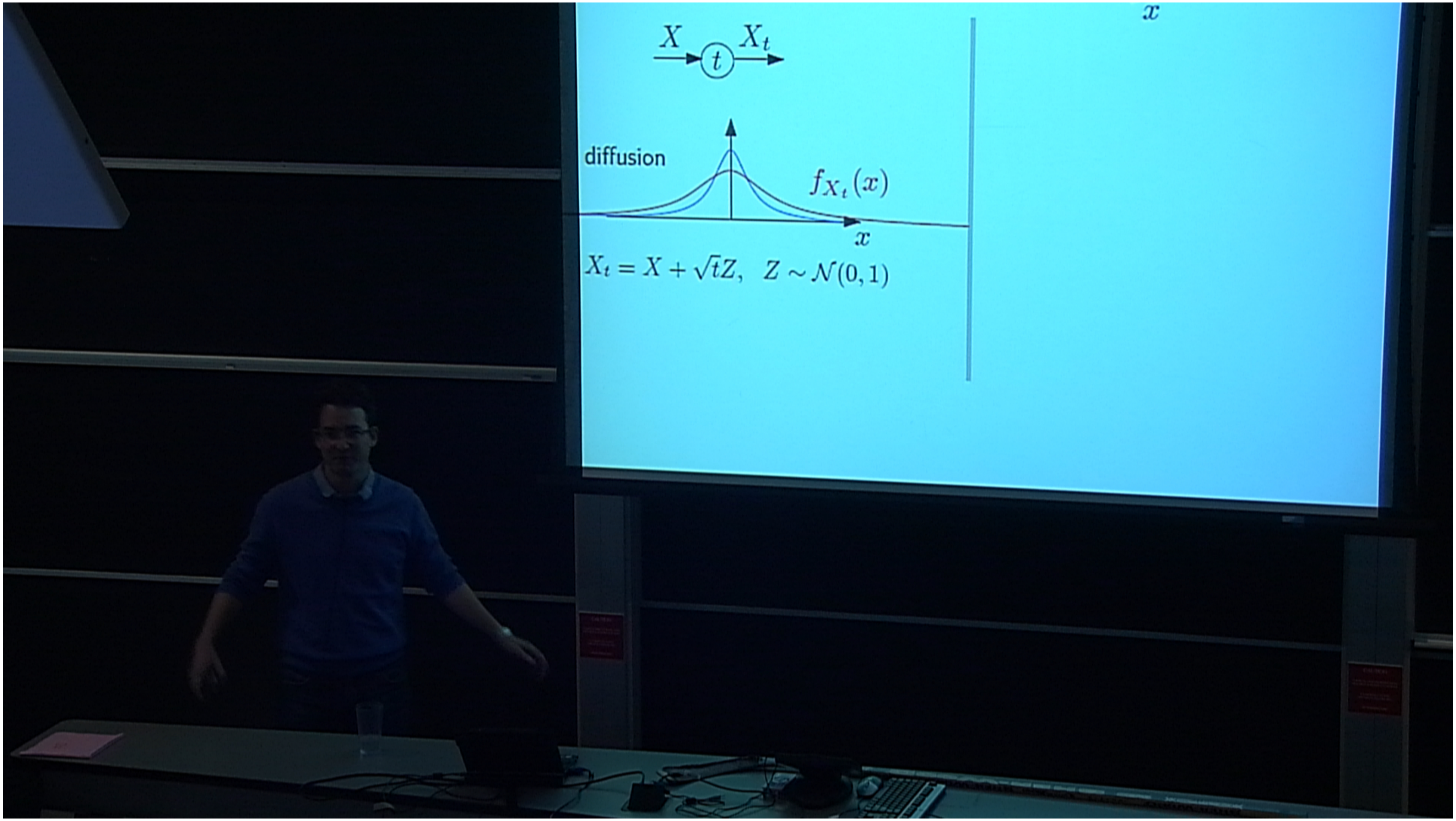
example: unknown translation

$$f^{(\theta)}(x) = f(x-\theta)$$



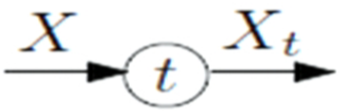


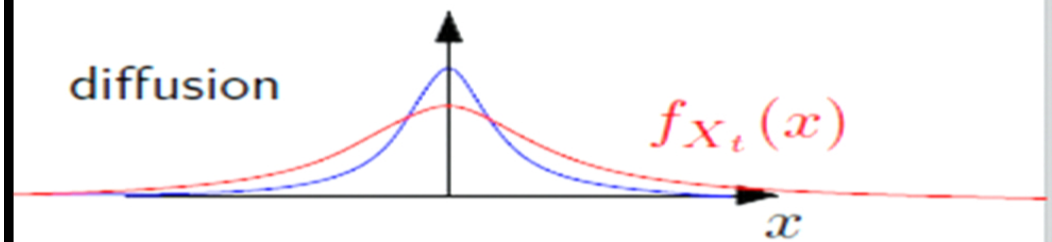
$$J(\{f^{(\theta)}\}_\theta)|_{\theta=0} = \int f'(x)^2/f(x)dx$$



x
X
t
X_t
diffusion
$f_{X_t}(x)$
x
$X_t = X + \sqrt{t}Z, \quad Z \sim \mathcal{N}(0,1)$

Entropy, Gaussian perturbations & Fisher information



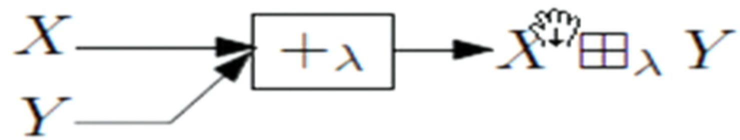


$X_t = X + \sqrt{t}Z, \;\; Z \sim \mathcal{N}(0,1)$

entropy derivative

$\frac{d}{dt} H(X + \sqrt{t}Z)$

Application: EPI $\Leftrightarrow$ Fisher information inequality



EPI: $$\boxed{\lambda H(X) + (1-\lambda) H(Y) \leq H(X \boxplus_\lambda Y)}$$

---

equivalent sufficient condition: $$\boxed{\lambda J(X) + (1-\lambda) J(Y) \geq J(X \boxplus_\lambda Y)}$$

Basic properties of the Fisher information

Fisher information: $$J(\{f^{(\theta)}\}_\theta)|_{\theta=\theta_0} = \int(\partial_\theta \log f^{(\theta)}(x))^2 dx|_{\theta=\theta_0}$$

trivial:

- **(linear) reparametrization:** for any $c > 0$:
  $$J(\{f^{(c\theta)}\}_\theta) = c^2 J(\{f^{(\theta)}\}_\theta)$$
- **additivity for product distributions:**
  $$J(\{f^{(\theta)} \times g^{(\theta)}\}_\theta) = J(\{f^{(\theta)}\}_\theta) + J(\{g^{(\theta)}\}_\theta)$$

data processing: for any channel $\mathcal{E}$

$$J(\{\mathcal{E}(f^{(\theta)})\}_\theta) \leq J(\{f^{(\theta)}\}_\theta)$$

Processing your observation can only decrease your information about the unknown parameter θ.

Entropy power inequality: proof tools

classical

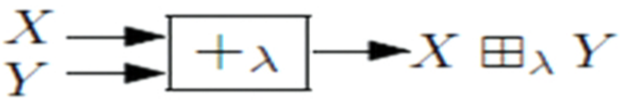
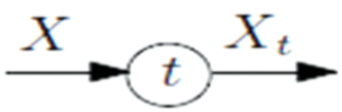
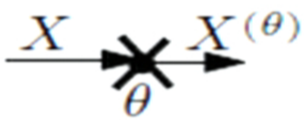
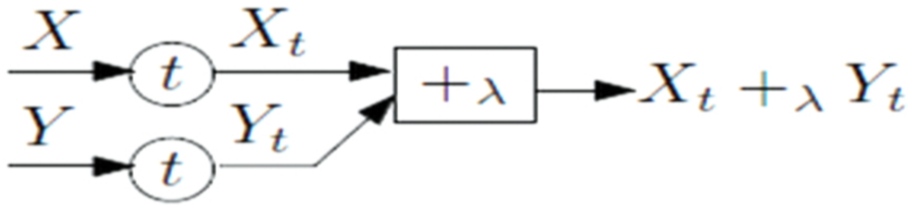
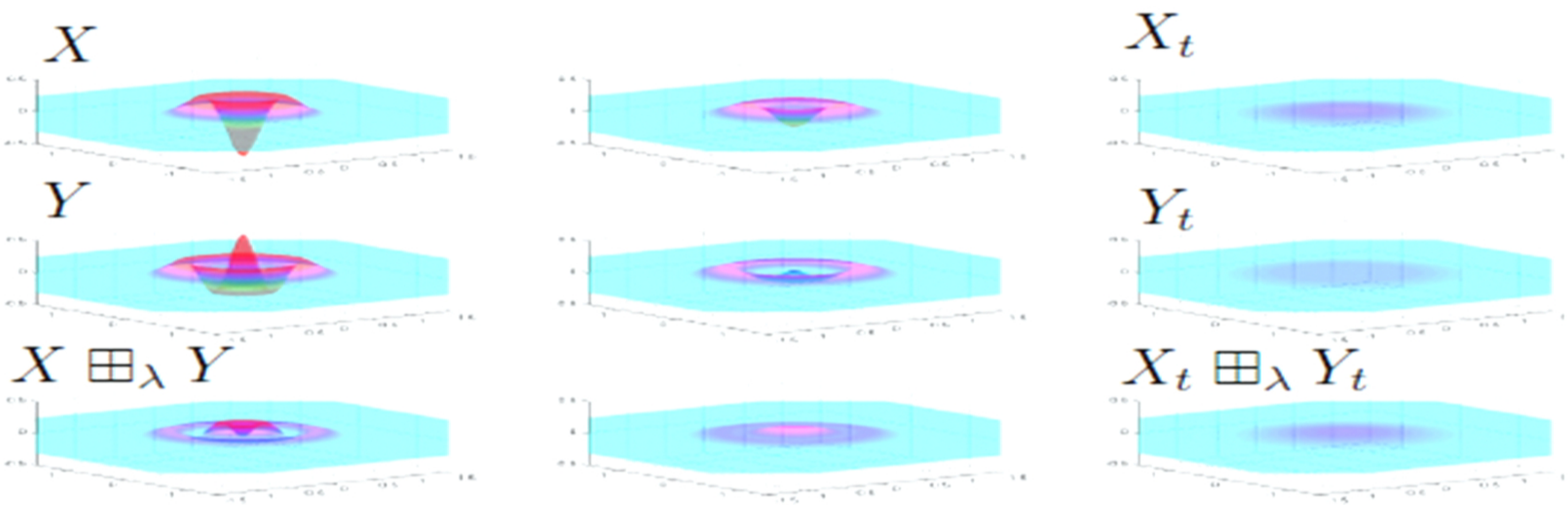
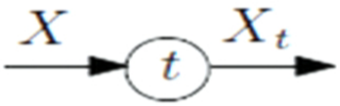
	rescaling by $\sqrt{\lambda}, \sqrt{1-\lambda}$ and convolution
	diffusion for time t $\frac{d}{dt} f_t = \Delta f_t$ $\Delta = \sum_j \frac{\partial^2}{\partial x_j^2}$
	translation by θ $f^{(\theta)}(x) = f(x-\theta)$
quantity J	Fisher information $\int (\partial_\theta \log f^{(\theta)}(x))^2 dx$

Illustration of proof



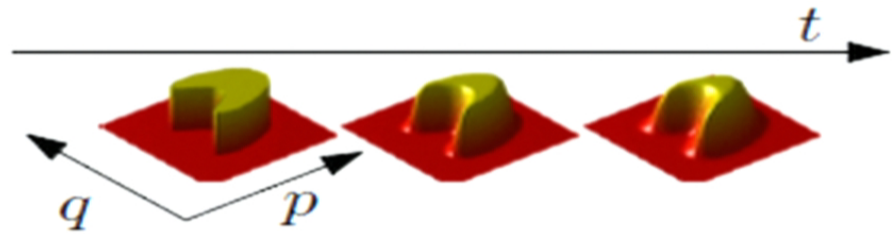


Wigner functions $W_\rho(q, p)$ in phase space



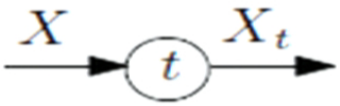
Quantum analog of the heat equation

classical diffusion on $\mathbb{R}^2$



$$\boxed{\frac{\partial}{\partial t} f_t = \left(\frac{\partial^2}{\partial q^2} + \frac{\partial^2}{\partial p^2} \right) f_t}$$

$$f_t = f_t(q,p) \in L^2(\mathbb{R}^2)$$



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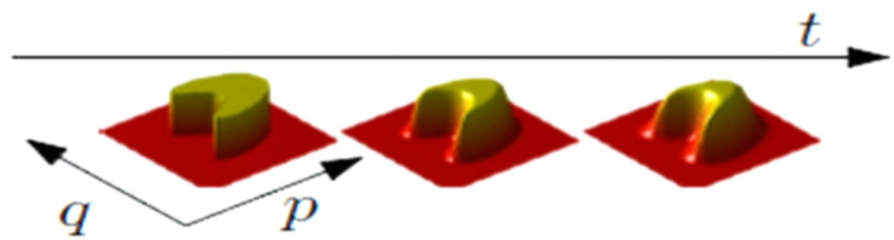


$$\boxed{\frac{\partial}{\partial t} f_t = \{q, \{q, f_t\}\} + \{p, \{p, f_t\}\}}$$

$$f_t = f_t(q, p) \in L^2(\mathbb{R}^2)$$

Poisson bracket $\{f, g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}$

use Dirac quantization prescription:

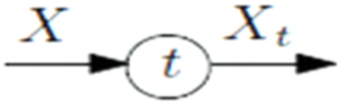
$$\{\cdot, \cdot\} \mapsto \frac{1}{i\hbar}[\cdot, \cdot]$$
$$(q, p) \mapsto (Q, P)$$

Hall, quant-ph 9912055

quantum diffusion equation

$$\boxed{\frac{\partial}{\partial t} \rho_t = -[Q, [Q, \rho_t]] - [P, [P, \rho_t]]}$$

$$\rho_t \in \mathcal{S}(\mathcal{H})$$



Quantum analog of the heat equation

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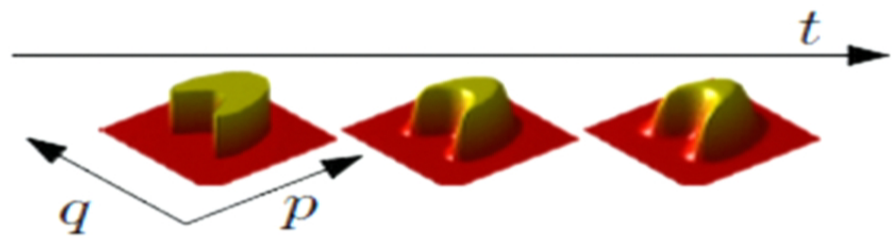

$$\boxed{\frac{\partial}{\partial t} f_t = \{q, \{q, f_t\}\} + \{p, \{p, f_t\}\}}$$

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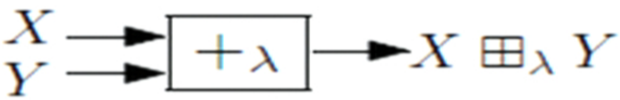
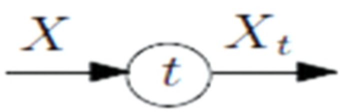
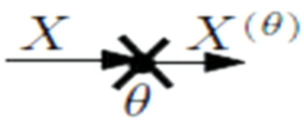
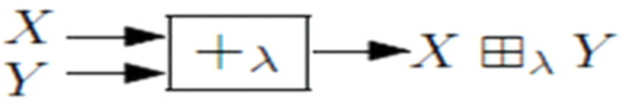
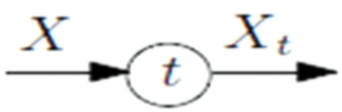
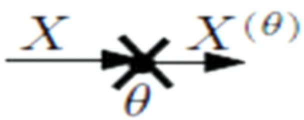
Entropy power inequality: proof tools

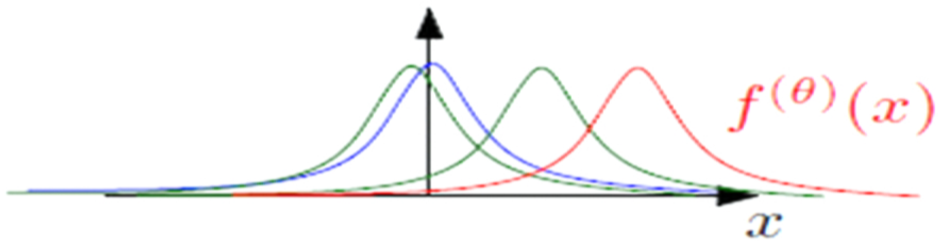
	classical	quantum
	rescaling by $\sqrt{\lambda}, \sqrt{1-\lambda}$ and convolution	transmissivity-λ beamsplitter followed by partial trace
	diffusion for time t $\frac{d}{dt} f_t = \Delta f_t$ $\Delta = \sum_j \frac{\partial^2}{\partial x_j^2}$	q-diffusion for time t $\frac{d}{dt}\rho_t = \mathcal{L}(\rho_t)$ $\mathcal{L}(\cdot) = -\sum_j [Q_j,[Q_j,\cdot]] + [P_j,[P_j,\cdot]]$
	translation by θ $f^{(\theta)}(x) = f(x-\theta)$	?
quantity J	Fisher information $\int (\partial_\theta \log f^{(\theta)}(x))^2 dx$	?

Entropy power inequality: proof tools

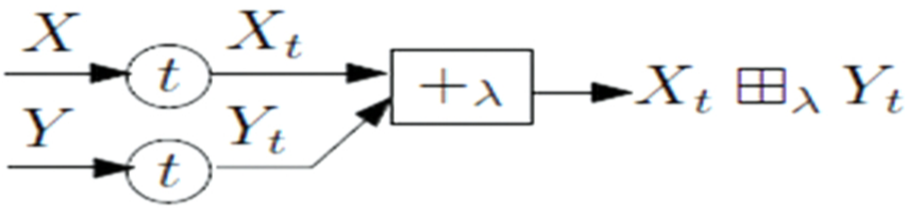
	classical	quantum
	rescaling by $\sqrt{\lambda}, \sqrt{1-\lambda}$ and convolution	transmissivity-λ beamsplitter followed by partial trace
	diffusion for time t $\frac{d}{dt} f_t = \Delta f_t$ $\Delta = \sum_j \frac{\partial^2}{\partial x_j^2}$	q-diffusion for time t $\frac{d}{dt}\rho_t = \mathcal{L}(\rho_t)$ $\mathcal{L}(\cdot) = -\sum_j [Q_j, [Q_j, \cdot]] + [P_j, [P_j, \cdot]]$
	translation by θ $f^{(\theta)}(x) = f(x-\theta)$	p-space translations by θ $\rho_Q^{(\theta)} = e^{i\theta Q/2} \rho e^{-i\theta Q/2}$ $\rho_P^{(\theta)} = e^{i\theta P/2} \rho e^{-i\theta P/2}$
quantity J	Fisher information $\int (\partial_\theta \log f^{(\theta)}(x))^2 dx$	?

A quantum generalization of classical Fisher information

Fisher information: $J(\{f^{(\theta)}\}_\theta)|_{\theta=\theta_0} = \int (\partial_\theta \log f^{(\theta)}(x))^2 dx|_{\theta=\theta_0}$



Proof details: see arXiv:1205.3409



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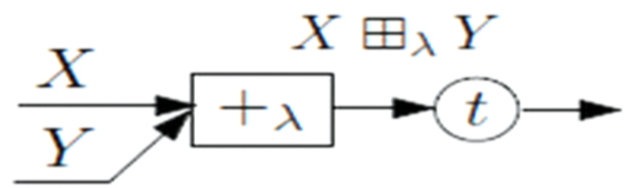
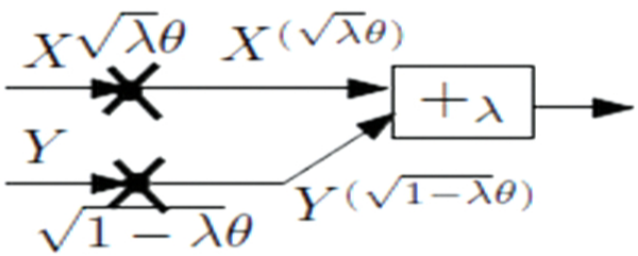


- *diffusing inputs....* *is equivalent to diffusing output*



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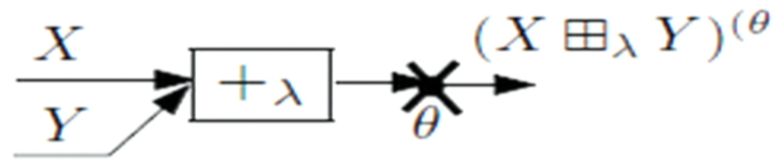

- *Translating inputs* *is equivalent to* *translating output*

- *asymptotic scaling* $S(e^{t\mathcal{L}}(\rho)) \sim g(t) \qquad t \to \infty$
- *Quantum de Bruijin:* $\frac{d}{dt}\Big|_{t=0} S(e^{t\mathcal{L}}(\rho)) = \frac{1}{2}J(\rho)$
- *data processing:* $J(\{\mathcal{E}(f^{(\theta)})\}_\theta) \leq J(\{f^{(\theta)}\}_\theta)$