Title: Universal Spin Duals to Majorana Networks, Fractionalization in Exactly Solvable Systems, and Holography

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Abstract:

Spin-Majorana (and other) Dualities, Holography, and Deconfinement

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Perimeter Institute - May 2012

The elusive Majorana fermion

Ettore Majorana: 1906-1938(?)

1937: "Real" counterpart to a Dirac fermion

The elusive Majorana fermion

$$\{c_{li}, c_{l'i'}\} = 2\delta_{l,l'}\delta_{i,i'}, \qquad c_{li}^{\dagger} = c_{li}.$$

The elusive Majorana fermion

The "real and imaginary parts" of a Dirac fermion are Majorana fermions.
A representation:

$$d_l \quad = \quad \frac{1}{\sqrt{2}}(c_{l1} + ic_{l2}), \;\; d_l^\dagger = \frac{1}{\sqrt{2}}(c_{l1} - ic_{l2})$$

Hilbert space dimension of N_s Majorana fermions scales as

$$2^{N_s/2}$$

The elusive Majorana fermion

High energy physics: neutrino(?), neutralino(??),
Condensed matter: p-wave superconductors(?), interface between topological insulators and s-wave superconductors(?), Quantum Hall states(?), semiconductor wires on s-wave superconductors

V. M. Mourik, K. Zou, S. M. Frolov, S. R. Plissard, E. P. A. M. Bakkers, and L. P. Kouwenhoven (Science 2012)

This talk: Majorana-Pauli spin dualities

Most of the work to date focuses on non-interacting Majorana fermions. We wish to map interacting Majorana systems in an arbitrary number of dimensions to Pauli spin systems for which much is known.

Intermezzo
the tool: the bond-algebraic approach to dualities

Exposing Dualities

Bonds are more fundamental objects than the elementary degrees of freedom

Space-time, momentum, spin (or other) coordinates are (generally non-unique) labels for bonds. Bonds can automatically be gauge or Lorentz invariant. Redundant degrees of freedom can be avoided.

The special character of various systems including statistics of their basic constituents [Bose, Fermi (Dirac or Majorana), spin, or other], etc. may be irrelevant. In the calculation of most physically measurable quantities such as various non-vanishing correlation functions, entropies, complexities, and free energies, **only composite quantities (the bonds) appear.**

When are two Hamiltonian dual?

H_1 and H_2 are dual if there is an

homomorphism between their bond algebras

DUALITIES are one-to-one, onto mappings between bond algebras that preserve every algebraic relation between bonds:

$$\mathcal{O}^1_{R_1} \leftrightarrow \mathcal{O}^2_{R_2}$$

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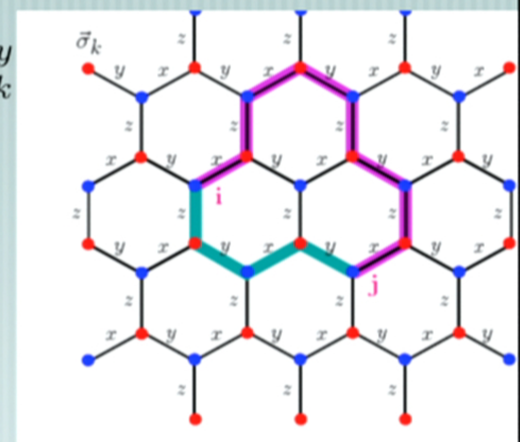
H_1 **and** H_2 **are dual if there is an homomorphism between their bond algebras**

DUALITIES **are one-to-one, onto mappings between bond algebras that preserve every algebraic relation between bonds:**

$$\mathcal{O}^1_{R_1} \leftrightarrow \mathcal{O}^2_{R_2}$$

A simple concrete example of a duality between Pauli spins and non-interacting Majorana fermions: Kitaev's honeycomb model

$$H_{K_h} = - J_x \sum_{x-\text{bonds}} \sigma_j^x \sigma_k^x - J_y \sum_{y-\text{bonds}} \sigma_j^y \sigma_k^y$$
$$- J_z \sum_{z-\text{bonds}} \sigma_j^z \sigma_k^z .$$



$$\mathcal{O}_{ij}^{\mu} = \sigma_i^{\mu} \sigma_j^{\mu} \; , \; \mu = x, y, z,$$

$$\dim \mathcal{H} = 2^{N_s}$$

A simple concrete example of a duality between Pauli spins and non-interacting Majorana fermions: Kitaev's honeycomb model

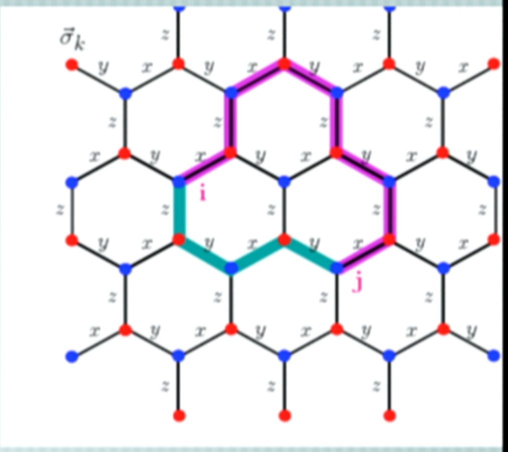
Bond Algebra of Pauli system:

$$\mathcal{O}_{ij}^{\mu} = \sigma_i^{\mu}\sigma_j^{\mu}\ ,\ \ \mu = x, y, z,$$

$$(\mathcal{O}_{ij}^{\mu})^2 = 1$$

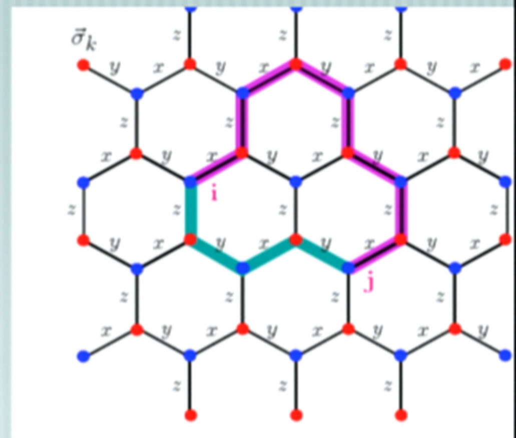
$[\mathcal{O}_{ij}^{\mu}, \mathcal{O}_{kl}^{\mu'}] = 0$ if two bonds share no common sites

$\{\mathcal{O}_{ij}^{\mu}, \mathcal{O}_{kl}^{\mu'}\} = 0$ if two bonds share a common site



A simple concrete example of a duality between Pauli spins and non-interacting Majorana fermions: Kitaev's honeycomb model

Bond Algebra of Pauli system:



$$\mathcal{O}^{\mu}_{ij} = \sigma^{\mu}_{i}\sigma^{\mu}_{j} \; , \;\; \mu = x, y, z,$$

$$(\mathcal{O}^{\mu}_{ij})^2 = 1$$

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$$\{\mathcal{O}^{\mu}_{ij}, \mathcal{O}^{\mu'}_{kl}\} = 0$$ if two bonds share a common site

A simple concrete example of a duality between Pauli spins and non-interacting Majorana fermions: Kitaev's honeycomb model

Bond algebra of a free Majorana system (i.e., a bilinear) on <u>any</u> lattice or graph:

Bonds of free Majorana system: $\bar{O}^{\mu}_{ij} = 2i\eta_{ij}c_i c_j$ with $\eta_{ij} = \pm 1$

$(\bar{O}^{\mu}_{ij})^2 = 1$ all bonds square to unity

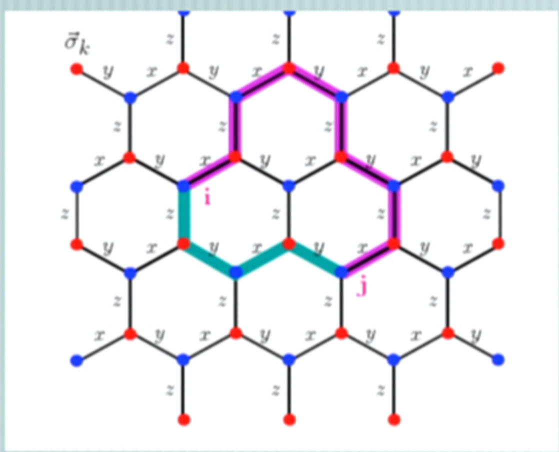
$[\bar{O}^{\mu}_{ij}, \bar{O}^{\mu'}_{kl}] = 0$ if two bonds share no common sites

$\{\bar{O}^{\mu}_{ij}, \bar{O}^{\mu'}_{kl}\} = 0$ if two bonds share a common site

A simple concrete example of a duality between Pauli spins and non-interacting Majorana fermions: Kitaev's honeycomb model

For any ordered product along a closed contour: $$\prod_{ij\in C} \bar{O}_{ij}^{\mu} = \prod_{ij\in C} \eta_{ij}$$

A simple concrete example of a duality between Pauli spins and non-interacting Majorana fermions: Kitaev's honeycomb model



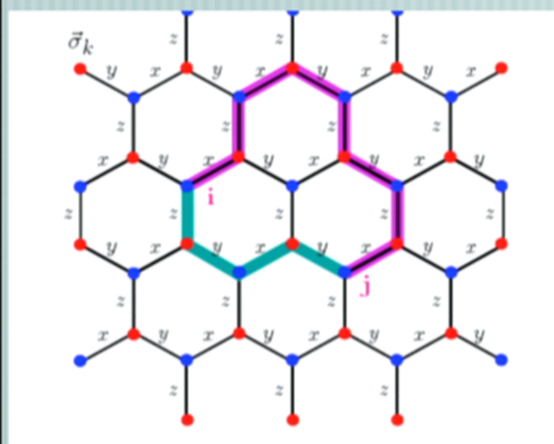
Any product along a closed contour (including elementary hexagons),

$$I_C \equiv \prod_{ij \in C} \mathcal{O}_{ij}^{\mu}$$

is a symmetry,

$$[I_C, \mathcal{O}_{i'j'}^{\mu'}] = [I_C, H_{K_h}] = 0.$$

A simple concrete example of a duality between Pauli spins and non-interacting Majorana fermions: Kitaev's honeycomb model



If for any hexagon h,

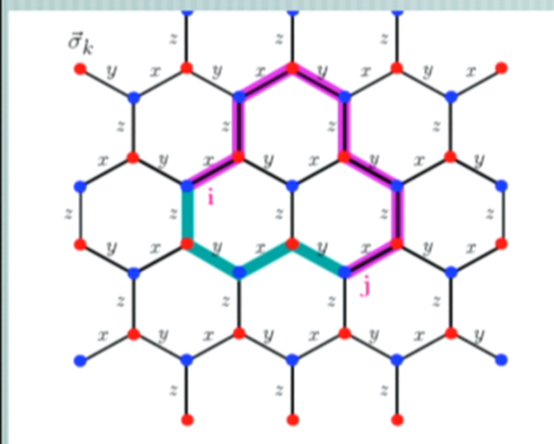
$$I_h = \prod_{ij \in h} \eta_{ij}$$

then in any such sector of the Pauli theory specified by $\{I_h\}$ we we will have the trivial duality

$$\mathcal{O}^{\mu}_{ij} \leftrightarrow \bar{O}^{\mu}_{ij}$$

$$Pauli\ spin \leftrightarrow Majorana$$

A simple concrete example of a duality between Pauli spins and non-interacting Majorana fermions: Kitaev's honeycomb model



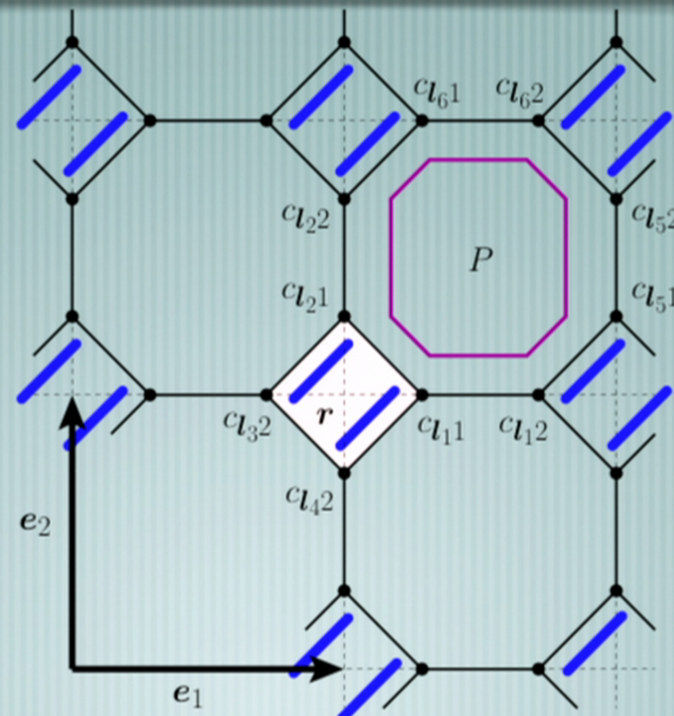
In any such sector of the Pauli theory specified by $\{I_h\}$ the Hilbert space dimension is $2^{N_s/2}$

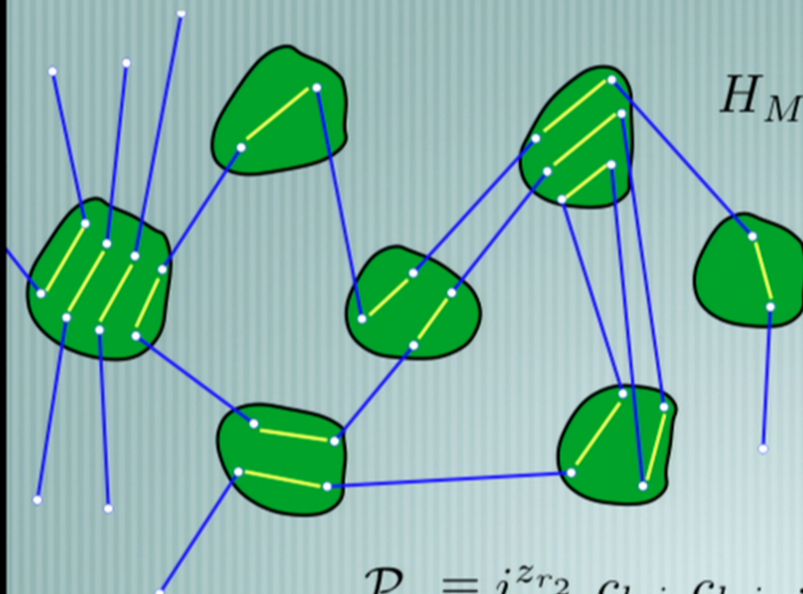
equivalent to the size of the state space of the Majorana system for fixed $\{\eta_{ij}\}$

Thus the system is that of free fermions and is trivially solvable.

Interacting Majorana fermion-Pauli spin dualities

Square lattice Majorana wire architecture
(B. Terhal, F. Hassler, and D. P. Divincenzo, arXiv: 1201.3757)





$$H_M = -i \sum_l J_l c_{l1} c_{l2} - \sum_r h_r \mathcal{P}_r;$$

Josephson tunneling

Charging energy

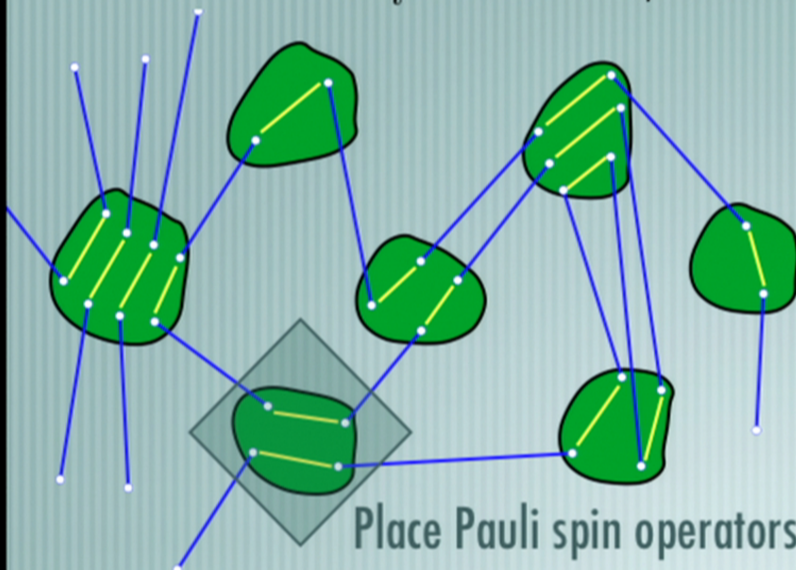
$$\mathcal{P}_r \equiv i^{z_{r2}} \; c_{l_1 i_1} c_{l_2 i_2} \cdots c_{l_{q_r} i_{q_r}}, \; r \in l_1, \cdots, l_{q_r}$$

Quantum Ising Gauge theories on general networks

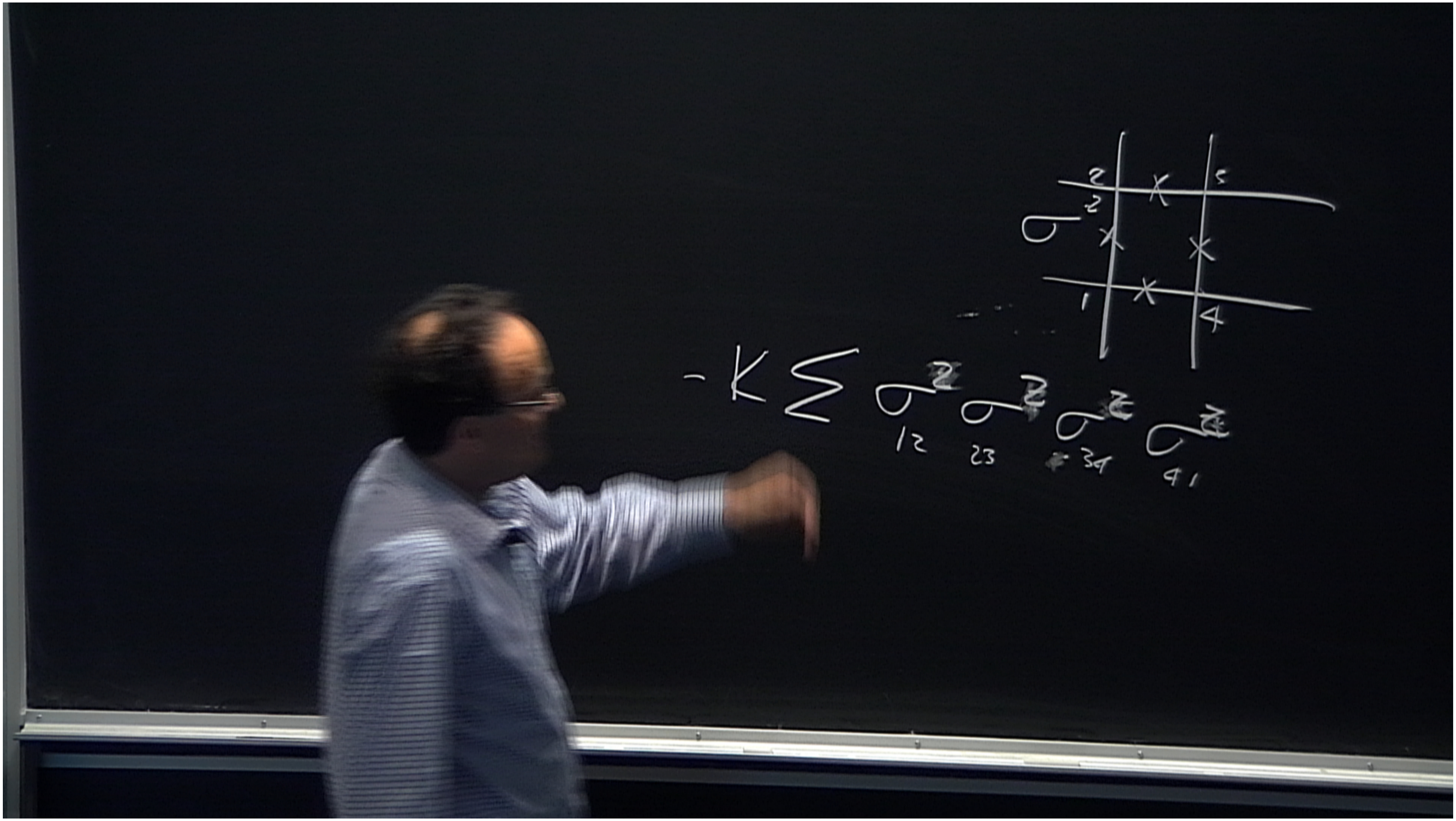
$$H_{QIG} = -\sum_{l} J_l \sigma_l^x - \sum_{r} h_r \widetilde{\mathcal{P}}_r$$

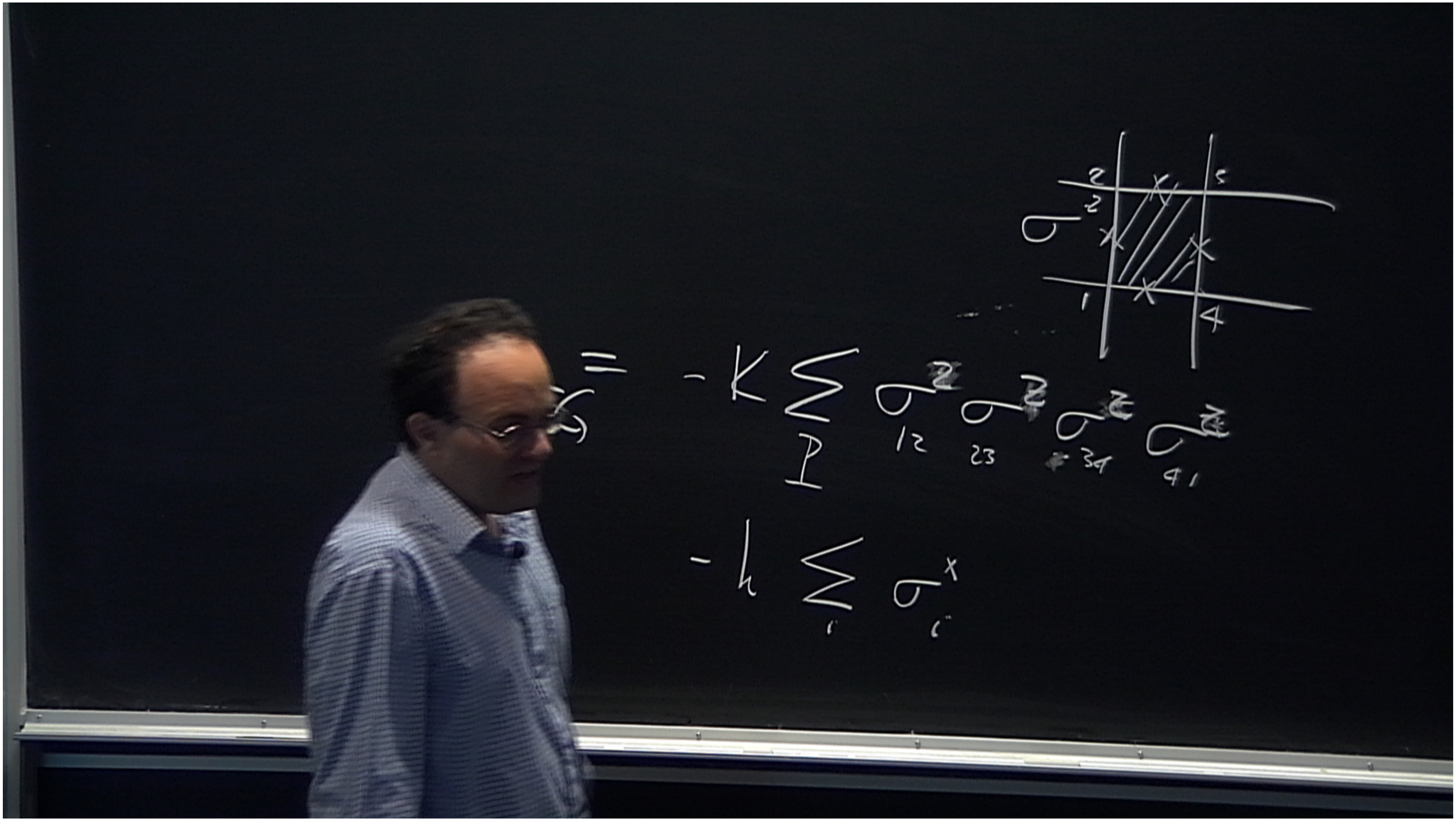
where

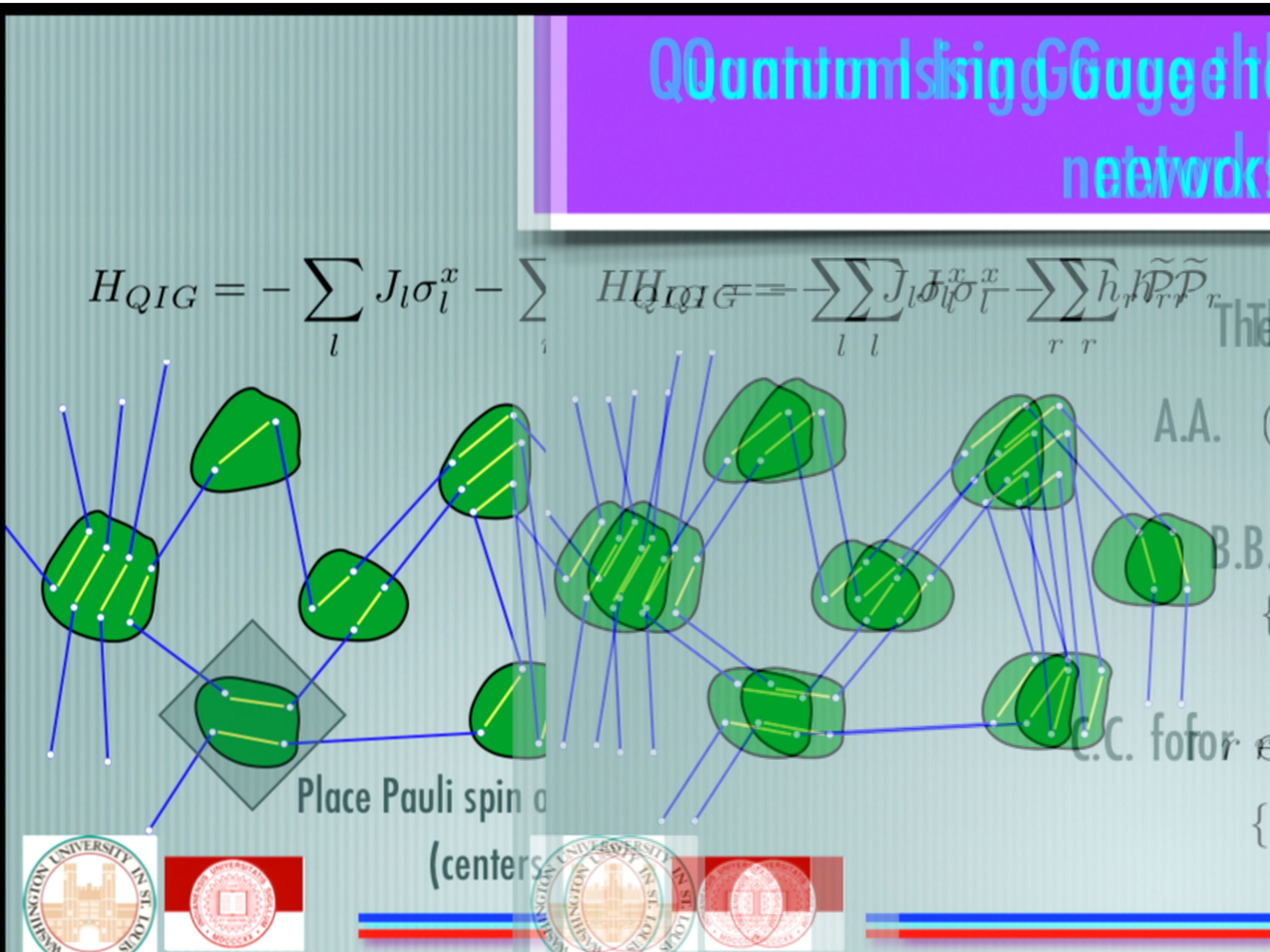
$$\widetilde{\mathcal{P}}_r = \prod_{\{l|r\in l\}} \sigma_l^z.$$



Place Pauli spin operators at the centers of all inter-grain links (centers of boundaries of dual Voronoi cells)


$-K \sum \sigma^z_{12} \sigma^z_{23} \sigma^z_{34} \sigma^z_{41}$


= -K Σ_P σ^z_12 σ^z_23 σ^z_34 σ^z_41
- h Σ_i σ^x_i


Quantum Ising Gauge th
networks
$H_{QIG} = -\sum_l J_l \sigma_l^x - \sum_r h_r \widetilde{\mathcal{P}}_r$
A.
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Quantum Ising Gauge theories on general networks

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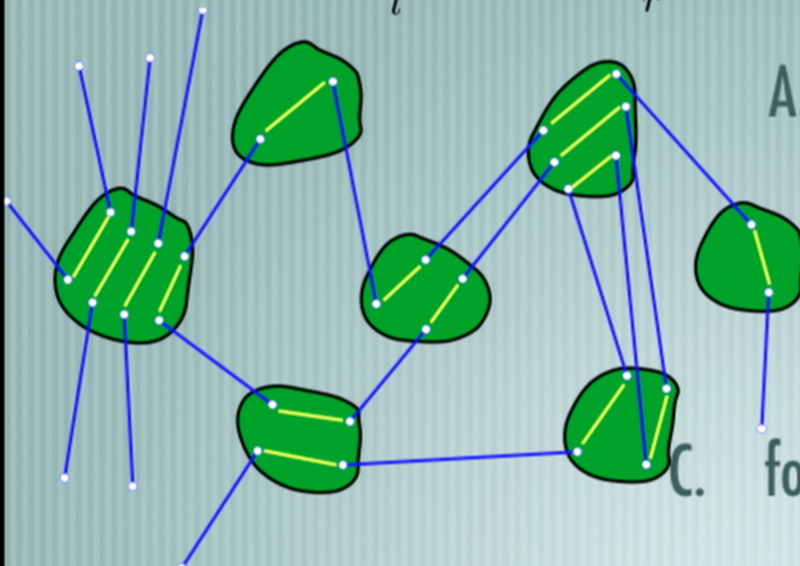


The bond algebra:

A. $(\sigma_l^x)^2 = 1 = (\widetilde{\mathcal{P}}_r)^2,$

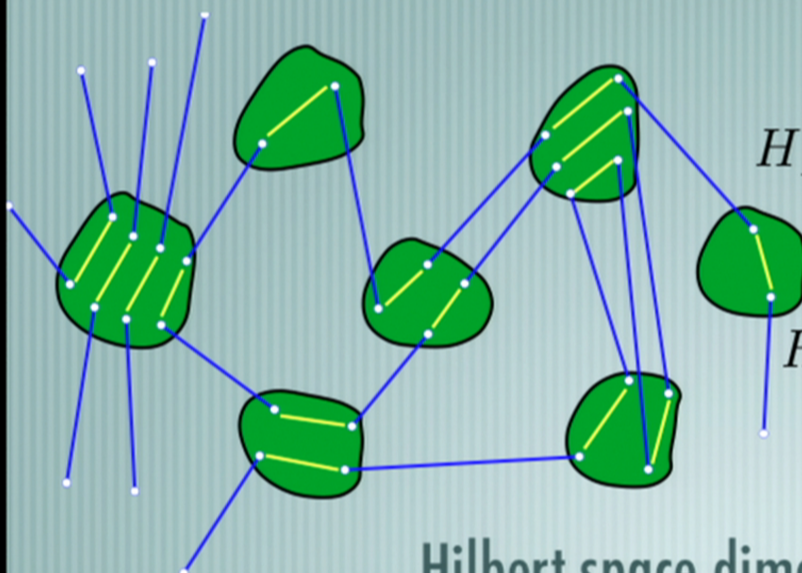
B. for $r, r' \in l,$

$\{\widetilde{\mathcal{P}}_r, \sigma_l^x\} = 0 = \{\widetilde{\mathcal{P}}_{r'}, \sigma_l^x\},$

C. for $r \in l_i,\ i = 1, 2, \cdots, q_r$

$\{\widetilde{\mathcal{P}}_r, \sigma_{l_i}^x\} = 0.$

Interacting Majorana fermion to Quantum Ising Gauge Theory dualities on general graphs



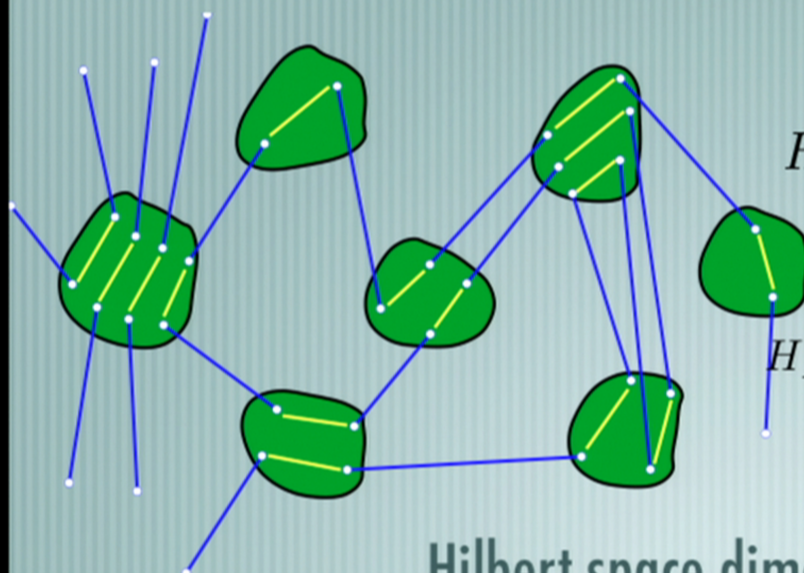
$$H_M = -i \sum_l J_l c_{l1} c_{l2} - \sum_r h_r \mathcal{P}_r;$$

$$\updownarrow \qquad \qquad \updownarrow$$

$$H_{QIG} = -\sum_l J_l \sigma_l^x - \sum_r h_r \widetilde{\mathcal{P}}_r$$

Hilbert space dimensions are the same

General dualities between the Quantum Ising gauge and annealed transverse field Ising model on general graphs



$$H_{QIG} = -\sum_l J_l \sigma^x_l - \sum_r h_r \widetilde{\mathcal{P}}_r$$

$$\updownarrow \qquad \updownarrow$$

$$H_{AI}\{\eta_l\} = -\sum_l J_l(\eta_l \sigma^z_r \sigma^z_{r'}) - \sum_r h_r \sigma^x_r$$

Hilbert space dimensions are different

Dualities between interacting Majorana fermions and Pauli spin gauge theory on the square lattice

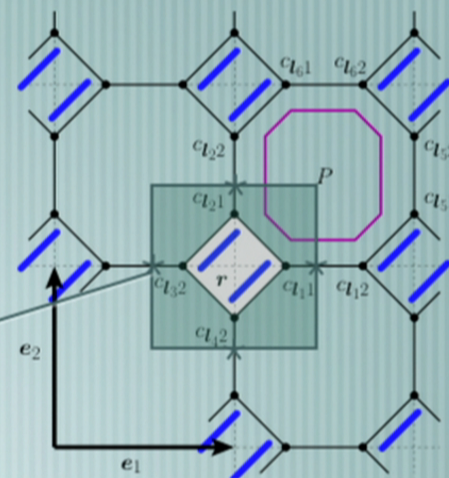
$$H_M = -\sum_l J_l (i c_{l1} c_{l2}) - \sum_r h_r c_{l_1 1} c_{l_2 1} c_{l_3 2} c_{l_4 2}$$

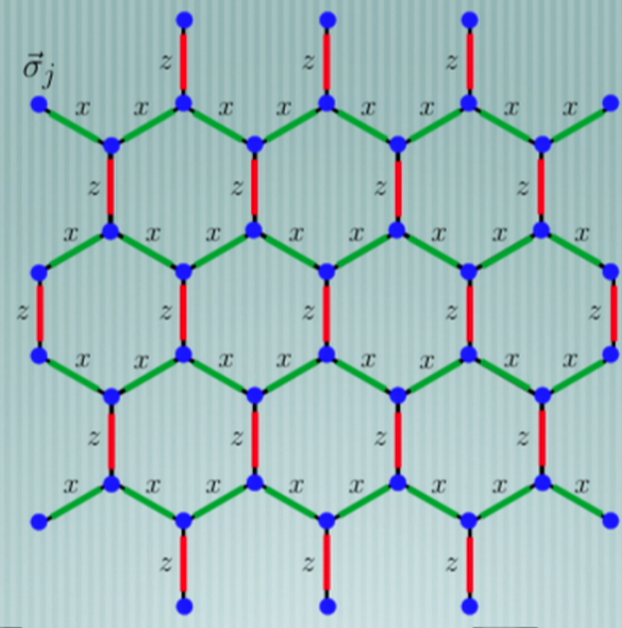
dual

$$H_{QIG} = -\sum_r h_r \sigma^z_{l_1} \sigma^z_{l_2} \sigma^z_{l_3} \sigma^z_{l_4} - \sum_l J_l \sigma^x_l$$

Transverse field

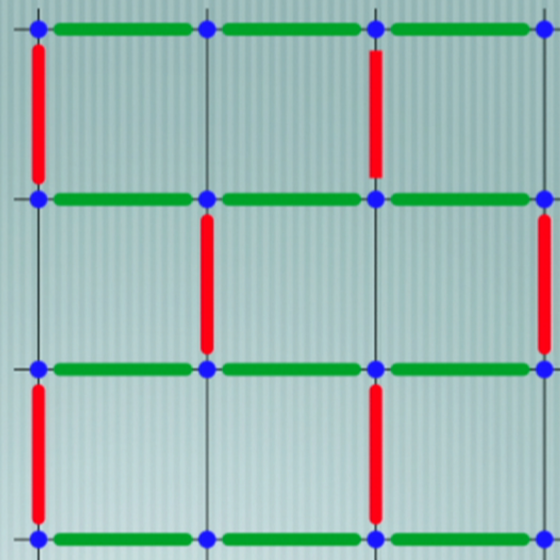
Standard plaquette term of lattice gauge theories





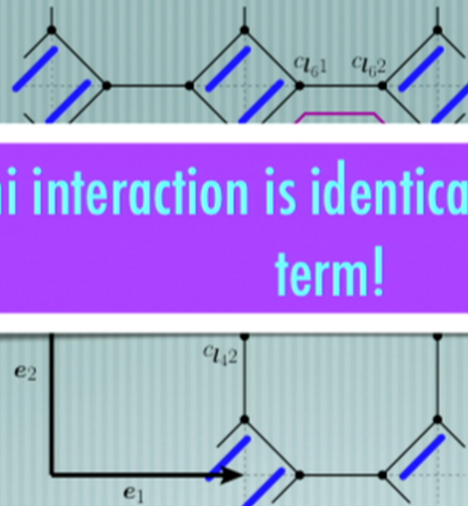
$$H_{XXZh} = -\sum_{\text{non-vertical links}} J_l\, \sigma^x_r \sigma^x_{r+\hat{e}_l} - \sum_{\text{vertical links}} h_r\, \sigma^z_r \sigma^z_{r+\hat{e}_z}$$

A brick wall rendition of the honeycomb lattice. Topologically, bricks= hexagons. The centers of vertical links form a square lattice on which the dual theory is constructed.



$$H_{XXZh} = - \sum_{\text{non-vertical links}} J_l \, \sigma^x_r \sigma^x_{r+\hat{e}_l} - \sum_{\text{vertical links}} h_r \, \sigma^z_r \sigma^z_{r+\hat{e}_z}$$

The Majorana fermi interaction is identically the same as the Hubbard term!

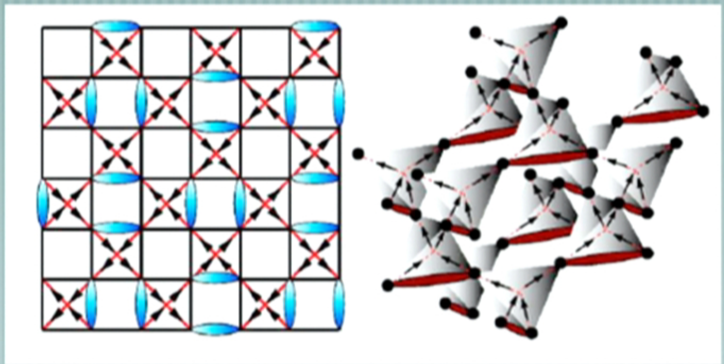


$$n_{r\sigma} = d^{\dagger}_{r\sigma} d_{r\sigma}$$

$$U_r(n_{r\uparrow} - 1)(n_{r\downarrow} - 1) = U_r(\mathcal{P}_r - 1)$$

Intra-grain Majorana fermi interaction

Fractionalization and deconfinement: the Hubbard model on the pyrochlore lattice



$$H_{ij}^{\alpha} = \vec{S}_i^{\alpha} \cdot \vec{S}_j^{\alpha}$$

$$H_{Klein} = J \sum_{\langle ij \rangle, \alpha} H_{ij}^{\alpha} + K \sum_{\alpha} (H_{ij}^{\alpha} H_{kl}^{\alpha} + H_{il}^{\alpha} H_{jk}^{\alpha} + H_{ik}^{\alpha} H_{jl}^{\alpha})$$

Fractionalization and deconfinement: the Hubbard model on the pyrochlore lattice

$$H_{\rm Hubb} = -t \sum_{\langle ij \rangle,\sigma} d^\dagger_{i\sigma} d_{j\sigma} + U \sum_i n_{i\uparrow} n_{i\downarrow},$$

$$\tilde{H}_{\rm Hubb} = H + J_3 \sum_{\langle\langle ij \rangle\rangle} \vec{S}_i \cdot \vec{S}_j$$ - effective 4th order Hamiltonian at half-filling

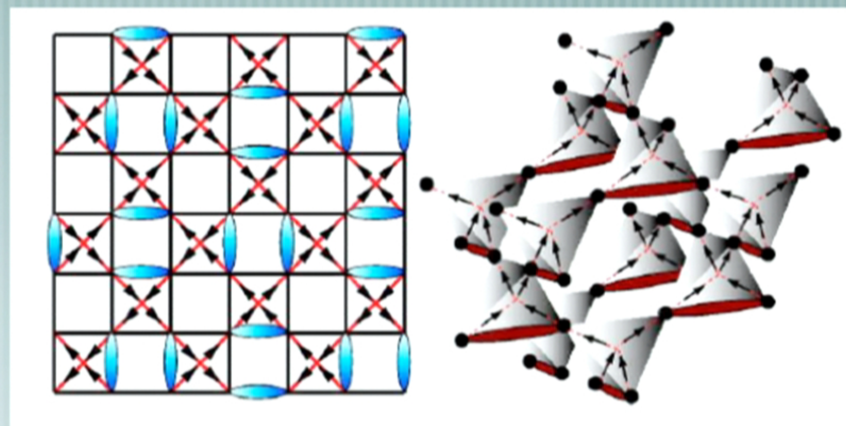
$$J_1 = \frac{4t^2}{U} - \frac{160t^4}{U^3} + \mathcal{O}\left(\frac{t^6}{U^5}\right), \quad J_3 = \frac{4t^4}{U^3} + \mathcal{O}\left(\frac{t^6}{U^5}\right)$$

$$J_2 = \frac{40t^4}{U^3} + \mathcal{O}\left(\frac{t^6}{U^5}\right).$$

Fractionalization and deconfinement: the Hubbard model on the pyrochlore lattice

All ground states are linear superpositions of dimer states.

$$H_K = \frac{12}{5} J \sum_{\boxtimes} \mathcal{P}^{\boxtimes}$$



Provable consequences: deconfined excitations, spin-charge separation,..., critical correlations (the latter assuming a gap and linear independence)

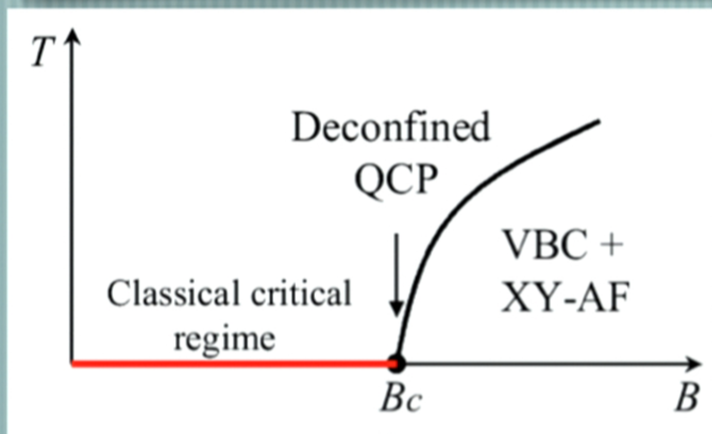
Extensive configurational entropy

The number of dimer coverings is exponential in the volume
(a consequence of local [gauge] symmetry):

Checkerboard lattice: $$S = \frac{3N}{4} \ln\left[\frac{4}{3}\right]$$

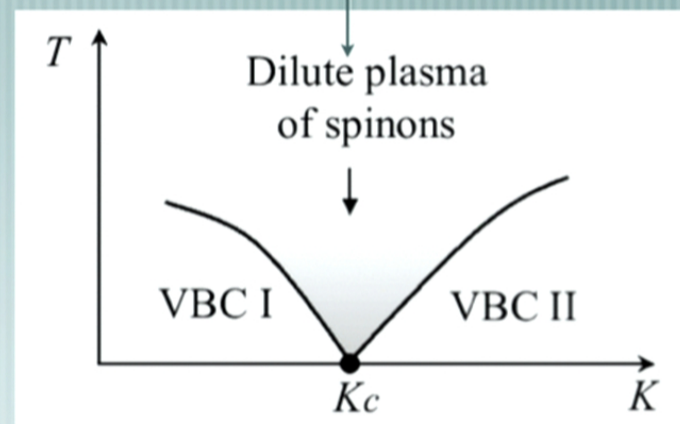
Pyrochlore lattice: $$S > \frac{N}{2} \ln\left[\frac{3}{2}\right]$$

Fractionalization and deconfinement: the Hubbard model on the pyrochlore lattice



Away from solvable point by applying field

Away from solvable point by varying the interaction strengths



Dimensional reduction inequalities

The average of any quasi-local quantity f is bounded from above by the absolute value of the mean of the same quantity when this quasi-local quantity is computed with a d-dimensional Hamiltonian that preserves the range of the interactions in the original D-dim system

$\mathcal{C}_j$

η_i

$$\phi_i = \begin{cases} \eta_i \text{ if } i \in \mathcal{C}_j \\ \psi_i \text{ if } i \notin \mathcal{C}_j \end{cases}$$

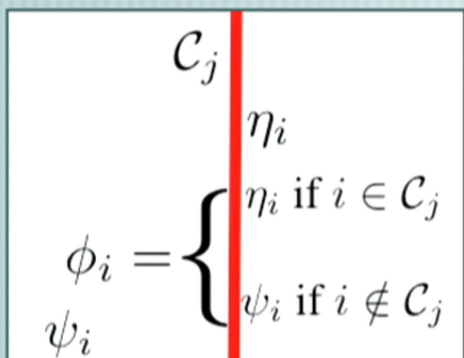
ψ_i

$$|\langle f(\phi_i(t))\rangle_{H_D}| \leq |\langle f(\eta_i(t))\rangle_{H_d}|$$

Dimensional reduction

Dimensional reduction inequalities

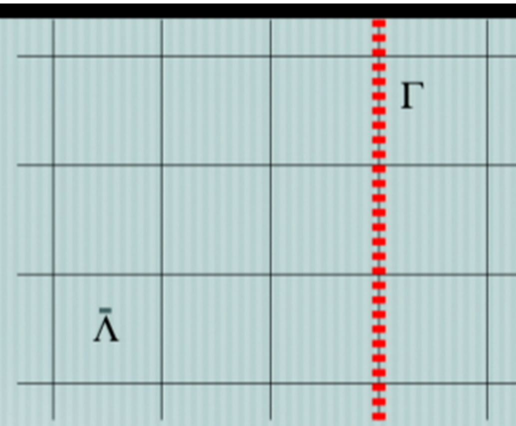
The average of any quasi-local quantity f is bounded from above by the absolute value of the mean of the same quantity when this quasi-local quantity is computed with a d-dimensional Hamiltonian that preserves the range of the interactions in the original D-dim system



$$|\langle f(\phi_i(t))\rangle_{H_D}| \leq |\langle f(\eta_i(t))\rangle_{H_d}|$$

Dimensional reduction

Dimensional reduction in classical systems:



$$\phi(\boldsymbol{x}) = \begin{cases} \phi_0(\boldsymbol{x}) & \text{if} \quad \boldsymbol{x} \in \Gamma \\ \psi(\boldsymbol{x}) & \text{if} \quad \boldsymbol{x} \in \bar{\Lambda} \end{cases}$$

$f[\phi] = f[\phi_0]$ localized observable

$$\langle f \rangle^D = \sum_{\{\psi\}} \sum_{\{\phi_0\}} f[\phi_0] \frac{e^{-\beta E[\phi_0, \psi]}}{\mathcal{Z}} = \sum_{\{\psi\}} \frac{z[\psi]}{\mathcal{Z}} \frac{\sum_{\{\phi_0\}} f(\phi_0) e^{-\beta E[\phi_0, \psi]}}{z[\psi]}$$

$$\langle f \rangle_l^d \equiv \mathsf{min}_\psi \, \langle f \rangle^d[\psi] = \langle f \rangle^d[\psi_{\mathsf{min}}], \qquad \langle f \rangle_u^d \equiv \mathsf{max}_\psi \, \langle f \rangle^d[\psi] = \langle f \rangle^d[\psi_{\mathsf{max}}]$$

$$\boxed{\langle f \rangle_l^d \leq \langle f \rangle^D \leq \langle f \rangle_u^d}$$

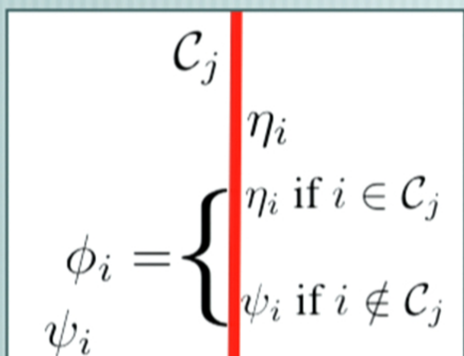
$$\langle f \rangle_l^d \;:\; E_l[\phi_0, \psi_{\mathsf{min}}] \quad \text{and} \quad \langle f \rangle_u^d \;:\; E_u[\phi_0, \psi_{\mathsf{max}}]$$

Local effective boundary theories

Dimensional reduction inequalities

In some cases, due to symmetries both upper and lower bounds scale in the same way.

In other systems (e.g., some surface codes), stringent upper bounds (on, e.g., autocorrelation functions measuring memory times) can be derived due to "lower dimensional symmetries". The effect of any additional symmetry breaking perturbations can be quantified with the bounds.



Dimensional Reduction/Holography

Similar inequalities can be proven for quantum systems. The correlation function **inequalities are general** and not specific to any model. In general they lead to:

- **Effective** dimensional reduction (bounds)
- **Exact** dimensional reduction:

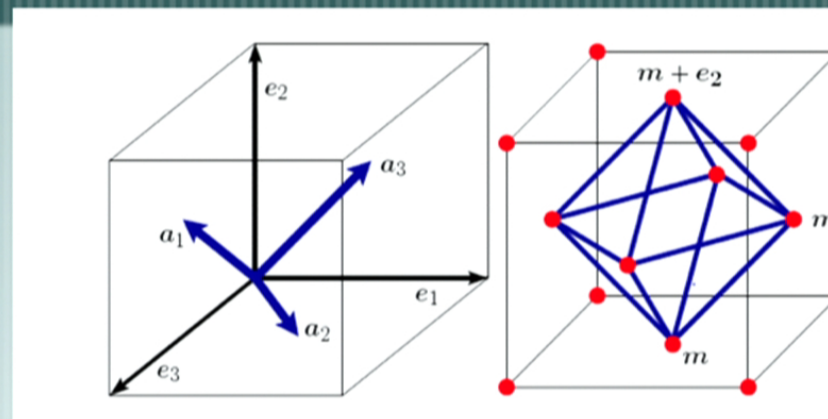
Inequalities become equalities

Exact Dimensional Reduction

XXYYZZ model (Chamon; Bravyi, Leemhuis, Terhal)

$$\mathbf{a}_1 = \frac{\hat{e}_2 + \hat{e}_3}{2}\ ,\ \mathbf{a}_2 = \frac{\hat{e}_1 + \hat{e}_3}{2}\ ,\ \mathbf{a}_3 = \frac{\hat{e}_1 + \hat{e}_2}{2}$$

$$O_m = \sigma^x_{m+a_1-a_2}\sigma^x_{m+a_3}\sigma^y_m\sigma^y_{m+a_2}\sigma^z_{m+a_3-a_2}\sigma^z_{m+a_1}$$



$$H_{XXYYZZ} = -J \sum_{m \in \Lambda^P_{fcc}} O_m$$

$$H_{4IP} = -J \sum_{\kappa=1}^{4} \sum_{m=1}^{N_s/4} \sigma^z_{\kappa,m}\sigma^z_{\kappa,m+1}$$

Exact Dimensional Reduction and holography in the large n limit

If two systems share the same density of states

$$\rho(\epsilon) = \int \frac{d^D k}{(2\pi)^D} \delta^{(D)}(\epsilon - J(\mathbf{k}))$$

then they will have identical self-energies $\Sigma^{(0)} = \int d\epsilon \frac{\rho(\epsilon)}{\epsilon + r}$

This enables a universal reduction to a one dimensional system with a kernel $J_{eff}(\mathbf{k})$:

$$\int \frac{d^D k}{(2\pi)^D} \delta(\epsilon - J(\mathbf{k})) = \left| \frac{dJ_{eff}}{dk} \right|^{-1}_{J_{eff}(k)=\epsilon}$$

Conclusions: There are exact spin-Majorana (and similar other) dualities, holography, and deconfinement

Further reading:

Z. Nussinov and G. Ortiz, "Autocorrelations and Thermal Fragility of Anyonic Loops in Topologically Quantum Ordered Systems", Physical Review B 77, 064302 (2008)

Zohar Nussinov, Gerardo Ortiz "Orbital order driven quantum criticality", Europhysics Letters 84 (2008) 36005

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G. Ortiz, E. Cobanera, an Z. Nussinov, "Dualities and the phase diagram of the p- clock model", Nuclear Physics B 854, 780 (2011)

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Most directly related to this talk:

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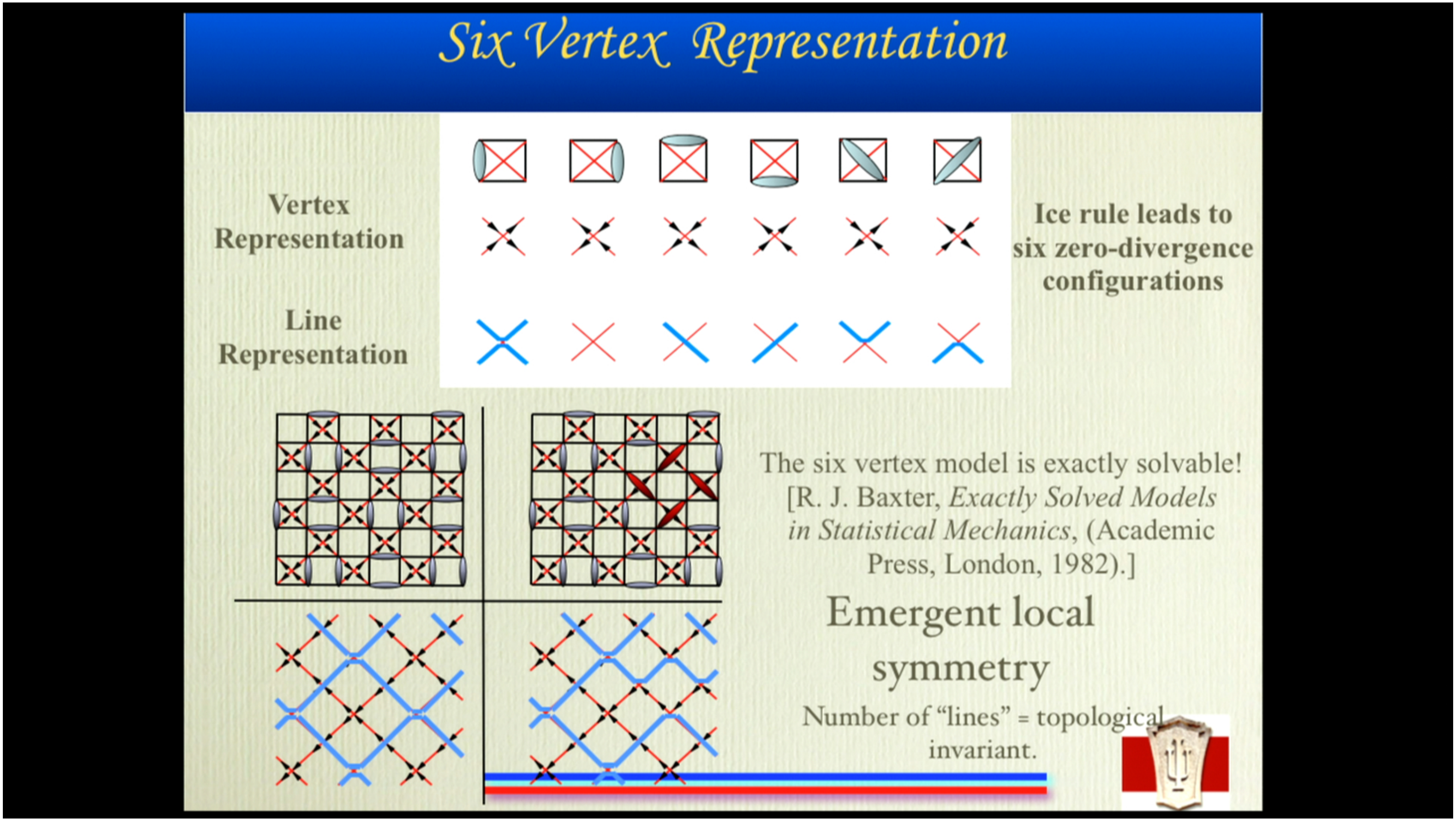
Six Vertex Representation
Vertex Representation
Line Representation
Ice rule leads to six zero-divergence configurations
The six vertex model is exactly solvable! [R. J. Baxter, Exactly Solved Models in Statistical Mechanics, (Academic Press, London, 1982).]
Emergent local symmetry
Number of "lines" = topological invariant.