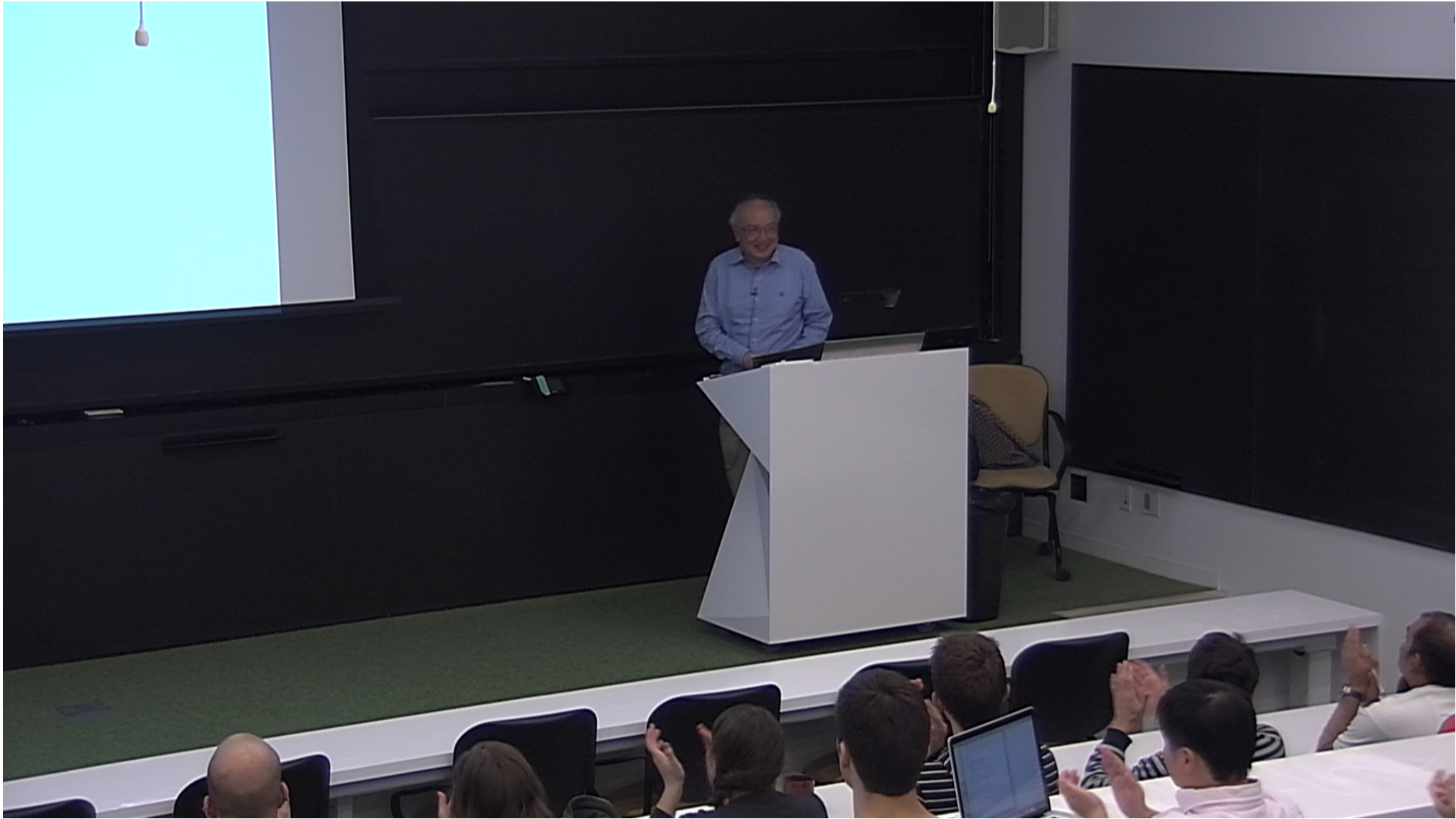


Title: In Search of Majorana

Date: May 02, 2012 02:00 PM

URL: <http://pirsa.org/12050005>

Abstract: Majorana disappeared under mysterious circumstances in 1938 and the particle that bears his name remains elusive to experiments. There is growing interests in realizing the Majorana bound state in the Laboratory because it is expected to possess unusual properties such as non-abelian statistics. I shall discuss various proposals to produce Majorana bound states and the associated topological superconductors which support them.



Ettore Majorana 5 August 1906 – 27 March 1938



Brilliant young Italian theorist who disappeared in 1938 en route by sea from Palermo to Naples.

“Majorana returns.” Frank Wilczek, *Nature Physics* **5**, 614 (2009)

Majorana found a representation of the Dirac equation which has real solution. The corresponding field theory describes a fermion which is its own anti-particle. It is suspected that neutrino's are such objects, but not proven.

Instead of a momentum eigenstate, we are interested in a **Majorana bound state**, ie a state which is localized in space.

Outline:

What are Majorana fermions?

Why are they interesting and/or useful?

How to create them in solid state physics?

How to detect them?



Some formal properties of a Majorana bound state: $\gamma = \gamma^\dagger$

$$\{\gamma, \gamma^\dagger\} = 2\gamma^2 = 2\gamma^\dagger\gamma = 1$$

“Half” of a fermion: $a_{12}^\dagger = \frac{\gamma_1 + i\gamma_2}{\sqrt{2}}$ $a_{12} = \frac{\gamma_1 - i\gamma_2}{\sqrt{2}}$

Fermion number parity: γ changes the fermion number between even and odd.

$$\gamma_1 = \frac{a_{12} + a_{12}^\dagger}{\sqrt{2}}$$

Fermion parity operator: $(1 - 2a_{12}^\dagger a_{12}) = 2i\gamma_1\gamma_2$

The only possible coupling between 2 Majorana fermions is:

$$H_\gamma = i\varepsilon\gamma_1\gamma_2.$$

Majorana fermion operator γ changes particle number by 1 (mod 2)

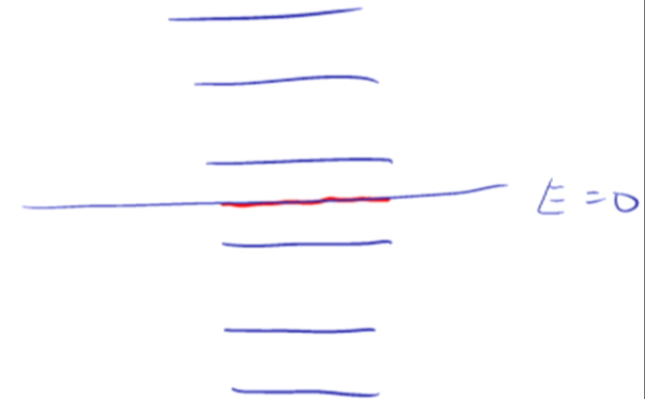
Arises naturally in solutions of Bogoliubov equation in superconductivity with spin polarized electrons. Then quasi-particles do not have spin degeneracy and is given by

$$\gamma = u(r) c_{\uparrow} + v(r) c_{\uparrow}^{\dagger}$$

$$\begin{bmatrix} H & \Delta \\ -\Delta^* & -H^* \end{bmatrix} \begin{pmatrix} c \\ c^{\dagger} \end{pmatrix} = E \begin{pmatrix} c \\ c^{\dagger} \end{pmatrix}$$

Particle-hole symmetry due to a **redundancy in the representation**: solutions come in pairs $\pm E$, but these states are not independent.

$$\gamma_{-E} = \gamma_E^{\dagger}$$



Odd number of zero energy solution is special:

1. It is a Majorana state. $\gamma = \gamma^{\dagger}$, $u = v^*$
2. It is protected, ie cannot be shifted away from zero energy by perturbation because it cannot split.

Majorana fermion operator γ changes particle number by 1 (mod 2)

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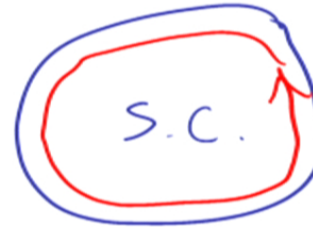
Odd number of zero energy solution is special:

1. It is a Majorana state. $\gamma = \gamma^{\dagger}$, $u = v^*$
2. It is protected, ie cannot be shifted away from zero energy by perturbation because it cannot split.

Prime candidate: 2 dim p-wave superconductor (spinless)

$$\Delta(\vec{p}) = \langle c_{\vec{p}} c_{-\vec{p}} \rangle \propto p_x + i p_y$$

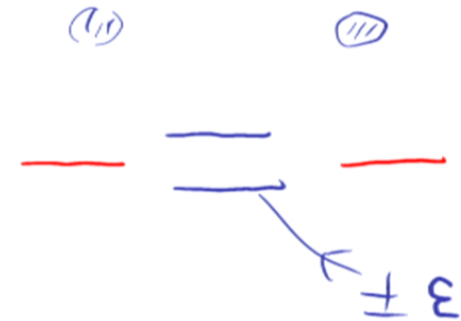
This state breaks time reversal symmetry and has chiral edge states, as well as a single zero mode in the vortex core. Jackiw and Rossi (1981), Read and Green.(2000)



When we bring 2 vortices close to each other, the Majorana states mix to form conventional Bogoliubov quasi-particle, ie a single fermion state which can be occupied or empty.

$$H_{\gamma} = i\varepsilon\gamma_1\gamma_2.$$

$$(1 - 2a_{12}^{\dagger}a_{12}) = 2i\gamma_1\gamma_2$$



2N Majorana fermions can be grouped into N ordinary fermions, each of which can be occupied or empty. The size of the Hilbert space is $2^{N/2}$.

However, the sectors with even and odd number of fermions cannot mix by exchange of Majorana's. Therefore the number of degrees of freedom is $2^{(N/2 - 1)}$.

The choice of grouping is arbitrary, and different grouping corresponds to a unitary rotation of basis in the Hilbert space.

Then perhaps it is not too surprising that exchanging (re-labeling) of majorana's leads to a rotation of the Hilbert space vectors, ie **non-abelian statistics**.

Statistics:

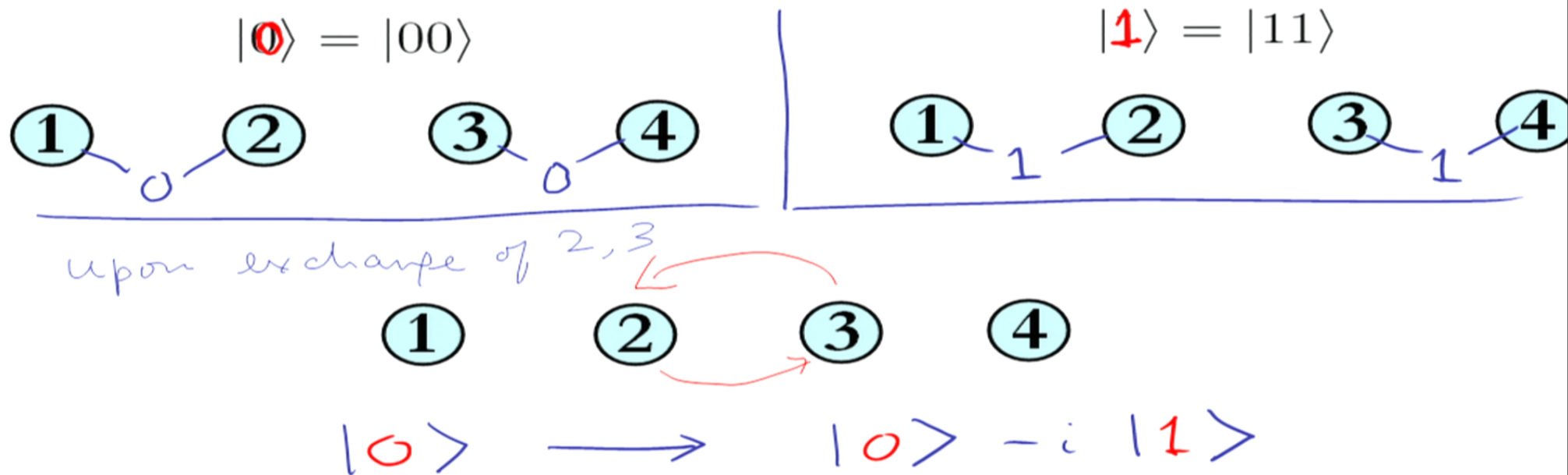
Boson and fermions \pm

Anyons in 2 dim: phase change upon exchange of particles. $e^{i\phi}$

Non-abelian statistics. Upon change of particles, the Hilbert space vector is rotated to a different vector!

Proposal for Quantum Computing with Fermion Parity Qubits (Kitaev)

The occ. or empty state can serve as q-bits which are highly nonlocal and protected from local perturbations. (Kitaev).

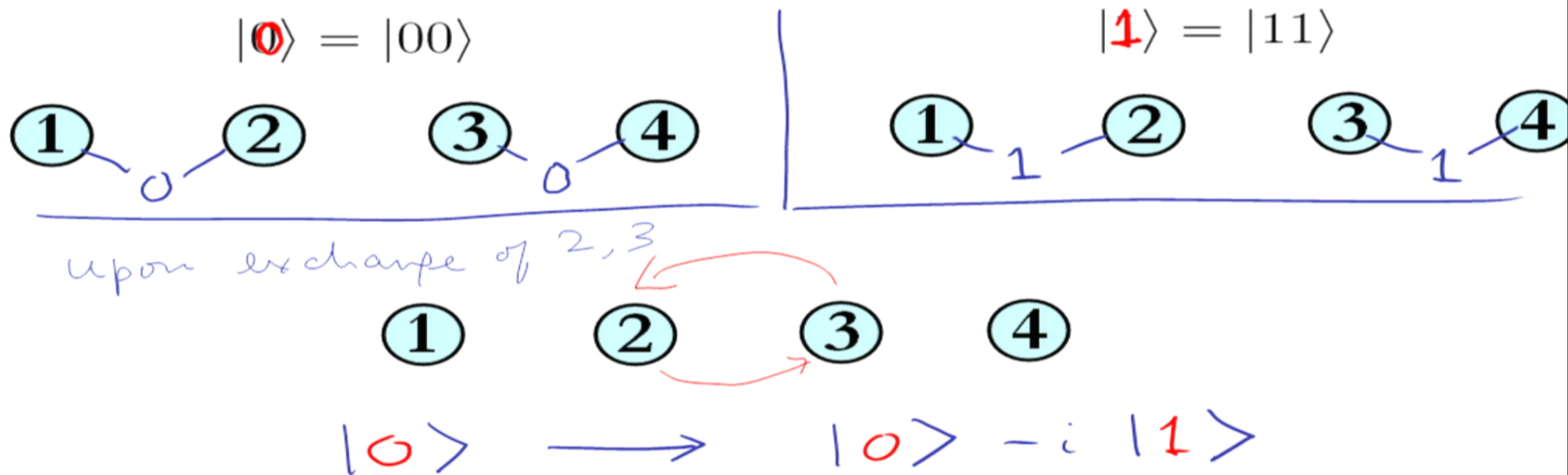


Caveat: for quantum computing, need to supplement braiding with (non-topological) phase gate:

$$U_{ph} = e^{i\pi/8\sigma_z}$$

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What are the experimental candidates to produce Majorana states?

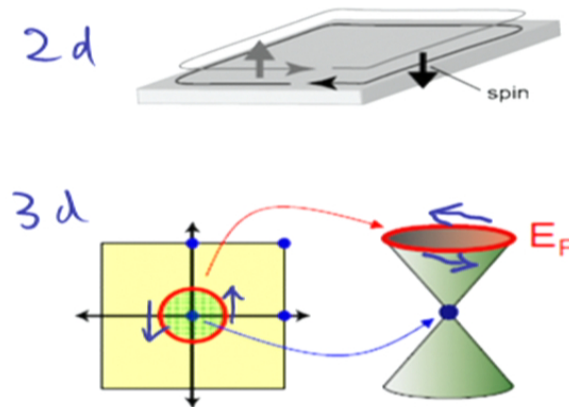
Need topological superconductor (All variants of p+ip superconductors.)

A. Naturally occurring:

1. **FQHE $\nu=5/2$ state**. Moore-Read state can be thought of as p+ip pairing of composite fermions. However, this state, if it exists, is very delicate.
2. A phase of superfluid He3.
3. Sr_2RuO_4 .
2 zero modes in $hc/2e$ vortex.
1 zero mode in $hc/4e$ vortex. (Ivanov 2001),
Recent expt by Budakian at Ill. succeeded to stabilize this in small samples.

B. Engineered systems:

4. **Topological insulator**. Fu and Kane.
Surface state of 3d TI is a Dirac state,
 $1/4$ of graphene!
5. Rashba split bands in semi-conductors
and surfaces of metallic films.



Fu and Kane:

Deposit conventional superconductor on top of 3D topological insulator.

Single component Helical surface state
(1/4 of graphene):

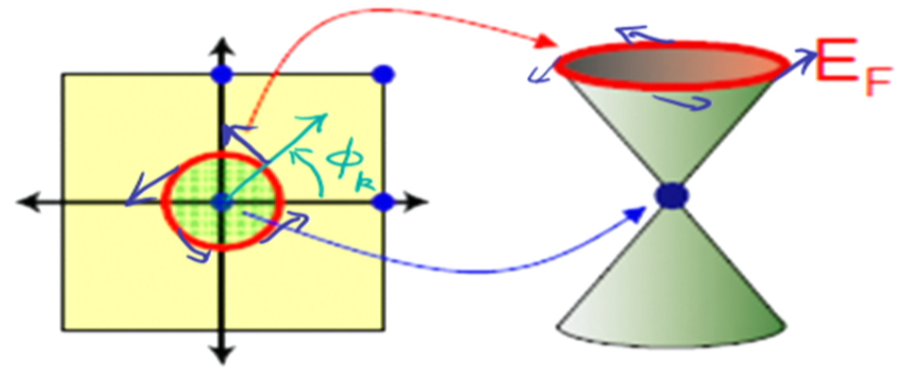
$$H_0 = \psi^\dagger (-iv\vec{\sigma} \cdot \nabla - \mu)\psi.$$

where $\psi = (\psi_\uparrow, \psi_\downarrow)^T$

The eigenstate is:

$$c_{\mathbf{k}} = (\psi_{\uparrow\mathbf{k}} + e^{i\theta_{\mathbf{k}}}\psi_{\downarrow\mathbf{k}})/\sqrt{2}$$

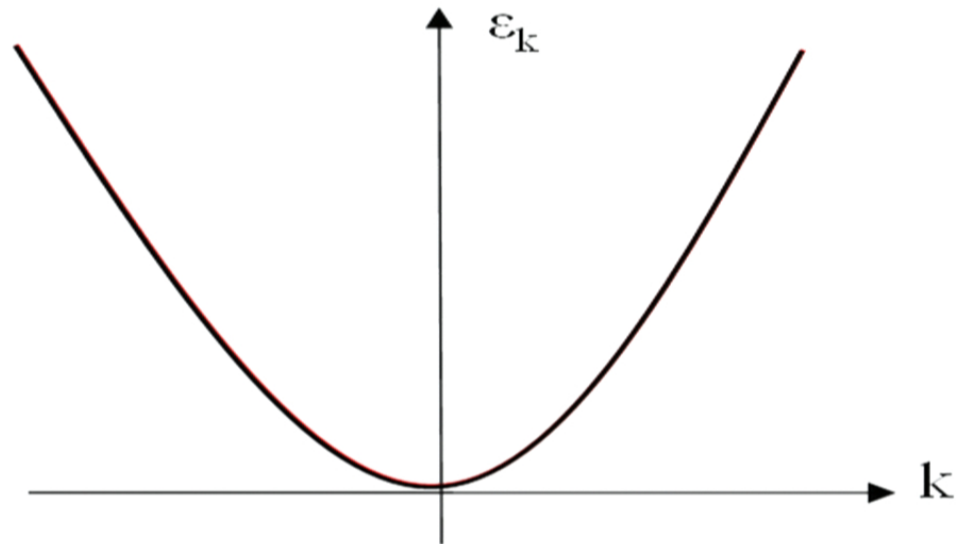
induced $\tilde{\Delta}_{\mathbf{k}} = \langle c_{\mathbf{k}} c_{-\mathbf{k}} \rangle$
 $= e^{i\theta_{\mathbf{k}}} \langle \psi_{\downarrow\mathbf{k}} \psi_{\uparrow-\mathbf{k}} \rangle$
 Δ_0



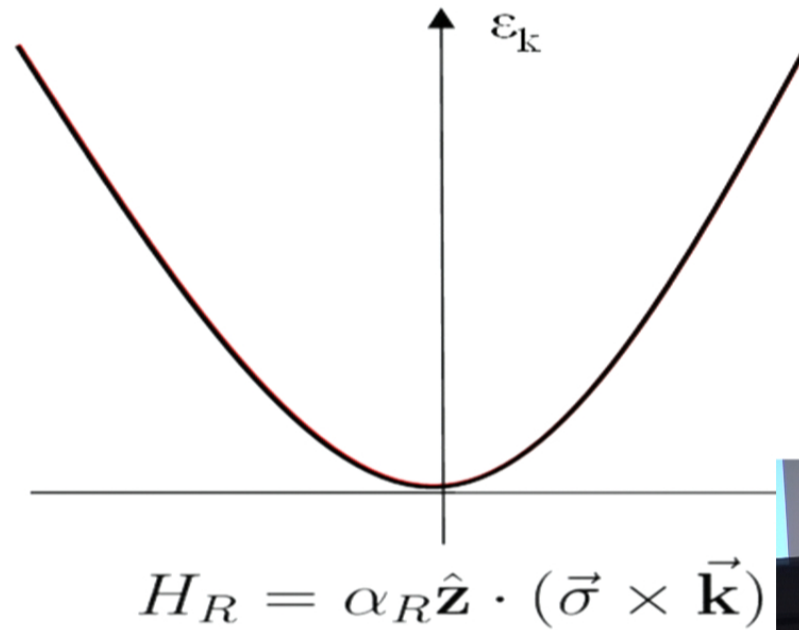
Engineering a p+ip SC

- Necessary Ingredients:
 - Induced s-wave pairing
 - 2D Helical spin band (convert s-wave->p-wave)
- second route:
 - Rashba+SC+T-breaking.
 - S. Fujimoto, PRB77 ,220501(2008), non centro-symmetric superconductor.
J. D. Sau et al , PRL 104, 040502 (2010), semiconductor realization.
Related idea, P. Lee, arXiv 0907.2681.

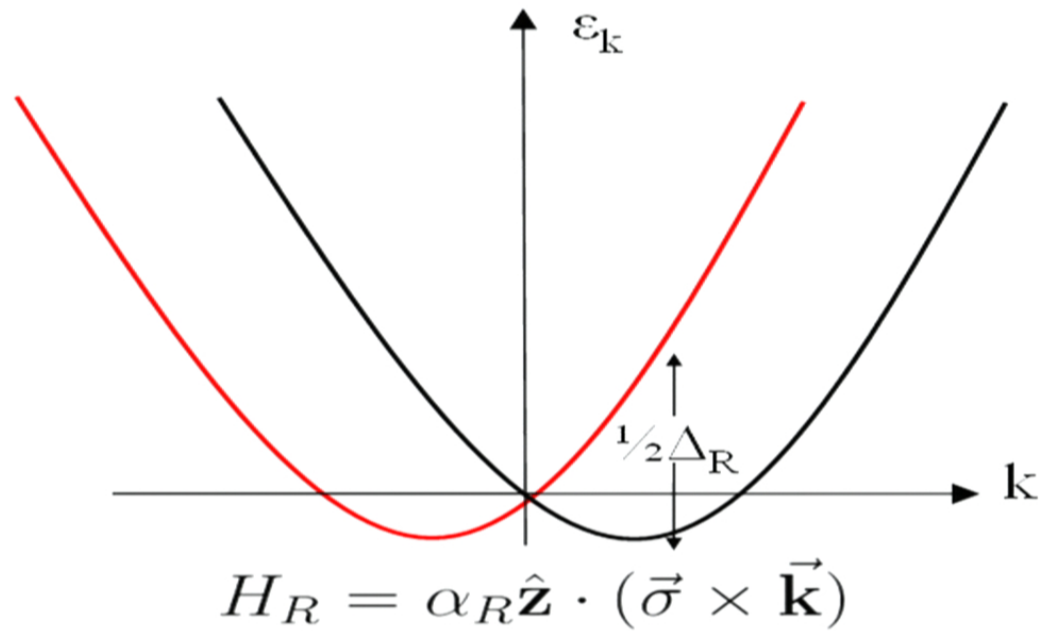
Rashba spin-orbit interaction + FM + SC.



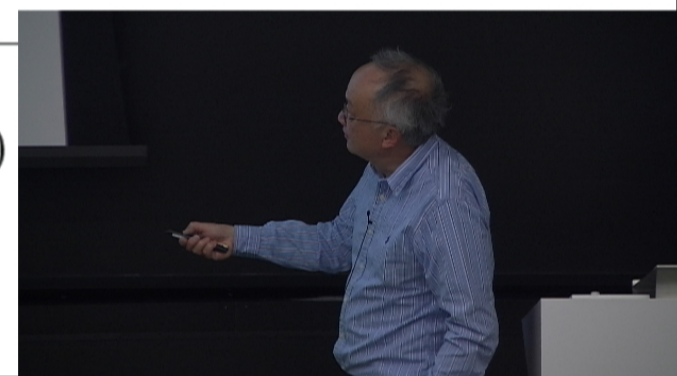
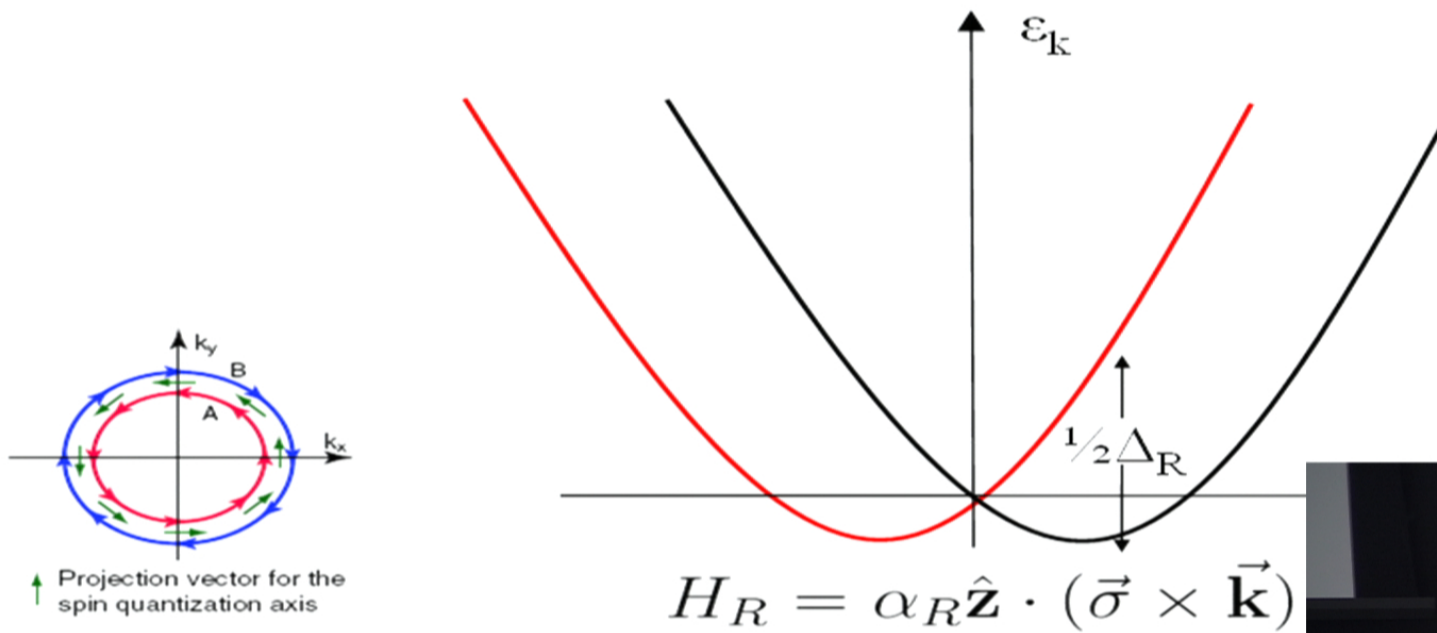
Rashba spin-orbit interaction + FM +SC.



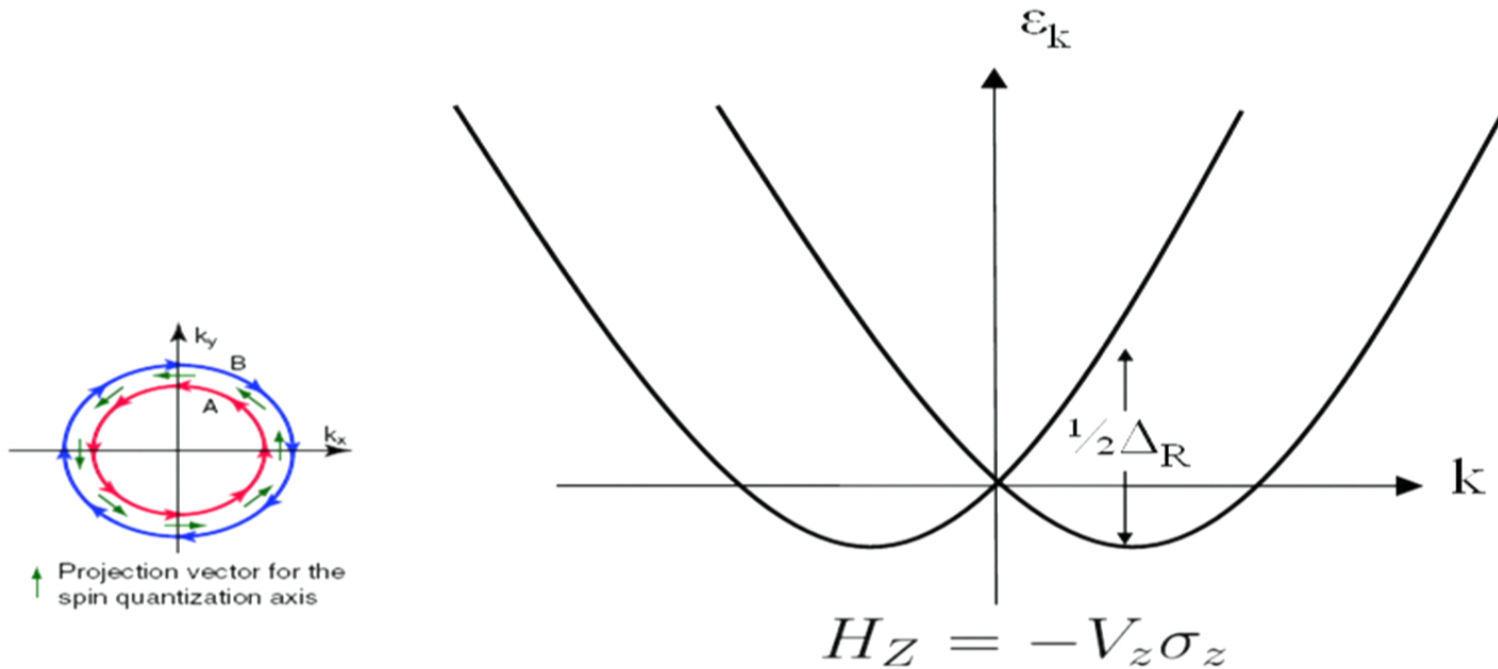
Rashba spin-orbit interaction + FM + SC.



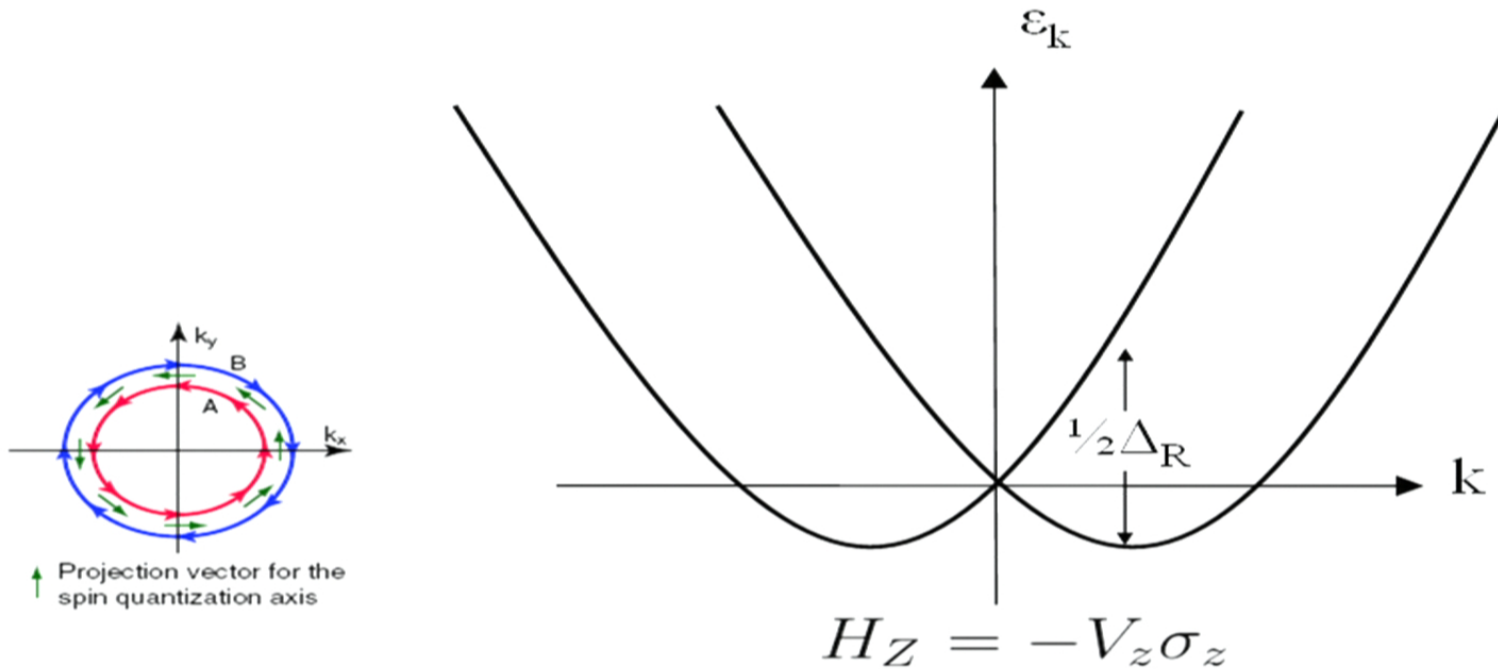
Rashba spin-orbit interaction + FM +SC.



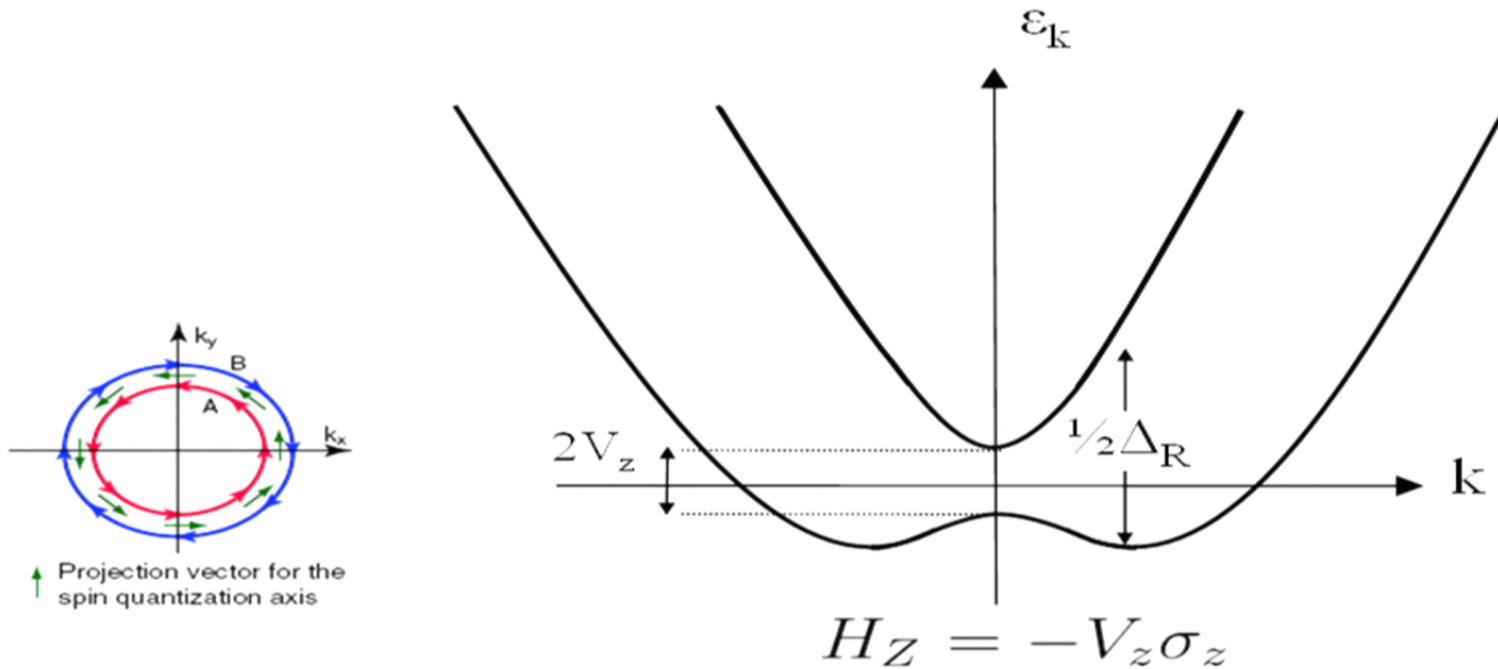
Rashba SOI + FM + SC



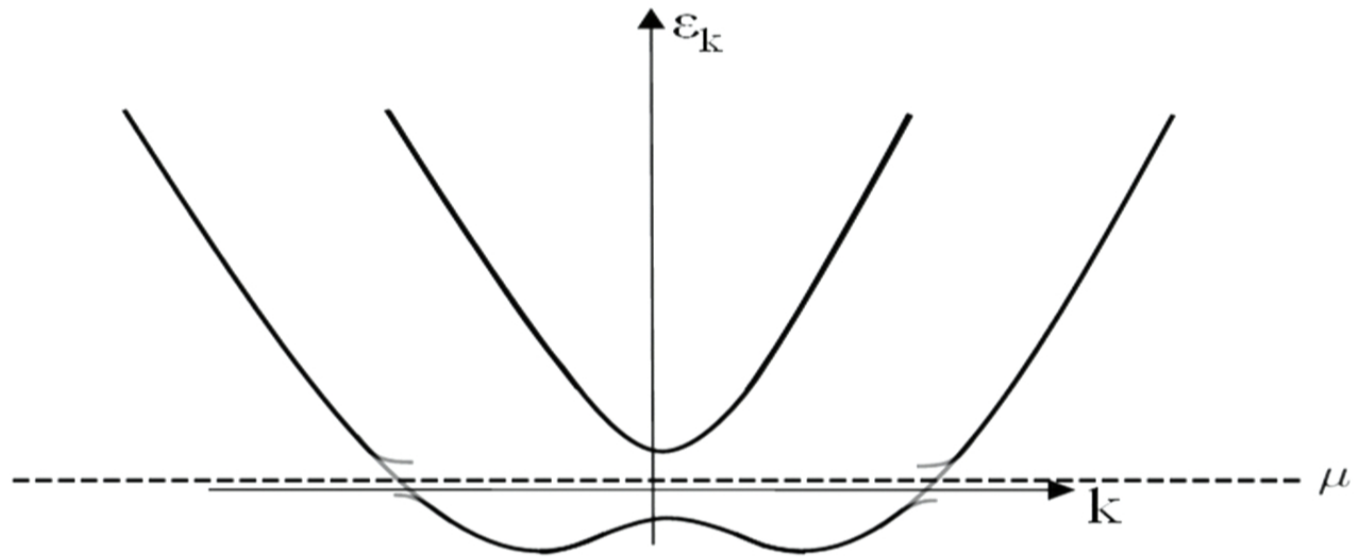
Rashba SOI + FM + SC



Rashba SOI + FM + SC

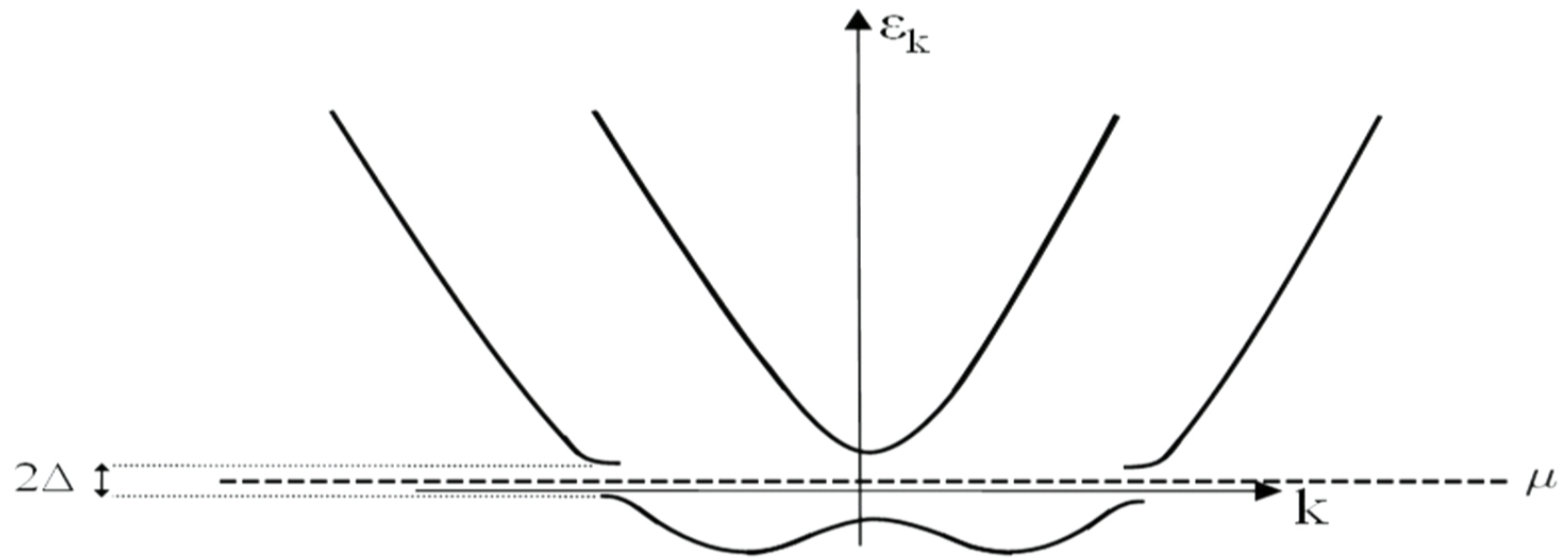


Rashba SOI + FM + SC



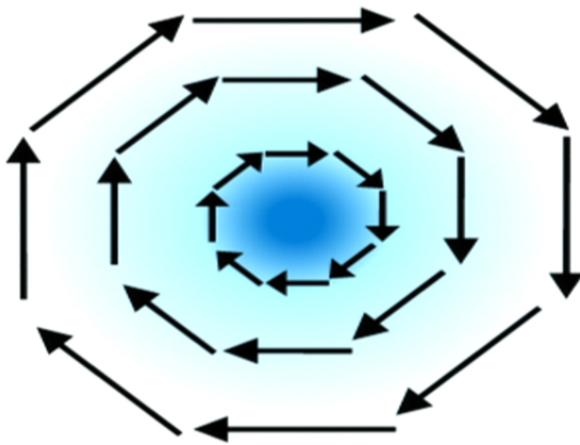
$$H_{BCS} = \sum_k \tilde{\Delta} c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger + h.c.$$

Rashba SOI + FM + SC

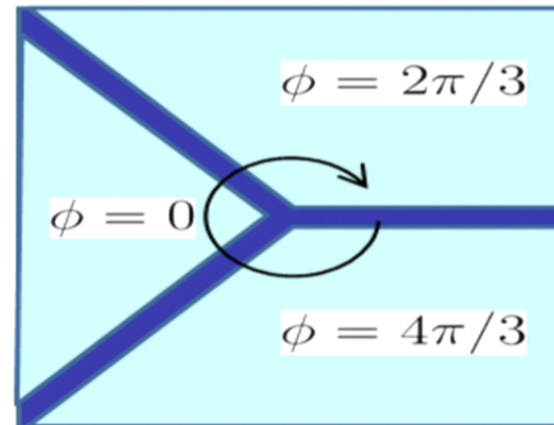


$$H_{BCS} = \sum_k \tilde{\Delta} c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger + h.c.$$

Practical Difficulties for Using Vortices



Abrikosov



Josephson

1. Controlling the motion of vortices may not be easy.
2. Mini-gaps in vortex core:

Mini gaps may not be an issue if vortices are kept far apart. Excitation of local qp's does not change the fermion parity of the vortex as a whole. (Akhmerov, PRB 2010)

1D Tight-Binding Model (Kitaev)

$$H = \sum_i \left[-t \left(c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1} \right) - \mu c_i^\dagger c_i + \Delta \left(c_i^\dagger c_{i+1}^\dagger + h.c. \right) \right]$$

Represent fermion on site i by 2 majorana fermions: $c = (b + ia)/2$

b

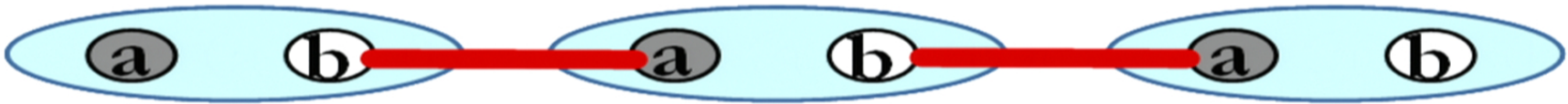
i

1D Tight-Binding Model (Kitaev)

$$H = \sum_i \left[-t \left(c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1} \right) - \mu c_i^\dagger c_i + \Delta \left(c_i^\dagger c_{i+1}^\dagger + h.c. \right) \right]$$

Represent fermion on site i by 2 majorana fermions: $c = (b + ia)/2$

Case 1: $\Delta = t \neq 0$
 $\mu = 0$ $H_1 = t \sum_i \left(c_i^\dagger + c_i \right) \left(\frac{c_{i+1}^\dagger - c_{i+1}}{2} \right) = -it \sum_i b_i a_{i+1}$

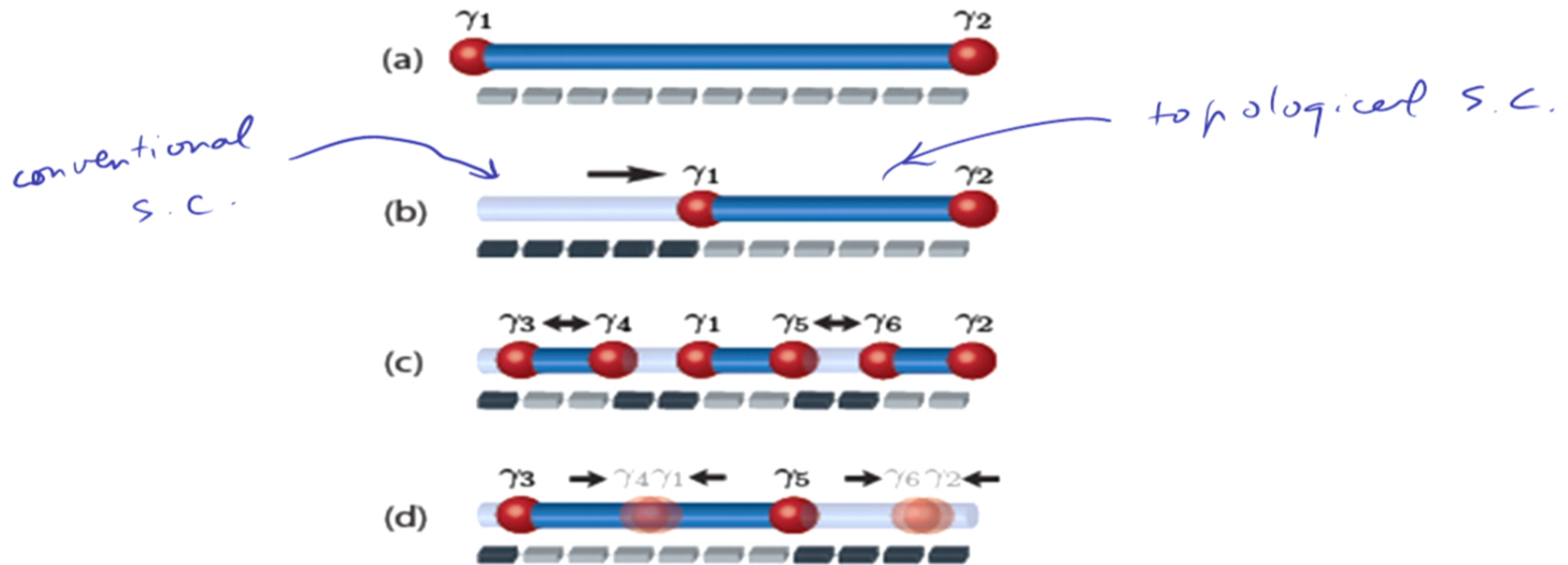


Case 2: $\Delta = t = 0$
 $\mu \neq 0$ $H_2 = -\mu \sum_i c_i^\dagger c_i = -i\mu \sum_i a_i b_i$

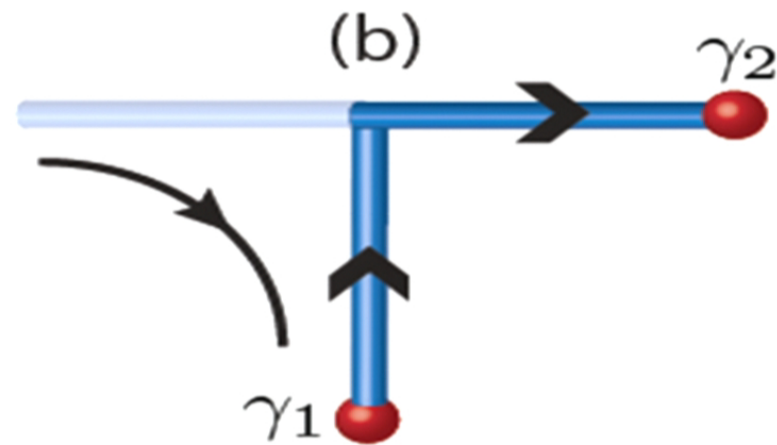


How Majorana end states can be manipulated and braided by gates.

J Alicea, Y Oreg, G Refael, F Von Oppen, MPA Fisher. PRB (2010)

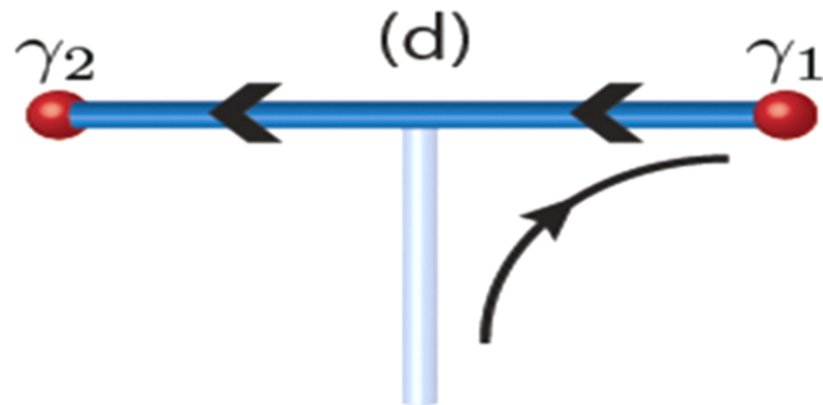


Majorana End-States Braiding in T-Junction



J Alicea, Y Oreg, G Refael, F Von Oppen, MPA Fisher. PRB(2010)

Majorana End-States Braiding in T-Junction

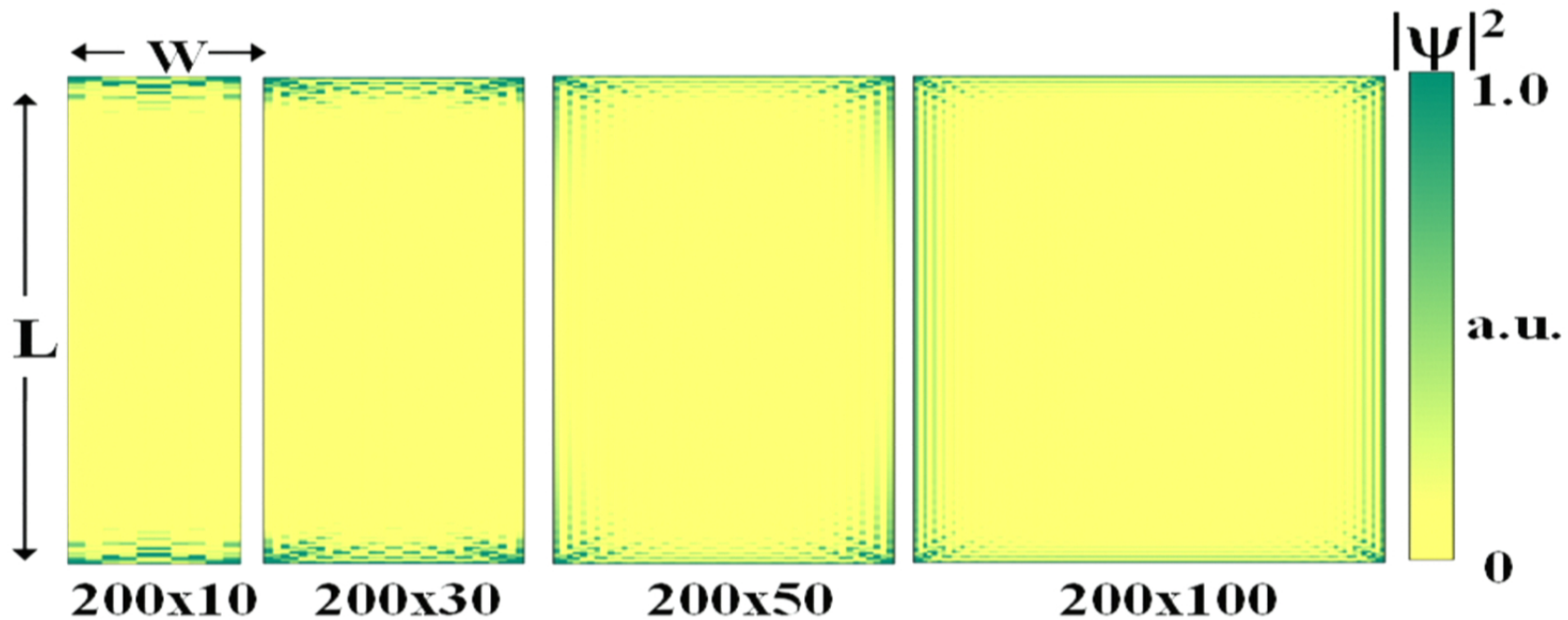


J Alicea, Y Oreg, G Refael, F Von Oppen, MPA Fisher. PRB(2010)

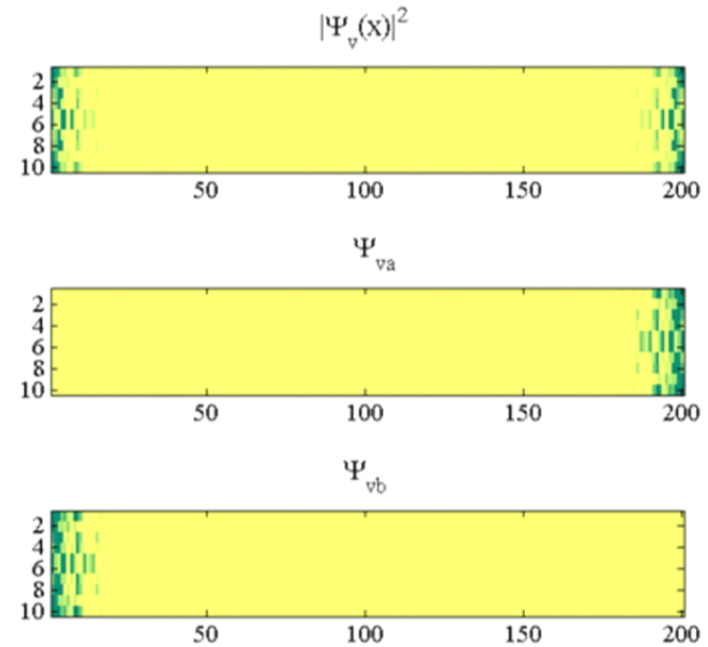
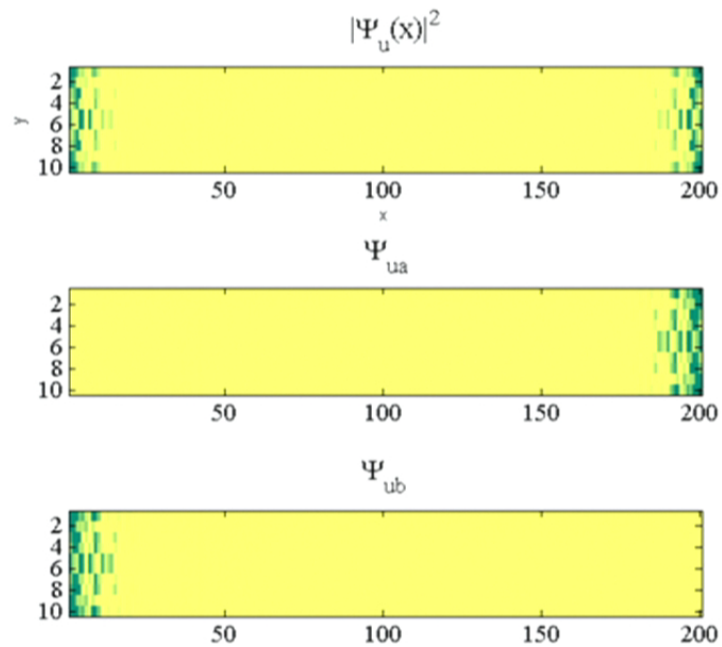
We consider tight binding model on a finite strip of width W and length L .

1. $p+ip$ superconductor.
2. Rashba surface band.

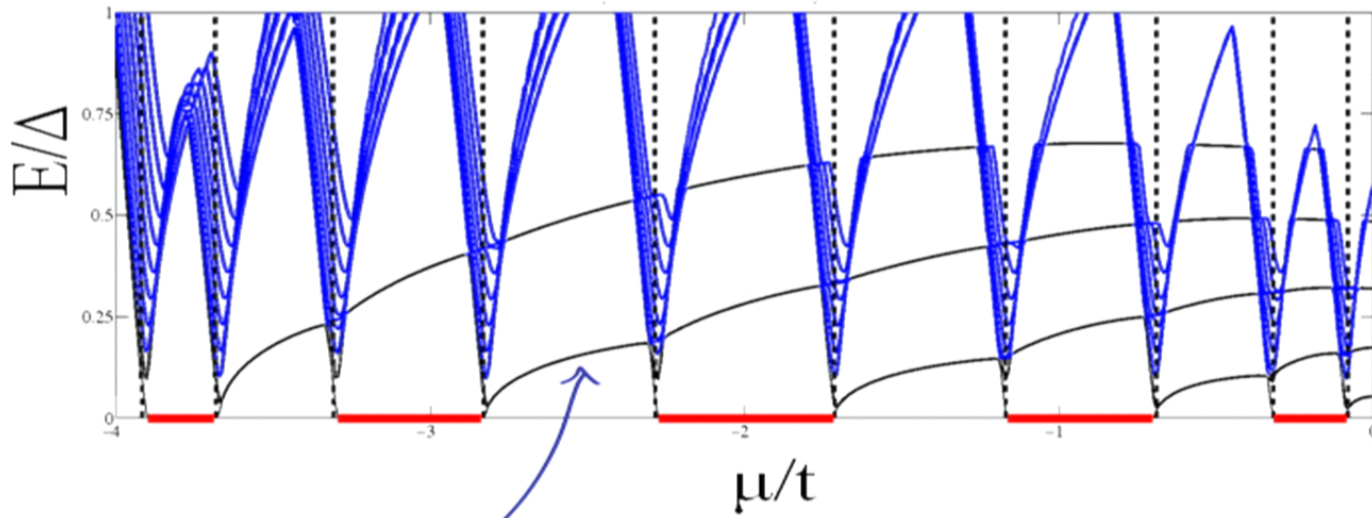
Question: how does chiral edge mode turn into Majorana edge states as W decreases?



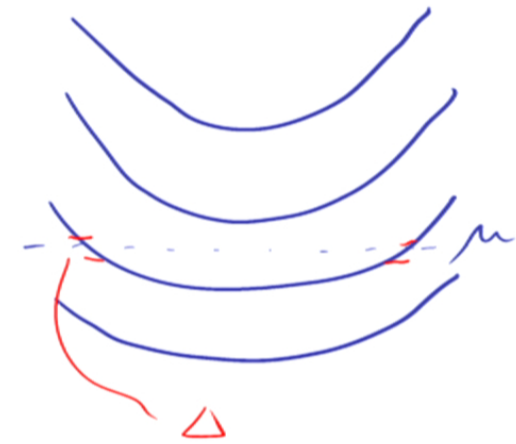
Potter and Lee, PRL 2010, PRB 2011.



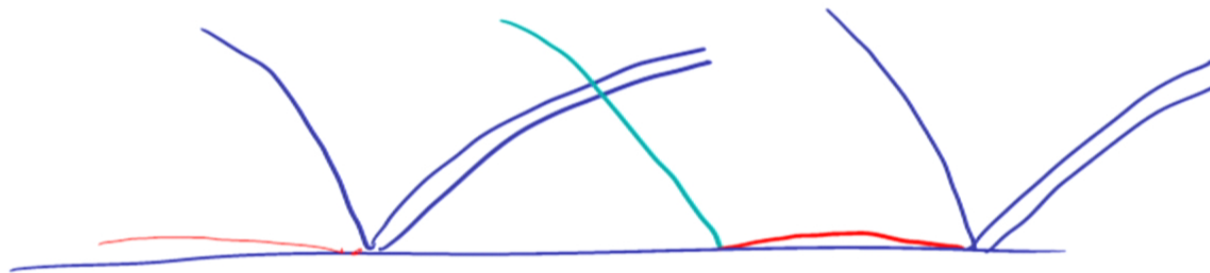
Disentangling of the two Majorana end states... Since the two Majorana end-states have weak overlap due to finite length, the simulations show complex fermion combinations of these two states. Top row shows wave-function from simulations for the infinitesimally positive energy combination of the two Majorana end-states. The middle and bottom row show that linear combinations $\psi_a = e^{i\phi}\psi_{e+} + e^{-i\phi}\psi_{e-}$, $\psi_b = e^{i\phi}\psi_{e+} - e^{-i\phi}\psi_{e-}$, of these two states are indeed real fermions confined to single end (where ϕ is an arbitrary overall relative phase factor from the simulation, which is found a posteriori by automated trial and error).



doubly degenerate
end fermions



doubly degenerate
end
fermions



An extra Majorana fermion appears and fuses with the existing Majorana to form fermion.

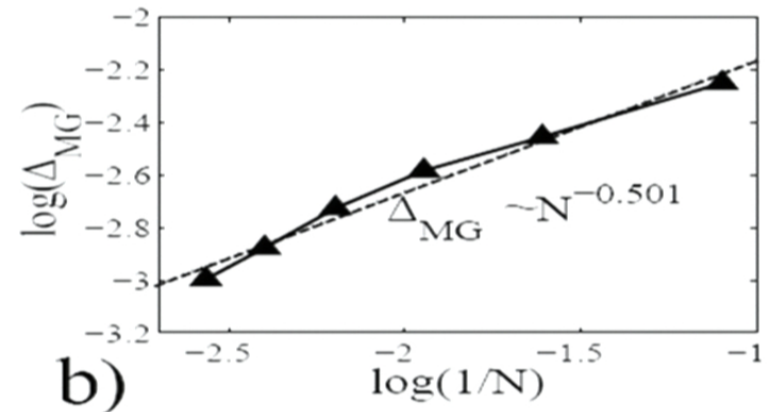


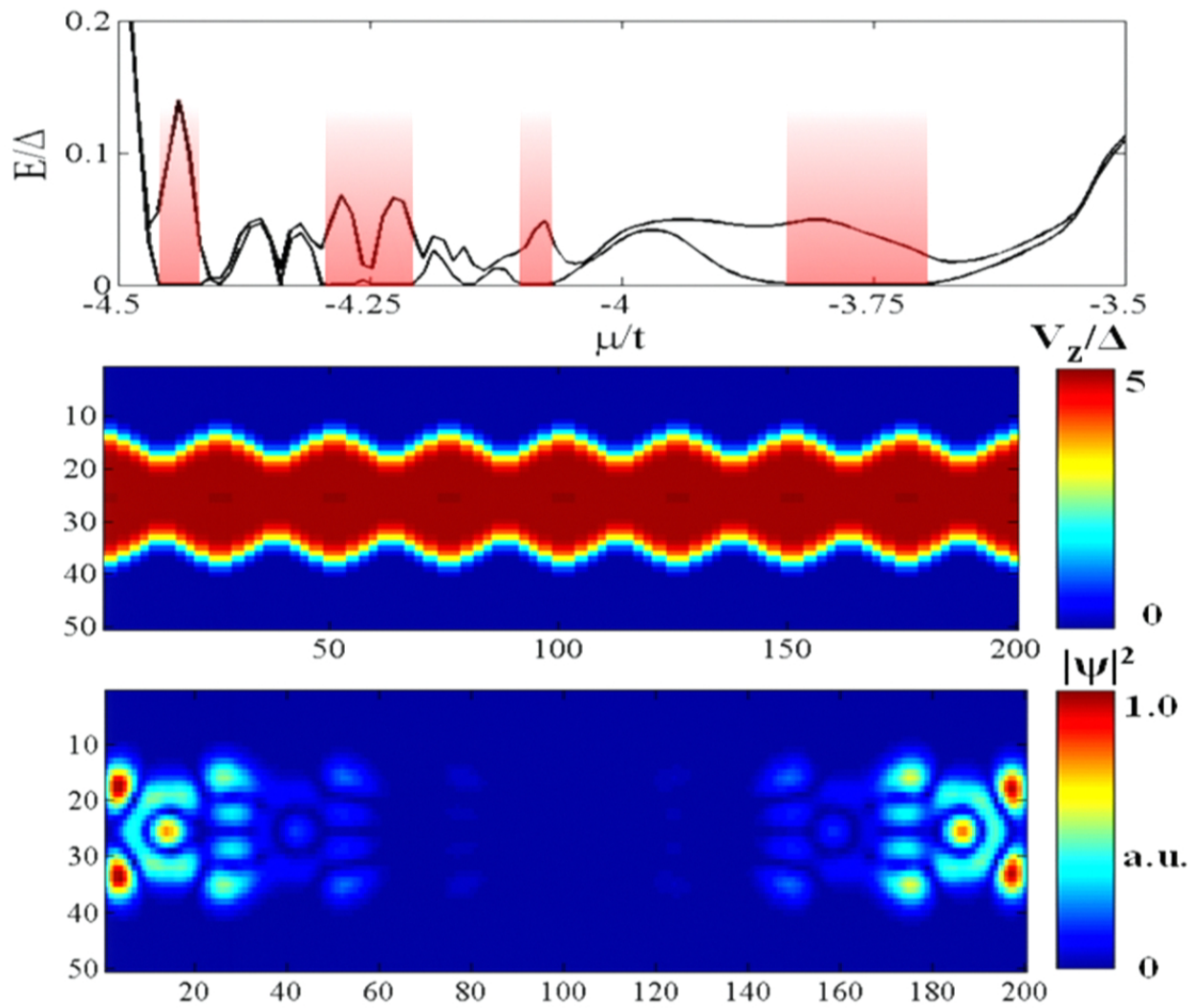
edge states admix with gap e^{-W/ξ_0}
and trap Majorana states at the ends.

As long as W is not much larger than the superconducting coherence length, the Majorana end states are robust.

Under these conditions, the energy interval in μ when the Majorana fermion is stabilized is of the order of the superconducting gap.

The “mini-gap” to fermions localized near the ends is given by Δ/\sqrt{N}





**Transverse quantization
not strictly needed.**

FM Domain with sinusoidal boundary –
 (top) Excitation spectrum vs. μ ,
 (middle) V_z profile, (bottom) Majorana
 end-state wave-function for $\mu = -4.25t$.
 Sample Geometry - $L=200$, avg width:
 20, sine-amplitude: 2 (i.e. top and
 bottom edge locations each vary by ± 2
 \Rightarrow total width variation is 20%)
 Hamiltonian Parameters – $t = 10$, $\Delta = 1$,
 $\alpha_R = 5$, $V_z = 5$

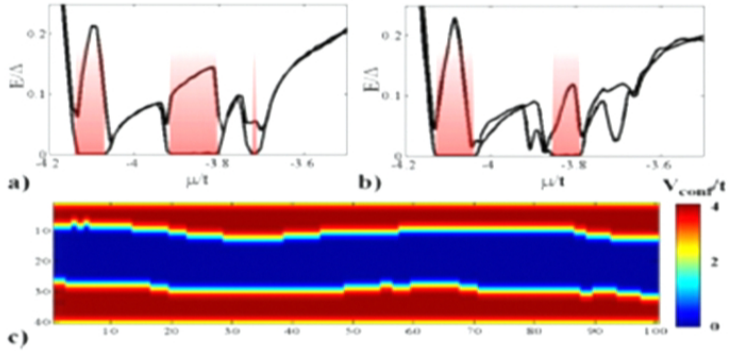


FIG. 8. (Color online) (a) and (b) show the two lowest energy in-gap excitations for an electrostatically confined strip with $L_x = 100$, $L_y = 20$, $t = 10$, $\Delta = 1$, $V_z = 2$, $\alpha_R = 2$. Red shading indicates the presence of isolated Majorana end-modes at zero-energy. The results in (a) are for straight edges ($\sigma_W = 0$), and those in (b) are for a random sample with $\sigma_W = 4$ and $\ell = 15$; (c) shows a colormap of the random edge geometry used to generate (b). Importantly the Majorana edge states survive, retaining a substantial excitation gap even for large edge variation (in this case $\sim 40\%$ of the average width L_y) and demonstrating that these states do not rely on the existence of transverse sub-bands.

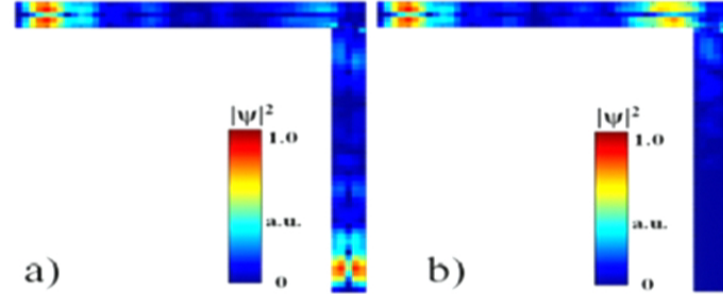
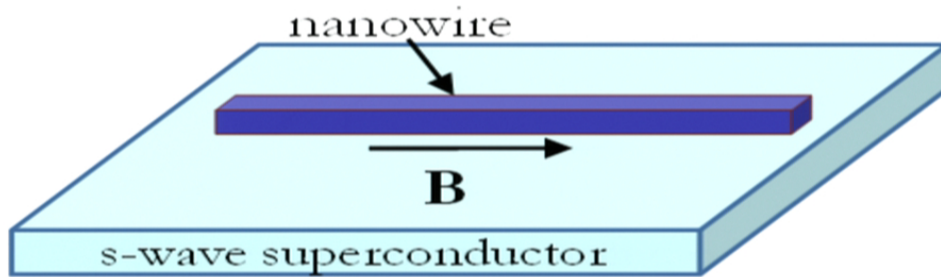


FIG. 9. (Color online) Spatial profile of the Majorana wave-function intensity, $|\Psi(x, y)|^2$, for L-shaped junction, for $t = 10$, $\Delta = 1$, $\alpha_R = 5$, and $V_z = 2$. The dimensions of each leg of the junction are 5×50 lattice sites. The left (a) shows the wave-function with three occupied sub-bands in each leg ($\mu = -3t$). In this case the Majorana states exist at the extreme ends of the the L. The right (b) shows the wave-function with three sub-bands occupied in the horizontal leg ($\mu = -3t$), and two sub-bands occupied in the vertical leg ($\mu = -3.4t$). In this case the second Majorana mode appears at the junction between topological and non-topological regions at the elbow of the L-junction.

Semiconductor quantum well or Nanowires with Rashba spin orbit coupling.

(Das Sarma group and Oreg et al)



- Advantage
 - Clean

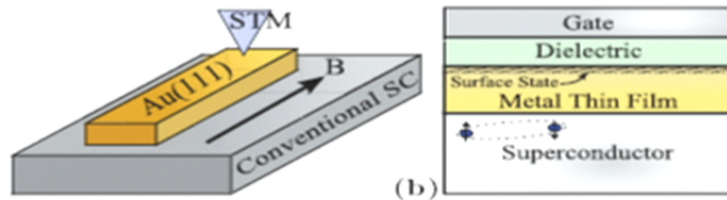
Disadvantages:

- Small Rashba energy: 0.5K
- Not-scalable

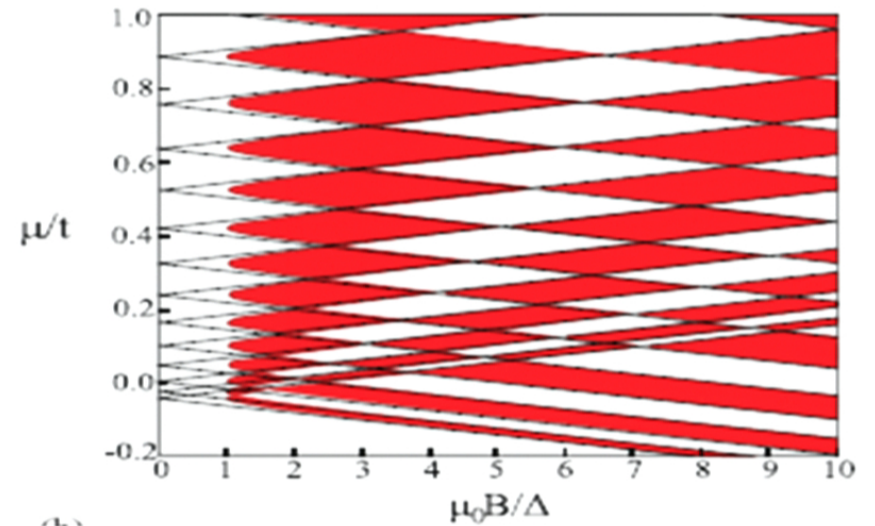
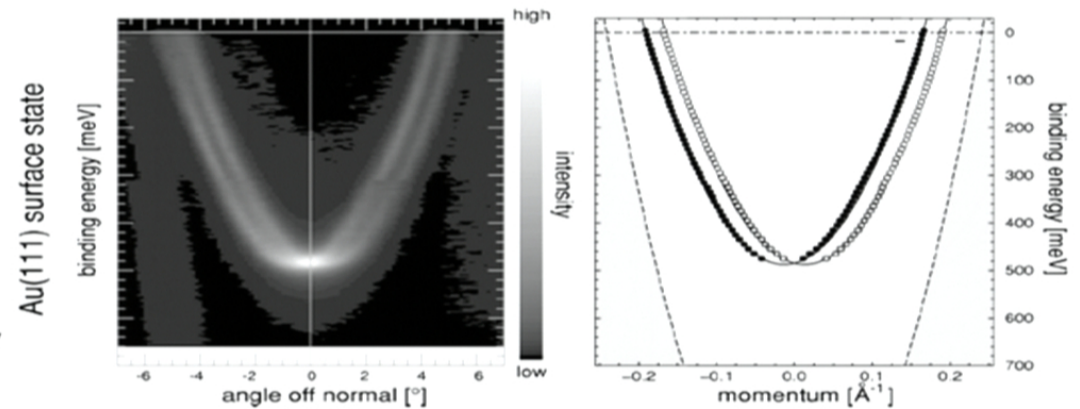
Instead of using semi-conductor quantum wells and nano-wires, our suggestion is to use the surface state on heavy metals. Rashba energy in the Au 111 surface state is about 10 meV, 200 times larger than semiconductor wire.

If we use narrow thin film, we can apply magnetic field parallel to the film. We can also operate near with many occupied bands.

Then we may be able to use the Au(111) surface state.

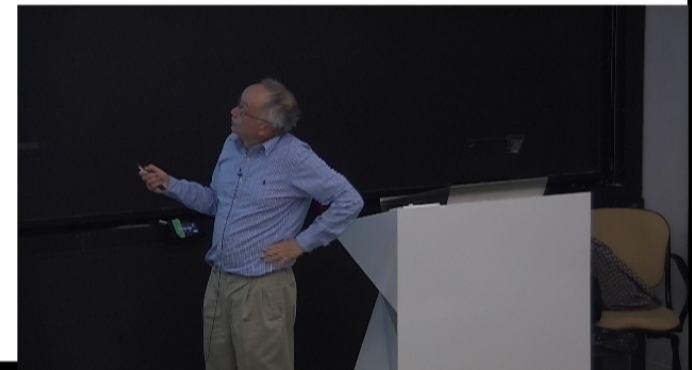
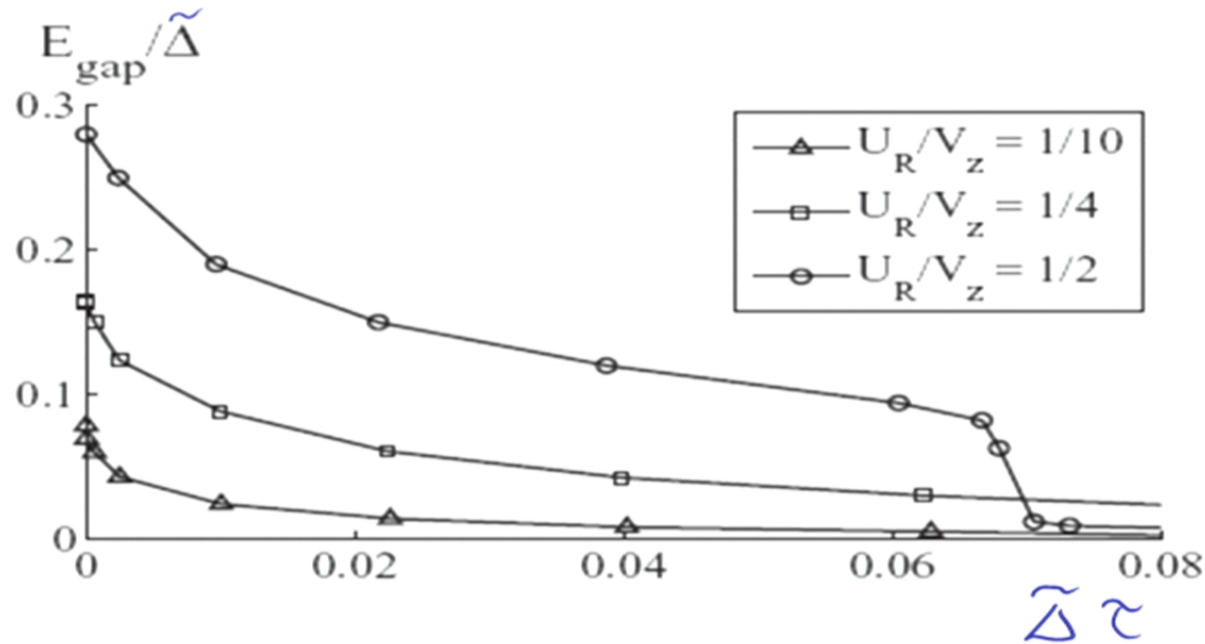


Potter and Lee, PRB 2012.



Due to time reversal symmetry, Fu-Kane scheme using TI is not sensitive to disorder. (generalized Anderson Theorem.)

However, for the Rashba scheme, disorder is pairing breaking, and strongly so if Rashba coupling is small compared with Zeeman coupling. This may be detrimental to the nano-wire scheme and surface state scheme may have an advantage.



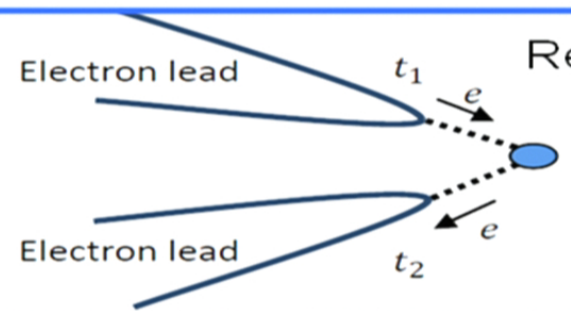
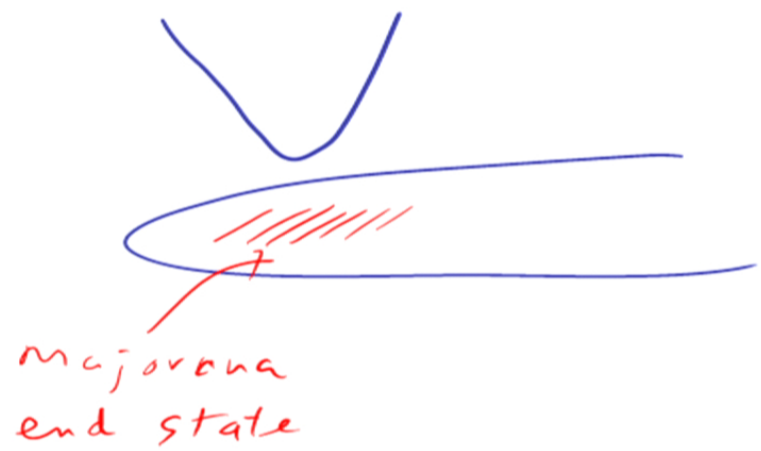
How to detect Majorana?

$$T(E) = \gamma_1^2 / [(E - E_l)^2 + \gamma_1^2],$$

$T \rightarrow 1$ at resonance, ie perfect Andreev reflection, ie perfect transmission of a Cooper pair, independent of tunneling strength t .

$$G = \frac{2e^2}{h}$$

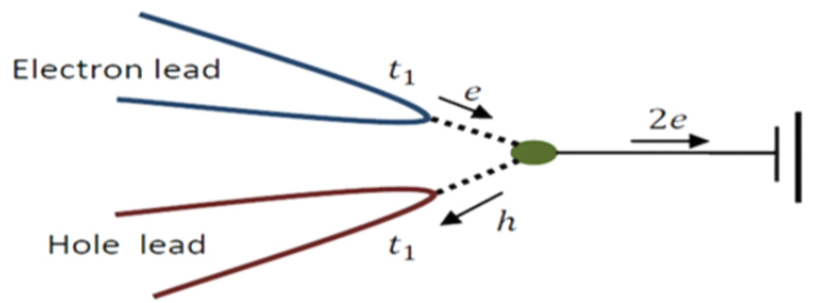
Law, Lee and Ng, PRL 2009.
Sengupta et al PRB 2001



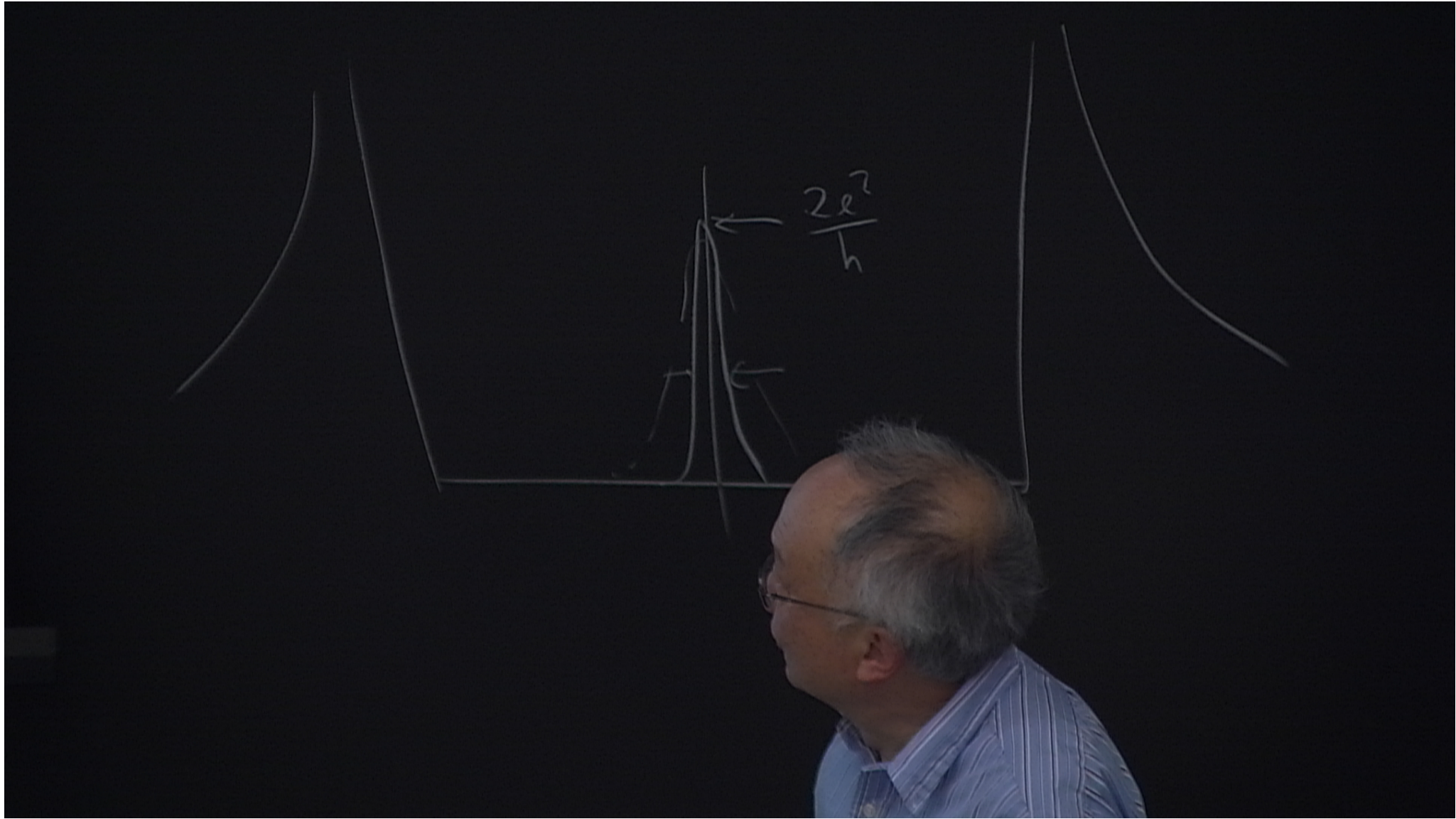
Resonant tunneling into a quantum dot.

$$T(E) = \frac{(2t_1t_2)^2}{(E - E_0)^2 + (t_1^2 + t_2^2)^2}$$

$$H_T = t_1\psi_D^+\psi_1 + t_2\psi_D^+\psi_2 + h.c.$$



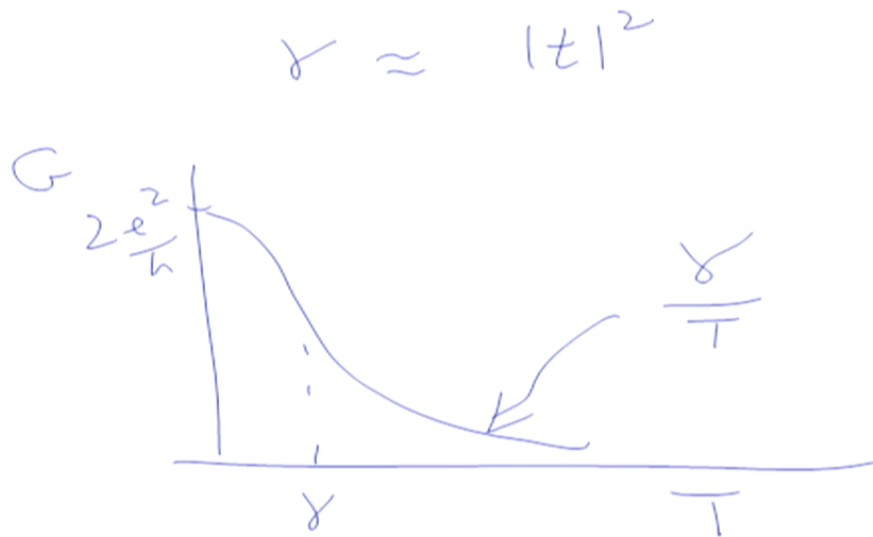
$$H_T = -it_1\eta(a)[\psi^+(0) + \psi(0)]$$



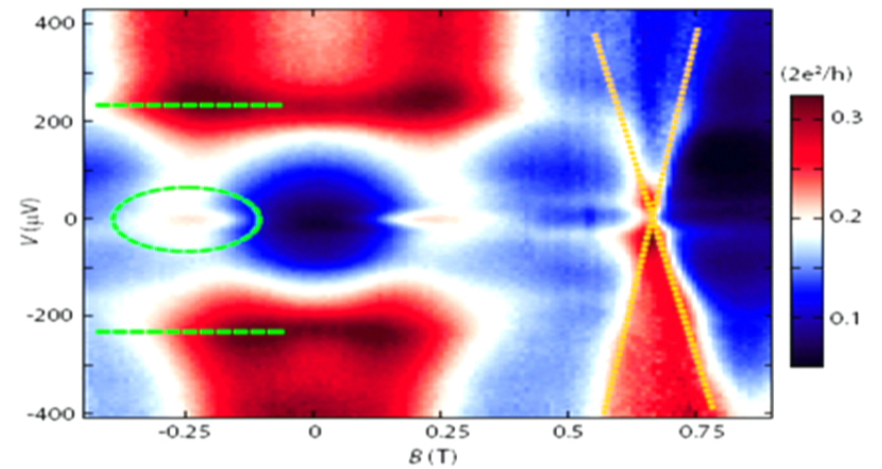
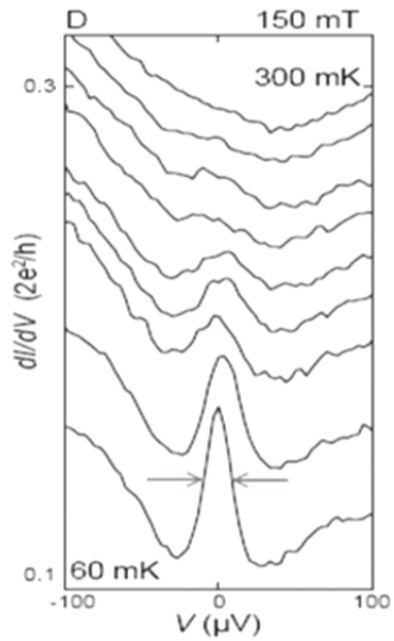
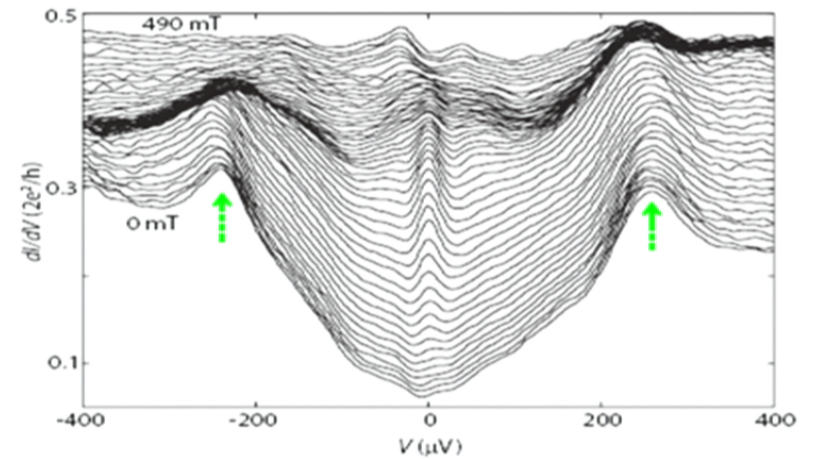
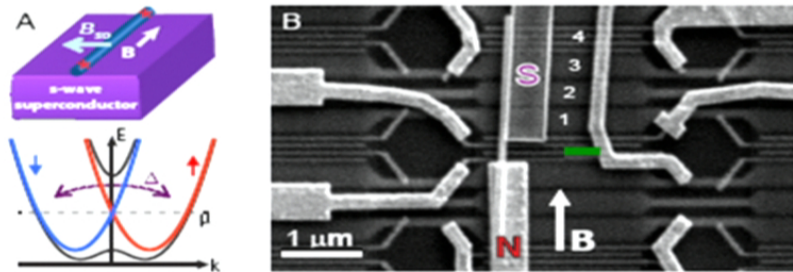
The result is quite general: tunneling via a Majorana state at energy zero gives a conductance peak at $E=0$ with height $2e^2/h$.

Finite temperature.

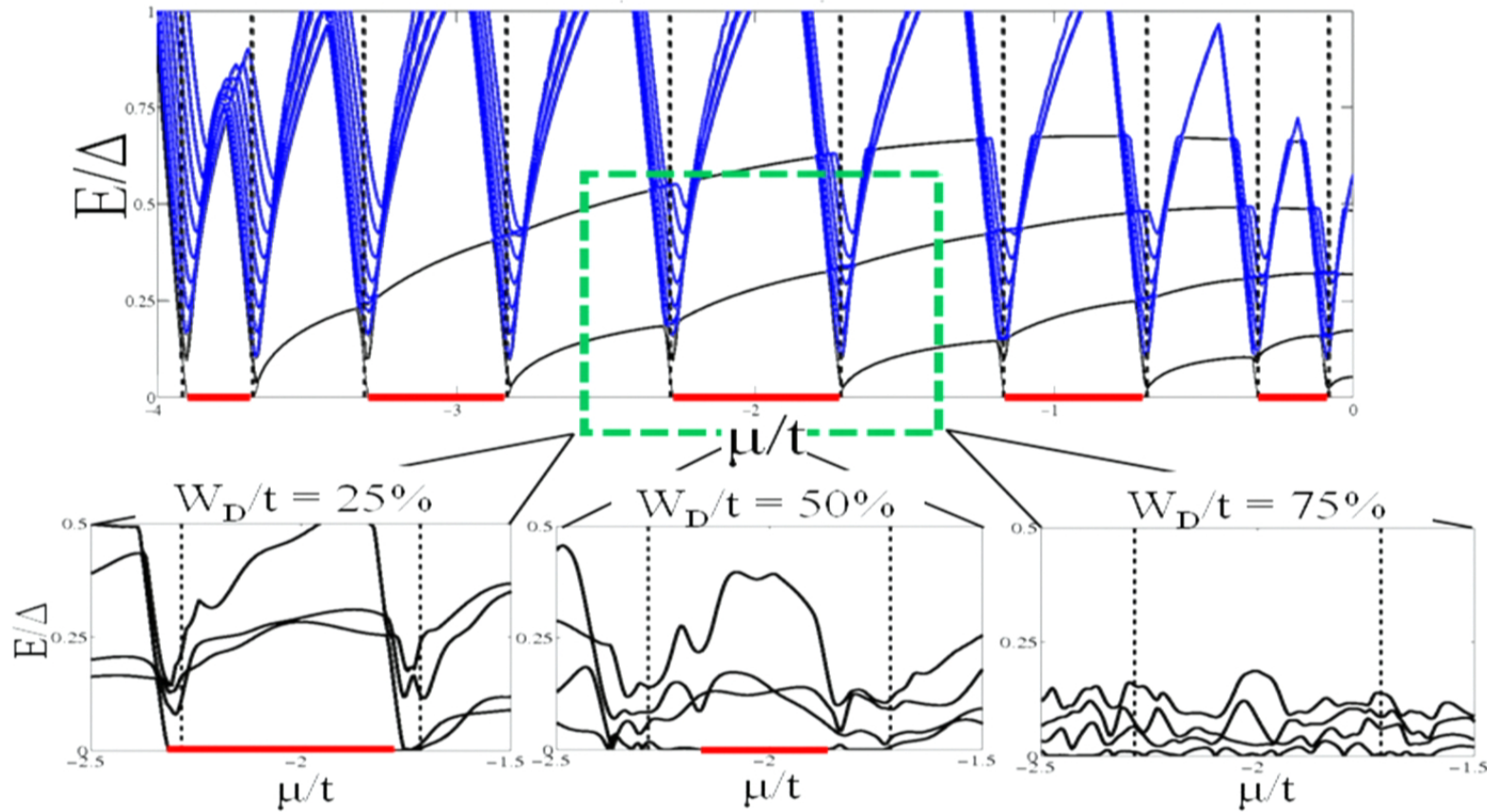
1. T needs to be less than excitation to mini-gap. (fraction of bulk gap.)
2. If T larger than level width γ , conductance $\sim \gamma/T$.



Recent experiment: Mourik et al (Delft group), science 2012.



1. Finite background tunneling: no gap in the bulk? Large (.25meV gap is likely from filled sub-band. Expt did not see gap closing.
2. Can disorder induce low lying states which can be mistaken for Majorana bound states?



Detection of fermion parity:
(Kitaev, Fu-Kane, Alicea et al)

4π period Josephson current.

Sign of this current distinguishes between occupied or empty fermion states.

K. T. Law and PAL, PRB,

1. This effect is robust to coupling with localized modes in the junction area.
2. Size of current is suppressed by thermal excitations to these localized modes. (mini-gap.)

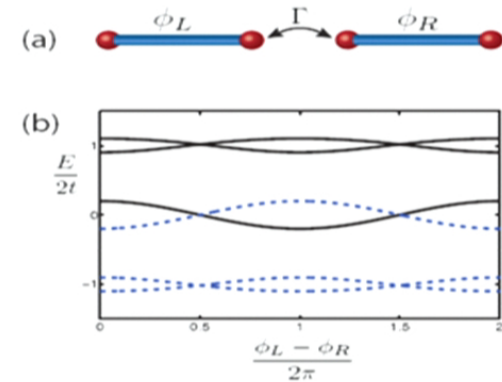
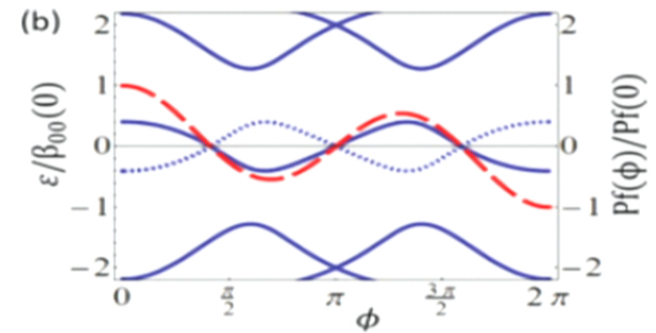


FIG. 9: (a) Schematic of the Josephson junction formed by two topological wires with p -wave superconducting phases $\phi_{L/R}$. The wires couple at the junction through an electron tunneling term with strength Γ . (b) Bogoliubov-de Gennes spectrum as a function of $(\phi_L - \phi_R)/(2\pi)$ for the effective Hamiltonian in Eq. (65) chosen to describe the junction. Only the solid lines denote physically distinct states. The states centered around $E/(2t) = 1$ represent ordinary bulk quasiparticles, while the state near zero energy represents the quasiparticle formed when the two end Majoranas at the junction fuse. The energy and hence Josephson current corresponding to the latter exhibit 4π periodicity in $\phi_L - \phi_R$. The ordinary bulk quasiparticle states, however, contribute only to the usual 2π -periodic Josephson effect.



Conclusion :

Majorana end states are surprisingly robust and may be more advantageous to implement than vortex based schemes.

The requirement is that the sample width is not larger than the SC coherence length.

The Fu-Kane scheme using TI + SC has advantage with respect to disorder.

The use of surface state may also be promising.

Perhaps the sighting of Majorana bound state is not too far in the future!